Chiral fermions and a multigrid algorithm

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Lattice regularization of chiral fermions is an important development of the theory of elementary particles. Nonetheless, brute force computer simulations are very expensive, if not prohibitive. In this Brief Report I exploit the noninteracting character of the lattice theory in flavor space and propose a multigrid approach for the simulation of the theory. Already a two-grid algorithm saves an order of magnitude of computer time for fermion propagator calculations.

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After many years of research in lattice QCD, it was possible to formulate QCD with chiral fermions on the lattice [1–4]. The basic idea is an expanded flavor space which may be seen as an extra dimension with left- and right-handed fermions defined in the two opposite boundaries or walls.

Let N be the size of the extra dimension, D_W the Wilson-Dirac operator, and m the bare fermion mass. Then, the theory with *domain wall fermions* is defined by the action [1,2]

$$S_{DW} := \bar{\Psi}\mathcal{M}\Psi = \sum_{i=1}^{N} \bar{\psi}_i[(D^{||}-1)\psi_i + P_+\psi_{i+1} + P_-\psi_{i-1}],$$
$$P_+(\psi_{N+1} + m\psi_1) = 0, \quad P_-(\psi_0 + m\psi_N) = 0, \tag{1}$$

where $\mathcal{M}$ is the five-dimensional fermion matrix of the regularized theory and $D^{||} = M - D_W$ with $M \in (0,2)$ being a mass parameter.

I define also a theory with *truncated overlap fermions* in complete analogy with the domain wall fermions by substituting [5]

$$P_+\psi_{i+1} \to (D^{||}+1)P_+\psi_{i+1},$$
$$P_-\psi_{i-1} \to (D^{||}+1)P_-\psi_{i-1} \tag{2}$$

while the boundary conditions remain the same as before.

Both theories can be compactified in the walls of the extra fifth dimension as low energy effective theories (see below) with the chiral Dirac operator D satisfying the Ginsparg-Wilson relation [6]

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = a\,\gamma_5 R, \tag{3}$$

where a is the lattice spacing and R is a local operator trivial in the Dirac space (see [5] for R-locality tests).

I defined truncated overlap fermions such that in the large N limit one obtains overlap fermions [3] with the Dirac operator given by [7]:

$$D_{OV} = \frac{1+m}{2} - \frac{1-m}{2}\,\gamma_5 \mathrm{sgn}(H) \tag{4}$$

where $H = \gamma_5 D^{||}$.

Until now computations with chiral fermions and standard algorithms have been very expensive. The extra fermion flavors introduce a large overhead. One multiplication with the fermion matrix costs $\mathcal{O}(n)$ D_W multiplications with $n \sim N$ for domain wall fermions and much larger for the overlap operator [8–10].

In this paper I propose a multigrid algorithm along the fifth dimension which makes these simulations much faster. The key observation is the lack of gauge connections along this dimension. It is well-known that the overhead of such algorithms scales like $N \log N$.

Here it is the *multigrid* algorithm: *ALGORITHM1 (Generic) for solving the system* $D_{OV}x = b$*:*

Given $N, x_0, r_0(=b), tol, tol_1$, set $tol_0 = 1$ and iterate:

$for \;\; i = 1, \ldots$

$tol_0 = tol_0 tol_1$

Solve $\boldsymbol{Dy = r_{i-1}}$ **within** $\boldsymbol{tol_0}$

$x_i = x_{i-1} + y$

$\boldsymbol{r_i = b - D_{OV} x_i}$

if $||r_i||_2 < tol$, $end \;\; for$ (5)

where o is denoted as a vector with zero entries and tol_1, tol are tolerances. tol_1 is typically orders of magnitude larger than tol such that the work per D_{OV} inversion is minimized.

Remark 1. Boldface equations represent the coarse grid solution and the fine grid correction of the right-hand side. The straightforward application of the *ALGORITHM1* gives a two-grid algorithm. By calling it again in solving the smaller system and iterating, one obtains a full multigrid algorithm.

Remark 2. The corresponding hybrid Monte Carlo (HMC) algorithm can be obtained by working with an approximate Hamiltonian in the coarse lattice and by a global correction on the fine lattice.

In Fig. 1, I compare the norm of the residual $r_i = b - D_{OV}x_i$ of the conjugate residual (CR) algorithm (which is optimal since D_{OV} is normal [11]) and *ALGORITHM1*. I gain about an order of magnitude (in average) on $30\ 4^4$ configurations at $\beta = 6.0$ and $m = 0.1$. For the coarse lattice I

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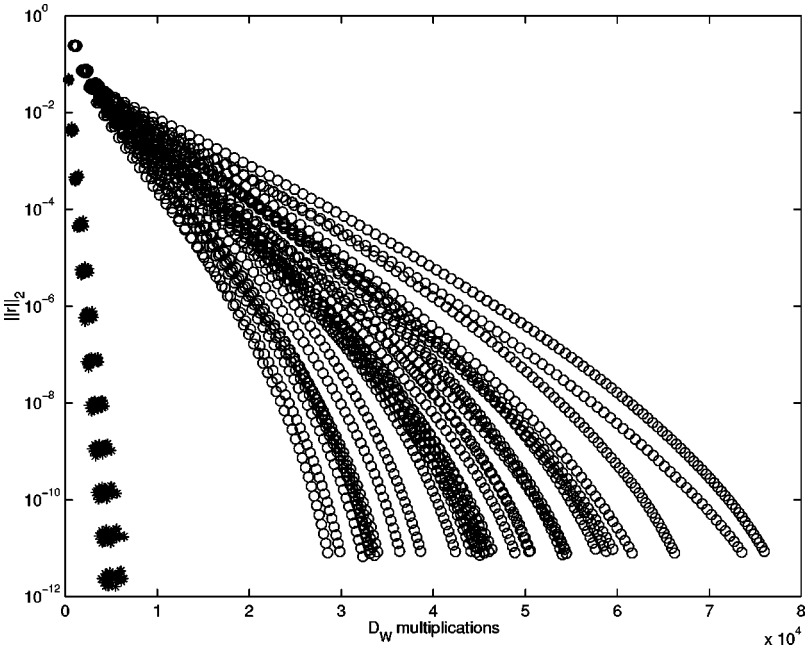

FIG. 1. Norm of the residual error vs the number of D_W multiplications on 30 configurations. Circles stand for the straightforward inversion with CR and stars for the *ALGORITHM1*.

used $N=6$ with the truncated overlap fermions and the Lanczos method to compute D_{OV} [8].

More on the compactification: I recall the action of the truncated overlap fermions

$$S_{TOV}:=\overline{\psi}_1[(D^{||}-1)\psi_1+(D^{||}+1)P_+\psi_2-m(D^{||}+1)P_-\psi_N]+\sum_{i=2}^{N-1}\overline{\psi}_i[(D^{||}-1)\psi_i+(D^{||}+1)P_+\psi_{i+1}+(D^{||}+1)P_-\psi_{i-1}]$$
$$+\overline{\psi}_N[(D^{||}-1)\psi_N-m(D^{||}+1)P_+\psi_1+(D^{||}+1)P_-\psi_{N-1}]. \tag{6}$$

Let P^T be the matrix representing the unitary transformation:

$$\chi_1=P_+\psi_1+P_-\psi_N,$$
$$\chi_i=P_+\psi_i+P_-\psi_{i-1},\quad i=2,\ldots,N, \tag{7}$$

and S the matrix representing the diagonal transformation: $\overline{\chi}_i=\overline{\psi}_i\gamma_5(H-1)$, $i=1,\ldots,N$. Let also the transfer matrix along the fifth dimension be defined by $T=(1+H)/(1-H)$.

In the new basis I obtain the following action:

$$S_{TOV}=\overline{\chi}_1[(P_+-mP_-)\chi_1-T\chi_2]+\sum_{i=1}^{N-1}\overline{\chi}_i(\chi_i-T\chi_{i+1})$$
$$+\overline{\chi}_N[\chi_N-T(P_--mP_+)\chi_1]. \tag{8}$$

Integrating over the Grassmann fields I obtain

$$\det\mathcal{M}=\det[(P_+-mP_-)-T^N(P_--mP_+)], \tag{9}$$

where I ignore the Jacobian factor coming from the diagonal transformation.

If $\mathcal{M}_1$ is the same matrix as $\mathcal{M}$ but with the special choice $m=1$, I define the effective low energy theory with the Dirac operator given by the equation:

$$D^{-1}=(P^T\mathcal{M}^{-1}\mathcal{M}_1P)_{1,1} \tag{10}$$

where the subscript 1,1 stands for the (1,1) block of an $N\times N$ partitioned matrix along the fifth dimension.

In terms of the transfer matrix the Dirac operator can be written as

$$D=\frac{1+m}{2}+\frac{1-m}{2}\gamma_5\frac{1-T^N}{1+T^N}. \tag{11}$$

I can repeat this derivation for the domain wall fermions with the obvious changes in the action and the transfer matrix, the rest of the formulas remaining the same.

Recently, the possibility of a multigrid algorithm along all dimensions is raised [12]. In this case a gauge fixing is needed.

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