Gauge invariant variational approach with fermions: The Schwinger model

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We extend the gauge invariant variational approach of I. Kogan and A. Kovner [Phys. Rev. D **51**, 1948 (1995)] to theories with fermions. As the simplest example we consider the massless Schwinger model in $1+1$ dimensions. We show that in this solvable model the simple variational calculation gives exact results.

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I. INTRODUCTION

This paper is an additional step in our exploration of the gauge invariant variational approach suggested in [1]. This approach was developed as an attempt to study analytically the nonperturbative infrared dynamics of non-Abelian gauge theory. The idea is that keeping gauge invariance exact at all stages of the calculation may be crucial for consistent understanding of the ground state structure of QCD. One therefore tries to account for the generation of nonperturbative QCD condensates by introducing them as variational parameters in the explicitly gauge invariant (Gaussian like) trial wave functional.

So far this variational approach has been applied to pure gluodynamics. There it seems to capture many of the essential features: mass generation, formation of the gluon condensate, [1], and asymptotic freedom [2–4]. Instanton transitions have been analyzed in [5] and are identified with the saddle points in the integration over the gauge group which projects the Gaussian wave functional onto the gauge invariant physical Hilbert space. Interestingly we found that in the best variational state instantons of large size are suppressed and the large size instanton problem that arises in the standard WKB calculation is completely avoided. The dynamically generated mass present in the best variational state stabilizes instantons at a size $\rho\sim 1/M$.

Although there remain many open questions regarding the reliability of these calculations, it is interesting to see how a simple ansatz of the same type would do in a theory with fermions. In particular one would like to see whether including fermions leads within the gauge invariant variational framework to the generation of fermionic condensates and chiral symmetry breaking.

As a first step towards this goal, in this paper we will study the simple toy model, $(1+1)$-dimensional Abelian theory with massless fermions, the so-called Schwinger model [6]. Our aim is not to further the understanding of the model itself (which has been solved exactly many times by different methods and is well understood) but to develop a workable generalization of our method to fermionic theory and test it in the simplest possible but nontrivial setting.

One of the most striking similarities between the Schwinger model and QCD is that in neither theory are the asymptotic states represented by the fields of the Lagrangian. The Schwinger model, with massless fermions, exhibits complete screening of charges and has one neutral asymptotic state: a ''meson'' of mass $e/\sqrt{\pi}$. The presence and interconnectedness of the axial anomaly, instantons and the massive asymptotic states of the theory have made the Schwinger model a rich field of study.

The outline of this paper is as follows. In Sec. II we generalize the gauge invariant Gaussian variational ansatz to include fermions and perform the variational calculation. We find that the best variational state in the fermionic sector does not contain new nontrivial structure. In Sec. III we cross-check our calculation using the bosonized form of the model. We find that our best trial state is in fact the exact ground state of the theory. In Sec. IV we conclude by calculating the fermionic condensate. We recover the correct value of the condensate and show thereby that the chiral anomaly is correctly represented in our calculation.

II. VARIATIONAL CALCULATION IN THE SCHWINGER MODEL

The Lagrangian density for the Schwinger model reads

$$\mathcal{L}=\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}\gamma^{\mu}(i\partial_{\mu}-eA_{\mu})\psi. \tag{1}$$

The theory is super-renormalizable and the coupling constant, e, has mass dimension $+1$. The 2×2 Dirac matrices obey the usual anti-commutation relation, $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}$, where spatial indices run from 0 to 1 and $g^{00}=1$. We write the Schwinger model in the Hamiltonian formalism in the temporal gauge, $A_0=0$. This leaves one degree of freedom for the photon field and all spatial indices are now suppressed: $A_1(x)=A(x)$ and $\partial_1=\partial$. The Hamiltonian is

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$$H=\int dx\left(\frac{1}{2}E^2(x)+\overline{\psi}(x)[i\partial-eA(x)]\gamma_1\psi(x)\right), \quad (2)$$

with the usual canonical relations,

$$[A(x),E(y)]=i\delta(x-y), \quad E(x)=\dot{A}(x),$$
$$[\psi(x),\pi_\psi(y)]=i\delta(x-y), \quad \pi_\psi(x)=i\psi^\dagger(x). \quad (3)$$

The gauge invariance of a physical state, in this case the vacuum wave functional, is ensured by requiring that it satisfy Gauss' law,

$$\mathcal{G}(x)\Psi[A,\psi]=[\partial E(x)-e\psi^\dagger(x)\psi(x)]\Psi[A,\psi]=0. \quad (4)$$

As an ansatz for the vacuum wave functional of the theory we take the product of the vacuum wave functionals in the gauge and fermion sectors, projected onto the gauge invariant subspace. Following [1] for the functional in the gauge sector we take a Gaussian with an arbitrary width $G(p)$ which is to be treated as a functional variational parameter.

Next we have to address the ansatz in the fermion sector. Since we may expect that the effects of chiral symmetry breaking show up in the fermion sector as dynamical generation of mass, we propose as an ansatz for the ground state a Dirac vacuum of free massive fermions, which we denote by $|m\rangle$. The mass, or rather the mass function $m(p)$, can depend on momentum and is to be treated as an additional variational function. In the absence of the gauge fields the fermionic operators that annihilate this state are related to the original fermionic operators by the well-known Bogolyubov transformation. The state itself is somewhat similar to the BCS state. The important difference is that the mixing is not between particles and antiparticles, but rather between left- and right-moving particles, so that the possible condensates in this state preserve the charge symmetry but break the chiral symmetry instead.

The product of the gauge and fermionic vacua is

$$\Psi[A,\psi]=\exp\left(-\frac{1}{2}\int dxdyA(x)G^{-1}(x-y)A(y)\right)|m\rangle. \quad (5)$$

The vacuum wave functional in the gauge theory must of course be gauge invariant. This is achieved by projecting Eq. (5) onto the gauge invariant Hilbert space. To do this note that Eq. (5) transforms under the gauge transformation generated by Gauss' law as

$$\Psi^\phi[A,\psi]=\exp\left(i\int dx\,\phi(x)\mathcal{G}(x)\right)\Psi[A,\psi]$$
$$=\exp\left(-\frac{1}{2}\int dx\,dy\,A^\phi(x)G^{-1}(x-y)A^\phi(y)\right)$$
$$\times\exp\left(-ie\int dx\,\phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle,$$

where

$$A^\phi(x)=A(x)-\partial\phi(x), \quad (6)$$

and ϕ is the gauge function. The gauge invariant ansatz for the vacuum wave functional is therefore written by integrating Ψ^ϕ over all possible gauge transformations,

$$\Psi[A,\psi]=\int D\phi\,\exp\left(-\frac{1}{2}\int dx\,dy\,A^\phi(x)G^{-1}(x-y)A^\phi(y)\right)$$
$$\times\exp\left(-ie\int dx\,\phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle. \quad (7)$$

In this formalism, one calculates expectation values of local operators with the ansatz for the ground state. Since we are interested in the calculation of physical quantities, the operators we shall consider are gauge invariant. The expectation of any such operator O can be written as

$$\langle\mathcal{O}(A,\overline{\psi},\psi)\rangle=\frac{1}{Z}\int D\phi DA\langle m|$$
$$\times\exp\left(-\frac{1}{2}\int dx\,dy\,A(x)G^{-1}(x-y)A(y)\right)\mathcal{O}(A,\overline{\psi},\psi)$$
$$\times\exp\left(-\frac{1}{2}\int dx\,dy\,A^\phi(x)G^{-1}(x-y)A^\phi(y)\right)$$
$$\times\exp\left(-ie\int dx\,\phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle. \quad (8)$$

First let us consider the norm of the state, $O=1$. We first integrate over A:

$$Z=Z_A\int D\phi\,\exp\left(-\frac{1}{4}\int dx\,dy\,\partial\phi(x)G^{-1}(x-y)\partial\phi(y)\right)$$
$$\times\langle m|\exp\left(-ie\int dx\,\phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle, \quad (9)$$

$$Z_A=\mathrm{Det}^{1/2}(2G^{-1}). \quad (10)$$

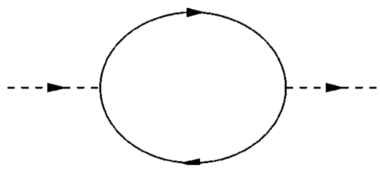

FIG. 1. Shows the quadratic term resulting from the expansion in small $e\phi$. Solid lines represent a fermion loop and the dashed lines represent the gauge parameter. Associated with each vertex is the factor $-ie\gamma^0\delta(t)$.

The integration over the fermions is a little less straightforward. Remembering that $|m\rangle$ is the vacuum of a free fermionic Hamiltonian with mass m one can use the standard trick to express the fermionic matrix element as a path integral. The gauge transformation operator in such a representation then has to be understood as an insertion at fixed ''time.'' This insertion generates an interaction between the fermions and the gauge parameter,

$$\langle m|\exp\left(-ie\int dx\ \phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle = \int D\bar{\psi}D\psi \exp\Big[iS_f - ie\int dt\,dx\ \delta(t)\phi(x)\bar{\psi}(x)\gamma_0\psi(x)\Big], \quad (11)$$

where S_f is the action for free fermions with momentum dependent mass,

$$S_f=\int dt\,dx\,dy\ \bar{\psi}(x,t)[i\delta(x-y)\gamma^\mu\partial_\mu - m(x-y)]\psi(y,t). \quad (12)$$

In the subsequent calculation we will treat $e\phi$ as small and expand to second order the logarithm in the exponential of the norm produced by integrating over ψ and $\bar{\psi}$. This results in a series of diagrams. The leading order is independent of ϕ and contributes a constant factor to the norm which will cancel. Represented by Fig. 1, the term quadratic in ϕ contains a fermion loop and is written

$$-\frac{e^2}{2}\int dx\,dy\ \phi(x)\phi(y)\mathrm{Tr}[\gamma^0 D_+(x-y)\gamma^0 D_-(y-x)], \quad (13)$$

where the equal time propagators are

$$D_\pm(k)=-\frac{1}{2}\left[\gamma^0 \pm \frac{k\gamma^1+m(k)}{E(k)}\right], \quad (14)$$

and $E^2(k)=k^2+m^2(k)$. We will see later that keeping only second order terms in $e\phi$ is sufficient to locate the best variational state. The contribution to the norm from fermion integration is

$$\langle m|\exp\left(-ie\int dx\ \phi(x)\psi^\dagger(x)\psi(x)\right)|m\rangle = \mathrm{const}\times\exp\left(-\frac{e^2}{4\pi}\int dx\,dy\ \partial\phi(x)X(x-y)\partial\phi(y) + O((e\phi)^3)\right), \quad (15)$$

where

$$X(k)=k^{-2}\int\frac{dp}{2\pi}\left(1-\frac{p(p-k)+m(p)m(p-k)}{E(p)E(p-k)}\right). \quad (16)$$

The norm then can be written as $Z=Z_A Z_\psi Z_\phi$, where Z_A and Z_ψ are constants which cancel in all calculations and

$$Z_\phi=\int D\phi\exp\left(-\frac{1}{4}\int dx\,dy\ \partial\phi(x)S(x-y)\partial\phi(y)\right). \quad (17)$$

The ''effective action'' for ϕ is given by

$$S(k)=G^{-1}(k)+\frac{e^2}{\pi}X(k). \quad (18)$$

We now compute the expectation value of the Hamiltonian in the variational state. Minimization of this quantity with respect to the variational parameters will yield the ground state of the theory. In doing so one should be careful with properly regularizing the ultraviolet singular operators that enter the Hamiltonian,

$$\mathcal{H}(x)=\frac{1}{2}E^2(x)+i\bar{\psi}(x)\gamma_1[\partial+ieA(x)]\psi(x). \quad (19)$$

The term to keep an eye on is the interaction term. In particular the current $j(x)=\bar{\psi}(x)\gamma\psi(x)$ has to be regularized using the point-splitting technique [7]. This point-splitting generates an additional term in the Hamiltonian which is quadratic in A [7,8],

$$i\bar{\psi}D_\mu\gamma^\mu\psi=-\frac{\Lambda^2}{2\pi}+i{:}\bar{\psi}\partial_\mu\gamma^\mu\psi{:}-e\,A{:}\bar{\psi}\gamma^1\psi{:}-\frac{e^2}{2\pi}A^2 + \frac{ie}{2\pi}\partial A, \quad (20)$$

where Λ is the UV cutoff. The first term is a constant and it does not affect our calculation. The last term is a total derivative and for this reason will also be dropped. We are therefore concerned with the calculation of the expectation of the regularized Hamiltonian,

$$H=\int dx\left(\frac{1}{2}E^2(x)+{:}i\bar{\psi}(x)\gamma_1\partial\psi(x){:} - eA(x){:}\bar{\psi}(x)\gamma_1\psi(x){:}+\frac{e^2}{2\pi}A^2(x)\right). \quad (21)$$

We proceed to calculate the expectation of Eq. (21) in the manner outlined above for the calculation of the norm. We find

$$\left\langle \frac{1}{2}\int dx E^2(x)\right\rangle = \frac{1}{4}\int \frac{dk}{2\pi}[G^{-1}(k)-G^{-2}(k)S^{-1}(k)], \tag{22}$$

$$\left\langle i\int dx : \bar{\psi}(x)\gamma_1\partial\psi(x): \right\rangle = \lim_{\Lambda\to\infty}\left(\int_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\frac{k^2}{\sqrt{k^2+m^2(k)}}-\frac{\Lambda^2}{2\pi}\right), \tag{23}$$

$$\left\langle -e\int dx A(x): \bar{\psi}(x)\gamma_1\psi(x): \right\rangle = -\frac{e^2}{2\pi}\int \frac{dk}{2\pi}S^{-1}(k), \tag{24}$$

$$\left\langle \frac{e^2}{2\pi}\int dx A^2(x)\right\rangle = \frac{e^2}{4\pi}\int \frac{dk}{2\pi}[G(k)+S^{-1}(k)]. \tag{25}$$

Functional variation of the expectation of the Hamiltonian with respect to the parameters $m(p)$ and $G^{-1}(p)$ yields two minimization equations,

$$\frac{\delta\langle H\rangle}{\delta m(p)} = \int \frac{dk}{8\pi}\left(G^{-2}(k)+\frac{e^2}{\pi}\right)S^{-2}(k)\frac{\delta X(k)}{\delta m(p)} - \frac{p^2 m(p)}{[p^2+m^2(p)]^{3/2}} = 0, \tag{26}$$

$$\frac{\delta\langle H\rangle}{\delta G^{-1}(p)} = \frac{e^4}{4\pi^2}S^{-2}(p)\left[\left(1-\frac{e^2}{\pi}G^2(p)\right)X^2(p) - 2X(p)G(p)\right] = 0. \tag{27}$$

Since $\delta X/\delta m \propto m$, the first equation is solved for $m(p)=0$. The second minimization equation has the solution

$$\left(\frac{e^2}{\pi}G(p)+X^{-1}(p)\right)^2 = \frac{e^2}{\pi}+X^{-2}(p), \tag{28}$$

where $X(p)$ is defined in Eq. (16). With $m(p)=0$ we have $X(p)=1/|p|$, and we can write the solution as

$$\left(\frac{e^2}{\pi}G(p)+|p|\right)^2 = \frac{e^2}{\pi}+p^2. \tag{29}$$

Summarizing, the result of our variational calculation is the vacuum wave functional (5) with the parameters

$$m(p)=0,\quad G(p)=\frac{\pi}{e^2}\left(\sqrt{\frac{e^2}{\pi}+p^2}-|p|\right). \tag{30}$$

The calculations in this section rely upon the perturbative expansion of the "effective action" for ϕ in Eq. (15). To improve on this we would have to resum this expansion. However, for $m(q)=0$ such a re-summation is trivial. It can be shown explicitly that all diagrams higher than second order vanish and therefore our result is exact at least in the sense that the solution we found is at least a local minimum. We will see in the next section that the result is indeed exact.

III. BOSONIZED FORMULATION OF THE SCHWINGER MODEL

As is well known the Schwinger model is exactly solvable by the bosonization technique [9]. In this section we will discuss the variational calculation in this bosonized form. We shall show that the solution obtained by the variational method in the bosonized form is exact. Further, we shall show that this solution is equivalent to the wave functional obtained in the previous section constructed in the fermionic Hilbert space.

The bosonized form of the Hamiltonian density in the temporal gauge ($A_0=0$) reads

$$\mathcal{H}(x) = \frac{1}{2}E^2(x)+\frac{1}{2}\left(p(x)-\frac{e}{\sqrt{\pi}}A(x)\right)^2+\frac{1}{2}[\partial\chi(x)]^2, \tag{31}$$

which is supplemented by a Gauss' law constraint on physical states,

$$\mathcal{G}_B(x) = E(x)-\frac{e}{\sqrt{\pi}}\chi(x)=0, \tag{32}$$

where $E(x)$ and $A(x)=A_1(x)$ are the electric and photon fields as defined in the previous section, $\chi(x)$ is the boson field, and $p(x)$ its canonical momentum. Since the Hamiltonian is quadratic in terms of bosonic fields and their conjugate momenta, the exact ground state wave functional can be readily found,

$$\Psi_0[A,\chi] = \exp\left[\frac{ie}{\sqrt{\pi}}\int dx\chi(x)A(x)-\frac{1}{2}\int dxdy\chi(x)\Sigma^{-1}(x-y)\chi(y)\right], \tag{33}$$

with $\Sigma^{-1}(k)=\sqrt{k^2+e^2/\pi}$ and the suffix 0 indicates the exact ground state.

We now want to show that this result is recovered within the variational calculation. We base our variational ansatz upon a Gaussian,

$$\exp\left[-\frac{1}{2}\int dxdyA(x)G^{-1}(x-y)A(y)\right.$$
$$\left.-\frac{1}{2}\int dxdy\chi(x)\tilde{\Sigma}^{-1}(x-y)\chi(y)\right], \tag{34}$$

with $G(x-y)$ and $\tilde{\Sigma}$ as variational parameters. Projecting this onto the gauge invariant subspace with the help of the Gauss' law, Eq. (32), we have the gauge invariant trial state,

$$\Psi_B[A,\chi]=\int D\phi\exp\left[-\frac{1}{2}\int dxdyA^{\phi}(x)G^{-1}(x-y)A^{\phi}(y)\right.$$
$$-\frac{1}{2}\int dx\,dy\,\chi(x)\tilde{\Sigma}^{-1}(x-y)\chi(y)$$
$$\left.+\frac{ie}{\sqrt{\pi}}\int dx\partial\phi(x)\chi(x)\right]. \tag{35}$$

Proceeding with the variational calculation in the usual way, we find the expectation value of the Hamiltonian to be

$$\langle H\rangle=\frac{1}{4}\int\frac{dk}{2\pi}\left(G^{-1}(k)+\frac{e^2}{\pi}\tilde{\Sigma}(k)\right)^{-1}$$
$$\times\left[\frac{e^2}{\pi}G^{-1}(k)\tilde{\Sigma}(k)+[\tilde{\Sigma}(k)G(k)]^{-1}+\frac{2e^2}{\pi}\right.$$
$$\left.+k^2\tilde{\Sigma}(k)G^{-1}(k)+\frac{e^4}{\pi^2}G(k)\tilde{\Sigma}(k)\right]$$
$$=\frac{1}{4}\int\frac{dk}{2\pi}\left(Y^2(k)+k^2+\frac{e^2}{\pi}\right)Y^{-1}(k), \tag{36}$$

where

$$Y(k)=\tilde{\Sigma}^{-1}(k)+\frac{e^2}{\pi}G(k). \tag{37}$$

Variational minimization of Eq. (36) with respect to $Y(k)$ yields the required solution,

$$Y^2(k)=\left(\tilde{\Sigma}^{-1}(k)+\frac{e^2}{\pi}G(k)\right)^2=\left(k^2+\frac{e^2}{\pi}\right). \tag{38}$$

After substituting into Eq. (35) and integrating over ϕ, we find that the best variational state is indeed Eq. (33), so the variational calculation gives the exact result.

Note that our initial parametrization of the trial state turned out to be somewhat redundant. The energy minimization determines only Y but not G and $\tilde{\Sigma}$ separately. It is in fact true that integrating over ϕ in Eq. (35) we obtain a Gaussian in χ with width Y. This redundancy however can be used to establish the equivalence of this calculation with the calculation in terms of the fermionic representation of the previous section. The point is that we can choose $\tilde{\Sigma}$ to be the width of the ground state in the free massless bosonic theory,

$$\tilde{\Sigma}^{-1}=\sqrt{k^2}. \tag{39}$$

The exact ground state is then represented as a gauge projected product of the vacuum wave functional of a theory of a massless boson and a Gaussian functional of the gauge potential A. In fact for this choice of $\tilde{\Sigma}$, Eq. (38) gives the same result for the width of the gauge field Gaussian G as the fermionic solution, Eq. (30). Also the free massless fermion by bosonization is equivalent to the free massless boson. Therefore the fermionic part of the variational state of the previous section for $m=0$ is equivalent to the free massless state for the bosonic field χ. We therefore see that our variational solution of the previous section is equivalent to the solution obtained here, and is therefore an exact ground state.

IV. CONCLUSIONS

In this paper we have presented a variational analysis of the massless Schwinger model. We have shown that there exists a simple and natural way to extend the gauge invariant variational ansatz of [1] to systems with fermions. We have also found that in this simple and well-understood system our variational ansatz in fact includes the exact ground state and therefore reproduces known exact results.

A noteworthy feature of the best variational state is that the dynamical mass in the fermionic sector is not generated, so that in terms of fermionic part the state seems to be trivial. In fact in 2 dimensional theory this is not entirely surprising. If we were to consider a theory with more than one flavor, this would have been an immediate corollary of the Coleman theorem. Since a continuous (chiral) symmetry cannot be broken in 2D, the state with nonvanishing dynamical mass should be strongly energetically disfavored. In the one flavor theory which we are considering here, such an *a priori* argument does not apply since the axial $U(1)$ symmetry is anomalous. Nevertheless, it is not unnatural that in terms of the dynamical mass the ground states in one flavor and multiflavor theories are similar.

On the other hand, in the one flavor case the fermionic bilinear condensate $\bar{\psi}\psi$ should be nonvanishing precisely due to the same anomaly. It is therefore interesting to see whether our best trial state leads to such a nonvanishing condensate.

It is simplest to calculate the condensate first in the bosonized version of the theory. The bosonization rules prescribe the identification [9]

$$\bar{\psi}(x)\psi(x)=-\frac{ce}{\sqrt{\pi}}\mathcal{N}_{e/\sqrt{\pi}}\cos[\sqrt{4\pi}\chi(x)], \tag{40}$$

where $\mathcal{N}_{e/\sqrt{\pi}}$ means normal ordering with respect to the free boson field of mass $e/\sqrt{\pi}$, and the prefactor is $c=\exp(\gamma)/2\pi$, with γ being Euler's constant [10]. We need the average of Eq. (40) over the wave functional (35). After integrating over A and χ we obtain

$$\langle\overline{\psi}(x)\psi(x)\rangle = -\frac{ce}{\sqrt{\pi}}\mathcal{N}_{e/\sqrt{\pi}}\int D\phi \times\exp\left[-\frac{1}{4}\int dydz\,\partial\phi(y) \times\left(G^{-1}(y-z)+\frac{e^2}{\pi}\tilde{\Sigma}(y-z)\right)\partial\phi(z)e \times\int dy\tilde{\Sigma}(x-y)\partial\phi(y)-\pi\tilde{\Sigma}(0)\right]. \quad (41)$$

Using the definition (37), a further integration over ϕ yields

$$\langle\overline{\psi}(x)\psi(x)\rangle = -\frac{ce}{\sqrt{\pi}}\mathcal{N}_{e/\sqrt{\pi}}\exp[-\pi Y^{-1}(x-x)]. \quad (42)$$

The exponential is UV divergent, but it cancels exactly with the normal ordering factor, so we finally obtain

$$\langle\overline{\psi}(x)\psi(x)\rangle = -\frac{e\exp(\gamma)}{2\pi^{3/2}}, \quad (43)$$

which is the exact result [11].

The same calculation can be performed in the fermionic formalism. We need to calculate

$$\langle\overline{\psi}(x)\psi(x)\rangle = \int D\phi D\overline{\psi}D\psi\overline{\psi}(x)\psi(x) \times\exp\left[-\frac{1}{4}\int dydz\,\partial\phi(y)G^{-1}(y-z)\partial\phi(z) +i\int dtdx\overline{\psi}(x,t)\gamma^{\mu}(i\partial_{\mu}-ea_{\mu})\psi(y,t)\right], \quad (44)$$

with $a_0(x,t)=\phi(x)\delta(t)$ and $a_1(x,t)=0$. Here we have used the path integral representation of the average over the massless fermion vacuum state, Eq. (11). To perform the integration over the fermions we can use the results of [12]. The fermionic integral is non-vanishing only for background fields with instanton number ± 1. Decomposing $a_{\mu}=\partial_{\mu}\theta-\epsilon_{\mu\nu}\partial_{\nu}\varphi$, we obtain

$$\langle\overline{\psi}(x)\psi(x)\rangle = \text{const}\times\int D\phi\exp\left[-\frac{1}{4}\int dydz\,\partial\phi(y) \times S(y-z)\partial\phi(z)+\frac{1}{2\pi}\int dy\log \times(x-y)\partial\phi(y)\right]. \quad (45)$$

Here we have used

$$\Box\varphi(x,t) = -\partial\phi(x)\delta(t)\Rightarrow\varphi(x,t=0) = \frac{1}{2\pi}\int dy\log(x-y)\partial\phi(y). \quad (46)$$

With $\tilde{\Sigma}$ given by Eq. (39) this coincides with Eq. (41) and so the subsequent integration over ϕ again reproduces Eq. (43).

We have thus seen that in this simple model the gauge invariant variational Gaussian approximation works well. Extension of this approach to four dimensional QCD with chiral symmetry breaking is the next challenging task.

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