Mass matrix for atmospheric, solar, and LSND neutrino oscillations

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We construct a mass matrix for the four neutrino flavors, three active and one sterile, needed to fit oscillations in all three neutrino experiments: atmospheric, solar, and LSND, simultaneously. It organizes the neutrinos into two doublets whose central values are about 1 eV apart, and whose splittings are of the order of 10^{-3} eV. Atmospheric neutrino oscillations are described as maximal mixing within the upper doublet, and solar as the same within the lower doublet. Then LSND is a weak transition from one doublet to the other. We comment on the Majorana versus Dirac nature of the active neutrinos and show that our mass matrix can be derived from an $S_2 \times S_2$ permutation symmetry plus an equal splitting rule.

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Neutrinos produced by the interaction of cosmic rays with the Earth's upper atmosphere provide the strongest evidence for neutrino oscillations [1], with $\nu_\mu \to \nu_\tau$ as the favored flavor transition [2]. If the additional evidence from solar [3] and Liquid Scintillation Neutrino Detector (LSND) [4] experiments is also confirmed, then it will be necessary to introduce a fourth light neutrino, a so-called ''sterile neutrino'' ν_s, in addition to the standard electron-, muon-, and tau-neutrinos to account for all the data [5]. The question then arises as to the mass spectrum and mixing scheme for these four particles.

In a two-flavor oscillation scenario, the atmospheric data suggest maximal mixing with mass difference $\Delta m^2 \approx 3 \times 10^{-3}$ eV^2 [6]. Of the three types of solution for the solar neutrino data, there are two, namely the large angle Mikheyev-Smirnov-Wolfenstein (LMSW) and the ''just-so'' *in vacuo* ones, which require close to maximal mixing [7], while the third, small angle MSW (SMSW), requires small mixing [8]. In all three cases, the mass difference Δm^2 is much smaller than in the atmospheric case. By contrast, the LSND data require small mixing, but with a relatively large Δm^2 as compared with the atmospheric case [4].

Recently, Bilenky *et al.* [9] have shown that the only way to account for these data in a four neutrino framework is to require a mass spectrum consisting of two doublets [10], with the splitting within each doublet being much smaller than the separation between them. Here we wish to propose a specific realization and mass matrix in which the members of the upper doublet are identified as maximal superpositions of ν_μ and ν_τ, and the members of the lower doublet are maximal superpositions of ν_e and ν_s. Atmospheric neutrino data can then be described as maximal oscillations between the levels of the upper doublet, and solar neutrino data as maximal oscillations between the levels of the lower doublet. LSND is then a weak transition from one doublet to the other.

In adopting this point of view, we recognize that there are some problems with the current data. The validity, or otherwise, of sterile neutrinos will be tested at SNO [11]. The main impetus for reviving the ''just-so'' solutions comes from the anomalous points at the high end of the solar electron recoil spectrum observed at SuperKamiokande [1]. As we have pointed out in another paper [12], a crucial test for this will be the measurement of the ^{7}Be neutrinos.

Our approach to the development of a mass matrix for a two-doublet model can be illustrated with the two-dimensional model

$$\bar{\Psi} M_2 \Psi = (\bar{\psi}_a \quad \bar{\psi}_b) \begin{pmatrix} m_s & m_k \\ m_k & m_s \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \tag{1}$$

in which the matrix M_2 is a linear combination of the unit (2×2) matrix I and the Pauli matrix σ_x:

$$M_2 = m_s I + m_k \sigma_x . \tag{2}$$

It has eigenvalues $(m_s \pm m_k)$ and eigenstates which are maximal mixtures of the basis states,

$$\psi_\pm = (\psi_a \pm \psi_b)/\sqrt{2}, \tag{3}$$

and thus it will lead to maximal mixing between neutrinos ν_a and ν_b. For future reference, we note that the matrix M_2 is symmetric under the permutation group S_2 of the two members of the doublet, and that the eigenvectors $\psi_\pm$ are respectively even and odd representations of S_2.

Now suppose we rotate M_2 through a small angle $(-2\delta\theta)$ about the y axis:

$$\begin{aligned} &\exp(+i\sigma_y \delta\theta) M_2 \exp(-i\sigma_y \delta\theta) \\ &= m_s I + m_k \sigma_x \cos 2\delta\theta + m_k \sigma_z \sin 2\delta\theta \\ &= \begin{pmatrix} m_s + m_k \sin 2\delta\theta & m_k \cos 2\delta\theta \\ m_k \cos 2\delta\theta & m_s - m_k \sin 2\delta\theta \end{pmatrix}. \end{aligned} \tag{4}$$

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It has the same eigenvalues as the original matrix, but its eigenstates are also rotated through the small angle $(-2\delta\theta)$,

$$\psi_{\pm}(\delta\theta)=\begin{pmatrix}\psi_{+}\cos\delta\theta+\psi_{-}\sin\delta\theta\\ \psi_{+}\sin\delta\theta-\psi_{-}\cos\delta\theta\end{pmatrix}, \tag{5}$$

and so it leads to small mixing oscillations between ψ_+ and ψ_-.

Guided by this analysis, we propose a four-flavor mass matrix which we construct by replacing m_s and m_k in the rotated form of M_2 by (2×2) matrices:

$$m_s\rightarrow M, \quad M=\begin{pmatrix} m_s & m_d\\ m_d & m_s\end{pmatrix}, \tag{6}$$

$$m_k\rightarrow K, \quad K=\begin{pmatrix} m_k & 0\\ 0 & m_k\end{pmatrix}. \tag{7}$$

Our model then takes the form

$$\bar{\Psi}M_4\Psi=(\bar{\Psi}_a\bar{\Psi}_b)\begin{pmatrix} M+K\sin 2\delta\theta & K\cos 2\delta\theta\\ K\cos 2\delta\theta & M-K\sin 2\delta\theta\end{pmatrix}\times\begin{pmatrix}\Psi_a\\ \Psi_b\end{pmatrix}, \tag{8}$$

where Ψ_a and Ψ_b are now two-dimensional column vectors:

$$(\Psi_a\Psi_b)=\begin{pmatrix}\psi_{a1} & \psi_{b1}\\ \psi_{a2} & \psi_{b2}\end{pmatrix}. \tag{9}$$

Next we rotate M_4 and Ψ into the forms

$$M_4\rightarrow\tilde{M}_4=\begin{pmatrix} M & K\\ K & M\end{pmatrix}, \tag{10}$$

$$\Psi\rightarrow\Phi=\begin{pmatrix}\Phi_a\\ \Phi_b\end{pmatrix}=\begin{pmatrix}\cos\delta\theta & -\sin\delta\theta\\ \sin\delta\theta & \cos\delta\theta\end{pmatrix}\begin{pmatrix}\Psi_a\\ \Psi_b\end{pmatrix}. \tag{11}$$

For future reference, we note that $\tilde{M}_4$ is symmetric under the permutation group $\tilde{S}_2$ which interchanges the two doublet pairs.

Now we rotate $\tilde{M}_4$ and Φ into

$$\tilde{M}_4\rightarrow\begin{pmatrix} M+K & 0\\ 0 & M-K\end{pmatrix}, \tag{12}$$

$$\Phi\rightarrow\frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_a & + & \Phi_b\\ -\Phi_a & + & \Phi_b\end{pmatrix}. \tag{13}$$

We now have to diagonalize the (2×2) matrices $(M\pm K)$, where

$$M\pm K=\begin{pmatrix} m_s\pm m_k & m_d\\ m_d & m_s\pm m_k\end{pmatrix}, \tag{14}$$

which have eigenstates $(\Phi_a+\Phi_b)/\sqrt{2}$ and $(-\Phi_a+\Phi_b)/\sqrt{2}$ respectively. The eigenvalues of $(M+K)$ are

$$M^{+}_{\pm}=m_s+m_k\pm m_d, \tag{15}$$

and those of $(M-K)$ are

$$M^{-}_{\pm}=m_s-m_k\pm m_d. \tag{16}$$

Thus we have two doublets whose mean masses are separated by $2m_k$, and whose splittings are both given by $2m_d$. The upper and lower components of $(\Phi_a+\Phi_b)/\sqrt{2}$ are maximally mixed, as are those of $(-\Phi_a+\Phi_b)/\sqrt{2}$. Finally, the eigenstates of $(M+K)$ are weakly mixed with those of $(M-K)$ via the relation between Φ and Ψ in Eq. (11) above.

We identify $(M+K)$ and its eigenstates with the atmospheric neutrino oscillations between ν_μ and ν_τ, and so the squared mass difference may be written

$$\Delta_A=(m_s+m_k+m_d)^2-(m_s+m_k-m_d)^2=4(m_s+m_k)m_d. \tag{17}$$

Similarly, we identify $(M-K)$ and its eigenstates with solar neutrino oscillations between ν_e and ν_s, and so

$$\Delta_S=(m_s-m_k+m_d)^2-(m_s-m_k-m_d)^2=4(m_s-m_k)m_d. \tag{18}$$

For reasons which will become apparent below, we write

$$m_s=m_0+\epsilon, \tag{19}$$

$$m_k=m_0-\epsilon \tag{20}$$

and so

$$\frac{\epsilon}{m_0}=\frac{\Delta_S}{\Delta_A}. \tag{21}$$

Since Δ_A is much greater than Δ_S, as discussed below, we conclude that ϵ is much smaller than m_0, and that m_s is only marginally greater than m_k:

$$\frac{m_s}{m_k}\approx(1+2\epsilon)\approx 1+2\frac{\Delta_S}{\Delta_A}. \tag{22}$$

For LSND, we assume that the $\overline{\nu_\mu}\rightarrow\overline{\nu_e}$ oscillation is dominated by the transition from the lower eigenvalue of $(M+K)$ to the upper eigenvalue of $(M-K)$

$$\Delta_L=(m_s+m_k-m_d)^2-(m_s-m_k+m_d)^2=4(m_k-m_d)m_s, \tag{23}$$

and so

$$8m_km_s=8(m_0^2-\epsilon^2)=2\Delta_L+\Delta_A+\Delta_S. \tag{24}$$

Since Δ_L is much bigger than either Δ_A or Δ_S, it follows that

$$2m_0 \approx \sqrt{\Delta_L}\left(1+\frac{\Delta_A}{4\Delta_L}\right). \tag{25}$$

We then find that m_d is much smaller than m_0:

$$2m_d \approx \frac{\Delta_A}{2\sqrt{\Delta_L}}\left(1-\frac{\Delta_A}{4\Delta_L}\right). \tag{26}$$

To gain a sense of the magnitude of the mass matrix elements, we assume the following values for the observed mass-squared differences:

$$\Delta_L \approx 1 \ \ \mathrm{eV}^2,$$
$$\Delta_A \approx 3\times 10^{-3} \ \ \mathrm{eV}^2,$$
$$\Delta_S \approx 10^{-5} \ \ \mathrm{eV}^2. \tag{27}$$

So the ratios of mass-squared differences are all the same, namely

$$\frac{\Delta_A}{\Delta_L} \approx \frac{\Delta_S}{\Delta_A} \approx 3\times 10^{-3}. \tag{28}$$

It is interesting to note that, for the above value of Δ_L, this is also the value of the weak mixing angle between upper and lower doublets needed to fit the LSND data [4]:

$$\sin^2 2\,\delta\theta \approx 3\times 10^{-3}. \tag{29}$$

The large parameter in the mass matrix, m_0, is close to 0.5 eV,

$$2m_0 \approx 1.001 \ \ \mathrm{eV}, \tag{30}$$

and the small parameters, ϵ and $2m_d$, are much smaller and roughly equal to one another:

$$\epsilon \approx 1.5\times 10^{-3} \ \ \mathrm{eV}, \quad 2m_d \approx 1.5\times 10^{-3} \ \ \mathrm{eV}. \tag{31}$$

Thus the upper doublet, corresponding to ν_τ and ν_μ, has a central value of 1.001 eV and a splitting of 1.5×10^{-3} eV, while the lower doublet, corresponding to ν_e and ν_s, has an almost zero central value, 3×10^{-3} eV, with the same splitting as the upper one.

We have not considered the Majorana versus Dirac nature of the four neutrinos and the constraints from no-neutrino double beta decay [13]. If the three active ones are all Majorana particles, then the sum of their masses times CP phase must not exceed the current bound of 0.2–0.6 eV [14]. In the above example, this is most easily achieved by giving the members of the upper doublet opposite CP phases, which make them ''pseudo-Dirac'' neutrinos because of the small mass difference $2m_d$. Whatever phase is assigned to the active member of the lower doublet, the sum of masses times phase will not exceed 6×10^{-3} eV, well within the experimental limit [15].

We may now ask whether the mass matrix M_4 can be derived from a symmetry principle. As we have noted above, the case of maximal mixing among the two members of a doublet corresponds to the permutation symmetry S_2 between them. Likewise the general structure of M_4 involves the permutation symmetry $\tilde{S}_2$ between the two doublets. It is not difficult to show that the most general 4×4 matrix H_4 which is invariant under $S_2\times\tilde{S}_2$ is given by

$$H_4 = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}. \tag{32}$$

The (2×2) submatrices X,Y are both of the same S_2 symmetric form as M_2 above.

Comparing M_4 with H_4, we see that it is of exactly the same form except that the off-diagonal submatrix K is a multiple of the unit (2×2) matrix whereas Y can have an off-diagonal matrix element. Physically, the absence of an off-diagonal matrix element in K means that the splitting between the members of the upper doublet is exactly the same as that between the members of the lower doublet—an ''equal splitting'' rule.

In conclusion, we have constructed a mass matrix which can simultaneously accommodate all three indications for neutrino oscillations. Its particular structure as a direct product of (2×2) matrices can be derived from an underlying $S_2\times\tilde{S}_2$ symmetry plus an equal splitting rule. It may be interesting to speculate that this symmetry might in turn be a subgroup of a larger permutation symmetry, for example S_4, and that the larger symmetry can be used to distinguish between the active and sterile neutrinos. For example, the three active neutrinos could belong to a triplet with respect to an S_3 subgroup of the larger group, while the sterile neutrino is a singlet.

We recognize that large mixing between a sterile neutrino and the electron-neutrino in the solar neutrino problem can disturb big bang nucleosynthesis [16], and we have no ready solution for this problem. Whether big bang nucleosynthesis can accommodate 3 or 4 light neutrino degrees of freedom will depend crucially on the amount of primordial deuterium in the universe; at the moment this is not well determined [17]. We do, however, regard the existence or non-existence of a sterile neutrino to be an experimental question which will eventually be settled by the observation of the neutral-current interactions of solar neutrinos, as in the SNO experiment [11].

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