Extended BRST invariance in topological Yang-Mills theory reexamined

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Extended BRST invariance (BRST plus anti-BRST invariances) provides in principle a natural way of introducing the complete gauge fixing structure associated with a gauge field theory in the minimum representation of the algebra. However, as happens in topological Yang-Mills theory, not all gauge fixings can be obtained from a symmetrical extended BRST algebra, where antighosts belong to the same representation of the Lorentz group of the corresponding ghosts. We show here that a field redefinition makes it possible to start with an extended BRST algebra with a symmetric ghost-antighost spectrum and arrive at the gauge fixing action of topological Yang-Mills theory.

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It was shown by Witten [1,2] how to build up a quantum field theory involving the vector gauge potential whose partition functional generates topological invariants such as the Donaldson polynomials. Soon after that, Baulieu and Singer [3] showed that this so-called topological Yang-Mills theory can be obtained by an appropriate gauge fixing of the topological invariant classical action

$$S_0=\int d^4x\left(-\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})\right), \tag{1}$$

where $\tilde{F}^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

Their approach was to quantize the action (1) along the standard Becchi-Rouet-Stora-Tyutin (BRST) prescriptions but writing out the BRST transformations of the gauge fields as the sum of the BRST transformations of standard Yang-Mills theory plus a topological ghost Ψ^μ that by itself removes the degrees of freedom associated with the gauge field A_μ:

$$\delta A_\mu=\Psi_\mu+D_\mu c. \tag{2}$$

Then the field content of the theory is enlarged in order to gauge fix the action, taking into account the reducibility of Eq. (2). Also the topological Yang Mills action was obtained from a BRST quantization procedure in an alternative way by Labastida and Pernici [4].

The quantization procedure of Ref. [3] is completely based on the sole BRST invariance of the gauge fixing action. The antighosts and auxiliary fields are introduced as trivial pairs. They do not belong to the minimum representation of the BRST algebra. It is known, however, [5] that one can associate to any classical gauge symmetry an extended BRST (BRST plus anti-BRST) algebra. One of the advantages of considering an extended algebra rather than a standard one is that the antighosts and auxiliary fields enter in this formulation as basic ingredients of the minimum representation of the algebra and not just as trivial pairs. The idea of introducing anti-BRST transformations in topological Yang-Mills theory was discussed in Refs. [6–8]. An extended BRST algebra associated with this theory was then presented in Ref. [9]. However, in this reference the connection with the action of Baulieu and Singer is only possible by means of a gauge fixing action involving terms with different scaling dimensions. They also use some field redefinitions that do not preserve the dimension and the number of degrees of freedom.

The standard way of introducing the anti-BRST transformations is to define them in a symmetric way with respect to the BRST ones. Following this approach one introduces antighosts in the same representation of the Lorentz group as the corresponding ghosts. This would correspond, in the case of the topological Yang-Mills theory to an extended BRST algebra that can be written as [9]

$$\begin{gathered}\delta_1A_\mu=\Psi_\mu+D_\mu c,\quad \delta_1c=\phi_1+\frac{1}{2}[c,c],\quad \delta_1\Psi_\mu=-D_\mu\phi_1-[c,\Psi_\mu],\quad \delta_1\phi_1=-[c,\phi_1],\quad \delta_1\bar{c}=b,\\ \delta_1b=0,\quad \delta_1\bar{\Psi}_\mu=-k_\mu,\quad \delta_1\lambda=\eta_1,\quad \delta_1\phi_2=\eta_2,\quad \delta_1k_\mu=0,\quad \delta_1\eta_1=0,\quad \delta_1\eta_2=0,\\ \delta_2A_\mu=\bar{\Psi}_\mu+D_\mu\bar{c},\quad \delta_2\bar{c}=\phi_2-\frac{1}{2}[\bar{c},\bar{c}],\quad \delta_2\bar{\Psi}_\mu=-D_\mu\phi_2-[\bar{c},\bar{\Psi}_\mu],\quad \delta_2\phi_2=-[\bar{c},\phi_2],\quad \delta_2c=\lambda-b-[c,\bar{c}],\\ \delta_2\Psi_\mu=D_\mu\lambda+k_\mu-[c,\bar{\Psi}_\mu]-[\bar{c},\Psi_\mu],\quad \delta_2\lambda=-\eta_2-[\bar{c},\lambda]-[c,\phi_2],\end{gathered} \tag{3}$$

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$$\delta_2\phi_1 = -\eta_1 - [\bar{c},\phi_1] - [c,\lambda'], \quad \delta_2\eta_2 = \delta_2(-[\bar{c},\lambda] - [c,\phi_2]), \quad \delta_2\eta = \delta_2(-[\bar{c},\phi] - [c,\lambda]),$$

$$\delta_2 b = \delta_2\lambda - \delta_2([c,\bar{c}]), \quad \delta_2 k_\mu = \delta_2(D_\mu\lambda + [c,\bar{\Psi}_\mu] + [\bar{c},\Psi_\mu]), \tag{4}$$

where we are representing BRST and anti-BRST transformations, respectively, as δ_1 and δ_2 and $[,]$ means a graded commutator.

This symmetric approach leads to a simpler algebraic structure for the extended algebra. However, in general not all gauge fixings of a gauge theory can be implemented by using just this kind of symmetrical ghost antighost pairs. This is precisely what happens in topological Yang Mills theory. In order to enforce the self-duality condition on $F_{\mu\nu}$ one has to introduce a self-dual antisymmetric antighost tensor $\overline{\chi}^{+\mu\nu}$ corresponding to the ghost vector Ψ^μ [3]. The difference between the so-called geometrical antighosts, such as $\bar{\Psi}^\mu$, that come from a symmetrical extended formulation and the actual gauge fixing antighosts, such as $\overline{\chi}^{+\mu\nu}$, normally introduced as parts of trivial BRST doublets was discussed in Ref. [10]. This reference introduces a general method for starting with a completely symmetrical set of geometrical ghost antighost pairs and then, by adding some trivial BRST doublets, arrive at a gauge fixing action corresponding to some particular gauge fixing. The idea there is to include some appropriate terms in the action that will have the role of eliminating the unwanted variables from the model by a supersymmetric compensation in the functional integration. This general mechanism is called a ''transmutation'' of geometrical antighosts into gauge fixing antighosts.

We will show here that it is possible to start with the symmetric ghost antighost set of algebras of Eqs. (3) and (4) and redefine the fields in such a way that an extended algebra involving the gauge fixing antighost emerges. The topological Yang-Mills action is then built up from a double (BRST-anti–BRST) variation of a gauge fixing boson. Let us consider the following field redefinitions:

$$\bar{\Psi}_\mu = \bar{\Psi}'_\mu + D^\nu \overline{\chi}^+_{\mu\nu} + D_\mu \bar{\rho},$$

$$k_\mu = k'_\mu + D^\nu b^+_{\mu\nu} + D_\mu d, \tag{5}$$

where $\overline{\chi}^+_{\mu\nu}$ and $b^+_{\mu\nu}$ are antisymmetric self-dual tensor fields carrying three independent components. The primed fields are necessary in order to find a representation for the extended BRST algebra at the interacting level and are assumed to vanish when the coupling constant goes to zero. (In other words, explicitely including the coupling constant g, omitted in the article for simplicity, we would have $\bar{\Psi}_\mu = g\bar{\Psi}'_\mu + D^\nu \overline{\chi}^+_{\mu\nu} + D_\mu \bar{\rho}$ and the same kind of thing for k_μ.) Equations (5) are just shifting the field $\bar{\Psi}_\mu$, separating the four components in three of $\overline{\chi}^+_{\mu\nu}$ plus one of $\bar{\rho}$. Observe that this shift in the first of Eq. (5) generates a new symmetry. The new ghosts of the second equation have precisely the role of fixing this symmetry.

Associated to this new set of fields we find the following extended algebra:

$$\delta_1 A_\mu = \Psi_\mu + D_\mu c, \quad \delta_1 c = \phi_1 + cc, \quad \delta_1\Psi_\mu = -D_\mu\phi_1 + [c,\Psi_\mu], \quad \delta_1\phi_1 = -[c,\phi_1], \quad \delta_1\bar{c} = b, \quad \delta_1 b = 0,$$

$$\delta_1\overline{\chi}^+_{\mu\nu} = -b^+_{\mu\nu}, \quad \delta_1\lambda = \eta_1, \quad \delta_1\eta_1 = 0, \quad \delta_1\phi_2 = \eta_2 - [\bar{c},b], \quad \delta_1 b^+_{\mu\nu} = 0, \quad \delta_1\bar{\rho} = d, \quad \delta_1 d = 0, \quad \delta_1\eta_2 = [b,b],$$

$$\delta_1\bar{\Psi}'_\mu = [\Psi^\nu + D^\nu c, \overline{\chi}^+_{\mu\nu}] + [\Psi_\mu + D_\mu c, \bar{\rho}] - k'_\mu - 2D_\mu d, \quad \delta_1 k'_\mu = [\Psi^\nu + D^\nu c, b^+_{\mu\nu}] + [\Psi_\mu + D_\mu c, d], \tag{6}$$

$$\delta_2 A_\mu = \bar{\Psi}'_\mu + D^\nu \overline{\chi}^+_{\mu\nu}, + D_\mu\bar{\rho} + D_\mu\bar{c}, \quad \delta_2\bar{c} = \phi_2 + \overline{cc}, \quad \delta_2\overline{\chi}^+_{\mu\nu} = [\bar{\rho} + \bar{c}, \overline{\chi}^+_{\mu\nu}], \quad \delta_2\bar{\rho} = -\phi_2 + \overline{\rho\rho} + [\bar{c},\bar{\rho}],$$

$$\delta_2\phi_2 = -[\bar{c},\phi_2],$$

$$\delta_2 c = \lambda - b - d + [c,\bar{c}], \quad \delta_2\Psi_\mu = k'_\mu + D^\nu b^+_{\mu\nu} + D_\mu d - D_\mu\lambda - [c,\bar{\Psi}^\mu] - [\bar{c},\Psi^\mu],$$

$$\delta_2\lambda = [c,\lambda + [c\bar{c}]] - [\bar{\rho},d] + [\lambda - \phi_1 - b - d - cc + [c,\bar{c}],\bar{c}], \quad \delta_2\phi_1 = -\eta_1 + [\phi_1 + cc,\bar{c}] - [c,\lambda + [c,\bar{c}]], \quad \delta_2\eta_2 = 0,$$

$$\delta_2\eta_1 = [c,[c,b] - [\phi_1 + cc,\bar{c}] + \eta_1] + [\phi_1 + cc,\lambda] - [d,d] + [\phi_1 - cc,b] - [[c,\phi_1 - cc],\bar{c}], \quad \delta_2 b = -\eta_2,$$

$$\delta_2 d = \eta_2 - [\bar{c} + \bar{\rho}, b + d], \quad \delta_2 b^+_{\mu\nu} = -[b + d, \overline{\chi}^+_{\mu\nu}] - [\bar{c} + \bar{\rho}, b^+_{\mu\nu}], \quad \delta_2\bar{\Psi}'_\mu = [D^m \overline{\chi}^{+\nu}_m + \bar{\Psi}'^\nu; \overline{\chi}^+_{\mu\nu}] + [\bar{\Psi}'_\mu, \bar{c} + \bar{\rho}], \tag{7}$$

and the anti-BRST transformation for k'_μ can be calculated from $\delta_1\delta_2\Psi_\mu=0$. Then we consider the gauge fixing of topological Yang-Mills action (1) by adding the gauge fixing action

$$S_{\rm GF}=-\int d^4x\,\delta_1\delta_2\frac{1}{2}\,{\rm Tr}[A_\mu A^\mu+(\bar{c}-\bar{\rho})\Box^{-1}\partial_\mu\Psi^\mu -c(\bar{c}+\bar{\rho})], \tag{8}$$

where all the terms have the appropriate scaling dimension. This action is different from the one in Ref. [9] that involves the BRST–anti-BRST variation of a sum involving terms $A^\mu A_\mu$ and $F^{\mu\nu}F_{\mu\nu}$ that have different scaling dimensions. This would make sense only if one includes additional dimensionfull parameters. Also our field redefinition (5) is different from the one in Ref. [9], where $\bar{\chi}^{+\mu\nu}$ was introduced as the exterior derivative of $\bar{\Psi}_\mu$, although these two quantities have different scaling dimensions.

Using Eqs. (6) and (7) , we find after redefining the fields as in Eq. (5):

$$\begin{aligned}S_{\rm GF}=\int d^4x\,{\rm Tr}\Big\{&A_\mu k'^\mu+A_\mu D^\mu(d-b)-A^\mu D^\nu b^+_{\mu\nu}-A_\mu[\Psi^\mu+D^\mu c,\bar{c}]-(\Psi_\mu+D_\mu c)(\bar{\Psi}'_\mu+D^\nu\bar{\chi}^+_{\mu\nu}+D_\mu\bar{\rho}+D_\mu\bar{c})\\
&-\frac{1}{2}(\bar{c}-\bar{\rho})\Box^{-1}\partial_\mu(-D^\mu\eta_1+[\Psi^\mu+D^\mu c]+[c,k^\mu]+[\phi_1+cc,\bar{\Psi}'_\mu+D^\nu\bar{\chi}^+_{\mu\nu}+D_\mu\bar{\rho}]-[\bar{c},-D^\mu\phi_1+[\Psi^\mu,c]]+[b,\Psi^\mu])\\
&+\frac{1}{2}(2\phi_2+\overline{cc}-\overline{\rho\rho}-[\bar{c},\bar{\rho}])\Box^{-1}\partial_\mu(-D^\mu\phi_1+[\Psi^\mu,c])-\eta_2\Box^{-1}\partial_\mu\Psi^\mu+\frac{1}{2}c[\bar{c}+\bar{\rho},b+d]\\
&+\frac{1}{2}(\phi_1+cc)(\overline{cc}+\overline{\rho\rho}+[\bar{c},\bar{\rho}])-\frac{1}{2}(\lambda-b-d)(b+d)+\eta_1(\bar{c}+\bar{\rho})\Big\}.\end{aligned}\tag{9}$$

This action represents a complete gauge fixing of the topological action (1). We can now see what happens in the limit of weak coupling constant. In this case all the interaction terms and also the fields $\bar{\Psi}'_\mu$and k'_μ vanish. In order to compare with Ref. [3] we can redefine the fields

$$\eta_1\equiv\eta,\quad \eta_2\equiv\Box\bar{\eta},\quad \phi_1\equiv\phi,\quad \phi_2\equiv\Box\bar{\phi} \tag{10}$$

the action then becomes

$$\begin{aligned}S_{\rm GF}=\int d^4x\,{\rm Tr}\Big[&\bar{\chi}^+_{\mu\nu}\partial^\mu\Psi^\nu-(\bar{\rho}+\bar{c})\partial_\mu\Psi^\mu+b^+_{\mu\nu}\partial^\mu A^\nu\\
&-\bar{\eta}\partial^\mu\Psi_\mu-(b+d)\partial_\mu A^\mu-\partial^\mu c\,\partial_\mu(\bar{c}+\bar{\rho})\\
&+\bar{\phi}\Box\phi-\bar{\rho}\eta-bd-\frac{b^2}{2}-\frac{d^2}{2}+\lambda d\Big].\end{aligned}\tag{11}$$

Path integration over the fields λ and η leads to functional deltas in d and $\bar{\rho}$. Thus, integrating over this four fields we arrive at the total action

$$\begin{aligned}S=S_0+\int d^4x\,{\rm Tr}\Big(&\bar{\chi}^+_{\mu\nu}\partial^\mu\Psi^\nu-\bar{c}\partial^\mu\Psi_\mu+b^+_{\mu\nu}\partial^\mu A^\nu\\
&-\bar{\eta}\partial^\mu\Psi_\mu-b\,\partial_\mu A^\mu-\partial^\mu c\,\partial_\mu\bar{c}+\bar{\phi}\Box\phi-\frac{b^2}{2}\Big).\end{aligned}\tag{12}$$

This is the weak coupling limit of the action found in Ref. [3]. Thus we see that the field redefinitions of Eq. (5) provide a simple way of separating the relevant variables, that implement the gauge fixing conditions (at weak coupling)

$$\partial^\mu A_\mu=0,\quad F_{\mu\nu}+\tilde{F}_{\mu\nu}=0,\quad \partial^\mu\Psi_\mu=0 \tag{13}$$

from the other unnecessary variables. It is interesting to note also the way that the λ and η fields cancel just the unwanted parts of $\bar{\Psi}_\mu$ and k_μ.

The consistency of our procedure can be checked by looking at the dimensions of the fields

$$\begin{aligned}&[A_\mu]=[c]=[\bar{c}]=[\bar{\chi}^+_{\mu\nu}]=[\bar{\eta}]=[\bar{\rho}]=1,\\
&[\Psi^\mu]=[\bar{\Psi}^\mu]=[\phi]=2,\quad [\bar{\phi}]=0,\\
&[\lambda]=[b]=[b^{\mu\nu}]=[d]=2,\quad [\eta]=[k^\mu]=3.\end{aligned}\tag{14}$$

Comparing our results with those of Ref. [10] one sees that here the gauge fixing action is both BRST and anti-BRST exact while there the action is only BRST exact. Also, one sees that here the antighost $\bar{\chi}^+_{\mu\nu}$ comes from a shift in $\bar{\Psi}_\mu$.

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[1]

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