Hydrodynamic transport functions from quantum kinetic field theory

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Starting from the quantum kinetic field theory [E. Calzetta and B. L. Hu, Phys. Rev. D **37**, 2878 (1988)] constructed from the closed-time-path (CTP), two-particle-irreducible (2PI) effective action we show how to compute from first principles the shear and bulk viscosity functions in the hydrodynamic-thermodynamic regime. For a real scalar field with $\lambda \Phi^4$ self-interaction we need to include four-loop graphs in the equation of motion. This work provides a microscopic field-theoretical basis to the ''effective kinetic theory'' proposed by Jeon and Yaffe [S. Jeon and L. G. Yaffe, Phys. Rev. D **53**, 5799 (1996)], while our result for the bulk viscosity reproduces their expression derived from linear-response theory and the imaginary-time formalism of thermal field theory. Though unavoidably involved in calculations of this sort, we feel that the approach using fundamental quantum kinetic field theory is conceptually clearer and methodically simpler than the effective kinetic theory approach, as the success of the latter requires a clever rendition of diagrammatic resummations which is neither straightforward nor fail-safe. Moreover, the method based on the CTP-2PI effective action illustrated here for a scalar field can be formulated entirely in terms of functional integral quantization, which makes it an appealing method for a first-principles calculation of transport functions of a thermal non-Abelian gauge theory, e.g., QCD quark-gluon plasma produced from heavy ion collisions.

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I. INTRODUCTION

In a recent series of papers [1–3], Jeon and Yaffe (JY) derived expressions for the transport functions for a real, self-interacting scalar field in flat space, from first principles, using the Kubo formulas [4] and the imaginary-time formalism of thermal field theory. A necessary step in evaluating transport functions using the Kubo formulas is taking the zero-frequency limit of the time-Fourier-transformed spatial correlator of the energy momentum tensor. In order to avoid infrared divergences which arise in taking the $\omega \to 0$ limit, a complicated set of resummations of ladder diagrams had to be performed [2]. However, it was observed in [3] that the same integral equations for the transport functions obtained from the Kubo formulas can be derived from an *effective kinetic theory* which takes into account the one-loop finite-temperature corrections to the mass of quasiparticles in the scalar theory, and to the effective vertices for quasiparticle scattering. In their treatment, the effective kinetic theory is presented as a physically well motivated, but phenomenological, theory, whose justification is taken to be the fact that it gives transport properties which agree with the leading-order nonperturbative results computed using thermal field theory.

We demonstrate in this work that, in fact, the effective kinetic equations for $\lambda \Phi^4$ theory proposed by Jeon and Yaffe are derivable from a kinetic theory of quantum fields constructed earlier by two of us [5] following the work of Kadanoff and Baym [6] and continued by many others [7]. In particular their effective kinetic equations are derivable in our approach from the closed-time-path (CTP), two-particle-irreducible (2PI) effective action [5] truncated at four loops for the $\lambda \Phi^4$ theory. Even though the calculation of higher loop effects is necessarily involved technically, we feel the quantum kinetic field theory approach is conceptually clearer and methodically simpler than the Jeon-Yaffe approach, as the success of the latter requires a clever rendition of diagrammatic resummations which is neither straightforward nor fail safe. For example, the usual form of the Boltzmann equation derived in, say, [5] assumes 2-2 particle scattering, which conserves particle number, but bulk viscosity arises from particle nonconserving processes (this was emphasized by JY). When we take into account 2-4 or 4-2 processes, we need to go to four-loop diagrams in the Boltzmann and the gap equations which will no longer assume the simpler form familiar in the usual derivations based on 2-2 processes. This extra effort is expected for tackling higher-order effects but within the same context of the same fundamental (not effective) kinetic theory. We show that the derivation of the effective kinetic equations from the CTP-2PI effective action requires only the basic assumptions of kinetic theory: namely, the existence of a separation of macroscopic and microscopic time scales[1] and the existence of

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[1]This means that the short wavelength quantum fluctuations of the field are in a state of near *local* thermal equilibrium, whose properties vary slowly on the scale of the correlation length for the field. We also assume a nearly Gaussian initial state, so that low-order correlation functions are sufficient to capture the dominant physical processes.

well-defined (perhaps modified) ''on-shell'' asymptotic particle states [8]. The assumption of weak coupling is also necessary in order to justify the neglect of yet higher-order scattering terms which arise in the collision integral for the derivation of a generalized Boltzmann equation.

The apparent reported ''failure'' of existing kinetic equations in the literature to account for the bulk viscosity of the scalar quantum field is due to the fact that previous work along this line entails perturbative expansions to insufficient accuracy, and not to any flaw or defect of the theory. When the calculation to a sufficient accuracy is performed, as done here, fundamental quantum kinetic field theory produces the ''effective kinetic theory'' of Jeon and Yaffe, in particular, the result for the bulk viscosity reproduces JY's expression derived from linear-response theory. More importantly perhaps, our method based on the CTP-2PI effective action illustrated here for a scalar field can be formulated entirely in terms of functional integral quantization, which makes it an appealing method for a first-principles calculation of transport functions of a thermal non-Abelian gauge theory, e.g., QCD quark-gluon plasma produced from heavy ion collisions. The first-principles approaches with a clear bearing on fundamental physics, involving nonlinear and nonperturbative effects, such as those employed here for this task, and elsewhere for related problems, are, in our opinion, essential for the successful establishment of a viable and useful quantum field theory of nonequilibrium processes. (For a sampling of current activities, see, e.g., [9].)

A. Statement of the problem

Let us begin by stating in simple terms what are the transport coefficients to be computed [10–12]. For a real scalar field, there are no other conserved quantities than energy and momentum. Since there is no fundamental concept of particle number (equivalently, since the field describes both particles and antiparticles), chemical potential must be set to zero identically in the grand canonical (equilibrium) density matrix. Equilibrium states are parametrized by the vector $\beta^\mu = \beta u^\mu$, where u^μ is a timelike unit vector ($u^\mu u_\mu = -1$) and $\beta = 1/T$ is the inverse temperature.

In the semiclassical limit, we define the energy momentum tensor $T^{\mu\nu}$ as the expectation value of the corresponding Heisenberg operator. $T^{\mu\nu}$ may be decomposed with respect to u^μ as

$$T^{\mu\nu} = \rho u^\mu u^\nu + p P^{\mu\nu}; \quad P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu; \quad P^{\mu\nu} u_\nu = 0, \tag{1}$$

where $\eta^{\mu\nu}$ is the Minkowsky metric diag $(-1,1,1,1)$ (for simplicity we shall work in flat spacetime, although a generally covariant formulation is readily available), thus defining the energy density ρ and pressure p. Observe that there is no *heat conduction*, namely, that in the rest frame there is no energy flux. In this case this shall be true even in nonequilibrium states, since there are no currents other than the four velocity to break the isotropy of space in the rest frame. In other words, we are forced to adopt the Landau-Lifshitz formulation over Eckart's. (See footnote below.)

The functional dependences $\rho = \rho(T)$ and $p = p(T)$ define the equation of state in parametric form. The macroscopic description is completed by giving a concrete expression for the entropy flux S^μ, which must be consistent with the first law of thermodynamics $dS^\mu = -\beta_\nu dT^{\mu\nu}$. Moreover, in equilibrium, vanishing entropy production $S^\mu_{,\mu} = 0$ must follow from energy-momentum conservation $T^{\mu\nu}_{,\nu} = 0$. If we introduce the thermodynamic potential $\Phi^\mu = S^\mu + \beta_\nu T^{\mu\nu}$, consistency requires $\Phi^\mu = p\beta^\mu$, and the identity

$$T\frac{dp}{dT} = p + \rho. \tag{2}$$

Energy-momentum conservation implies the identities (recall that $u_\mu u^\mu_{,\nu} = 0$)

$$\rho_{,t} - (\rho + p)u^\nu_{;\nu} = 0; \quad -(\rho + p)u^\mu_{,t} + P^{\mu\nu}p_{,\nu} = 0, \tag{3}$$

where

$$X_{,t} \equiv -u^\mu X_{,\mu}. \tag{4}$$

Since ρ and p become space dependent only through their temperature dependence, we may write $\rho_{,t} = \rho_{,T} T_{,t}$, and similarly for p. Using the identity Eq. (2), Eqs. (3) simplify to

$$\frac{1}{T}T_{,t} - c_s^2 u^\nu_{;\nu} = 0; \quad -u^\mu_{,t} + \frac{1}{T}P^{\mu\nu}T_{,\nu} = 0, \tag{5}$$

where $c_s = \sqrt{p_{,T}/\rho_{,T}}$ is the speed of sound.

Let us now consider a near equilibrium state, meaning that the properties of the actual state remain close to that of a conveniently chosen fiducial equilibrium state. There is some arbitrariness in this choice,[2] but, following the so-called Landau-Lifshitz prescription [10], we shall choose the fiducial equilibrium state as that having the same four velocity and energy density as the actual state. More precisely, we define the four velocity as the only timelike unit eigenvector of the actual energy momentum tensor $T^{\mu\nu}$ (assumed to satisfy suitable energy conditions), and then the energy density is defined as $\rho = T^{\mu\nu} u_\mu u_\nu$. Knowing ρ, we may compute the temperature T and pressure p of the equilibrium state, and thus the departure of the actual $T^{\mu\nu}$ from its value $T_0^{\mu\nu}$ in the fiducial equilibrium state. Observe that if we write $T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}$, then by construction $u_\nu \delta T^{\mu\nu} = 0$. $\delta T^{\mu\nu}$ is usually parametrized in terms of the bulk τ and shear $\tau^{\mu\nu}$ stresses, as $\delta T^{\mu\nu} = \tau P^{\mu\nu} + \tau^{\mu\nu}$, $\tau^\mu_\mu = 0$.

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[2]There are two common choices of fiducial states. In the Eckart prescription, one chooses an equilibrium state with the same particle current and energy density as the actual state, and reads out the equilibrium pressure from the equation of state. Thus in the Eckart frame there may be energy flux relative to the particle flux, which is interpreted as heat. In the Landau-Lifshitz prescription, the fiducial state has the same energy density, energy flux, and particle number density as the actual state. Thus heat is read out of the particle current.

Remaining within the so-called *first-order formalism* [12,13], we may write the entropy production in the nonequilibrium evolution as $S^\mu_{,\mu} = -\beta_{\mu,\nu} \delta T^{\mu\nu}$ (we refer the reader to the literature for a thorough discussion of the hypothesis involved in this formula). Decompose

$$\beta_{\mu,\nu} = -\frac{1}{T} u_\nu u_\mu c_s^2 u^\nu_{;\nu} + \frac{1}{T^2} [P^\lambda_\mu u_\nu T_{,\lambda} - P^\lambda_\nu T_{,\lambda} u_\mu]$$
$$-\frac{1}{T} u_\nu u_\mu \left[\frac{T_{,t}}{T} - c_s^2 u^\nu_{;\nu} \right] + \frac{1}{T} P^\sigma_\mu u_\nu \left[u_{\sigma,t} - \frac{1}{T} P^\lambda_\sigma T_{,\lambda} \right]$$
$$+ \frac{1}{T} \tilde{H}_{\mu\nu} + \frac{1}{T} H_{\mu\nu} + \frac{1}{3T} P_{\mu\nu} u^\rho_{,\rho}, \tag{6}$$

where

$$H^{\mu\nu} = \frac{1}{2} P^{\mu\lambda} P^{\nu\sigma} \left[u_{\lambda,\sigma} + u_{\sigma,\lambda} - \frac{2}{3} P_{\lambda\sigma} u^\rho_{,\rho} \right]; \tag{7}$$

$$\tilde{H}^{\mu\nu} = \frac{1}{2} P^{\mu\lambda} P^{\nu\sigma} [u_{\lambda,\sigma} - u_{\sigma,\lambda}].$$

The condition that entropy is created rather than destroyed leads us to parametrize

$$\tau^{\mu\nu} = -\eta H^{\mu\nu}; \quad \tau = -\zeta u^\rho_{,\rho}; \quad \eta, \zeta \geq 0, \tag{8}$$

where η and ζ are the *shear* and *bulk* viscosity coefficients, respectively. We wish to compute these coefficients as functions of the temperature and other parameters in the theory.

B. Transport coefficients from kinetic equations

Since thermodynamics alone cannot determine the dependence of the viscosity coefficients on temperature, to proceed we must place the macroscopic description within a more basic and comprehensive framework, i.e., kinetic theory, where there is a well-known method to extract the transport functions [14,15].

The framework is a system described by a one-particle distribution function f. There is a known prescription to compute the energy momentum tensor $T^{\mu\nu}$ from the moments of the distribution function. In equilibrium, f depends only on the inverse temperature four-vector field β_μ. The starting point is the transport equation for f:

$$\left[p^\mu \frac{\partial}{\partial X^\mu} - \frac{1}{2} M^2_{,\mu} \frac{\partial}{\partial p_\mu} \right] f = I_{col}[f], \tag{9}$$

where I_{col} is the collision integral and M^2 is the mass of the particle (with possible position dependence). f is assumed to be of the form $f = f_0 + \delta f$, where f_0 is the local equilibrium distribution

$$f_0 = \frac{1}{e^{|\beta^0_\mu p^\mu|} - 1}, \tag{10}$$

where $\beta^0_\mu = u_\mu/T_0$ and δf is a perturbation. Since the collision integral vanishes identically for local thermal equilibrium, we can write the collision integral as a linear integral operator $\hat{Q}$ acting on δf

$$I_{col}[f_0 + \delta f] = \hat{Q}[\delta f]. \tag{11}$$

On the other hand, if we neglect δf on the left-hand side (lhs) of the transport equation, we can write it as some differential operator acting on β^0_μ, thus obtaining a linear integral equation for δf

$$\hat{Q}[\delta f] = Q_E[\partial_t, \partial_i](\beta^0_\mu). \tag{12}$$

The $\hat{Q}$ operator satisfies four constraints, which follow from energy-momentum conservation, namely

$$\int \frac{d^4 p}{(2\pi)^4} \theta(p^0) \delta(\Omega_0) p^\mu \hat{Q} = 0, \tag{13}$$

where $\Omega_0 = p^2 + M^2$ enforces the on-shell condition. Thus the equation for δf requires four integrability conditions

$$\int \frac{d^4 p}{(2\pi)^4} \theta(p^0) \delta(\Omega_0) p^\mu Q_E[\partial_t, \partial_i](\beta^0_\mu) = 0. \tag{14}$$

The integrability conditions reduce to a system of differential equations for β^0_μ, which are in fact the conservation laws for the energy-momentum tensor $T^{\mu\nu}$ Eqs. (5). These equations allow us to eliminate time derivatives from the transport equation, which simplifies to

$$\hat{Q}[\delta f] = Q_E[\partial_i](\beta^0_\mu). \tag{15}$$

On solving this equation, we determine the correction δf to the distribution function, and thereby the correction to the energy momentum tensor. In general, the terms containing δf will contribute a term $\delta\rho$ to the energy density; thus we define the physical temperature T from the condition $\rho(T) = \rho(T_0) + \delta\rho$, or equivalently $T_0 = T + \delta T$, where $\delta T = -\delta\rho/\rho_{,T}$. Knowing the temperature T we may compute the energy momentum tensor $T^{\mu\nu}_0$ in the fiducial state, subtract it from the physical $T^{\mu\nu}$ to determine the nonequilibrium part $\delta T^{\mu\nu}$, and then read out the viscosity coefficients by matching it to the form given in Eq. (8).

C. Transport functions in quantum kinetic field theory

From the discussion above, we may identify the main steps involved in computing the transport functions: namely, (1) find a description of the system in terms of a one-particle distribution function f, and the corresponding transport equation; (2) find the structure of equilibrium states, including the expression of conserved currents in terms of f, and the equilibrium equation of state; (3) solve the linearized transport equation to obtain the response of the system to gradients in the hydrodynamical variables, and read out the nonequilibrium stresses.

Step 1 is done in detail in [5], where the self-energy is computed to two-loops accuracy, giving as a result that, for a $\lambda \phi^4$ type interaction, the transport equation for f is simply the relativistic Boltzmann equation for Bose particles, with the only modification of allowing for a variable mass as in Eq. (9). This Vlasov-type correction takes into account the fact that the physical mass M^2 of the particles is connected to the temperature through the gap equation, and thereby M^2 becomes position dependent if T is. The (only) conserved current $T^{\mu\nu}$ is defined as the expectation value of the corresponding Heisenberg operator, and the hydrodynamic variables are read out from it, so step (2) does not present great difficulty.

The problem arises in step (3), because the Boltzmann collision operator satisfies, besides the four conservation laws associated to energy-momentum, a fifth constraint

$$\int \frac{d^4p}{(2\pi)^4} \theta(p^0) \delta(\Omega_0) \hat{Q} = 0 \qquad (16)$$

associated to the conservation of the particle number in binary collisions. There is, therefore, a fifth integrability condition, and the system of macroscopic equations for T_0 and u^μ becomes overdetermined. One could hope that the fifth constraint would be true just as a consequence of the other four, but we shall show below that in an interacting theory this is not the case.

Continuing on this route, the linearized transport equation built out of the Boltzmann collision operator is not integrable, and the calculation grinds to a halt. If we are going to compute the bulk viscosity out of quantum kinetic theory, then the collision operator cannot be just Boltzmann's derived from 2-2 collision processes. A generalized collision operator including particle number changing terms besides the usual binary scattering terms is needed. Thus the fifth constraint has to be lifted to eliminate the inconsistency. However, in the ''effective kinetic theory'' of Jeon and Yaffe these new terms are not derived but rather postulated to match an independent calculation of cross sections from linear-response theory. We feel that it is conceptually and methodically more gratifying if these terms can be derived *ab initio* from a kinetic theory of quantum fields. This is indeed possible, as our present work aims to demonstrate.

Since these particle number changing interactions are higher order in the coupling constant (for pure $\lambda \phi^4$ theory they appear at λ^4th order), it is to be expected that they may be retrieved by simply carrying the calculation in [5] to a higher loop order. However, there appears a matter of principle: if we are going to work to high (eventually, arbitrarily high) order in perturbation theory, we cannot *assume* that the Green functions will look anything like those of the free theory. Thus we must first confront the need to provide a nonperturbative definition of the one-particle distribution function, (which should of course reduce to the one used in [5] at first order in perturbation theory). In equilibrium, this problem is solved by the Kubo-Martin-Schwinger (KMS) theorem ([16]), which implies the proportionality of the Fourier transforms of the Hadamard and Jordan propagators (see below). Off equilibrium, following [6], we shall define the

one-particle distribution function from the ratio of the partial Fourier transforms of these propagators. Familiarity with the KMS theorem and the Kadanoff-Baym equations should not blind us to the highly nontrivial nature of this definition. With this we will then have come to a full circle of deliberations for consistency.

For the specific goal laid out for this investigation, the main technical difficulty lies in the analysis of the collision term giving rise to the bulk viscosity, as it is due to the particle-changing processes which even in the leading order already involve four-loop self-energy diagrams. This is one of the main tasks we need to overcome.

D. Summary of the paper

The outline above provides us with a step by step guide to computing transport functions in quantum kinetic field theory, which we shall execute in the following sections. As noted above, the first step is the precise definition of the one-particle distribution function, which is discussed in Sec. II. In Sec. III we derive the transport equation. For simplicity, after showing that to lowest nontrivial order the Boltzmann collision operator is recovered, we shall write down only the terms related to particle number changing interactions. Section IV is dedicated to studying the equilibrium states of the field, with the aim of finding the precise equation of state. The results of Secs. III and IV amount to a first-principles derivation of Jeon and Yaffe's effective kinetic theory from quantum field theory. Finally, in Sec. V we go through the actual calculation of the bulk viscosity, which in the appropriate limit reproduces JY's estimates from linear-response theory.

II. NONPERTURBATIVE QUANTUM KINETIC THEORY

Our specific goal is to show that by consistently extending the existing methods of quantum kinetic field theory (see e.g., [5]) to four-loop order, it is possible to account both for the shear and bulk viscosity of an interacting scalar field, as computed by Jeon and Yaffe. We consider a purely quartic interaction, although for the application to gauge theories cubic plus quartic would seem closer to what is needed. Since bulk viscosity entails particle number changing scattering, and these processes appear for the first time at $O(\lambda^4)$, we must push the calculation through to five loops in the closed time path (CTP) two-particle irreducible (2PI) effective action (EA) [17,18], which will yield four loops in the equations of motion for the propagators. We assume that the background field vanishes identically, so we shall look at the 2PI-EA as a functional of the propagators alone [19,20].

A. The model

Let us begin with the classical action for a quartically self-interacting scalar field in Minkowski space. Using a modification of DeWitt's notation in which capital letters denote both spacetime (x^μ) and time branch (1,2) indices [21], the action can be written

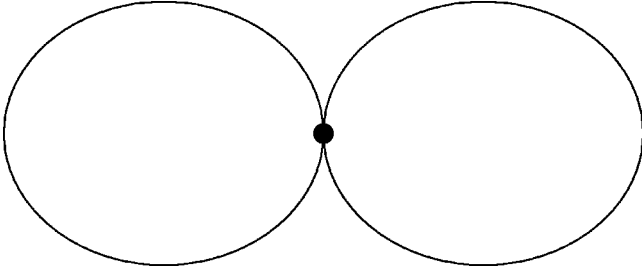$$S = \frac{1}{2} \phi^A D_{AB} \phi^B + S_i, \qquad (17)$$

FIG. 1. Two-loops contribution to the CTP effective action.

$$D_{AB} = [Z_b \Box - m_b^2] c_{AB};$$

$$S_i[\phi] = \frac{-1}{4!} \lambda_b c_{ABCD} \phi^A \phi^B \phi^C \phi^D, \qquad (18)$$

where m_b is the bare ''mass'' of the field, λ_b is the bare coupling constant and ϕ^A is the scalar field. With the benefit of hindsight, we set the wave-function renormalization factor $Z_b = 1$, but it should be generally included. The two- and four-index objects c_{AB} and c_{ABCD} are defined by their contraction into the scalar field:

$$c_{AB} \phi^A \psi^B = \int d^4x [\phi^1 \psi^1 - \phi^2 \psi^2](x), \qquad (19)$$

$$c_{ABCD} \phi^A \phi^B \phi^C \phi^D = \int d^4x [(\phi^1)^4 - (\phi^2)^4](x). \qquad (20)$$

We use c_{AB} and its inverse c^{AB} to raise and lower indices, and with the use of the Einstein convention of summing over repeated indices, their appearance may be implicit.

We wish to derive an effective kinetic description of this theory valid at arbitrary temperature, for sufficiently weak coupling λ, in the case of unbroken symmetry. This assumes that the expectation value of the Heisenberg field operator Φ_H vanishes. The two-point function $\langle \Phi_H(x) \Phi_H(y) \rangle$ is the lowest-order nonvanishing correlation function for the space of initial conditions with which we are presently concerned. Therefore, let us couple an external, c-number, nonlocal source K_{AB} to the scalar field as follows:

$$S[\phi] \rightarrow S[\phi] + \frac{1}{2} K_{AB} \phi^A \phi^B, \qquad (21)$$

and construct a quantum generating functional

$$Z[K] \equiv \int D\phi \exp\left[\frac{i}{\hbar}\left(S[\phi] + \frac{1}{2} K_{AB} \phi^A \phi^B\right)\right], \qquad (22)$$

whose functional power series expansion contains all the n-point functions of the theory. The generating functional of normalized expectation values is given by

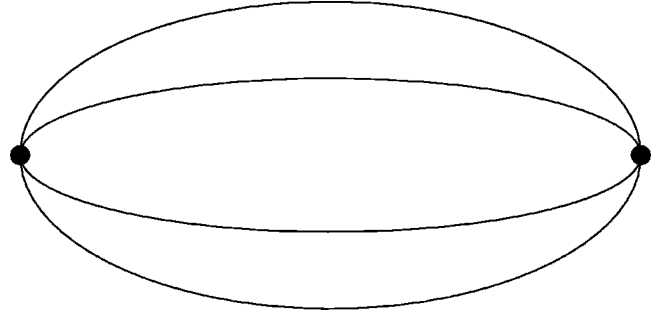$$W[K] \equiv -i\hbar \ln Z[K]. \qquad (23)$$

Now, we define

FIG. 2. Three-loops contribution to the CTP effective action.

$$\hbar G^{AB} \equiv 2 \frac{\delta W[K]}{\delta K_{AB}}, \qquad (24)$$

and construct a new functional $\Gamma[G]$ which is the Legendre transform of $W[K]$,

$$\Gamma[G] \equiv W[K] - \frac{\hbar}{2} K_{AB} G^{AB}. \qquad (25)$$

It follows immediately from the above definition that

$$\frac{\delta \Gamma[G]}{\delta G^{AB}} = -\frac{\hbar}{2} K_{AB}, \qquad (26)$$

and Γ obeys the integrodifferential equation

$$\Gamma[G] = -i\hbar \ln\left[\int D\phi \exp\left\{\frac{i}{\hbar}\left(S[\phi] - \frac{1}{\hbar}\frac{\delta \Gamma[G]}{\delta G^{AB}}\right.\right.\right.$$
$$\left.\left.\left. \times (\phi^A \phi^B - \hbar G^{AB})\right)\right\}\right]. \qquad (27)$$

By expanding Γ in a functional power series in $\hbar$, this equation can be solved [18]. The solution has the form

$$\Gamma[G] = -\frac{i\hbar}{2} \mathrm{Tr} \ln G + \frac{i\hbar}{2} D_{AB} G^{AB} + \Gamma_2[G], \qquad (28)$$

where the functional Γ_2 is $-i\hbar$ times the sum of all two-particle-irreducible diagrams with lines given by $\hbar G$ and vertices given by the quartic interaction.

The functional $\Gamma[G]$ is the two-particle-irreducible (2PI) effective action whose variation with respect to G gives the equation of motion for the two-point function. Because we
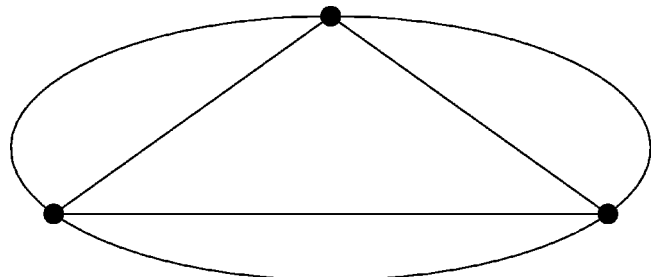

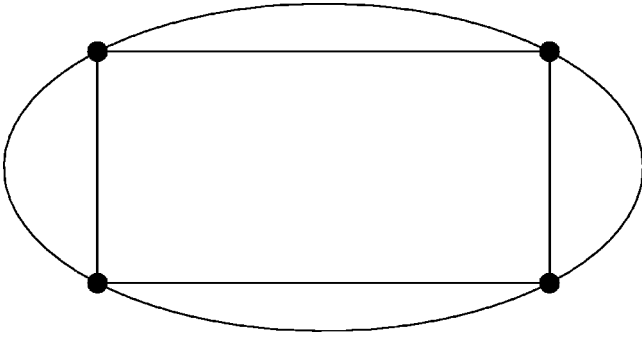FIG. 3. Four-loops contribution to the CTP effective action.

FIG. 4. Five-loops contribution to the CTP effective action.

FIG. 6. One-loop contribution to the self-energy (tadpole graph).

are interested in computing transport properties of this theory, we will need to include those terms in the perturbative expansion for $\Gamma[G]$ which will contribute to the bulk and shear viscosity in the weak-coupling, near-equilibrium limit. In order to have binary scattering of quasiparticles in the effective kinetic theory, we need to have a term with four propagators, which appears at $O(\lambda^2)$. In order to have number-changing processes such as two quasiparticles scattering into four (and vice versa), we need to include a term with *eight* (six asymptotic on-shell propagators and two internal lines [2]), which appears at $O(\lambda^4)$ in the 2PI effective action. This means taking into account the Feynman graphs in Figs. 1–5 [22]. Taking the functional derivative with respect to G_{AB} yields a formal equation for the two-point function of the theory:

$$\hbar D_{AB} - i\hbar (G^{-1})_{AB} - \frac{1}{2}\mathbf{T}_{AB} + \Pi_{AB} = 0, \qquad (29)$$

where we have singled out the tadpole term **T**, Fig. 6. The remainder of the self-energy (which we shall refer to as the self-energy, for short) is given by the sum of the graphs in Figs. 7–11 (Note: observe that in the graphs Figs. 1–4 all internal lines are equivalent; in Fig. 5 we have instead two sets of equivalent lines, marked a and b in the figure. Thus this last graph gives rise to two different graphs upon variation, i.e., Figs. 10 and 11). This is just the Dyson equation for the inverse propagator, where the self-energy is already
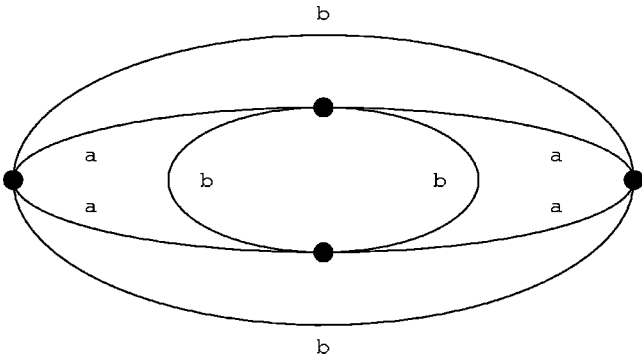
expressed in terms of the propagators themselves. In the sense of [19,20] we say the self-energy has been *slaved* to the propagators.

There are two ways we can proceed. We can either right-multiply G^{BC}, or left-multiply G^{CA} into the equation, obtaining the right-multiplied and left-multiplied Dyson equations, respectively. Note that only the tadpole term is invariant under simultaneous translations of the A and B spacetime indices. The higher-order terms all violate translation invariance in the equation of motion for the two-point function as a consequence of slaving—they describe the dissipative processes by which the system approaches equilibrium [5]. From now on we set $\hbar = 1$.

B. Nonperturbative properties of the propagators

Our strategy is as follows. In equilibrium, the propagators are translation invariant, and their Fourier transform are simply proportional [Kubo-Martin-Schwinger (KMS) theorem [16]]. Out of equilibrium, we write

$$G(x,x') = \int \frac{d^4 p}{(2\pi)^4} e^{ipu} G(X,p) \qquad (30)$$

with $u = x - x'$ and $X = (x+x')/2$. We assume that $G(X,p)$ is slowly varying with respect to the center-of-mass variable X.

Before we start, it is useful to display the properties of the propagators which actually follow from their definition as path ordered products of field operators. We consider eight different propagators.

FIG. 5. The other five-loops contribution to the CTP effective action. Observe the two sets of inequivalent lines, marked a and b.

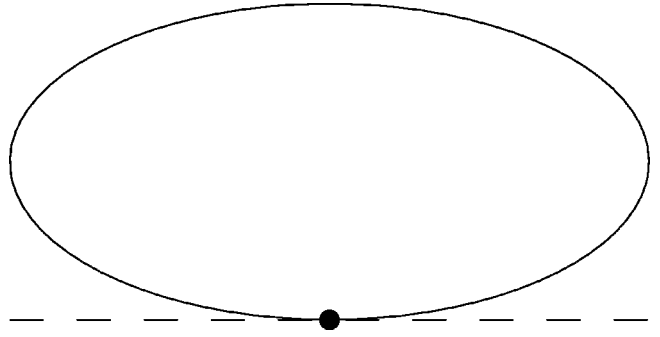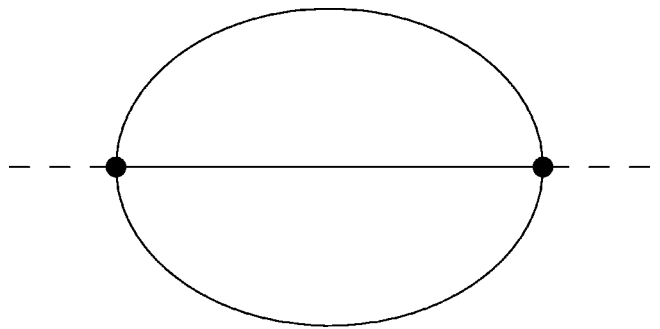FIG. 7. Two-loops contribution to the self-energy (setting sun graph).
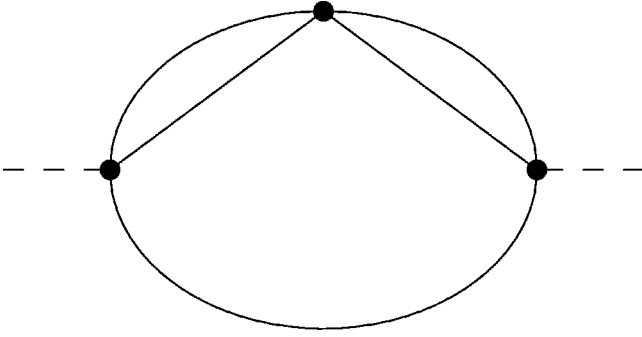
FIG. 8. Three-loops contribution to the self-energy.

(a) The four basic propagators, appearing in equation (24): Feynman $G^{11} \equiv \langle T(\Phi_H(x)\Phi_H(x')) \rangle$, where T stands for time ordering, Dyson $G^{22} \equiv \langle \tilde{T}(\Phi_H(x)\Phi_H(x')) \rangle$, where $\tilde{T}$ stands for antitime ordering, positive frequency $G^{21} \equiv \langle \Phi_H(x)\Phi_H(x') \rangle$ and negative frequency $G^{12} \equiv \langle \Phi_H(x')\Phi_H(x) \rangle$. The Feynman and Dyson propagators are even. We also have

$$G^{11} = G^{22*}; \quad G^{12} = G^{21*}; \quad G^{12}(x,x') = G^{21}(x',x). \tag{31}$$

As a consequence, G^{11} and $G^{22}(X,p)$ are even functions of momentum, while $G^{12}(X,p) = G^{21}(X,-p)$. Moreover, G^{12} and $G^{21}(X,p)$ are real, and $G^{22}(X,p)* = G^{11}$. Finally, we have the identity

$$G^{11} + G^{22} = G^{12} + G^{21}, \tag{32}$$

which follows from the path-ordering constraints

$$G^{11} = \theta(t-t')G^{21} + \theta(t'-t)G^{12}, \tag{33}$$

$$G^{22} = \theta(t-t')G^{12} + \theta(t'-t)G^{21}. \tag{34}$$

(b) The Hadamard propagator $G_1 = G^{21} + G^{12} \equiv \langle \{\Phi_H(x), \Phi_H(x')\} \rangle$ is real and even and, therefore, also is $G_1(X,p)$. The Jordan propagator $G = G^{21} - G^{12} \equiv \langle [\Phi_H(x), \Phi_H(x')] \rangle$ is imaginary and odd, and so $G(X,p)$ is odd but real.

(c) The advanced and retarded propagators

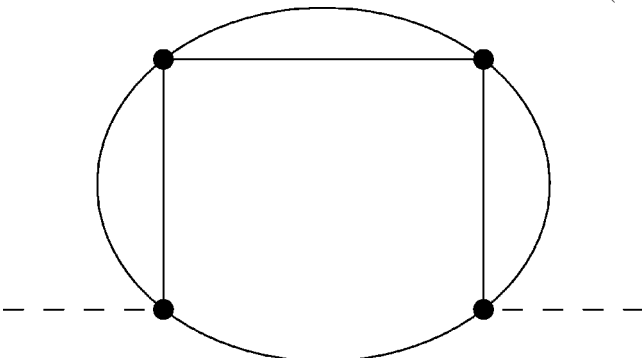$$G_{\mathrm{adv}}(x,x') = -iG(x,x')\theta(t'-t), \tag{35}$$

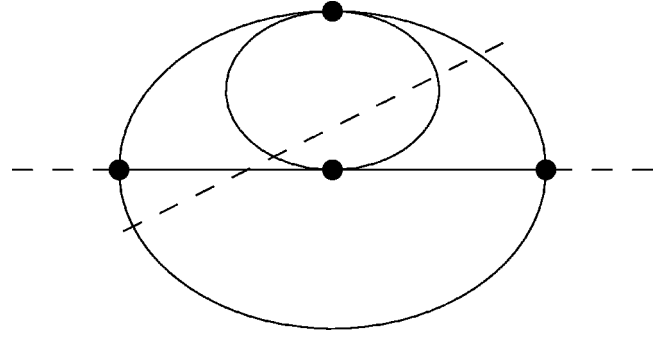FIG. 9. Four-loops contribution to the self-energy.

FIG. 10. Another four-loops contribution to the self-energy. Cutting as shown, we go across five internal lines. The symmetric cut also goes across five lines.

$$G_{\mathrm{ret}}(x,x') = G_{\mathrm{adv}}(x',x) = iG(x,x')\theta(t-t')$$

or else

$$G_{\mathrm{ret}} = i(G^{11} - G^{12}); \quad G_{\mathrm{adv}} = i(G^{22} - G^{12}). \tag{36}$$

Once G_{ret} is known, we can reconstruct G as

$$G(x,x') = (-i)[G_{\mathrm{ret}}(x,x') - G_{\mathrm{ret}}(x',x)] \tag{37}$$

So,

$$G(p) = (-i)[G_{\mathrm{ret}}(p) - G_{\mathrm{ret}}(-p)] = 2\,\mathrm{Im}\,G_{\mathrm{ret}}(p), \tag{38}$$

where we have used that $G_{\mathrm{ret}}(x,x')$ is real, so $G_{\mathrm{ret}}(-p) = G_{\mathrm{ret}}(p)^*$. Also observe that $G_{\mathrm{adv}}(p) = G_{\mathrm{ret}}(p)^*$.

Since the retarded propagator is causal, it satisfies the equation

$$G_{\mathrm{ret}} = \theta(t-t')G_{\mathrm{ret}}. \tag{39}$$

Therefore, the real and imaginary parts of its transform are Hilbert transforms of each other

$$G_{\mathrm{ret}}(p) = \frac{i}{2\pi}\int \frac{d\omega}{p^0 - \omega + i\varepsilon} G_{\mathrm{ret}}(\omega,\vec{p})$$

$$= \frac{1}{2}G_{\mathrm{ret}}(p) + \frac{i}{2\pi}PV\int \frac{d\omega}{p^0 - \omega} G_{\mathrm{ret}}(\omega,\vec{p}), \tag{40}$$
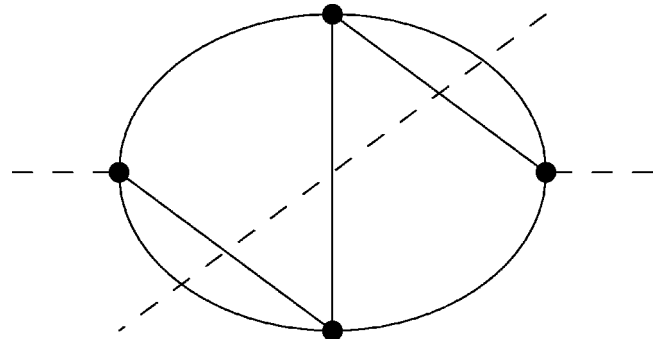

FIG. 11. The final four-loops contribution to the self-energy. Cutting as shown, we go across five internal lines.

$$\text{Re } G_{\text{ret}}(p) = \frac{1}{\pi} PV \int \frac{d\omega}{\omega - p^0} \text{Im } G_{\text{ret}}(\omega, \vec{p}). \tag{41}$$

This implies in particular that the real and imaginary parts are orthogonal to each other

$$\int d\omega \, \text{Im } G_{\text{ret}}(\omega, \vec{p}) \text{Re } G_{\text{ret}}(\omega, \vec{p}) = 0. \tag{42}$$

All other propagators can be decomposed in a similar way. For example, since

$$G^{11}(x,x') = \frac{1}{2}[G_1(x,x') + G(x,x')\text{sgn}(t-t')]$$

$$= \frac{1}{2}\{G_1(x,x') - i[G_{\text{ret}}(x,x') + G_{\text{ret}}(x',x)]\} \tag{43}$$

so

$$G^{11}(X,p) = \frac{1}{2}[G_1 - 2i \, \text{Re } G_{\text{ret}}]; $$

$$G^{22}(X,p) = \frac{1}{2}[G_1 + 2i \, \text{Re } G_{\text{ret}}]. \tag{44}$$

To give a nonperturbative definition of the one-particle distribution function f, which is the focus of attention in quantum kinetic theory, we assume that the partial Fourier transforms of the Hadamard and Jordan propagators are proportional

$$G_1 = \text{sgn}(p^0)[1+2f]G. \tag{45}$$

Introducing a density of states function $\Delta(p)$

$$G(p) \equiv 2\pi \, \text{sgn}(p^0)\Delta(p), \tag{46}$$

then

$$G_1 = 2\pi[1+2f]\Delta, \tag{47}$$

$$G^{21} = 2\pi[\theta(p^0)+f]\Delta = 2\pi F^{21}\Delta, \tag{48}$$

$$G^{12} = 2\pi[\theta(-p^0)+f]\Delta = 2\pi F^{12}\Delta. \tag{49}$$

In equilibrium, f is the Bose-Einstein distribution function (KMS theorem). It can be assumed that Eq. (45) serves as the definition of the function f, valid to all orders in perturbation theory. Observe that, since the relevant Fourier transforms are distributions (e.g., in free theories), this definition may only be applied if both Fourier transforms have the same singularity structure, which in the last analysis is a restriction on allowed quantum states. In what follows, we assume these restrictions are met.

C. The nonperturbative retarded and Jordan propagators

In the approximation where only terms linear in the gradients of the Fourier transforms of propagators are retained,

it is possible to write down a nonperturbative (in the coupling constant) expression for the retarded and Jordan propagators.

Let us obtain an equation for G_{ret} from, say, the equations for G^{11} and G^{12}, namely

$$G_{\text{ret}} = i(G^{11} - G^{12}) = i(G^{21} - G^{22}) \tag{50}$$

leading to

$$-1 = DG_{\text{ret}} - \frac{1}{2}\mathbf{T}_{11}G_{\text{ret}} + \Pi_{\text{ret}}G_{\text{ret}}, \tag{51}$$

where we have used that $\mathbf{T}_{12} = 0$, and

$$\Pi_{\text{ret}} = \Pi_{11} + \Pi_{12}. \tag{52}$$

Next we perform the Fourier transform. Since we are only interested in computing the transport coefficients, we only need to keep terms which are first order in gradients (this approximation which is formally invoked in the derivation of kinetic theory may not be always useful when dealing with realistic physical conditions, see, e.g., [23]). Therefore, in computing the transforms, we drop all second derivative terms. The free term $D = \Box - m_b^2$ transforms into

$$D = -p^2 + ip\frac{\partial}{\partial X} + \frac{1}{4}\Box_X - m_b^2. \tag{53}$$

We drop the D'Alembertian as it contains second derivatives:

$$D \sim -p^2 + ip\frac{\partial}{\partial X} - m_b^2. \tag{54}$$

The tadpole (for a generic propagator G) reads in position space

$$\mathbf{T}_{11}G = \lambda G^{11}(x,x)G(x,x'). \tag{55}$$

We write

$$\frac{\lambda}{4}G_1(x,x)G(x,x') = \frac{\lambda}{4}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{ipu}G_1(x,q)G(X,p) \tag{56}$$

and retain only terms linear in gradients to obtain

$$\frac{\lambda}{4}G_1(x,x)G(x,x') = \frac{\lambda}{4}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{ipu}G(X,p)$$

$$\times\left[G_1(X,q) + \frac{u}{2}\frac{\partial}{\partial X}G_1(X,q)\right]. \tag{57}$$

The contribution to the equation has the form

$$\left[-\delta M^2(X) - \frac{i}{2}\frac{\partial(\delta M^2)}{\partial X}\frac{\partial}{\partial p}\right]G(X,p), \tag{58}$$

where

$$\delta M^2(X) = \frac{\lambda}{4}\int \frac{d^4q}{(2\pi)^4}G_1(x,q). \tag{59}$$

Let us write the remaining term collectively as

$$\Pi G = \int d^4y\, \Pi(x,y)G(y,x'), \tag{60}$$

which transforms into

$$\int d^4y \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{ip(y-x')}e^{iq(x-y)}$$
$$\times \Pi\left(\frac{x+y}{2},q\right)G\left(\frac{x'+y}{2},p\right). \tag{61}$$

Keeping only first terms in gradients, this transforms into

$$\int d^4y \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{ip(y-x')}e^{iq(x-y)}$$
$$\times \left\{\Pi(X,q)G(X,p)+\frac{y-x'}{2}(\partial_X\Pi)(q)G(p)\right.$$
$$\left.+\frac{y-x}{2}\Pi(q)\partial_X G(p)\right\} \tag{62}$$

and then into

$$\int d^4y \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{ip(y-x')}e^{iq(x-y)}$$
$$\times \left\{\Pi(X,q)G(X,p)+\frac{i}{2}(\partial_X\Pi)(q)\partial_p G(p)\right.$$
$$\left.-\frac{i}{2}\partial_q\Pi(q)\partial_X G(p)\right\}, \tag{63}$$

which contributes a term

$$\left[\frac{i}{2}[(\partial_X\Pi_{\text{ret}})\partial_p-(\partial_p\Pi_{\text{ret}})\partial_X]+\Pi_{\text{ret}}(p)\right]G_{\text{ret}} \tag{64}$$

to the equation of motion.

Introducing the Poisson bracket

$$\{f,g\}=\partial_p f\partial_X g-\partial_X f\partial_p g, \tag{65}$$

we may write the equation for G_{ret} as

$$-1=-\Omega G_{\text{ret}}+\frac{i}{2}\{\Omega,G_{\text{ret}}\}, \tag{66}$$

where

$$\Omega=p^2+M^2-\Pi_{\text{ret}}(p),\quad M^2=m_b^2+\delta M^2, \tag{67}$$

and we get the *exact* (formal) solution

$$G_{\text{ret}}=[Z_b(p+i\varepsilon)^2+M^2-\Pi_{\text{ret}}(p)]^{-1}=\frac{1}{\Omega}\Bigg|_{\text{Im }p^0\to 0^+}, \tag{68}$$

where

$$(p+i\varepsilon)^2=-(p^0+i\varepsilon)^2+\vec{p}^2 \tag{69}$$

(we have displaced p^0 into the complex plane to account for the retarded boundary conditions). Now we write

$$G_{\text{ret}}=\text{Re }G_{\text{ret}}+\frac{i}{2}G. \tag{70}$$

Then

$$G=\frac{-2\,\text{Im }\Omega}{|\Omega|^2};\quad \Delta=\frac{-\text{sgn}(p^0)\text{Im }\Omega}{\pi|\Omega|^2}. \tag{71}$$

D. Equation for the negative frequency propagator

The equation for the negative frequency propagator is

$$DG^{12}-\frac{1}{2}\mathbf{T}_{1B}G^{B2}+\Pi_{1B}G^{B2}=0. \tag{72}$$

Recall that

$$G^{22}=G^{12}+iG_{\text{adv}},\quad G^*_{\text{ret}}=\frac{1}{\Omega^*}=\frac{\Omega}{|\Omega|^2}. \tag{73}$$

After Fourier transforming, we obtain

$$0=-\Omega\left[G^{12}-\frac{i\Pi_{12}}{|\Omega|^2}\right]+\frac{i}{2}\{\Omega,G^{12}\}+\frac{1}{2\Omega^{*2}}\{\Omega^*,\Pi_{12}\}. \tag{74}$$

In keeping with the stipulation to consider only first-order corrections to local thermal equilibrium, we shall neglect all terms containing both derivatives and radiative corrections. So the equation is equivalent to

$$0=-\Omega\left[G^{12}-\frac{i\Pi_{12}}{|\Omega|^2}\right]+\frac{i}{2}\{\Omega,G^{12}\}. \tag{75}$$

To separate this equation into real and imaginary parts, we must notice that the combination $i\Pi_{12}$ is actually real (see Appendix A).

E. The unperturbed theory

The unperturbed theory is obtained by neglecting the $O(\lambda^2)$ terms in our equations. The unperturbed equations are

$$\Omega_0=p^2+M^2 \tag{76}$$

and

$$\Delta(p)=\delta(p^2+M^2)+O(\lambda^2). \tag{77}$$

These are exact solutions of the above equations.

Concerning the distribution function, the real part of Eq. (75) above shows that G^{12} is concentrated on the zeroes of Ω, as required by Eq. (45), and, since $\{\Omega_0, \Delta\} = 0$, the imaginary part becomes the unperturbed transport equation

$$0 = \Delta \left[p \frac{\partial}{\partial X} - \frac{1}{2} \partial_X M^2 \partial_p \right] F^{12},$$

which is in the form of a Vlasov equation.

III. THE TRANSPORT EQUATION

The nonperturbative (in the coupling constant) equation we have derived for G^{12}, plus the decomposition Eq. (49) lead in a straightforward way to the transport equation. Neglecting $\{\Omega, \Delta\}$ as before, we write Eq. (75) as

$$0 = -\Omega \left[G^{12} - \frac{i\Pi_{12}}{|\Omega|^2} \right] + i\pi\Delta\{\mathrm{Re}\,\Omega, F^{12}\}. \quad (78)$$

Since $\mathrm{Im}\,\Omega = -|\Omega|^2 G/2$, its imaginary part reduces to

$$0 = \Delta \left[-F^{12}\,\mathrm{Im}\,\Omega - \frac{i}{2}\Pi_{12}\,\mathrm{sgn}(p^0) + \frac{1}{2}\{\mathrm{Re}\,\Omega, F^{12}\} \right], \quad (79)$$

which is the Boltzmann equation. To simplify it even further, we observe that since $\mathrm{sgn}(p^0) = F^{21} - F^{12}$,

$$\mathrm{Im}\,\Omega = -\mathrm{Im}\,\Pi_{\mathrm{ret}} = -\mathrm{Im}\,\Pi_{11} + i\Pi_{12} = (i/2)(\Pi_{12} - \Pi_{21}) \quad (80)$$

(see the Appendix), so

$$0 = \Delta \left[\frac{1}{2}\{\mathrm{Re}\,\Omega, F^{12}\} - \frac{i}{2}(\Pi_{12}F^{21} - \Pi_{21}F^{12}) \right]. \quad (81)$$

This equation is formally valid to all orders in the coupling constant. However, it is convenient to consider the loop expansion of Π to reduce this equation to a more familiar form.

A. The collision term

In this subsection we consider the expansion of the self-energy Π in terms of Feynman graphs of increasing loop order, as a means to obtain a definite expression for the collision term in the kinetic equation (81). Since we have the relationship $\Pi_{21}(p) = \Pi_{12}(-p)$ (see Appendix) it is enough to analyze only the expansion of Π_{12}. Physically this means considering only the gain processes, which produce a particle within a given phase-space cell. The collision term is then obtained by subtracting the loss processes, which remove a particle therein.

The first term in the expansion is the single two-loop graph Fig. 7. To this order,

$$\Pi_{12}(x,y) = \frac{-i}{6}\lambda^2\Sigma^{12}(x,y) = \frac{-i}{6}\lambda^2[G^{12}(x,y)]^3. \quad (82)$$

In momentum space, dealing with the propagators as if they were translation invariant,

$$\Sigma^{12}(p) = (2\pi)^4 \int \frac{d^4r}{(2\pi)^4} \frac{d^4s}{(2\pi)^4} \frac{d^4t}{(2\pi)^4} \delta(p - r - s - t) \\ \times G^{12}(r)G^{12}(s)G^{12}(t), \quad (83)$$

and, using the definition Eq. (49), we get

$$\Sigma^{12}(p) = (2\pi)^4 \int \frac{d^4r\,\Delta(r)}{(2\pi)^3} \frac{d^4s\,\Delta(s)}{(2\pi)^3} \frac{d^4t\,\Delta(t)}{(2\pi)^3} \\ \times \delta(p - r - s - t)F^{12}(r)F^{12}(s)F^{12}(t). \quad (84)$$

If we substitute Δ by its lowest order value $\Delta_0 = \delta(p^2 + M^2)$, this yields the collision term given earlier in [5]. This represents binary collisions, which conserves the particle number. For the reasons discussed in the Introduction, it leads to an inconsistency when one tries to compute the bulk viscosity coefficient.

The first correction to Eq. (84), within the two-loop theory, comes from the radiative corrections to the density of states, as given by Eqs. (71) and (80). We write

$$\Sigma^{12} = \Sigma_0^{12} + \delta\Sigma^{12}, \quad (85)$$

where Σ_0^{12} is the lowest order result just discussed, and (writing $\Omega_{0s} = s^2 + M^2$ for short)

$$\delta\Sigma^{12}(p) = 3i(2\pi)^3 \int \frac{d^4r\,\mathrm{sgn}(r^0)}{(2\pi)^3} \frac{d^4s\,\delta(\Omega_{0s})}{(2\pi)^3} \\ \times \frac{d^4t\,\delta(\Omega_{0t})}{(2\pi)^3} \delta(p - r - s - t) \\ \times G_{\mathrm{ret}}(r)G_{\mathrm{adv}}(r)F^{12}(r)F^{12}(s)F^{12}(t) \\ \times [\Pi_{21} - \Pi_{12}](r). \quad (86)$$

We use again Eq. (82) to get

$$\delta\Sigma^{12}(p) = \frac{-\lambda^2(2\pi)^4}{2} \int \frac{d^4s\,\delta(\Omega_{0s})}{(2\pi)^3} \frac{d^4t\,\delta(\Omega_{0t})}{(2\pi)^3} \\ \times \frac{d^4u\,\delta(\Omega_{0u})}{(2\pi)^3} \frac{d^4v\,\delta(\Omega_{0v})}{(2\pi)^3} \frac{d^4w\,\delta(\Omega_{0w})}{(2\pi)^3} \\ \times \delta(p + u + v + w + s + t)\sigma(u + v + w) \\ \times F^{21}(s)F^{21}(t)[F^{12}(u)F^{12}(v)F^{12}(w) \\ - F^{21}(u)F^{21}(v)F^{21}(w)], \quad (87)$$

where

$$\sigma(r) = \mathrm{sgn}(r^0)G_{\mathrm{ret}}(r)G_{\mathrm{adv}}(r)F^{21}(r).$$

A successful contribution to the gain part of the collision term describing scattering of two into four particles (this be-

ing the simplest particle number nonconserving process in this theory) must involve, besides the factor $(1+f_p)$ already explicit in Eq. (81), five other factors f or $1+f$ evaluated on on-shell momenta adding up to p. But Eq. (87) cannot contain a term like this, because of the interference between the two terms in brackets. After all cancellations, we are left with radiative corrections to the already known binary collision term. We conclude that to order λ^4 there are no contributions to a particle number nonconserving collision term arising from the setting sun graph. We must consider instead the higher loop graphs, Figs. 8–11.

B. Higher loops

Generally speaking, we expect the collision term to describe both particle number conserving $(2\rightarrow2)$ and changing $(2\rightarrow4)$ scattering. Because of parity, we do not expect transitions between an even and odd number of particles. The $2\rightarrow2$ scattering is already present in the two-loop theory,

and any further correction to it will not contribute to the transport functions. So from the three and four loop contributions we seek only terms related to $2\rightarrow4$ scattering.

Since we only seek the lowest order contribution to the bulk viscosity, we may substitute the density of states Δ by a delta function concentrated on mass shell, so the notions of on and off shell recover their usual meaning. It is then possible to ascertain from the momentum flow in the graph whether the condition of five on-shell momenta adding to p may be fulfilled: this is just the question of whether it is possible to cut the graph by going across five internal lines [24]. The three-loop contribution cannot satisfy this criterium, and we shall not analyze it further (it only renormalizes the binary scattering amplitude). For the same reason, we discard the graph in Fig. 9, and concentrate on the graphs in Figs. 10 and 11, which *a priori* pass the test.

The complete contribution to Π_{12} from the graph in Fig. 10 reads

$$\frac{-i\lambda^4}{4}(2\pi)^{12}\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4u}{(2\pi)^4}\frac{d^4v}{(2\pi)^4}\frac{d^4w}{(2\pi)^4}\delta(q+r+s-p)\delta(t+u+v-q)\delta(u+t+r-w)$$

$$\times\{2G^{11}(q)G^{12}(r)G^{12}(s)G^{12}(t)G^{12}(u)G^{12}(v)G^{22}(w)-G^{11}(q)G^{11}(r)G^{12}(s)G^{11}(t)G^{11}(u)G^{12}(v)G^{12}(w)$$

$$-G^{12}(q)G^{12}(r)G^{12}(s)G^{22}(t)G^{22}(u)G^{22}(v)G^{22}(w)\}. \tag{88}$$

In a true contribution to $2\rightarrow4$ scattering, the six on-shell momenta involved (including p) are *irreducible*, in the sense that there are no other linear relations among them than overall momentum conservation. If we look at the three terms in curly brackets in Eq. (88), we see that in the second term the three momenta s, v, and w are on shell, but they satisfy the linear relation $s+v+w-p=0$, irrespective of the other momenta. Thus this term is not irreducible, and does not contribute to $2\rightarrow4$ scattering; it is another radiative correction to the binary collision term. The same analysis disposes of the third term, since here the on-shell momenta q, r, and s are constrained to satisfy $q+r+s-p=0$. We will disregard these two terms.

The graph in Fig. 11 contributes

$$\frac{-i\lambda^4}{4}(2\pi)^{12}\int \frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4u}{(2\pi)^4}\frac{d^4v}{(2\pi)^4}\frac{d^4w}{(2\pi)^4}\delta(q+r+s-p)\delta(t+u+v-q)\delta(t+r+s-w)$$

$$\times\{G^{11}(q)G^{12}(r)G^{12}(s)G^{12}(t)G^{12}(u)G^{12}(v)G^{22}(w)+G^{12}(q)G^{11}(r)G^{11}(s)G^{21}(t)G^{22}(u)G^{22}(v)G^{12}(w)$$

$$-G^{11}(q)G^{11}(r)G^{11}(s)G^{11}(t)G^{12}(u)G^{12}(v)G^{12}(w)-G^{12}(q)G^{12}(r)G^{12}(s)G^{22}(t)G^{22}(u)G^{22}(v)G^{22}(w)\}. \tag{89}$$

Only the first term in curly brackets is irreducible. Retaining only the irreducible contributions from both graphs, we get the prospective particle number nonconserving collision term as

$$\frac{-i\lambda^4}{4}(2\pi)^4\int \frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\frac{d^4u}{(2\pi)^4}\frac{d^4v}{(2\pi)^4}\delta(r+s+t+u+v-p)\sigma^2(-p,r,s,t,u,v)$$

$$\times G^{12}(r)G^{12}(s)G^{12}(t)G^{12}(u)G^{12}(v),$$

$$\sigma^2=G^{11}(p+r+s)[2G^{22}(p+v+s)+G^{22}(p+u+v)]. \tag{90}$$

It is clear that only the totally symmetric (as a function of r, s, t, u, and v) part σ_s^2 of σ^2 contributes to the integral, so we shall assume that σ^2 has been symmetrized.

To reduce Eq. (90) to a more familiar form, let us assume that $p^0>0$, and restrict the integral to future oriented momenta (that is, when a momentum is past oriented, we reverse its sign). Because of momentum conservation, some momenta must be future oriented, but because they are all on mass shell, they cannot be all future oriented at the same time; the number of future

oriented momenta can only be 4, 3 or 2. The terms with three future oriented momenta describe $3 \rightarrow 3$ scattering, which conserves particle number, so they are not related to the bulk viscosity. With these considerations we finally get the particle number nonconserving collision term as

$$I_{2\rightarrow4} = \int \frac{d^4r\,\theta(r^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4s\,\theta(s^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4t\,\theta(t^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4u\,\theta(u^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4v\,\theta(v^0)\,\delta(\Omega)}{(2\pi)^3}$$
$$\times\{R_1\delta_1[(1+f_p)(1+f_r)(1+f_s)(1+f_t)f_uf_v-(1+f_u)(1+f_v)f_pf_rf_sf_t]$$
$$+R_2\delta_2[(1+f_p)(1+f_r)f_sf_tf_uf_v-(1+f_s)(1+f_t)(1+f_u)(1+f_v)f_pf_r]\}, \tag{91}$$

where

$$R_1 \equiv \frac{5\lambda^4}{4}(2\pi)^4\sigma_s^2(-p,-r,-s,-t,u,v); \tag{92}$$

$$R_2 \equiv \frac{5\lambda^4}{8}(2\pi)^4\sigma_s^2(-p,-r,s,t,u,v), \tag{}$$

$$\delta_1 \equiv \delta(p+r+s+t-u-v); \tag{93}$$

$$\delta_2 \equiv \delta(p+r-s-t-u-v). \tag{}$$

IV. THERMODYNAMICS FROM QUANTUM KINETIC THEORY

Our goal in this section is to investigate the thermodynamic and hydrodynamic properties of a quantum field, particularly the equation of state and the speed of sound. Our starting point is the on-shell Boltzmann equation (79). To render the Poisson bracket manageable, we keep only the unperturbed Ω, equation (76), where M^2 is given self-consistently by Eqs. (59) and (67), namely

$$M^2 = m_b^2 + \delta M^2; \tag{94}$$

$$\delta M^2(X) = \frac{\lambda_b}{2}\int\frac{d^4p}{(2\pi)^3}\delta(\Omega_0)\left[\frac{1}{2}+f(X,p)\right]. \tag{}$$

The kinetic equation can be written as

$$\frac{1}{2}\{\Omega_0,f\} = I_{col}(X,p), \tag{95}$$

where I satisfies the constraint

$$\int\frac{d^4p}{(2\pi)^4}\theta(p^0)\delta(\Omega_0)p^\mu I_{col}(X,p)=0, \tag{96}$$

which expresses energy-momentum conservation. Our concern is to investigate this (only) conservation law, but first, we need to express the gap equation (94) in terms of finite quantities.

A. The gap equation

Let us write the gap equation as

$$M^2 = m_b^2 + m_V^2 + \frac{\lambda_b}{2}M_T^2, \tag{97}$$

where

$$M_T^2 = \int\frac{d^4p}{(2\pi)^3}\delta(\Omega_0)f(X,p), \tag{98}$$

$$m_V^2 = \frac{\lambda_b}{4}\int\frac{d^4p}{(2\pi)^3}\delta(\Omega_0). \tag{99}$$

This second quantity is actually divergent, so to evaluate it we need to regularize it first. We shall use dimensional regularization, writing

$$m_V^2 = \frac{\lambda_b}{2}\mu^\varepsilon\int\frac{d^dp}{(2\pi)^d}\frac{(-i)}{p^2+M^2-i\varepsilon}, \tag{100}$$

where the dimensionality $d=4-\varepsilon$. We also go to Euclidean momenta, $p^0\rightarrow ip^0$, so

$$m_V^2 = \frac{\lambda_b}{2}\mu^\varepsilon\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+M^2}. \tag{101}$$

We obtain

$$m_V^2 = -\frac{\lambda_bM^2}{16\pi^2}\left(\frac{M^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\frac{\Gamma\left[1+\frac{\varepsilon}{2}\right]}{\varepsilon\left[1-\frac{\varepsilon}{2}\right]}.$$

Then, we write

$$\frac{\Gamma\left[1+\frac{\varepsilon}{2}\right]}{\varepsilon\left[1-\frac{\varepsilon}{2}\right]} = \frac{1}{\varepsilon}+\frac{1}{2}(1-\gamma)+\cdots\equiv z \tag{102}$$

$(\gamma=.5772\ldots)$. We get

$$m_V^2 = -\frac{\lambda_B M^2}{16\pi^2}\left[z - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right].\qquad(103)$$

We go back to the gap equation and write it as

$$ZM^2 = m_b^2 + \frac{\lambda_b}{2}\left[\frac{M^2}{16\pi^2}\ln\left(\frac{M^2}{4\pi\mu^2}\right) + M_T^2\right],\qquad(104)$$

where

$$Z = 1 + \frac{z\lambda_b}{16\pi^2}.\qquad(105)$$

We now renormalize the bare couplings

$$m_b^2 = Zm^2;\quad \lambda_b = Z\lambda\qquad(106)$$

(the actual coupling in d dimensions being $\mu^\varepsilon\lambda_b$) to obtain the physical gap equation

$$M^2 = m^2 + \frac{\lambda}{2}M_f^2,\qquad(107)$$

where

$$M_f^2 = \frac{M^2}{16\pi^2}\ln\left(\frac{M^2}{4\pi\mu^2}\right) + M_T^2.\qquad(108)$$

With these we turn our attention to the energy-momentum tensor.

B. Energy-momentum tensor

To define the energy-momentum tensor, we write the effective action in a general curved background, and then use the customary definition [25]

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}^{(1)}},\qquad(109)$$

where only the derivative with respect to the metric in the first time branch is taken. The effective action itself is given by Eq. (28). The first term $\mathrm{Tr}\ln G$ does not depend on the metric. Written in full, the second term reads

$$\frac{1}{2}\int d^4x\{\sqrt{-g^{(1)}}(\Box_x^{(1)} - m_b^2)G^{11}(x,x')|_{x'=x} - (1\to2)\}.\qquad(110)$$

As usual

$$\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu};\quad \frac{\delta g^{\mu\nu}}{\delta g_{\rho\sigma}} = -g^{\mu\rho}g^{\nu\sigma},\qquad(111)$$

and so the contribution from this term to $T^{\mu\nu}$ is

$$\left[-\partial^\mu\partial^\nu + \frac{1}{2}\eta^{\mu\nu}(\Box_x - m_b^2)\right]G^{11}(x,x')|_{x'=x}.\qquad(112)$$

In the third term, the metric appears through the $\sqrt{-g}$ factors multiplying the coupling constants. Therefore, the contribution to $T^{\mu\nu}$ takes the form

$$\eta^{\mu\nu}\lambda_B\frac{\delta}{\delta\lambda_{1111}}\Gamma_2 = -\frac{\lambda_b}{8}\eta^{\mu\nu}[G^{11}(x,x)]^2 + \Lambda_b\eta^{\mu\nu},\qquad(113)$$

where Λ_b contains all the higher-order contributions. To the accuracy desired, Λ_b is position independent, and we shall not analyze it further. Adding the two nontrivial contributions we get

$$T^{\mu\nu}(X) = -\left[\partial^\mu\partial^\nu - \frac{1}{2}\eta^{\mu\nu}\Box_x\right]G^{11}(x,x')|_{x'=x}$$
$$-\frac{1}{2}\eta^{\mu\nu}\left[m_b^2 + \frac{\lambda_b}{4}G^{11}\right]G^{11} + \Lambda_b\eta^{\mu\nu}.\quad(114)$$

To write the first term in terms of the distribution function, observe that $\partial_x\to ip + \frac{1}{2}\partial_X$. We must neglect second derivative terms, and observe that terms involving $p\,\partial_X$ eventually vanish because $G^{11}(X,p)$ is even in p. So

$$T^{\mu\nu}(X) = \int\frac{d^4p}{(2\pi)^4}\left[p^\mu p^\nu - \frac{1}{2}\eta^{\mu\nu}p^2\right]G^{11}(X,p)$$
$$-\frac{1}{2}\eta^{\mu\nu}\left[m_b^2 + \frac{\lambda_b}{4}G^{11}\right]G^{11} + \Lambda_b\eta^{\mu\nu}.\quad(115)$$

We are entitled to use the unperturbed approximation for G^{11}:

$$G^{11} = \frac{(-i)}{p^2 + M^2 - i\varepsilon} + 2\pi\delta(\Omega_0)f(X,p).\qquad(116)$$

The expressions that appear in $T^{\mu\nu}$ are divergent and we must regularize them. Let us consider

$$T_V^{\mu\nu} = -i\int\frac{d^dp}{(2\pi)^d}\frac{\left[p^\mu p^\nu - \frac{1}{2}\eta^{\mu\nu}p^2\right]}{p^2 + M^2 - i\varepsilon}$$
$$= \left(\frac{i(d-2)}{2d}\right)\eta^{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\frac{p^2}{p^2 + M^2 - i\varepsilon}.\quad(117)$$

We rotate the integral into the Euclidean domain and compute the integral in $d = 4 - \varepsilon$ dimensions, so finally

$$T_V^{\mu\nu} = -\frac{M^4\eta^{\mu\nu}}{32\pi^2}\left[z - \frac{1}{4} - \frac{1}{2}\ln\left(\frac{M^2}{4\pi\mu^2}\right)\right].\qquad(118)$$

We also need

$$G^{11}(x,x) = \frac{2}{\lambda_B}\delta M^2 = Z^{-1}\left[M_f^2 - Z\frac{zm^2}{8\pi^2}\right].$$

Therefore,

$$m_b^2 + \frac{\lambda_b}{4} G^{11}(x,x) = Z \left[m^2 + \frac{\lambda}{4Z} \left(M_f^2 - Z \frac{zm^2}{8\pi^2} \right) \right]$$

$$= Z \left[m^2 + \frac{\lambda}{4} \left(M_f^2 - \frac{zm^2}{8\pi^2} \right) \right] + O(\lambda^2)$$

(119)

and

$$\left[m_b^2 + \frac{\lambda_b}{4} G^{11} \right] G^{11} = -m^2 \left(\frac{zm^2}{8\pi^2} - M_f^2 \right)$$

$$- \frac{\lambda}{4} \left[\left(\frac{zm^2}{8\pi^2} \right)^2 + \frac{zm^2 M_f^2}{4\pi^2} - M_f^4 \right]$$

$$+ O(\lambda^2).$$

So far, we get

$$T_V^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} \left[m_B^2 + \frac{\lambda_B}{4} G^{11} \right] G^{11}$$

$$= -\eta^{\mu\nu} \left\{ -\frac{\lambda}{8} \left(\frac{zm^2}{8\pi^2} \right)^2 - \frac{zm^4}{32\pi^2} - \frac{m^4}{2\lambda} \right.$$

$$\left. + \frac{M^2 m^2}{2\lambda} - \frac{M^4}{128\pi^2} + \frac{M^2 M_T^2}{4} + O(\lambda^2) \right\}.$$

(120)

Next, we call

$$T_T^{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} p^\mu p^\nu 2\pi \delta(\Omega_0) f(X,p), \qquad (121)$$

and observe that

$$\int \frac{d^4 p}{(2\pi)^4} \left[p^\mu p^\nu - \frac{1}{2} \eta^{\mu\nu} p^2 \right] 2\pi \delta(\Omega_0) f(X,p)$$

$$= T_T^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} M^2 M_T^2, \qquad (122)$$

so

$$T^{\mu\nu} = T_0^{\mu\nu} + T_f^{\mu\nu} + T_T^{\mu\nu}, \qquad (123)$$

where

$$T_0^{\mu\nu} = \eta^{\mu\nu} \left\{ \frac{\lambda}{8} \left(\frac{zm^2}{8\pi^2} \right)^2 + \frac{zm^4}{32\pi^2} + \frac{m^4}{2\lambda} \right\}, \qquad (124)$$

$$T_f^{\mu\nu} = -\Lambda_f \eta^{\mu\nu};$$

$$\Lambda_f = \frac{M^2 m^2}{2\lambda} - \frac{M^4}{128\pi^2} - \frac{M^2 M_T^2}{4} + O(\lambda^2). \qquad (125)$$

Here, $T_0^{\mu\nu}$ is infinite, but state independent and conserved. It belongs to the theory of the renormalization of the gravitational action (see [25,26] and references therein), and we shall not consider it further.

Consistency requires that we actually neglect the $O(\lambda^2)$ terms in Λ_f, or at least that we consider them as a true (temperature independent) constant. Then we can establish the following identity, which will be useful later on. First, we write

$$M_T^2 = \frac{2}{\lambda} (M^2 - m^2) - \frac{M^2}{16\pi^2} \ln \left(\frac{M^2}{4\pi\mu^2} \right),$$

$$\Lambda_f = \frac{M^2 m^2}{\lambda} - \frac{M^4}{128\pi^2} - \frac{M^4}{2\lambda} + \frac{M^4}{64\pi^2} \ln \left(\frac{M^2}{4\pi\mu^2} \right) + \text{const.}$$

(126)

Then observe that

(a) Λ_f depends on temperature only through the physical mass M^2, and

(b)

$$\frac{d\Lambda_f}{dM^2} = \frac{m^2}{\lambda} - \frac{M^2}{\lambda} + \frac{M^4}{32\pi^2} \ln \left(\frac{M^2}{4\pi\mu^2} \right) = \frac{-1}{2} M_T^2.$$

(127)

This is the identity we need below. This expression for the energy momentum tensor is equivalent to that given by Jeon and Yaffe. In particular, Eq. (127) implies that energy momentum conservation follows from the transport equation.

C. Entropy flux and the *H* theorem

Let us mention also the entropy flux

$$S^\mu = 2 \int \frac{d^4 p}{(2\pi)^4} \theta(p^0) p^\mu 2\pi \delta(\Omega_0) \{ (1+f) \ln(1+f) - f \ln f \}.$$

(128)

Associated with this, entropy generation is given by

$$S_{;\mu}^\mu = 2 \int \frac{d^4 p}{(2\pi)^4} \theta(p^0) 2\pi \delta(\Omega_0) \left[\ln \frac{(1+f)}{f} \right] I_{\text{col}}.$$

(129)

The positivity of this integral expresses the *H* theorem. Let us write

$$I_{\text{col}} = I_{2 \to 2} + I_{2 \to 4}, \qquad (130)$$

where the first term is the usual binary collision term

$$I_{2\to2}=\sigma\int\frac{d^4r\,\theta(r^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4s\,\theta(s^0)\,\delta(\Omega)}{(2\pi)^3}$$

$$\times\frac{d^4t\,\theta(t^0)\,\delta(\Omega)}{(2\pi)^3}\delta(p+r-s-t)$$

$$\times\{(1+f_p)(1+f_r)f_sf_t-(1+f_s)(1+f_t)f_pf_r\}$$

$$(131)$$

and the second term involves the number changing interactions, already given in Eq. (91).

When inserted in Eq. (129), we find

$$S^\mu_{;\mu}=H_{2\to2}+H_{2\to4},\qquad(132)$$

where $H_{2\to2}$ is the usual result [14]

$$H_{2\to2}=\frac{1}{2}\int\frac{d^4p\,\theta(p^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4r\,\theta(r^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4s\,\theta(s^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4t\,\theta(t^0)\,\delta(\Omega)}{(2\pi)^3}\left[\ln\frac{(1+f_p)(1+f_r)f_sf_t}{(1+f_s)(1+f_t)f_pf_r}\right]$$

$$\times\delta(p+r-s-t)\{(1+f_p)(1+f_r)f_sf_t-(1+f_s)(1+f_t)f_pf_r\},\qquad(133)$$

whereas [from $I_{2\to4}$ in Eq. (91)]

$$H_{2\to4}=\frac{1}{3}\int\frac{d^4p\,\theta(p^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4r\,\theta(r^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4s\,\theta(s^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4t\,\theta(t^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4u\,\theta(u^0)\,\delta(\Omega)}{(2\pi)^3}\frac{d^4v\,\theta(v^0)\,\delta(\Omega)}{(2\pi)^3}$$

$$\times(R_1-R_2)\left[\ln\frac{(1+f_p)(1+f_r)(1+f_s)(1+f_t)f_uf_v}{(1+f_u)(1+f_v)f_pf_rf_sf_t}\right]\delta(p+r+s+t-u-v)$$

$$\times[(1+f_p)(1+f_r)(1+f_s)(1+f_t)f_uf_v-(1+f_u)(1+f_v)f_pf_rf_sf_t]\qquad(134)$$

is new. Thus the H theorem demands the inequality

$$R_1\geq R_2.\qquad(135)$$

We expect that the integral will be dominated by grazing collisions, where one of the reactants and one of the products carry essentially all the momentum. In this limit, $R_1\sim2R_2$ [see Eq. (92)], so the H theorem is satisfied.

D. (Local) thermal equilibrium states

Our next concern is to investigate the equation of state, for a local equilibrium state described by a Planckian distribution function f_0 as in Eq. (10). The energy momentum tensor is decomposed as in Eq. (1). The thermal component $T_T^{\mu\nu}$ admits a similar decomposition:

$$T_{0T}^{\mu\nu}=\int\frac{d^4p}{(2\pi)^4}p^\mu p^\nu2\pi\delta(\Omega_0)f_0(X,p)=\rho_Tu^\mu u^\nu+p_TP^{\mu\nu},\qquad(136)$$

where

$$\rho_T=\int\frac{d^4p}{(2\pi)^4}(up)^22\pi\delta(\Omega_0)f_0(X,p).\qquad(137)$$

Since ρ_T and M^2 are scalars, we may compute them in the rest frame:

$$\rho_T=\frac{1}{\pi^2}\int_M^\infty d\omega\,\frac{\omega^2}{e^{\beta\omega}-1}\sqrt{\omega^2-M^2},$$

$$(138)$$

$$M_T^2=\frac{1}{\pi^2}\int_M^\infty d\omega\,\frac{1}{e^{\beta\omega}-1}\sqrt{\omega^2-M^2}.$$

For the thermal pressure, we find $3p_T-\rho_T=-M^2M_T^2$, so

$$p_T=\frac{1}{3}(\rho_T-M^2M_T^2).\qquad(139)$$

The total energy density and pressure are then

$$\rho=\rho_T+\Lambda_f;\quad p=p_T-\Lambda_f.\qquad(140)$$

The equilibrium entropy flux takes the form $S_0^\mu=p\beta^\mu-T_0^{\mu\nu}\beta_\nu=(\rho+p)\beta^\mu=(\rho_T+p_T)\beta^\mu$. On the other hand, Eq. (128) yields $S_0^\mu=\Phi_{0T}^\mu-T_{0T}^{\mu\nu}\beta_\nu$, where

$$\Phi_{0T}^\mu=-2\int\frac{d^4p}{(2\pi)^4}\theta(p^0)p^\mu2\pi\delta(\Omega_0)\ln[1-e^{-|\beta_\mu p^\mu|}].$$

$$(141)$$

This form of the thermodynamic potential brings to our attention other equivalent expressions for the thermal pressure

$$\frac{p_T}{T} = \frac{-1}{\pi^2} \int_M^\infty d\omega\, \omega \sqrt{\omega^2 - M^2}\, \ln[1 - e^{-\beta\omega}] \qquad (142)$$

and

$$p_T = \frac{1}{3\pi^2} \int_M^\infty d\omega [\omega^2 - M^2]^{3/2} f_0. \qquad (143)$$

Observe that Eqs. (127) and (142) imply the thermodynamic relationship Eq. (2) (here and henceforth, we shall use d/dT to denote a total temperature derivative, that is with respect to the explicit temperature dependence through f_0 as well as the implicit dependence through M^2. We shall use $\partial/\partial T$ when we mean only the former). Indeed, Eq. (142) implies

$$T\frac{dp_T}{dT} = p_T + \rho_T - \frac{M_T^2}{2} T \frac{dM^2}{dT}. \qquad (144)$$

But $p_T + \rho_T = \rho + p$, and

$$T\frac{dp}{dT} = T\frac{dp_T}{dT} - T\frac{d\Lambda_f}{dT}. \qquad (145)$$

So Eq. (2) follows from Eq. (127). This concludes our study of the local equilibrium states.

V. LINEARIZED TRANSPORT EQUATION

Under local thermal equilibrium, the transport equation is violated. We have $I_{\text{col}} = 0$, while the transport part (for $p^0 > 0$)

$$\left[p^\mu \frac{\partial}{\partial X^\mu} - \frac{1}{2} M^2_{,\mu} \frac{\partial}{\partial p_\mu} \right] f_0$$

$$= f_0(1 + f_0) \left[p^\mu p^\nu \beta_{\mu,\nu} - \frac{1}{2} M^2_{,\mu} \beta^\mu \right]. \qquad (146)$$

Recalling the decomposition Eq. (6) and assuming the macroscopic equations Eq. (5), the lhs of the transport equation becomes

$$f_0(1 + f_0) \left[\frac{1}{T} p_\mu p_\nu H^{\mu\nu} - \frac{1}{T} \left\{ (p \cdot u)^2 \left[c_s^2 - \frac{1}{3} \right] \right. \right.$$

$$\left. \left. + \frac{M^2}{3} - \frac{c_s^2}{2} T M^2_{,T} \right\} u^\lambda_{,\lambda} \right]. \qquad (147)$$

This plays the role of the Q_E differential operator in Eq. (15).

A. The linearized collision term

At this point we need to shift our attention to the righthand side of the transport equation, Eqs. (130), (131), and (91). The collision term vanishes identically under local thermal equilibrium, so we need to consider a distribution function deviating from it. Write

$$f = f_0 + f_0(1 + f_0)\chi. \qquad (148)$$

Since $I_{\text{col}}[f_0] \equiv 0$, only the deviation contributes to the collision integral. We keep only linear terms, and write, by analogy with Eq. (147)

$$\delta I_{\text{col}} = f_{0p}(1 + f_{0p})[\delta I_{2\to2} + \delta I_{2\to4}], \qquad (149)$$

where, upon introducing the momentum space volume element

$$Dp = \frac{d^4p\, \theta(p^0)\, \delta(\Omega)}{(2\pi)^3} f_{0p}(1 + f_{0p}), \qquad (150)$$

we have

$$\delta I_{2\to2} = \sigma \int Dr Ds Dt\, \delta(p + r - s - t)$$

$$\times \frac{\{-\chi_p - \chi_r + 2\chi_s\}}{[(1 + f_{0p})(1 + f_{0r})f_{0s}f_{0t}]}, \qquad (151)$$

and similarly

$$\delta I_{2\to4} = \int Dr Ds Dt Du Dv$$

$$\times \left\{ R_1 \delta_1 \frac{[-\chi_p - 3\chi_r + 2\chi_u]}{[(1 + f_p)(1 + f_r)(1 + f_s)(1 + f_t)f_u f_v]} \right.$$

$$\left. + R_2 \delta_2 \frac{[-\chi_p - \chi_r + 4\chi_s]}{[(1 + f_p)(1 + f_r)f_s f_t f_u f_v]} \right\}, \qquad (152)$$

where $R_{1,2}$ and $\delta_{1,2}$ were defined in Eqs. (92) and (93).

B. The method of moments

Given Eqs. (147) and (149), the linearized transport equation can be rewritten as

$$\frac{1}{T} p_\mu p_\nu H^{\mu\nu} - \frac{u^\lambda_{,\lambda}}{T} \left\{ (p \cdot u)^2 \left[c_s^2 - \frac{1}{3} \right] + \frac{M^2}{3} - \frac{c_s^2}{2} T M^2_{,T} \right\}$$

$$= K[\chi], \qquad (153)$$

where K is a Hermitian operator in the space of functions defined on the positive energy mass shell. We further introduce an inner product in this space by defining

$$\langle s, \chi \rangle = \int Dp\, s^*(p)\chi(p); \quad \langle\chi\rangle \equiv \langle 1, \chi \rangle. \qquad (154)$$

For our purposes, it will be enough to forfeit a rigorous solution, and to seek instead a solution using the method of moments. This entails first writing Eq. (153) in the orthogonal basis built out of the monomials 1, p^μ, $p^\mu p^\nu$, etc. [always with respect to the inner product Eq. (154), with Dp defined as in Eq. (150)], and then truncating it to only the first few moments.

To simplify our notation, let us adopt the local rest frame, and write $\omega = p^0 = -u \cdot p$. Let $\chi_0 = 1$ be the first element of our basis. The remaining functions are ($i = 1,2,3$):

$$\chi_1 = \omega - \frac{\langle \omega \rangle}{\langle 1 \rangle};$$

$$\chi_2 = \omega^2 + \omega \frac{\langle \omega^2 \rangle \langle \omega \rangle - \langle \omega^3 \rangle \langle 1 \rangle}{\langle 1 \rangle \langle \omega^2 \rangle - \langle \omega \rangle^2} + \frac{\langle \omega^3 \rangle \langle \omega \rangle - \langle \omega^2 \rangle^2}{\langle 1 \rangle \langle \omega^2 \rangle - \langle \omega \rangle^2},$$

$$q_1^i = p^i; \quad q_2^i = p^i \left[\omega - \frac{\langle \omega \vec{p}^2 \rangle}{\langle \vec{p}^2 \rangle} \right]. \tag{155}$$

To this we must add five independent functions built out of the binary products $p^i p^j$ (there are only five independent functions, because $\vec{p}^2 = \omega^2 - M^2$ is not independent of the above). The simplest procedure is to think of these monomials as the composition of two spin 1 objects; the spin zero component of the composition is precisely $\vec{p}^2$, and the spin 1 part, being antisymmetric, will vanish, so our functions are the five $l = 2$ spherical harmonics. For example, calling $p_{\pm} = p_x \pm i p_y$, we may choose

$$Y_m = (p_+^2, p_+ p_z, p_z^2 - p_+ p_-, p_- p_z, p_-^2); \quad 2 \geq m \geq -2. \tag{156}$$

We also have the relationships (see Appendix B)

$$\langle \omega \rangle = T^2 \frac{d\rho}{dT} \frac{[1 - 3c_s^2]}{\left[M^2 - \frac{1}{2} T M_{,T}^2 \right]}, \tag{157}$$

$$\langle \omega^3 \rangle = T^2 \frac{d\rho}{dT} \frac{\left[M^2 - \frac{3}{2} T M_{,T}^2 c_s^2 \right]}{\left[M^2 - \frac{1}{2} T M_{,T}^2 \right]}, \tag{158}$$

$$\langle \omega \vec{p}^2 \rangle = 3 c_s^2 T^2 \frac{d\rho}{dT}. \tag{159}$$

In terms of the new functions, Eq. (153) reads

$$\frac{1}{T} \Gamma_{ij}^m Y_m H^{ij} + \Gamma \left\{ \langle \omega \rangle \chi_2 + c \left[\frac{\langle \omega \rangle \chi_1}{\langle \omega, \chi_1 \rangle} - 1 \right] \right\} = K[\chi], \tag{160}$$

where

$$\Gamma = \frac{u_{,\lambda}^\lambda \left[M^2 - \frac{1}{2} T M_{,T}^2 \right]}{3 T^3 \frac{d\rho}{dT}}; \quad \Gamma_{ij}^m = \frac{\left\langle Y_m, p_i p_j - \frac{1}{3} \vec{p}^2 \right\rangle}{\langle Y_m, Y_m \rangle};$$

$$c = \frac{\langle \omega^3 \rangle \langle 1 \rangle - \langle \omega^2 \rangle \langle \omega \rangle}{\langle 1 \rangle}. \tag{161}$$

When we expand the operator $K[\chi]$ we notice that, when truncated to the subspace spanned by the functions Eq. (155), the operator matrix acquires a block form, with one block corresponding to the χ_a functions ($a = 0$, 1 or 2), another to the q_a^i, and yet another to the Y_m functions. Since there are no q_a^i functions in the left-hand side of Eq. (160), we may as well write

$$\chi = \frac{-1}{T} b_m Y^m - \Gamma [A + B\chi_1 + C\chi_2]. \tag{162}$$

Since $K[\omega] = 0$, the B coefficient will remain undetermined [the left-hand side of Eq. (160) is orthogonal to ω, so the system is integrable]. We will set $B = 0$ for the time being, and postpone further discussion until we enforce the Landau-Lifshitz conditions.

To determine the b_m coefficients, we must solve the linear system

$$\Gamma_{ij}^m H^{ij} = X^{mn} b_n, \tag{163}$$

where

$$X^{mn} = -\frac{\langle Y^m, K[Y^n] \rangle}{\langle Y_m, Y_m \rangle}. \tag{164}$$

By symmetry, the X matrix must be diagonal

$$X^{mn} = b \delta^{mn}; \quad b \geq 0 \tag{165}$$

(for the positivity of b, see Israel [15]) leading to

$$b^m = \frac{1}{b} \Gamma_{ij}^m H^{ij}. \tag{166}$$

To find the A and C coefficients let us expand

$$K[1] = \frac{\langle K[1] \rangle}{\langle 1 \rangle} \left[1 - \frac{\langle \omega \rangle \chi_1}{\langle \omega, \chi_1 \rangle} \right] + \beta \frac{\chi_2}{\langle \chi_2^2 \rangle},$$

$$K[\chi_2] = \frac{\beta}{\langle 1 \rangle} \left[1 - \frac{\langle \omega \rangle \chi_1}{\langle \omega, \chi_1 \rangle} \right] + \gamma \frac{\chi_2}{\langle \chi_2^2 \rangle}, \tag{167}$$

where we have used $\langle \omega, K[\chi] \rangle = 0$. If only binary scattering is considered, then also $\langle K[\chi] \rangle = 0$ and $\langle K[1] \rangle = \beta = 0$. In general, then, $\langle K[1] \rangle \sim \beta \ll \gamma$. Therefore, these equations admit an approximate solution with $C = 0$, yielding

$$\chi = \frac{-1}{bT} \Gamma_{ij}^m H_{ij} Y_m + c_0; \quad c_0 = \Gamma \frac{\{\langle \omega^3 \rangle \langle 1 \rangle - \langle \omega^2 \rangle \langle \omega \rangle\}}{|\langle K[1] \rangle|}, \tag{168}$$

where we have used $\langle K[1] \rangle \leq 0$, as follows from the inequality Eq. (135) and the identity

$$\langle K[1] \rangle = 4 \lambda^4 \int \frac{Dp\,Dr\,Ds\,Dt\,Du\,Dv\,(R_2 - R_1) \delta_1}{[(1 + f_p)(1 + f_r)(1 + f_s)(1 + f_t) f_u f_v]}. \tag{169}$$

C. The temperature shift and the bulk stress

It can be seen from Eq. (168) that the correction to the distribution function has two components. The one associated with the $H^{\mu\nu}$ tensor contributes to shear stress, but it does not induce a change in the energy density, and therefore it is compatible with the Landau-Lifshitz matching conditions. The constant shift of χ by c_0, on the other hand, affects in principle both the energy density and the thermal mass M_T. So, to enforce the Landau-Lifshitz conditions, it must be partially compensated by a temperature shift. Concretely, if we call T the temperature of the fiducial equilibrium state, such that $\rho(T)$ is equal to the energy density in the nonequilibrium state, then the temperature appearing in the local equilibrium distribution function f_0 must be $T_0 = T + \delta T$. The effect of this temperature shift is the same as that in the coefficient B in Eq. (162).

The distribution function and temperature shifts in turn produce a shift δM^2 in the physical mass, which likewise does not affect the transport equation. However, both δT and δM^2 are relevant to the bulk stress. Observe that there is no shift in the four velocity u^μ.

The three displacements c_0, δT, and δM^2 are related by the constraints that the gap equation must hold, and the total energy density in the nonequilibrium state must be the same as in the equilibrium state. We write the gap equation as

$$M^2 - \varphi(M^2, \mu^2) = \frac{\lambda}{2} M_T^2. \quad (170)$$

The linearized equation then reads

$$\left[1 - \varphi' - \frac{\lambda}{2}\frac{\partial M_T^2}{\partial M^2}\right]\delta M^2 = \frac{\lambda}{2}\left[\frac{\partial M_T^2}{\partial T}\delta T + c_0\langle 1\rangle\right]. \quad (171)$$

As a matter of fact,

$$\frac{\partial M_T^2}{\partial T} = \frac{\langle \omega \rangle}{T^2}. \quad (172)$$

So, finally,

$$\delta M^2 = M_{,T}^2 \delta T + M_{,c}^2 c_0, \quad (173)$$

where

$$M_{,c}^2 = T^2 M_{,T}^2 \frac{\langle 1 \rangle}{\langle \omega \rangle}. \quad (174)$$

Since the gap equation is enforced, we can look at the (cosmological) constant Λ as a function of M^2, and

$$\delta\Lambda_f = \frac{-1}{2} M_T^2 \delta M^2, \quad (175)$$

then

$$\delta\rho = \rho_{,T}\delta T + \left[\frac{\partial \rho_T}{\partial M^2} - \frac{1}{2}M_T^2\right]M_{,c}^2 c_0 + \langle \omega^2 \rangle c_0. \quad (176)$$

Actually,

$$\frac{\partial \rho_T}{\partial M^2} = \frac{1}{2}M_T^2 - \frac{\langle \omega \rangle}{2T}, \quad (177)$$

so

$$\delta\rho = \rho_{,T}\delta T + \left[\langle \omega^2 \rangle - \frac{\langle 1 \rangle}{2}TM_{,T}^2\right]c_0. \quad (178)$$

And since the total energy remains the same,

$$\rho_{,T}\delta T = -c_0\left[\langle \omega^2 \rangle - \frac{\langle 1 \rangle}{2}TM_{,T}^2\right]. \quad (179)$$

Let us apply the same reasoning to the bulk stress, which results from both the departure of the pressure from $p(T)$ and the direct contribution from the new terms in the distribution function

$$\tau = c_s^2 \rho_{,T}\delta T + \left[\frac{\partial p_T}{\partial M^2} + \frac{1}{2}M_T^2\right]M_{,c}^2 c_0 + \frac{1}{3}[\langle \omega^2 \rangle - M^2\langle 1 \rangle]c_0. \quad (180)$$

Now,

$$\frac{\partial p_T}{\partial M^2} = \frac{-1}{2}M_T^2, \quad (181)$$

so

$$\tau = -c_0\left\{\left[c_s^2 - \frac{1}{3}\right]\langle \omega^2 \rangle + \left[\frac{M^2}{3} - \frac{c_s^2}{2}TM_{,T}^2\right]\langle 1 \rangle\right\}. \quad (182)$$

Using Eqs. (168), (157), and (158), we get

$$\tau = -\frac{u_{,\lambda}^\lambda\left[M^2 - \frac{1}{2}TM_{,T}^2\right]^2}{3T^5\left(\frac{d\rho}{dT}\right)^2}\frac{\{\langle \omega^3 \rangle\langle 1 \rangle - \langle \omega^2 \rangle\langle \omega \rangle\}^2}{|\langle K[1]\rangle|}. \quad (183)$$

D. Shear stress and viscosity

The shear stress can be read directly out of the new terms in $T_T^{\mu\nu}$. In the rest frame, we get

$$\tau^{ij} = \frac{-1}{bT}\Gamma_{kl}^m H^{kl}\langle p^i p^j, Y_m \rangle$$

$$= \frac{-1}{bT}H_{kl}\left\langle p^i p^j - \frac{1}{3}\delta^{ij}\vec{p}^2, p^k p^l - \frac{1}{3}\delta^{kl}\vec{p}^2\right\rangle$$

$$= \frac{-R}{bT}H^{ij} \quad (184)$$

from which we can read out the shear viscosity

$$\eta = \frac{R}{bT}. \qquad (185)$$

To estimate η, it is enough to keep only the leading (binary scattering) contributions, so $\eta \sim \lambda^{-2}$. On dimensional grounds, $b \sim T^2$ and $R \sim T^6$, so we recover the usual result, $\eta \sim T^3 / \lambda^2$.

E. Bulk viscosity

As expected, things are not so simple with the bulk viscosity. We can read it out from Eq. (183) as

$$\zeta = \frac{\left[M^2 - \frac{1}{2}TM_{,T}^2\right]^2}{3T^5\left(\frac{d\rho}{dT}\right)^2} \frac{\{\langle\omega^3\rangle\langle1\rangle - \langle\omega^2\rangle\langle\omega\rangle\}^2}{|\langle K[1]\rangle|}. \qquad (186)$$

However, in evaluating it we must consider that $\langle 1 \rangle$ is logarithmically divergent in the massless limit, so we must correct the sheer dimensional estimate to $\langle 1 \rangle \sim T^2 \ln(M/T)$. As for the size of $|\langle K[1]\rangle|$, observe that the integral is dominated by the Rayleigh-Jeans tail, where $f_0 \sim T/\omega \gg 1$. Thus $|\langle K[1]\rangle| \sim \lambda^4 T^6 F(M^2)$. Since the overall units are Mass^4, it must be $|\langle K[1]\rangle| \sim T^6/M^2$. For the remaining elements we may use the conventional estimates $\langle \omega^3 \rangle \sim T^5$, $\rho \sim T^4$, and thus obtain

$$\zeta \sim \frac{M^2}{\lambda^4 T^3}\left[M^2 - \frac{1}{2}TM_{,T}^2\right]^2 \ln^2(M/T), \qquad (187)$$

which reproduces JY's Eq. (5.6) [3].

In the limit in which the bare mass vanishes, or equivalently in the $T \to \infty$ limit, we may write on dimensional grounds

$$M^2 - \frac{1}{2}TM_{,T}^2 \equiv \frac{1}{2}\mu M_{,\mu}^2 \sim \lambda M^2 \qquad (188)$$

and since $M^2 \sim \lambda T^2$ itself, Eq. (187) reduces to $\zeta \sim \lambda T^3 \ln^2(\lambda)$, again in agreement with JY [3].

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APPENDIX A: Π_{12} IS PURELY IMAGINARY

To show this, observe that Π_{12} is the sum of all 1PI Feynman graphs with two external vertices, one carrying a 1 index and the other a 2 index (this follows from it being the result of opening one 12 leg in each 2PI vacuum bubble). It can also be represented as

$$\Pi_{12} = \frac{\partial\Gamma_1}{\partial\phi^1\partial\phi^2}\bigg|_{\phi=0}, \qquad (A1)$$

where Γ_1 is the usual (1PI) effective action, and ϕ^a is the background field. The effective action has the structure

$$\Gamma_1 = \frac{1}{2}\int dx\,dy\{[\phi](x)D(x,y)\{\phi\}(y)$$
$$+ i[\phi](x)N(x,y)[\phi](y)\} + O(\phi^3), \qquad (A2)$$

where $\{\phi\} = \phi^1 + \phi^2$, $[\phi] = \phi^1 - \phi^2$; both D and N are real, N is even, and D is causal. Therefore,

$$\frac{\partial\Gamma_1}{\partial\phi^1(x)} = \frac{1}{2}\int dy\{D(x,y)\{\phi\}(y) + [\phi](y)D(y,x)$$
$$+ 2iN(x,y)[\phi](y)\} + O(\phi^2), \qquad (A3)$$

and

$$\Pi_{12} = -iN(x,y) + \frac{1}{2}[D(x,y) - D(y,x)]. \qquad (A4)$$

The real part of Π_{12} is odd, and its imaginary part even, which shows that its Fourier transform is purely imaginary. Then we write

$$\Pi_{12}(x,y) = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\Pi_{12}(p), \qquad (A5)$$

then the identity $\Pi_{12}^*(x,y) = -\Pi_{12}(y,x)$ becomes indeed $\Pi_{12}^*(p) = -\Pi_{12}(p)$.

We may use the same argument to find that $\Pi_{21}(x,y) = \Pi_{12}(y,x)$, so $\Pi_{21}(p) = \Pi_{12}(-p)$ is also imaginary. We also find

$$\Pi_{11} = iN(x,y) + \frac{1}{2}[D(x,y) + D(y,x)] \qquad (A6)$$

from where $\text{Im}\,\Pi_{11}(p) = (i/2)(\Pi_{12} + \Pi_{21})$.

APPENDIX B: $\langle\omega\rangle$ AND $\langle\omega^3\rangle$

Our objective is to compute

$$\langle\omega^3\rangle = \frac{1}{2\pi^2}\int_M^\infty d\omega\,\omega^3[\omega^2 - M^2]^{1/2}f_0(1+f_0),$$

$$\langle\omega\rangle = \frac{1}{2\pi^2}\int_M^\infty d\omega\,\omega[\omega^2 - M^2]^{1/2}f_0(1+f_0).$$

Recall the identity

$$\omega f_0 (1 + f_0) = T^2 \frac{\partial f_0}{\partial T}. \tag{B1}$$

This and Eq. (2) may be used to establish the identity

$$\langle \omega^3 \rangle - M^2 \langle \omega \rangle = 3T(p + \rho) = 3T^2 \frac{d\rho}{dT} c_s^2. \tag{B2}$$

Similarly,

$$\langle \omega^3 \rangle - \frac{1}{2} T M^2_{,T} \langle \omega \rangle = T^2 \frac{d\rho}{dT}. \tag{B3}$$

So

$$\langle \omega^3 \rangle = T^2 \frac{d\rho}{dT} \frac{\left[M^2 - \frac{3}{2} T M^2_{,T} c_s^2 \right]}{\left[M^2 - \frac{1}{2} T M^2_{,T} \right]};$$

$$\langle \omega \rangle = T^2 \frac{d\rho}{dT} \frac{\left[1 - 3 c_s^2 \right]}{\left[M^2 - \frac{1}{2} T M^2_{,T} \right]}. \tag{B4}$$

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