Monopole dynamics and BPS dyons in $N=2$ super-Yang-Mills theories

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We determine the low-energy dynamics of monopoles in pure $N=2$ Yang-Mills theories for points in the vacuum moduli space where the two Higgs fields are not aligned. The dynamics is governed by a supersymmetric quantum mechanics with potential terms and four real supercharges. The corresponding superalgebra contains a central charge but nevertheless supersymmetric states preserve all four supercharges. The central charge depends on the sign of the electric charges and consequently so does the BPS spectrum. We focus on the SU(3) case where certain BPS states are realized as zero modes of a Dirac operator on Taub-NUT space twisted by the triholomorphic Killing vector field. We show that the BPS spectrum includes hypermultiplets that are consistent with the strong- and weak-coupling behavior of Seiberg-Witten theory.

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I. INTRODUCTION

The Bogomol'nyi-Prasad-Sommerfield (BPS) spectrum of monopoles and dyons is an important nonperturbative feature of supersymmetric Yang-Mills theories. At weak coupling one can determine the BPS spectrum using semiclassical techniques. Following [1,2], the BPS spectrum of $N=2$ and $N=4$ theories was studied in a number of papers [3–6] at points in the moduli space of vacua where only a single Higgs field was involved or, more precisely, where all of the Higgs fields were aligned. In these cases one studies certain supersymmetric quantum mechanics models with the target manifold given by the moduli space of classical BPS monopole solutions.

New features arise when one studies the spectrum at points in the moduli space where the Higgs fields are not aligned [7–12]. For theories with $N=4$ supersymmetry, the BPS bound is determined by two complex central charges that appear in the supersymmetry algebra. For aligned Higgs fields the two charges are necessarily equal and a BPS state preserves 1/2 of the supersymmetry. When the six Higgs fields are not aligned the central charges can be different and then the BPS states preserve 1/4 of the supersymmetry.

The low-energy dynamics of monopoles for nonaligned Higgs fields in $N=4$ theories was recently studied by Bak *et al.* [13]. The supersymmetric quantum mechanics is still based on the same BPS monopole moduli space, but is now supplemented by a supersymmetric potential term which is constructed from a set of triholomorphic Killing vector fields that generate unbroken U(1) gauge symmetries. It was noticed in Ref. [12] that this potential naturally appears in the expression for the energy of BPS states, while Bak *et al.* later showed how the same potential occurs in the low-energy dynamics, albeit with an important multiplicative factor of 1/2, and used the resulting Lagrangian to study the spectrum of $N=4$ Yang-Mills, including 1/4 BPS states.

In this paper we will analyze analogous issues for pure $N=2$ supersymmetric Yang-Mills theories. Since the $N=2$ supersymmetry algebra has one complex central charge, there can only be BPS states preserving 1/2 of the supersymmetry. Since the pure $N=2$ Yang-Mills theory can be embedded in the $N=4$ theory, it is not surprising that the central charge is one of the central charges that appear in the $N=4$ theory. It is interesting that the other $N=4$ central charge also appears as a bound on the classical mass of dyons, but it is no longer related to the preservation of supersymmetry. If this latter bound is stronger than the BPS bound for a given set of charges, then no BPS state can exist with those charges.

At points in the vacuum moduli space of pure $N=2$ Yang-Mills theories where the Higgs fields are aligned, the low-energy dynamics is a supersymmetric quantum mechanics on the moduli space of BPS monopoles with four real supersymmetries [1]. The BPS states correspond to harmonic spinors on the monopole moduli space (or, equivalently, on a hyper-Kähler manifold, harmonic holomorphic forms). This follows from the simple fact that one of the low-energy supercharges Q is proportional to a Dirac operator on the moduli space,

$$\mathcal{D}=-i\gamma^{\mu}\nabla_{\mu}, \tag{1}$$

with covariant derivative on the moduli space ∇, and its square gives the supersymmetric sigma-model Hamiltonian:

$$Q^2=\frac{1}{2}\mathcal{D}^2=\mathcal{H}_0. \tag{2}$$

With two Higgs fields active, we will argue that the low-energy dynamics includes a supersymmetric potential term as in the $N=4$ theories [13]. The only difference from the $N=4$ case is that the number of fermions and the number of supercharges is reduced by half. We will write down this low-energy Lagrangian explicitly in Sec. III. The super-

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charge is now given by the Dirac operator twisted by a triholomorphic vector field G. The superalgebra then has the general form

$$Q^2=\frac{1}{2}(\mathcal{D}-\gamma^\mu G_\mu)^2=\mathcal{H}-\mathcal{Z}, \tag{3}$$

where $\mathcal{H}$ is the modified Hamiltonian, and $\mathcal{Z}$ is a real central charge defined by the Lie derivative along G,

$$\mathcal{Z}=-i\mathcal{L}_G, \tag{4}$$

and measures a linear combination of electric charges.

The BPS states with $\mathcal{H}=\mathcal{Z}$ preserve not only the supercharge Q but, as we will show, all four supercharges. This is consistent with preservation of 1/2 of the spacetime supersymmetry. It is interesting to note that if we flip the signs of the electric charges so that $\mathcal{Z}\rightarrow-\mathcal{Z}$, the state will no longer be BPS. This should be contrasted with the $N=4$ theory, where BPS states with electric charges of both signs may occur and break a further half of the supersymmetries in general.

We will analyze in some detail the simplest case of SU(3) broken to $U(1)\times U(1)$ by two adjoint Higgs fields. In particular we will focus on BPS states with a (1,1) magnetic charge. The BPS monopole moduli space for this case is given by $R^3\times(R\times M)/Z$ where M is Taub-Newman-Unti-Tamburino (Taub-NUT) space [5,14]. The BPS spectrum is then determined by solving the Dirac equation on the Taub-NUT manifold twisted by the triholomorphic Killing vector field and we will be able to utilize the results of Pope who studied precisely the same operator in [16].

An early analysis of the BPS spectrum of $N=2$ SU(3) Yang-Mills theories in the weak-coupling regime was carried out in the context of Seiberg-Witten theories [17] by Fraser and Hollowood [18]. Acting with semiclassical monodromy transformations on purely magnetic states, they argued that in a certain part of the vacuum moduli space there should exist hypermultiplets with magnetic charge (1,1) and electric charge $(n,n-1)$ with arbitrary integer n. Since the monodromy cannot alter the supermultiplet structures, all of these dyons fill out hypermultiplets. By solving the low-energy dynamics of two distinct monopoles, we will find that these are particular cases of more general states with electric charges (m,l), where integers m and l are such that $m>l$. The size of supermultiplet of the BPS state grows linearly with the positive integer $m-l$.

This paper is organized as follows. Section II will briefly summarize the classical energy bound of the pure $N=2$ Yang-Mills theory. We will show that there are two bounds on the classical energy and only one of them corresponds to a supersymmetric BPS bound. In Sec. III we will present the supersymmetric quantum mechanics with potential that should describe the low-energy dynamics of monopoles and dyons. We analyze the conditions for preserved supersymmetry and use this in Sec. IV to analyze the BPS spectrum for the case of SU(3). In Sec. V, we summarize some of the previously known results on the spectrum of pure SU(3) Seiberg-Witten theory from monodromies as well as strong-coupling singularities in the vacuum moduli space, and show that the results are consistent with those in Sec. IV. We conclude in Sec. VI.

II. BPS BOUND

The $N=2$ super-Yang-Mills Lagrangian is given by

$$L=\frac{1}{2}\mathrm{tr}\bigg\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+D_\mu\phi^I D^\mu\phi^I+e^2[\phi^1,\phi^2]^2$$
$$+i\bar{\chi}\Gamma^\mu D_\mu\chi-e\bar{\chi}[\phi^1,\chi]-ie\bar{\chi}\gamma_5[\phi^2,\chi]\bigg\}, \tag{5}$$

where ϕ^I, $I=1,2$, denote the two real Higgs fields, $D_\mu\phi^I=\partial_\mu\phi^I-ie[A_\mu,\phi^I]$, and χ is a Dirac spinor and all fields are in the adjoint representation of the gauge group G. The classical vacuum moduli space demands that $[\phi^1,\phi^2]=0$; we may choose the asymptotic values of the Higgs fields along the positive z axis, say, to be in the Cartan subalgebra, $\phi^I=\boldsymbol{\phi}^I\cdot\mathbf{H}$, where $\boldsymbol{\phi}^I$ are vectors of dimension $r=\mathrm{rank}(G)$. This does not completely fix the gauge transformations as one has the freedom to perform discrete gauge transformations by elements of the Weyl group. These can be fixed by demanding, for example, that $\boldsymbol{\phi}^1\cdot\boldsymbol{\beta}^a\geqslant0$ for a given set of simple roots $\boldsymbol{\beta}^a$ of the Lie algebra $\mathcal{G}$ of G. We will only consider points in the moduli space of vacua where the symmetry is maximally broken to $U(1)^r$.

For a given vacuum we can define electric and magnetic charge two-vectors

$$\vec{Q}_e=\mathrm{tr}\oint dS^iE_i\vec{\phi},\quad \vec{Q}_m=\mathrm{tr}\oint dS^iB_i\vec{\phi}, \tag{6}$$

with $i=1,2,3$ and $\vec{\phi}=(\phi^1,\phi^2)$. These can be written as

$$Q_e^I=\boldsymbol{\phi}^I\cdot\mathbf{q},\quad Q_m^I=\boldsymbol{\phi}^I\cdot\mathbf{g}, \tag{7}$$

where we have introduced the electric and magnetic charge vectors given by

$$\mathbf{q}=en_e^a\boldsymbol{\beta}^a,$$
$$\mathbf{g}=\frac{4\pi}{e}n_m^a\boldsymbol{\beta}_a^*, \tag{8}$$

respectively, where $\boldsymbol{\beta}^a$ are the simple roots, $\boldsymbol{\beta}_a^*$ are the simple coroots of $\mathcal{G}$, n_m^a are the topological winding numbers, and n_e^a are, in the quantum theory, the electric quantum numbers.

By determining the central charges that appear in the supersymmetry algebra as in [15] we can determine the BPS bound:

$$M\geqslant|Z_-=(Q_e^1-Q_m^2)+i(Q_m^1+Q_e^2)|. \tag{9}$$

Note that if we introduce a complex rescaled Higgs vector $\mathbf{A}=e(\boldsymbol{\phi}^1+i\boldsymbol{\phi}^2)$ and rescale the charge vectors via $\hat{\mathbf{q}}=\mathbf{q}/e$ and $\hat{\mathbf{g}}=(e/4\pi)\mathbf{g}$, then the BPS condition becomes $M=|\mathbf{A}\cdot\hat{\mathbf{q}}+\mathbf{A_D}\cdot\hat{\mathbf{g}}|$ where $\mathbf{A_D}=(i4\pi/e^2)\mathbf{A}$ which is the form familiar from Seiberg-Witten theory (for vanishing θ) [17].

It is illuminating to rederive the BPS bound using Bogomol'nyi's method of rewriting the energy as a sum of squares plus conserved charges. Indeed we will see that this gives rise to two bounds on the classical energy. Since the bosonic part of the $N=2$ Lagrangian differs from the $N=4$ theory only in the fact that there are two Higgs fields instead of six Higgs fields, one can immediately adapt the derivation of the general BPS bound for the $N=4$ theory [7,10] to the $N=2$ case. One finds that the most stringent bound on the mass is given by

$$\begin{aligned} M &\geqslant \sqrt{|\vec{Q}_e|^2+|\vec{Q}_m|^2+2|\vec{Q}_e||\vec{Q}_m|\sin\xi} \\ &= \mathrm{Max}(\sqrt{|\vec{Q}_e|^2+|\vec{Q}_m|^2 \pm 2[Q_m^2 Q_e^1 - Q_m^1 Q_e^2]}), \end{aligned} \tag{10}$$

where $0\leqslant\xi\leqslant\pi$ is the angle between the two 2-vectors $\vec{Q}_e$ and $\vec{Q}_m$. This is equivalent to

$$M \geqslant \mathrm{Max}|Z_\pm = (Q_e^1 \pm Q_m^2) + i(Q_m^1 \mp Q_e^2)|. \tag{11}$$

In $N=4$ theories, $Z_\pm$ appear as central charges in the supersymmetry algebra. If a state saturates the BPS bound (11), it will preserve 1/4 of the supersymmetry. In cases where $Z_+ = Z_-$, which occurs when the angle between $\vec{Q}_e$ and $\vec{Q}_m$ vanishes, the state will preserve 1/2 of the supersymmetry. By contrast, in $N=2$ theories there is only one complex central charge that appears in the supersymmetry algebra,[1] Z_-, giving rise to the BPS bound (9). A state saturating this bound will preserve 1/2 of the supersymmetry. A classical soliton can only saturate the larger of the two bounds, $|Z_\pm|$. Thus, if it so happens that $|Z_-|<|Z_+|$, then there can be no classical BPS soliton with such charges in such a vacuum. In particular, suppose that a state of charge $(\mathbf{g},\mathbf{q})$ saturates the BPS bound $|Z_-|>|Z_+|$. Then, for a state of charge $(\mathbf{g},-\mathbf{q})$, the BPS bound $|Z_-|$ will be smaller than the classical energy bound $|Z_+|$. In the asymptotic region of vacuum moduli space where a semiclassical analysis is suitable the quantum corrections to the classical soliton mass will be small and we conclude that that the latter state cannot be BPS saturated. This asymmetry with respect to the sign of the electric charge is a generic feature of the $N=2$ dyon spectrum. This feature will be manifest in the low energy superalgebra derived in Sec. III and will be further analyzed for the specific example of SU(3) in Sec. IV.

With this knowledge in mind, let us continue exploring the energy bound further. Defining the linear combinations of Higgs fields via

$$\begin{aligned} a &= \cos\alpha\,\phi^1 - \sin\alpha\,\phi^2, \\ b &= \sin\alpha\,\phi^1 + \cos\alpha\,\phi^2, \end{aligned} \tag{12}$$

using the arguments of [7,10] the mass bound is saturated when

$$E_i = \pm D_i a,$$

$$B_i = D_i b, \tag{13}$$

and the angle α is constrained to be

$$\tan\alpha = \frac{Q_m^1 \mp Q_e^2}{Q_m^2 \pm Q_e^1}. \tag{14}$$

In addition, in the gauge $A_0 = -a$ all fields are static and Gauss' law becomes

$$D^2 a - e^2[b,[b,a]] = 0. \tag{15}$$

Note that the second equation in Eq. (13) is the usual BPS equation for a single Higgs field.

In terms of the vectors $\mathbf{a},\mathbf{b}$, the mass bound can then be written

$$M \geqslant \mathrm{Max}(\pm\mathbf{a}\cdot\mathbf{q} + \mathbf{b}\cdot\mathbf{g}), \tag{16}$$

and the constraint (14) is replaced with

$$\mathbf{a}\cdot\mathbf{g} = \pm\mathbf{b}\cdot\mathbf{q}. \tag{17}$$

It should be emphasized that $\boldsymbol{\phi}^I$ and not $\mathbf{a},\mathbf{b}$ specify the point in vacuum moduli space where a semiclassical analysis is relevant[2] since the latter depend on $\mathbf{g},\mathbf{q}$ via the angle α.

Note that for gauge group SU(2), in order that $[\phi^1,\phi^2]=0$, ϕ^1 must be proportional to ϕ^2. For finite energy configurations we then deduce that $\vec{Q}_e$ is proportional to $\vec{Q}_m$ and hence the only bound on the mass is the BPS bound given by $M^2 \geqslant (\vec{Q}_e)^2 + (\vec{Q}_m)^2$ as in [15]. It is perhaps worth commenting that even for gauge group U(1) there are infinite energy configurations with $\vec{Q}_e$ not proportional to $\vec{Q}_m$ [19].

III. LOW-ENERGY DYNAMICS OF MONOPOLES AND DYONS

For a single adjoint Higgs field, it is well known that classical bosonic dyons can be described as monopoles with some internal momentum excited. The low-energy dynamics is determined by a sigma model whose classical orbits are geodesics on the moduli space of monopole solutions. In the case of maximal symmetry breaking, the moduli space has $U(1)^r$ symmetry, arising from global gauge transformations, and the corresponding momenta are the conserved electric charges.

It was argued in [10] that one can similarly analyze solutions of Eqs. (13),(15) by constructing a modified low-energy dynamics on monopole moduli spaces. Given a solution of the BPS equation $B = Db$, the other BPS equation (15) is solved by any gauge-zero mode of the solution. In the case of widely separated fundamental monopoles [20] with respect to b, the solution can be thought of as classically bound dyons (with respect to b). Using this information, it

[1]If $\phi^2 \to -\phi^2$ in Eq. (5), the central charge would be Z_+.

[2]This can be further illustrated for $N=4$ Yang-Mills theory. To ensure a duality invariant BPS mass formula, $\boldsymbol{\phi}^I$ are invariant while $\mathbf{a},\mathbf{b}$ transform under SL(2,$\mathbb{Z}$) duality because the angle α transforms.

was argued that the correct low-energy dynamics is determined by the sigma model supplemented with a potential term. The arguments presented in [10] were in the context of $N=4$ theories, but since only two Higgs fields were involved, the arguments can be immediately adapted to the $N=2$ case. In particular, the bosonic Lagrangian should be given by

$$\mathcal{L}=\frac{1}{2}g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu-\frac{1}{2}g_{\mu\nu}G^\mu G^\nu, \tag{18}$$

where g is the metric on the monopole moduli space, and G is a triholomorphic Killing vector field on the moduli space which is associated with a certain unbroken U(1) gauge symmetry. More precisely, G is given by

$$G=e\,\mathbf{a}\cdot\mathbf{K}, \tag{19}$$

where the r Killing vectors K_a are generated by the $\mathrm{U}(1)^r$ unbroken gauge group acting on the moduli space.

This low-energy dynamics is a nonrelativistic approximation, so we always assume slow motion in the moduli space of monopoles. A related but independent condition that is needed to justify the above dynamics is that the potential energy contribution be small compared to the rest mass of the monopoles. In particular, when we realize dyons as bound states of monopoles, the low-energy approximation is valid only if the following condition holds:

$$\mathbf{a}\cdot\mathbf{q}\ll\mathbf{b}\cdot\mathbf{g}, \tag{20}$$

which is satisfied for weak coupling, since the left-hand side $\sim e$ while the right-hand side $\sim 1/e$.

The above bosonic dynamics must be generalized to include fermions and supersymmetry. Monopoles preserve half of the $N=2$ supersymmetry in four dimensions, so the low-energy dynamics will have exactly four real supercharges. In the absence of the potential term (i.e., when only one Higgs field is active), the dynamics has been derived and takes the following form [1]:

$$\mathcal{L}=\frac{1}{2}(g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu+ig_{\mu\nu}\lambda^\mu D_t\lambda^\nu), \tag{21}$$

where $D_t\lambda^\mu=\dot{\lambda}^\mu+\Gamma^\mu_{\nu\rho}\dot{z}^\nu\lambda^\rho$. The addition of the bosonic potential $G^2/2$ induces a term involving fermions, and the full supersymmetric Lagrangian with potential is given by

$$\mathcal{L}=\frac{1}{2}(g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu+ig_{\mu\nu}\lambda^\mu D_t\lambda^\nu-g^{\mu\nu}G_\mu G_\nu-iD_\mu G_\nu\lambda^\mu\lambda^\nu). \tag{22}$$

Assuming that the target is hyper-Kähler and that the Killing vector field G is triholomorphic, the action is invariant under the following four supersymmetry transformations:

$$\delta z^\mu=-i\epsilon\lambda^\mu+i\epsilon_a J^{(a)\mu}{}_\nu\lambda^\nu,$$

$$\delta\lambda^\mu=(\dot{z}^\mu-G^\mu)\epsilon+J^{(a)\mu}{}_\nu(\dot{z}^\nu-G^\nu)\epsilon_a -i\epsilon_a\lambda^\rho\lambda^\nu J^{(a)\sigma}{}_\rho\Gamma^\mu_{\sigma\nu}, \tag{23}$$

where ϵ,ϵ_a are constant one-component Grassmann odd parameters. Note that the two-form dG is (1,1) with respect to all complex structures when G is triholomorphic. This in turn implies that dG is anti-self-dual. The commutator of two different supersymmetry transformations vanishes, while those of like supersymmetry transformations give rise to a combination of a time translation and the symmetry generated by the Killing vector field G:

$$\delta z^\mu=kG^\mu,$$

$$\delta\lambda^\mu=kG^\mu{}_{,\nu}\lambda^\nu. \tag{24}$$

This supersymmetric quantum mechanics thus has all the features we require and on this basis we will assume that it is in fact the correct description of the low energy dynamics.

To quantize we first introduce a frame e^A_μ and define $\lambda^A=\lambda^\mu e^A_\mu$ which commute with all bosonic variables. The remaining canonical commutation relations are then given by

$$[z^\mu,p_\nu]=i\delta^\mu_\nu,$$

$$\{\lambda^A,\lambda^B\}=\delta^{AB}. \tag{25}$$

We can realize this algebra on spinors on the moduli space by letting $\lambda^A=\gamma^A/\sqrt{2}$, where γ^A are gamma matrices. Since the moduli space is hyper-Kähler, an equivalent quantization is obtained using holomorphic differential forms. The supercovariant momentum operator defined by

$$\pi_\mu=p_\mu-\frac{i}{4}\omega_{\mu AB}[\lambda^A,\lambda^B], \tag{26}$$

where $\omega^A_{\mu B}$ is the spin connection, then becomes the covariant derivative acting on spinors $\pi_\mu=-iD_\mu$. Note that

$$[\pi_\mu,\lambda^\nu]=i\Gamma^\nu_{\mu\rho}\lambda^\rho,$$

$$[\pi_\mu,\pi_\nu]=-\frac{1}{2}R_{\mu\nu\rho\sigma}\lambda^\rho\lambda^\sigma. \tag{27}$$

The supersymmetry charges take the form

$$Q=\lambda^\mu(\pi_\mu-G_\mu),$$

$$Q_a=\lambda^\mu J^{(a)\nu}_\mu(\pi_\nu-G_\nu). \tag{28}$$

Introducing the spin charges

$$S^a=\frac{1}{2}\lambda^\mu\lambda^\nu J^{(a)}_{\mu\nu}, \tag{29}$$

satisfying

$$[S^a,S^b]=4\epsilon_{abc}S^c, \tag{30}$$

we have

$$Q^a=[S^a,Q],$$

$$[Q^a,S^b]=\delta^{ab}Q+\epsilon^{abc}Q^c. \tag{31}$$

The algebra of supercharges is given by

$$\{Q,Q\}=2(\mathcal{H}-\mathcal{Z}),$$

$$\{Q_a,Q_b\}=2\delta_{ab}(\mathcal{H}-\mathcal{Z}), \tag{32}$$

$$\{Q,Q_a\}=0,$$

where the Hamiltonian $\mathcal{H}$ and the central charge $\mathcal{Z}$ are given by

$$\mathcal{H}=\frac{1}{2}\left(\frac{1}{\sqrt{g}}\pi_\mu\sqrt{g}g^{\mu\nu}\pi_\nu+G_\mu G^\mu+i\lambda^\mu\lambda^\nu D_\mu G_\nu\right), \tag{33}$$

$$\mathcal{Z}=G^\mu\pi_\mu-\frac{i}{2}\lambda^\mu\lambda^\nu(D_\mu G_\nu). \tag{34}$$

Note that the operator $i\mathcal{Z}$ is the Lie derivative $\mathcal{L}_G$ acting on spinors (see, e.g., [21]),

$$\mathcal{L}_G\equiv D_G+\frac{1}{8}[\gamma^\mu,\gamma^\nu]D_\mu G_\nu. \tag{35}$$

Although the algebra of supercharges contains a central charge $\mathcal{Z}$, we see that the states will either preserve all four supersymmetries of the supersymmetric quantum mechanics if $\mathcal{H}=\mathcal{Z}$ or none. This is entirely consistent with the fact that the parent $N=2$ field theory has a complex central charge and hence BPS states preserve 1/2 of the eight field theory supercharges, while generic states preserve none of the supersymmetry (of course the vacuum preserves all of the supersymmetry). The BPS bound states satisfying $\mathcal{H}=\mathcal{Z}$ are obtained by finding the normalizable zero modes of the following Dirac operator on the moduli space:

$$Q\mathbf{\Psi}=\frac{1}{\sqrt{2}}\gamma^\mu(-i\nabla_\mu-G_\mu)\mathbf{\Psi}=0. \tag{36}$$

The BPS states of the $N=2$ theory are obtained by solving this equation on the monopole moduli space specified by the Higgs field b. The spin content of the supermultiplets will be the tensor product of that of a half-hypermultiplet, $(0,0,\pm 1/2)$, which comes from the noninteracting center-of-mass fermions, with the spin of the bound states on the relative moduli space. In the simplest case of a singlet bound state we get a full hypermultiplet when combined with the corresponding states from the antimonopole sector.

Note that BPS states are only possible if the $\mathcal{Z}$ eigenvalue is non-negative, since $\mathcal{H}$ is non-negative.[3] From Eq. (19) we see that this eigenvalue is given by a linear combination of the electric charges. Since classical bosonic dyonic bound states should exist for both signs of the electric charge (recall the BPS bound and the mass bound discussed in Sec. II), we expect ''wrong-sign'' non-BPS dyons as quantum bound states, unless the potential $G^2/2$ is too weak. Such states will solve only the second order Schrödinger equation,

$$\mathcal{H}\mathbf{\Psi}=\mathcal{E}\mathbf{\Psi}, \tag{37}$$

and will break all of the supersymmmetries. Because of this, these states will form longer $N=2$ supermultiplets. For example, the smallest possible non-BPS multiplet has degeneracy 16 arising from the four states coming from the center-of-mass fermions with an additional factor of 4 arising from the supercharges acting on the bound states on the relative moduli space. This multiplet has highest spin 1. It is identical to the $N=4$ vector multiplet and is a long multiplet with respect to $N=2$ supersymmetry algebra. In the rest of paper, we will consider BPS bound states only.

IV. BPS DYONS IN $N=2$ SU(3) YANG-MILLS THEORY

We now use the supersymmetric monopole dynamics to analyze the special case of two distinct monopoles in pure $N=2$ SU(3) Yang-Mills theory. As we discussed, when two Higgs fields are involved one considers this the monopole moduli space determined by $\mathbf{b}$ and the effects of $\mathbf{a}$ are incorporated via the potential terms. Recall that for the case of a single Higgs field classical SU(3) monopoles can be built out of two distinct species of monopoles, known as fundamental monopoles. The magnetic charges of these fundamental monopoles correspond to the two simple roots of SU(3) Lie algebra which are defined by the asymptotic behavior of the Higgs field [20]. When two Higgs fields are involved we use the expectation value of $\mathbf{b}$ to specify the simple roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ by demanding $\mathbf{b}\cdot\boldsymbol{\alpha}\geqslant 0$ and $\mathbf{b}\cdot\boldsymbol{\beta}\geqslant 0$. This is illustrated in the root diagram in Fig. 1. We take the normalization such that $\boldsymbol{\alpha}^2=\boldsymbol{\beta}^2=1$ and thus $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}=-1/2$.

Dyons built on a single $\boldsymbol{\alpha}$ or a single $\boldsymbol{\beta}$ magnetic charge are easy to find. The moduli space is flat, $R^3\times S^1$, and one obtains integral electric charges by exciting momentum along the internal U(1) angle, which give rise to integral electric charges parallel to the magnetic charge. The possible charges $(\hat{\mathbf{g}},\hat{\mathbf{q}})\equiv(\mathbf{g}e/4\pi,\mathbf{q}/e)$ are

$$(\boldsymbol{\alpha},n\boldsymbol{\alpha}),\quad(\boldsymbol{\beta},m\boldsymbol{\beta}), \tag{38}$$

for integers n and m. The potential term in the quantum mechanics is constant and just contributes to the BPS mass. The quantization of the free fermions gives a half-

[3]This can easily be seen by introducing a complex conjugated operator $Q^*=\gamma^\mu(+i\nabla_\mu-G_\mu)/\sqrt{2}$. It satisfies the identity $(Q^*)^2=\mathcal{H}+\mathcal{Z}$, so that we have $2\mathcal{H}=Q^2+(Q^*)^2$, which is clearly non-negative.

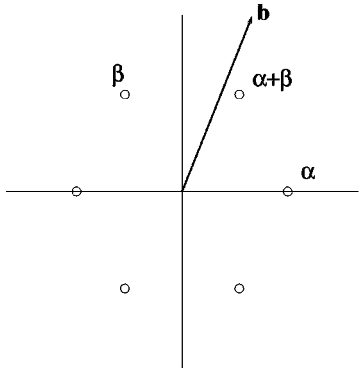

FIG. 1. The root diagram of SU(3) Lie algebra. Considering the Higgs expectation value **b** as a vector in the root space defines $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be the simple roots.

hypermultiplet with spin content $(0,0,\pm 1/2)$ which combines with the charge conjugate states to form a full hypermultiplet.

Dyons with magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$ are more difficult to analyze since the moduli space is now eight dimensional. The exact metric is known [14,5] and it factors into a center-of-mass piece and a relative moduli space. The center-of-mass part is flat, with metric given by

$$ds^2_{\rm c.m.}=(\mu_1+\mu_2)d\vec{X}^2+\frac{16\pi^2}{e^4(\mu_1+\mu_2)}d\xi^2, \qquad (39)$$

where $\vec{X}$ is a three-vector that encodes the center-of-mass position of the two monopoles, and ξ is an internal phase. Here we introduced the masses of the two fundamental monopoles $\mu_1=4\pi\mathbf{b}\cdot\boldsymbol{\alpha}/e$ and $\mu_2=4\pi\mathbf{b}\cdot\boldsymbol{\beta}/e$.

The relative moduli space is more complicated and is given by the Taub-NUT metric

$$ds^2_{\rm rel}=\left(\mu+\frac{2\pi}{e^2r}\right)d\vec{r}^2+\frac{4\pi^2/e^4}{\mu+2\pi/e^2r}[d\psi+\vec{w}(\vec{r})\cdot d\vec{r}]^2, \qquad (40)$$

where $\vec{r}$ is the relative position vector, while ψ is an angular coordinate of period 4π. The reduced mass μ is defined as $\mu_1\mu_2/(\mu_1+\mu_2)$. The three-vector $\vec{w}(\vec{r})$ is the Dirac potential such that $\nabla\times\vec{w}(\vec{r})=-\vec{r}/r^3$. The eight-dimensional, total moduli space is then given by

$$\mathcal{M}=R^3\times\frac{R^1\times\mathcal{M}_{TN}}{\mathbb{Z}}, \qquad (41)$$

where $\mathcal{M}_{TN}$ is the Taub-NUT manifold. The identification map $\mathbb{Z}$,

$$(\xi,\psi)=\left(\xi+2\pi,\psi+\frac{4\pi\mu_2}{\mu_1+\mu_2}\right), \qquad (42)$$

arises from the relationships

$$\xi=\xi_1+\xi_2,\quad \psi=2(\mu_1\xi_2-\mu_2\xi_1)/(\mu_1+\mu_2), \qquad (43)$$

where ξ_i are internal U(1) angles of the two fundamental monopoles, respectively, with each ξ_i having period 2π.

In the special case that $\mu_1=\mu_2$, ξ becomes periodic by itself with the range of $[0,4\pi)$ and the moduli space gets simplified a bit,

$$\mathcal{M}=R^3\times\frac{S^1\times\mathcal{M}_{TN}}{\mathbb{Z}_2}. \qquad (44)$$

The $\mathbb{Z}_2$ action shifts ψ and ξ by 2π simultaneously. The half-integer-quantized momentum along ξ corresponds to the overall U(1) charge in units of $e(\boldsymbol{\alpha}+\boldsymbol{\beta})$, while the half-integer-quantized momentum along ψ corresponds to the relative U(1) charge in units of $e(\boldsymbol{\alpha}-\boldsymbol{\beta})$. Because of the $\mathbb{Z}_2$ action, under which the wave function should be invariant, the two momenta are correlated such that either both are integers or both are half integers.

When the second Higgs expectation **a** is turned on, the low-energy dynamics is twisted by the triholomorphic vector fields. The moduli space has two triholomorphic isometries K_a's, generated by $\partial/\partial\xi_i$ or, equivalently, generated by $\partial/\partial\psi$ and $\partial/\partial\xi$. The relevant triholomorphic vector field G can be decomposed into two orthogonal pieces

$$G=e\,\mathbf{a}\cdot\mathbf{K}=e\left(\overline{a}_T\frac{\partial}{\partial\xi}+\overline{a}\frac{\partial}{\partial\psi}\right), \qquad (45)$$

where $\overline{a}_T$ and $\overline{a}$ are defined as $\mathbf{a}\cdot(\boldsymbol{\alpha}+\boldsymbol{\beta})$ and $\mathbf{a}\cdot(\boldsymbol{\alpha}-\boldsymbol{\beta})$, respectively. The potential associated with $\partial/\partial\xi$ is a constant and accounts for an important contribution to the total electric (or excitation) BPS energy when an electric $\boldsymbol{\alpha}+\boldsymbol{\beta}$ charge is excited. The potential associated with $\partial/\partial\psi$ is position dependent, and leads to interesting dyonic bound states. Thus, the interaction between the two fundamental monopoles is dictated by the low energy supersymmetric quantum mechanics on Taub-NUT space, twisted by the triholomorphic vector field $\tilde{G}\equiv e\overline{a}\,\partial/\partial\psi$.

Given this and the results of the previous section, to determine the BPS states with magnetic charge $\mathbf{g}=4\pi(\boldsymbol{\alpha}+\boldsymbol{\beta})/e$ we need to find normalizable solutions to the Dirac equation

$$\gamma^\mu(-i\nabla_\mu-\tilde{G}_\mu)\boldsymbol{\Psi}=0, \qquad (46)$$

where ∇ is a covariant derivative with respect to Taub-NUT space. This particular problem has been solved by Pope in a different context [16] and we can summarize his results as follows. Let ν be the half-integral charge of the state $\boldsymbol{\Psi}$ with respect to $-i\mathcal{L}_{\partial_\psi}$. The spectrum is different depending on the sign of $\overline{a}\nu$: for negative or vanishing $\overline{a}\nu$, no normalizable bound state exists; for positive $\overline{a}\nu$, there exists a set of $|2\nu|$ normalizable states, for $|\nu|<(4\pi^2/e^3\mu)|\overline{a}|$, which form an angular momentum $|\nu|-1/2$ multiplet.[4]

[4]The restriction of the BPS spectrum to $|\nu|<(4\pi^2/e^3\mu)|\overline{a}|$ is reminiscent of 1/4 BPS dyon spectrum in $N=4$ theories [8,10].

TABLE I. The number of states with various eigenvalues of J_3.

J_3	$\|\nu\|$	$\|\nu\|-1/2$	$\|\nu\|-1$	$\cdots$	$-\|\nu\|+1/2$	$-\|\nu\|$
Degeneracy	1	2	2	$\cdots$	2	1

One must consider the full moduli space to construct the actual wave function. The electric charges of the state are determined by momenta along ψ and ξ. Taking into account the $\mathbb{Z}_2$ identification in Eq. (44) in the case that $\mu_1=\mu_2$, the general electric charge is given by

$$\mathbf{q}=e(k+\nu)\boldsymbol{\alpha}+e(k-\nu)\boldsymbol{\beta}, \tag{47}$$

where the ξ momentum k is (half-)integer whenever ν is (half-)integer. In other words,

$$\mathbf{q}=e\,n(\boldsymbol{\alpha}+\boldsymbol{\beta})-2e\nu\,\boldsymbol{\beta}. \tag{48}$$

Although we derived this for the special case when the masses of the two fundamental monopoles are equal, this form is valid more generally.

The center-of-mass part of the moduli space also gives rise to an additional degeneracy factor. The four free fermionic partners to $\vec{X}$ and ξ generate a half-hypermultiplet structure of degeneracy 4. Thus, for each bound state wave function of charge (ν,k), a single BPS supermultiplet of the highest spin $|\nu|$ is formed. The degeneracy of these supermultiplets is summarized in Table I.

In particular, the smallest possible multiplet is the half-hypermultiplet associated with $\nu=1/2$ (for positive $\bar{a}$) or $\nu=-1/2$ (for negative $\bar{a}$). Combined with charge conjugate states, it forms a full hypermultiplet of the $N=2$ theory. We thus find an infinite tower of hypermultiplets with electric charge given by $\mathbf{q}=e\,n(\boldsymbol{\alpha}+\boldsymbol{\beta})\mp e\,\boldsymbol{\beta}$, for $\bar{a}$ positive or negative, respectively, as a subset of the more general BPS states with electric charge (48).

In the limit of $\bar{a}=0$ it is known that a purely magnetic BPS bound state of one $\boldsymbol{\alpha}$ and one $\boldsymbol{\beta}$ monopole does not exist. Indeed such a state corresponds to a holomorphic harmonic form on Taub-NUT space and its existence would have contradicted the $N=4$ S-duality prediction of a unique anti-self-dual harmonic form. Here we have found that this feature persists for more general vacua when $\bar{a}\neq 0$.

V. CONSISTENCY WITH SEIBERG-WITTEN THEORY

In Sec. IV we found the complete BPS spectrum of dyons for magnetic charges $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\alpha}+\boldsymbol{\beta}$, for values of b where where the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ monopoles are approximately fundamental. These states include dyons in hypermultiplets and we now comment how this subsector relates to the known results of Seiberg-Witten theory.

The weak-coupling spectrum of pure SU(N) theory was studied by Fraser and Hollowood by analyzing monodromy transformations on the vacuum moduli space [18]. Starting with the hypermultiplets corresponding to fundamental monopoles in regions where only a single Higgs field is nonvanishing, the monodromy transformations predict additional hypermultiplets. For the case of SU(3), the vacuum moduli space splits into two disjoint regions separated by a curve of marginal stability. In one region there are hypermultiplets with charge vectors $(\hat{\mathbf{g}},\hat{\mathbf{q}})$ given by

$$\begin{aligned}&(\boldsymbol{\alpha},n\boldsymbol{\alpha}),\\&(\boldsymbol{\beta},n\boldsymbol{\beta}),\\&(\boldsymbol{\alpha}+\boldsymbol{\beta},n(\boldsymbol{\alpha}+\boldsymbol{\beta})-\boldsymbol{\beta}),\end{aligned} \tag{49}$$

for any integer n. In the second region the states have identical charges except that the electric charge of the third set of states has the form $n(\boldsymbol{\alpha}+\boldsymbol{\beta})+\boldsymbol{\beta}$. Consistency then requires that the states with magnetic charge $\boldsymbol{\alpha}+\boldsymbol{\beta}$ decay as they cross the curve of marginal stability. This is entirely consistent with the spectrum of hypermultiplets that we found in Sec. IV.

The curious asymmetry of the spectrum with respect to the sign of the electric charge also manifests itself in the strong-coupling behavior of Seiberg-Witten vacuum moduli space. For example, the asymmetry shows up in the charges of those BPS states that become massless at strong-coupling singularities. The electromagnetic charges of such states in pure SU(3) have been initially worked out explicitly on a symplectic basis [23] and later in terms of weight vectors [18].

Let us make some additional observations concerning the singularity structure of the SU(3) theory. Recall that the moduli space of vacua can be described in terms of a spectral curve of genus 2, which can be written in the Weierstrass form as follows [22]:

$$y^2=\prod_{i=1}^{3}(x-\phi_i)^2-\Lambda^6. \tag{50}$$

Here the ϕ_i's satisfy the traceless condition $\Sigma_i\phi_i=0$ and the vacuum moduli space is parametrized by the gauge-invariant combinations $u\equiv\phi_1\phi_2+\phi_2\phi_3+\phi_3\phi_1$ and $v\equiv\phi_1\phi_2\phi_3$.

The x plane has a set of three square-root branch cuts, which, for a small UV cutoff Λ, are located near the ϕ_i's. Some of the six end points of these branch cuts can meet pairwise for special values of the ϕ_i's. Along such hypersurfaces, a certain charged state becomes massless and generates a singularity of complex codimension 1 on the moduli space. With $\Lambda\neq 0$, there are six different ways that this can happen in terms of the ϕ_i coordinates, and therefore six different charged states that can become massless along such hypersurfaces. The six homology cycles of the Riemann surface have some mutual intersection numbers, which are related to the Schwinger products between the corresponding dyons. From this one can deduce that the electric and magnetic charges of these six dyons are given in a symplectic basis by [23],

$$\begin{aligned} &(1,0;1,0), \quad (1,0;-1,1), \\ &(0,1;-1,1), \quad (0,1;0,-1), \\ &(1,1;0,1), \quad (1,1;-1,0). \end{aligned} \tag{51}$$

Here $(m_1,m_2;n_1,n_2)$ represents a charge of the form $m_1b_1 + m_2b_2 + n_1a_1 + n_2a_2$ with a_i and b_i a charge basis whose pairwise Schwinger products are such that $2(a_i \odot b_j) = \delta_{ij}$ and $a_1 \odot a_2 = 0 = b_1 \odot b_2$. This does not fix the electromagnetic charges uniquely. First of all, there are monodromies on the moduli space which must be fixed by hand. Furthermore, the same intersection matrix shows up regardless of the number of flavors in the theory, which implies that there must be certain ambiguity in translating the symplectic basis to its counterpart in the weight lattice. Part of the ambiguity is resolved by noting that near each pair of singularities that extend far out into the asymptotic region, the local physics should resemble that of SU(2). Each of the states must thus have a unit magnetic charge that corresponds to one of three positive roots $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}+\boldsymbol{\beta}$. Finally, when we discuss theories with adjoint matter only, the electric charges must also fall onto the root lattice, and this knowledge determines the charges up to the monodromies.

Let us choose the monodromy so that (1,0;1,0) is a pure monopole with charge $(\boldsymbol{\alpha},0)$ and that $(0,1;0,-1)$ is a pure monopole with charge $(\boldsymbol{\beta},0)$. This is possible because

$$(\boldsymbol{\alpha},0)\odot(\boldsymbol{\beta},0) = \boldsymbol{\alpha}\cdot 0 - 0\cdot\boldsymbol{\beta} = 0. \tag{52}$$

Using the above arguments, one then finds the following minimal set of charge vectors on the root lattice:

$$\begin{aligned} &(\boldsymbol{\alpha},0), \quad (\boldsymbol{\alpha},\boldsymbol{\alpha}), \\ &(\boldsymbol{\beta},-\boldsymbol{\beta}), \quad (\boldsymbol{\beta},0), \\ &(\boldsymbol{\alpha}+\boldsymbol{\beta},-\boldsymbol{\beta}), \quad (\boldsymbol{\alpha}+\boldsymbol{\beta},\boldsymbol{\alpha}). \end{aligned} \tag{53}$$

In this set of charges, we again find the prominent feature we found in the low-energy dynamics, namely, the asymmetry with respect to the sign of electric charges. Furthermore, assuming that there is no marginal stability domain wall for the last two states in passing to the asymptotic region we worked in, this also tells us that there must be dyonic bound states of charges $(\boldsymbol{\alpha}+\boldsymbol{\beta},-\boldsymbol{\beta})$ and $(\boldsymbol{\alpha}+\boldsymbol{\beta},\boldsymbol{\alpha})$ in hypermultiplets, which are exactly the lowest-lying bound states found in Sec. IV.

VI. CONCLUSIONS

We have presented a low-energy effective dynamics that allows one to study the weak-coupling spectrum of dyons for pure $N=2$ Yang-Mills gauge theory for general gauge groups in a systematic manner. For aligned Higgs fields it has been known for some time that one should study a supersymmetric quantum mechanics on the BPS monopole moduli space [1]. For generic vacua, we have argued that supersymmetric quantum mechanics is supplemented by a potential term constructed from the triholomorphic Killing vectors on the moduli space. The BPS states correspond to normalizable zero modes of a Dirac operator twisted by the Killing vectors. It would be interesting to derive this dynamics directly by generalizing the arguments of [1].

We used the formalism to study the semiclassical BPS spectrum for the pure $N=2$ SU(3) gauge theory. The vector **b**, defined by the Higgs vacuum expectation values (VEVs) and by the choice of the Weyl chamber, specifies a basis of simple roots in the algebra. When the magnetic charge is given by a simple root $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, there is a tower of dyons in hypermultiplets with parallel electric charges. A more interesting structure emerges for a magnetic charge given by $\boldsymbol{\alpha}+\boldsymbol{\beta}$. First, the electric charge vector is necessarily not parallel to the magnetic charge, and in particular, there is no purely magnetic BPS state of this charge. Second, there is a tower of hypermultiplets with electric charge that are consistent with previous results in Seiberg-Witten theory in that they are in accordance with semiclassical monodromy and strong-coupling singularities. More generally, the hypermultiplets are a special case of an infinite tower of BPS states with electric charge (m,l), with either $m>l$ or $m<l$, depending on the sign of the second Higgs **a**, with maximal spin $|m-l|/2$.

It would be interesting to extend these results to a more general magnetic charge, but as a result of a lack of understanding of the relevant SU(3) monopole moduli spaces, this appears difficult. A more accessible problem would be to generalize the results of this paper to analyze the BPS spectrum for an SU(N) gauge group with a magnetic charge given by $(1,1,\ldots,1)$.

The effective Lagrangian in this paper is for the case when hypermultiplets in the $N=2$ Yang-Mills theory can be ignored. If there are light hypermultiplets, one must introduce new quantum mechanical degrees of freedom associated with them. Presumably, there is a way to couple them to our Lagrangian while preserving the four real supercharges. However, it goes beyond the scope of this paper, and will be worked out elsewhere.

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