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Exact renormalization group and loop equation

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We propose a gauge-invariant formulation of the exact renormalization-group equation for nonsupersymmetric pure U(N) Yang-Mills theory, based on the construction by Morris. In fact we show that our renormalization-group equation amounts to a regularized version of the loop equation, thereby providing a direct relation between the exact renormalization group and the Schwinger-Dyson equations. We also discuss a possible implication of our formulation to the holographic correspondence of the bulk gravity and the boundary gauge theory.

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I. INTRODUCTION

The discovery of certain limits in string theory and M theory can probably provide a reasoning for the dualities between quantum gravity and gauge theory. The celebrated examples are the light like limit [1] of M theory, which at least literally justifies the matrix conjecture [2], and the near horizon limits [3,4] of D branes and M branes [5] that led us to recent excitement of the anti–de Sitter (AdS) conformal field theory (CFT) duality [3,6,7]. Both the matrix conjecture and the AdS–CFT duality intimately originates from the old *s-t* channel duality in open string loop amplitudes. It is, however, quite noteworthy that the discovery of the limits of [1,3,4] elaborated and complemented our intuitive but naive anticipation of a dual description of quantum gravity by a certain gauge theory, which may be based on the above-mentioned duality of open and closed strings.

The IR-UV relation, pointed out in [8,9], itself is not surprising from the viewpoint of the *s-t* channel duality. But a rather conceptual payoff of this relation in the near horizon limit seems quite remarkable. In particular, the identification of the radial coordinate U in the near horizon geometry with the energy or the cutoff scale Λ of the gauge theory clarifies the holographic nature of the AdS-CFT duality, in which it is quite plausibly assumed that the gauge theory contains only one degree of freedom per cell of the cutoff size.[1] This postulate and the resultant holographic property actually fit in the basic idea of the Wilsonian renormalization group (Wilsonian RG) [10].[2] The assumption concerning degrees of freedom of the gauge theory is almost assured, if one refers to, for example, a lattice regularization. Once we regard the cutoff Λ of the boundary gauge theory as the radial dimension in the bulk space, degrees of freedom of the bulk gravity may well be constituted solely from those of the boundary gauge theory, and the coarsening procedure or the RG transformation gives the dynamics in the interior space, which will likely be in general more complicated than that on the boundary. Thus we hope in this respect that the RG flow may correspond to the holographic mapping of the boundary data.

The motivation of this paper is to invent a formulation, on the gauge theory side, that might be a useful setting for discussing this hope. Since in the AdS-CFT duality gravitational modes in the bulk correspond to the gauge invariant operators on the boundary gauge theory, a gauge invariant formulation, in which the gauge redundancy is eliminated, may be well suited for discussing this sort of duality. In this regard we would like to respect a technique of collective field theory developed in [11], and apply it to a gauge invariant formulation of the exact RG equation.

There exist several papers [12], in which the authors discussed the RG interpretation of the AdS-CFT duality and some of them suggested the equivalence of the equation of motion in the AdS space (and its slight generalization) with the RG equation.[3] We hope that our formulation adds a new perspective along the line of their arguments.

II. A GAUGE-INVARIANT FORMULATION OF THE EXACT RG EQUATION

Recently Tim Morris [14] proposed an exact RG equation for the Yang-Mills theory in a manifestly gauge-invariant way, by constructing it in terms of a gauge-invariant variable, Wilson loop, just like the philosophy of the collective field method [11]. His formulation is based on an observation inspired from a simple derivation [15] of the exact RG, that the exact RG equation is related to a particular field redefinition of the theory. For example, in scalar field theories in D dimensions, the exact RG equation may be written in the form

$$\frac{\partial}{\partial\Lambda}e^{-S}=\int d^D\mathbf{x}\frac{\delta}{\delta\phi(\mathbf{x})}(\Psi[\phi(\mathbf{x})]e^{-S}), \tag{1}$$

where $\Psi[\phi(\mathbf{x})]$ is induced from an infinitesimal change of the scalar field, $\phi(\mathbf{x})\to\phi(\mathbf{x})+\delta\Lambda\Psi[\phi(\mathbf{x})]$. The exact RG

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[1]There is in fact a subtlety in the relation between the radial coordinate U and the cutoff scale Λ [9]. The existence of two distinct relations is emphasized there. In both cases, however, there is a universal property that the cutoff scale Λ increases with the radial scale U up to dimension 5 of the boundary gauge theory.

[2]The scale-invariant theories, of course, do not have a nontrivial RG flow. We would, however, like to emphasize this point, suppose that the argument in [8] can be applied to more general nonconformal cases.

[3]S.-J. Rey also has an idea to develop this line of argument [13].

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equation [15], which was employed to prove the renormalizability of the $\lambda\phi^4$ theory in four dimensions, is obtained by choosing the field redefinition as

$$\Psi[\phi(\mathbf{x})]=\frac{1}{2}\int d^D\mathbf{y}\left[\dot{G}_\Lambda(\mathbf{x}-\mathbf{y})\frac{\delta S}{\delta\phi(\mathbf{y})}-2(\dot{G}_\Lambda\cdot G_\Lambda^{-1})\times(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\right], \tag{2}$$

where $G_\Lambda(\mathbf{x}-\mathbf{y})$ is the cutoff propagator of a massless scalar, defined by

$$G_\Lambda(\mathbf{x}-\mathbf{y})=\int d^D\mathbf{p}\frac{1}{p^2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}K(p^2/\Lambda^2). \tag{3}$$

in which $K(p^2/\Lambda^2)$ is a cutoff function that will take the value 1 for $p^2<\Lambda^2$ and vanish rapidly at infinity. Also $\dot{G}_\Lambda$ is the derivative of the propagator with respect to the cutoff Λ, i.e., $\dot{G}_\Lambda=(\partial/\partial\Lambda)G_\Lambda$.

In pure Yang-Mills theory in D dimensions we may write the exact RG equation as

$$\frac{\partial}{\partial\Lambda}e^{-S}=\mathrm{Tr}\int d^D\mathbf{x}\frac{\partial}{\partial A^\mu(\mathbf{x})}(\Psi^\mu[A^\mu(\mathbf{x})]e^{-S}). \tag{4}$$

Here we introduced a standard convention, $A^\mu(\mathbf{x})=T^aA_a^\mu(\mathbf{x})$ and $\delta/\delta A^\mu(\mathbf{x})=T^a\delta/\delta A_a^\mu(\mathbf{x})$. A straightforward adaptation of the above regularization scheme (2), however, spoils the gauge symmetry. One way to avoid it is to look for some other form of the field redefinition (2) which maintains the gauge invariance. Indeed one such a choice was found in [14], in which a trick, the introduction of a pair of Wilson lines into the field redefinition, seemed to play an essential role. We make use of this trick, but we propose somehow a similar but different formulation of the gauge-invariant exact RG equation. In fact our formulation can be directly connected with the loop equation [16], as is rather different from the one proposed in [14].

Our choice of the field redefinition is

$$\Psi_\mu[A^\mu(\mathbf{x})]=-\frac{1}{N\Lambda^3}\int d^D\mathbf{y}\int d^D\mathbf{p}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Bigg[K'(p^2/\Lambda^2)\Phi[\Gamma_{\mathbf{xy}}]\frac{\delta S}{\delta A^\mu(\mathbf{y})}\Phi^{-1}[\Gamma_{\mathbf{xy}}]+2K'(p^2/\Lambda^2)K^{-1}(p^2/\Lambda^2)\Phi[\Gamma_{\mathbf{xy}}]\frac{1}{2g_b^2}D^\nu F_{\nu\mu}(\mathbf{y})\Phi^{-1}[\Gamma_{\mathbf{xy}}]-K'(p^2/\Lambda^2)\frac{\delta}{\delta A_a^\mu(\mathbf{y})}(\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\Bigg]. \tag{5}$$

This corresponds to the Fourier expansion of Eq. (2). Here $\Phi[\Gamma_{\mathbf{xy}}]$ is an advertised Wilson line, and defined by $\Phi[\Gamma_{\mathbf{xy}}]=Pe^{i\oint_{\Gamma_{\mathbf{xy}}}d\mathbf{x}'\cdot\mathbf{A}(\mathbf{x}')}$, where the contour $\Gamma_{\mathbf{xy}}$ is a line from $\mathbf{x}$ to $\mathbf{y}$. Also $K'(x)$ denotes the derivative of $K(x)$, i.e., $K'(x)=(d/dx)K(x)$, and g_b is the bare Yang-Mills coupling. In this form it is not clear whether the exact RG equation (4) is gauge invariant. We will, however, see below that it is indeed the case.

Now we can formally integrate the RG equation (4), and it takes the form

$$e^{-S}=e^{\mathcal{H}[A,\delta/\delta A;\Lambda]}e^{-S_b}, \tag{6}$$

with the bare action S_b and

$$\mathcal{H}\left[A,\frac{\delta}{\delta A};\Lambda\right]=\frac{1}{2N}\int d^D\mathbf{x}\int d^D\mathbf{y}\int d^D\mathbf{p}\frac{1}{p^2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\frac{\delta}{\delta A_b^\mu(\mathbf{x})}\Bigg[-[K(p^2/\Lambda^2)-1]\mathrm{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta}{\delta A_a^\mu(\mathbf{y})}+\ln K(p^2/\Lambda^2)\mathrm{Tr}\left(T^b\Phi[\Gamma_{\mathbf{xy}}]\frac{1}{g_b^2}D^\nu F_{\nu\mu}(\mathbf{y})\Phi^{-1}[\Gamma_{\mathbf{xy}}]\right)-[K(p^2/\Lambda^2)-1]\times\frac{\delta}{\delta A_a^\mu(\mathbf{y})}\mathrm{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\Bigg]. \tag{7}$$

Note that we chose the boundary condition appropriately. In fact $\mathcal{H}[A,\delta/\delta A;\Lambda]\to 0$ at $\Lambda\to\infty$, since $K(p^2/\Lambda^2)\to 1$ at $\Lambda\to\infty$. Also the first functional derivative in the right-hand side (rhs) operates passing through the square brackets as well.

Next let us consider the generating functional for Wilson loop correlators, as we are interested only in correlation functions of the gauge-invariant operators. The generating functional is given by[4]

$$Z[J]=\int \mathcal{D}A^{\mu}\exp\left(-S+\sum_{C} J(C)W(C)\right), \tag{8}$$

with the definition of the Wilson loop $W(C)=(1/N)\mathrm{Tr}Pe^{i\oint_C d\mathbf{x}\cdot \mathbf{A}(\mathbf{x})}$. Using the integrated expression (6) of the exact RG equation and performing the integration by parts, we can rewrite it as

$$Z[J]=\int \mathcal{D}A^{\mu}e^{-S_b}(e^{\tilde{\mathcal{H}}[W,\delta/\delta W;\Lambda]}e^{\Sigma_C J(C)W(C)}), \tag{9}$$

where $\tilde{\mathcal{H}}[W,\delta/\delta W;\Lambda]$ is written only in terms of Wilson loops. This shows the manifestation of gauge invariance in our formulation. We will relegate the detailed calculation of the operator $\tilde{\mathcal{H}}[W,\delta/\delta W;\Lambda]$ to the Appendix. It finally comes up with a rather suggestive form which looks like the string field Hamiltonian:

$$\begin{aligned}\tilde{\mathcal{H}}\left[W,\frac{\delta}{\delta W};\Lambda\right]=\frac{1}{2}\int d^D\mathbf{p}\frac{1}{p^2}\Bigg[&[K(p^2/\Lambda^2)-1]\\ &\times\Bigg\{\frac{1}{N^2}\sum_{C,C'}\int_0^{2\pi}ds\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}W(C_M)\frac{\delta}{\delta W(C')}\frac{\delta}{\delta W(C)}\\ &+\sum_C\int_0^{2\pi}ds\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}W(C_B)W(C_{\bar{B}})\frac{\delta}{\delta W(C)}\Bigg\}\\ &-\frac{1}{g_b^2N}\int d^D\mathbf{y}\ln K(p^2/\Lambda^2)\sum_C\int_0^{2\pi}se^{i\mathbf{p}\cdot(\mathbf{x}(s)-\mathbf{y})}\frac{\delta^2W(C)}{\delta x(s_0)^2}\bigg|_{\mathbf{x}(s_0)=\mathbf{y}}\frac{\delta}{\delta W(C)}\Bigg].\end{aligned} \tag{10}$$

As explained in the Appendix, a loop C_M denotes the merging of two loops C and C', and is given by a product of four line contours as $C_M=C_{\mathbf{x}(s)\mathbf{x}(s)}\Gamma_{\mathbf{x}(s)\mathbf{x}(s')}C_{\mathbf{x}(s')\mathbf{x}(s')}\Gamma_{\mathbf{x}(s')\mathbf{x}(s)}$. Also a pair of loops C_B and $C_{\bar{B}}$ is broken up from a loop C, and they are expressed as $C_B=\Gamma_{\mathbf{x}(s')\mathbf{x}(s)}C_{\mathbf{x}(s)\mathbf{x}(s')}$ and $C_{\bar{B}}=C_{\mathbf{x}(s')\mathbf{x}(s)}\Gamma_{\mathbf{x}(s)\mathbf{x}(s')}$. Here we distinguished a pair of line $\Gamma_{\mathbf{xy}}$ introduced in the field redefinition (5) from the other lines $C_{\mathbf{xy}}$ which are parts of loops C and C'.

We will discuss later a possible implication of this result in the holographic correspondence of the bulk gravity and the boundary gauge theory.

III. LOOP EQUATION FROM THE EXACT RG EQUATION

The Schwinger-Dyson equation is supposed to contain as much information as the exact RG equation, in a sense that, if we were to solve either of these two equations nonperturbatively, we could in principle obtain all the physical information of the quantum field theory. Thus we will be expected to have an intrinsic way to translate the Schwinger-Dyson equation into the exact RG equation, and vice versa, implicitly or explicitly. Actually we need to check if our proposed formulation really works, by, say, rederiving a known result obtained from a reliable formulation. In this respect it turns out interestingly enough that our exact RG equation amounts to a regularized version of the loop equation of [16], while this result is not so surprising and in fact rather reasonable, as we just mentioned. To see it, let us note that Eq. (9) can be further rewritten into the form

$$Z[J]=e^{\tilde{\mathcal{H}}[\delta/\delta J,J;\Lambda]}Z_b[J], \tag{11}$$

where $Z_b[J]$ is the generating functional of the bare form, i.e.,

$$Z_b[J]=\int \mathcal{D}A^{\mu}\exp\left(-S_b+\sum_C J(C)W(C)\right). \tag{12}$$

Also in the operator $\tilde{\mathcal{H}}[\delta/\delta J,J;\Lambda]$ the derivatives $\delta/\delta J$'s are ordered on the right side of the sources J's.

Now the exact RG equation implies

$$\frac{d}{d\Lambda}Z[J]=0. \tag{13}$$

This is equivalent to

[4]In the standard formulation of the exact RG, the source J for a local operator is chosen in such a way that $J(p)$ is vanishing for higher momentum $p^2>\Lambda^2$ [15]. But the source $J(C)$ introduced here is the one for a nonlocal operator, and so it is unclear what choice is appropriate for the exact RG. We will not make any restrictions on the source at this stage, instead it will be constrained by the consistency of the exact RG equation, as we will see in the next section.

$$0=\int d^D\mathbf{p}\frac{1}{p^2}\left(\frac{\partial}{\partial\Lambda}K(p^2/\Lambda^2)\right)K^{-1}(p^2/\Lambda)\sum_C J(C)\int_0^{2\pi}ds$$
$$\times\left[\frac{1}{N^2}\sum_{C'}J(C')\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))K(p^2/\Lambda^2)e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}\frac{\delta Z[J]}{\delta J(C_M)}\right.$$
$$\left.+\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))K(p^2/\Lambda^2)e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}\frac{\delta^2 Z[J]}{\delta J(C_B)\delta J(C_{\bar{B}})}-\frac{1}{g_b^2N}\int d^D\mathbf{y}e^{i\mathbf{p}\cdot(\mathbf{x}(s)-\mathbf{y})}\frac{\delta^2}{\delta x(s_0)^2}\frac{\delta Z[J]}{\delta J(C)}\bigg|_{\mathbf{x}(s_0)=\mathbf{y}}\right]. \tag{14}$$

Therefore, we may recognize that the quantity in the square bracket is vanishing, and then integrating over the momentum **p**, we come up to the loop equation with a regularization,

$$\frac{1}{g_b^2N}\frac{\delta^2}{\delta x(s)^2}\frac{\delta Z[J]}{\delta J(C)}=\frac{1}{N^2}\sum_{C'}J(C')\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\int d^D\mathbf{p}K(p^2/\Lambda^2)e^{i\mathbf{p}\cdot(\mathbf{x}(s)-\mathbf{x}(s))}\frac{\delta Z[J]}{\delta J(C_M)}$$
$$+\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\int d^D\mathbf{p}K(p^2/\Lambda^2)e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}\frac{\delta^2 Z[J]}{\delta J(C_B)\delta J(C_{\bar{B}})}. \tag{15}$$

Note that interestingly the cutoff function $K(p^2/\Lambda^2)$ enters in an expected way. In fact $\int d^D\mathbf{p}K(p^2/\Lambda^2)e^{i\mathbf{p}\cdot(\mathbf{x}(s')-\mathbf{x}(s))}$ can be thought of as a smeared δ function, which is necessary to regularize the loop equation. Also loops are ordinarily closed by this δ function, and so the smearing of the δ function might undergo a potential breakdown of the gauge symmetry. It is, however, obvious from our explicit computation that loops are closed in spite of the smearing of the δ function in our formulation, due to a pair of Wilson lines introduced in the field redefinition (5).

IV. DISCUSSION

As emphasized in the introduction, the cutoff scale Λ of the gauge theory can be regarded as the radial scale U in the AdS space, or more generally in the near horizon geometries of Dp branes [9].[5] This simple but significant observation led us to contemplate the RG interpretation of the bulk-boundary duality, and pursue a gauge-invariant formulation of the exact RG equation for the Yang-Mills theory. In particular, to discuss the AdS-CFT duality, we apparently need to consider the supersymmetric extension of our formulation. For this purpose the analysis in [18,19] and a supersymmetric Wilson loop in [20] are certainly of importance. We would, however, like to discuss a possibility implied by our formulation in a rather wider context of the holographic correspondence between the bulk gravity and the boundary gauge theory.

According to the wisdom of the AdS-CFT duality, a disturbance on the boundary will be responded to by the bulk gravity as a gravitational fluctuation, in such a way that

$$\left\langle\exp\left(\int_{\partial M}d^D\mathbf{x}\phi_0(\mathbf{x})\mathcal{O}(\mathbf{x})\right)\right\rangle=\exp(-S_{\text{grav}}[\phi(\mathbf{x},U)]), \tag{16}$$

where ∂M is the boundary of the bulk space M, and $\mathcal{O}(\mathbf{x})$ is a local operator on the boundary field theory. Also $\phi(\mathbf{x},U)$ is a gravitational mode on the bulk space, and it becomes $\phi_0(\mathbf{x})$ at the boundary. Furthermore, in the gravity action S_{grav} the gravitational mode $\phi(\mathbf{x},U)$ is subject to the equation of motion, so that the rhs depends only on the boundary value $\phi(\mathbf{x},\infty)=\phi_0(\mathbf{x})$ of a gravitational mode.

If we insist on a stronger conjecture [3] on this duality that the string theory in the bulk is dual to the finite-N boundary gauge theory, we may consider the correlation functions of Wilson loops $W(C)$ instead of local operators $\mathcal{O}(\mathbf{x})$. Then the gravity action might be replaced by the string field action. From this viewpoint we would like to recall our result (11) on the generating functional of Wilson loop correlators:

$$\left\langle\exp\left(\sum_C J(C)W(C)\right)\right\rangle=Z[J]=\exp\left(\tilde{\mathcal{H}}\left[\frac{\delta}{\delta J},J;\Lambda\right]\right)Z_b[J]. \tag{17}$$

Now let us set the RG interpretation of the bulk-boundary duality into this argument. The bulk physics at some scale U may be given by the boundary gauge theory in which degrees of freedom at higher momentum modes than the cutoff scale Λ are integrated out. In this regard $Z_b[J]$ denotes the unintegrated form of the generating functional, and so it corresponds to the bulk physics at the IR limit or the boundary. Then the operator $\tilde{\mathcal{H}}[\delta/\delta J,J;\Lambda]$ gives the RG flow from UV to IR of the boundary gauge theory, that corresponds to a mapping from the boundary to the interior of the bulk gravity. In this sense we would like to regard the RG flow opera-

[5]Again we should be careful with the distinction between holographic and D-brane probes. See a footnote in the introduction.

tor $\widetilde{\mathcal{H}}[\delta/\delta J,J;\Lambda]$ as the holographic mapping of the boundary data. Moreover, as we discussed in the last section, the RG equation implies $(d/d\Lambda)Z[J]=0$, which is tantamount to a regularized version of the loop equation. Thus we might interpret the regularized loop equation as the string field equation of motion in the bulk space. Remember also that the RG flow operator $\widetilde{\mathcal{H}}[\delta/\delta J,J;\Lambda]$ has a form of the string field Hamiltonian which consists of the terms that describe the joining and the splitting of strings and the string propagation. These facts fit at least literally in a stronger conjecture on the bulk-boundary duality.

Apart from the RG interpretation of the bulk-boundary duality, we would like to mention a similarity of our formulation with the stochastic quantization of the Yang-Mills theory. In fact the authors in [21] proposed an interpretation of the Fokker-Planck Hamiltonian of the Yang-Mills theory as the string field Hamiltonian, initially as the one in the temporal gauge, just as discussed in the noncritical string theory, and later speculated an alternative interpretation that the fictitious time τ of the stochastic quantization can be thought of as the radial coordinate U of the AdS space.[6] Our formulation is close to their latter speculation. This similarity originates from the fact that the exact RG and the Fokker-Planck equations are quite similar diffusion equations with the cutoff Λ and the fictitious time τ, respectively, as the time. It, however, seems hard to give a physical meaning to the finite value of the fictitious time τ in the stochastic quantization.

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APPENDIX

The computation of $\widetilde{\mathcal{H}}[W,\delta/\delta W;\Lambda]$ can be done by mixture of the technique of the collective field method in [11] and that of [17] presented in a derivation of the loop equation. In terms of the gauge fields $A^\mu(\mathbf{x})$, it is expressed as

$$\widetilde{\mathcal{H}}\left[W,\frac{\delta}{\delta W};\Lambda\right]=\frac{1}{2N}\int d^D\mathbf{x}\int d^D\mathbf{y}\int d^D\mathbf{p}\,\frac{1}{p^2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Bigg[-[K(p^2/\Lambda^2)-1]\mathrm{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta^2}{\delta A_a^\mu(\mathbf{y})\,\delta A_b^\mu(\mathbf{x})}$$
$$-\ln K(p^2/\Lambda^2)\mathrm{Tr}\left(T^b\Phi[\Gamma_{\mathbf{xy}}]\frac{1}{g_b^2}D^\nu F_{\nu\mu}(\mathbf{y})\Phi^{-1}[\Gamma_{\mathbf{xy}}]\right)\frac{\delta}{\delta A_b^\mu(\mathbf{x})}\Bigg]. \tag{A1}$$

The second derivative term consists of two pieces, the joining and the splitting of strings, due to the chain rule, as in (the first paper of) [11]. The one that describes the joining of strings is

$$\sum_{C,C'}\mathrm{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta W(C)}{\delta A_a^\mu(\mathbf{y})}\frac{\delta W(C')}{\delta A_b^\mu(\mathbf{x})}\frac{\delta}{\delta W(C')}\frac{\delta}{\delta W(C)}. \tag{A2}$$

Actually the joining property can be easily understood from an explicit calculation.

$$N^2\mathrm{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta W(C)}{\delta A_a^\mu(\mathbf{y})}\frac{\delta W(C')}{\delta A_b^\mu(\mathbf{x})}=-\int_0^{2\pi}ds\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\delta^D(\mathbf{x}(s)-\mathbf{y})\delta^D(\mathbf{x}(s')-\mathbf{x})$$
$$\times\mathrm{Tr}[(Pe^{i\int_0^s ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi^{-1}[\Gamma_{\mathbf{xy}}](Pe^{i\int_{s'}^{2\pi}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})$$
$$\times(Pe^{i\int_0^{s'}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi[\Gamma_{\mathbf{xy}}](Pe^{i\int_s^{2\pi}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})]$$
$$=-N\int_0^{2\pi}ds\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\delta^D(\mathbf{x}(s)-\mathbf{y})\delta^D(\mathbf{x}(s')-\mathbf{x})W(C_M), \tag{A3}$$

where a loop C_M denotes the merging of two loops C and C', and it is composed of four lines, i.e., $C_M=C_{\mathbf{x}(s)\mathbf{x}(s)}\Gamma_{\mathbf{yx}}C_{\mathbf{x}(s')\mathbf{x}(s')}\Gamma_{\mathbf{xy}}$. Here we define the product of line contours as an oriented contour in which each line is connected

[6] Actually the string field theory in the temporal gauge of noncritical strings was first proposed in [22], and subsequently it was reconstructed in [23] as a collective field theory of stochastic quantization of matrix models in the double scaling limit. Also the intrinsic equivalence of the Fokker-Planck Hamiltonian and the loop operator was pointed out in [24]. The authors in [21] patched those ideas together, and added a new interpretation in the context of Polyakov's noncritical strings [25] and also of the Ads-CFT duality.

at a common point. Also we used different symbols, $C_{\mathbf{xy}}$ and $\Gamma_{\mathbf{xy}}$, for line contours, in order to distinguish a pair of lines $\Gamma_{\mathbf{xy}}$ introduced in the field redefinition (5) from the other lines $C_{\mathbf{xy}}$, which are segments of loops C and C'.

The part corresponding to the splitting of strings is given by

$$\sum_C \operatorname{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta^2 W(C)}{\delta A_b^\mu(\mathbf{x})\,\delta A_a^\mu(\mathbf{y})}\frac{\delta}{\delta W(C)}. \tag{A4}$$

Similarly, it can be rewritten in terms of Wilson loops as

$$\begin{aligned}
N\operatorname{Tr}(T^b\Phi[\Gamma_{\mathbf{xy}}]T^a\Phi^{-1}[\Gamma_{\mathbf{xy}}])\frac{\delta^2 W(C)}{\delta A_b^\mu(\mathbf{x})\,\delta A_a^\mu(\mathbf{y})} =& -\int_0^{2\pi} ds\Bigg[\int_0^s ds\operatorname{Tr}\{(Pe^{i\int_0^{s'}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi[\Gamma_{\mathbf{xy}}](Pe^{i\int_s^{2\pi}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\}\\
&\times\operatorname{Tr}\{(Pe^{i\int_{s'}^{s}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi^{-1}[\Gamma_{\mathbf{xy}}]\}\\
&+\int_s^{2\pi} ds\operatorname{Tr}\{(Pe^{i\int_0^{s}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi^{-}[\Gamma_{\mathbf{xy}}](Pe^{i\int_{s'}^{2\pi}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\}\\
&\times\operatorname{Tr}\{(Pe^{i\int_s^{s'}ds''\dot{\mathbf{x}}(s'')\cdot\mathbf{A}(\mathbf{x}(s''))})\Phi[\Gamma_{\mathbf{xy}}]\}\Bigg](\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\delta^D\\
&\times(\mathbf{x}(s)-\mathbf{y})\delta^D(\mathbf{x}(s')-\mathbf{x})\\
=& -N^2\int_0^{2\pi}ds\int_0^{2\pi}ds'(\dot{\mathbf{x}}(s)\cdot\dot{\mathbf{x}}(s'))\delta^D(\mathbf{x}(s)-\mathbf{y})\delta^D\\
&\times(\mathbf{x}(s')-\mathbf{x})W(C_B)W(C_{\bar{B}}),
\end{aligned} \tag{A5}$$

where a loop C is broken into two loops C_B and $C_{\bar{B}}$, and they are, respectively, given by $C_B=\Gamma_{\mathbf{xy}}C_{\mathbf{x}(s)\mathbf{x}(s')}$ and $C_{\bar{B}}=C_{\mathbf{x}(s')\mathbf{x}(s)}\Gamma_{\mathbf{yx}}$.

Finally, the second term in Eq. (A1) corresponds to the kinetic term of the string field,

$$\begin{aligned}
\sum_C \operatorname{Tr}\left(T^b\Phi[\Gamma_{\mathbf{xy}}]\frac{1}{g_b^2}D^\nu F_{\nu\mu}(\mathbf{y})\Phi^{-1}[\Gamma_{\mathbf{xy}}]\right)\frac{\delta W(C)}{\delta A_b^\mu(\mathbf{x})}\frac{\delta}{\delta W(C)} =& \sum_C \frac{i}{g_b^2 N}\int_0^{2\pi}ds\,\dot{x}^\mu(s)\,\delta^D(\mathbf{x}(s)-\mathbf{x})\\
&\times\operatorname{Tr}\{(Pe^{i\int_0^s ds'\dot{\mathbf{x}}(s')\cdot\mathbf{A}(\mathbf{x}(s'))})\Phi[\Gamma_{\mathbf{xy}}]D^\nu F_{\nu\mu}(\mathbf{y})\Phi^{-1}[\Gamma_{\mathbf{xy}}]\\
&\times(Pe^{i\int_s^{2\pi}ds'\dot{\mathbf{x}}(s')\cdot\mathbf{A}(\mathbf{x}(s'))})\}\frac{\delta}{\delta W(C)}\\
=& \frac{1}{g_b^2}\int_0^{2\pi}ds\,\delta^D(\mathbf{x}(s)-\mathbf{x})\frac{\delta^2 W(C)}{\delta x(s_0)^2}\bigg|_{\mathbf{x}(s_0)=\mathbf{y}}\frac{\delta}{\delta W(C)},
\end{aligned} \tag{A6}$$

where we introduced a local derivative of the loop space [17],

$$\frac{\delta^2}{\delta x(s)^2}=\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}dt\,\frac{\delta^2}{\delta x^\mu(s+t/2)\,\delta x_\mu(s-t/2)}. \tag{A7}$$

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