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Stochastic semiclassical fluctuations in Minkowski spacetime

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The semiclassical Einstein-Langevin equations which describe the dynamics of stochastic perturbations of the metric induced by quantum stress-energy fluctuations of matter fields in a given state are considered on the background of the ground state of semiclassical gravity, namely, Minkowski spacetime and a scalar field in its vacuum state. The relevant equations are explicitly derived for massless and massive fields arbitrarily coupled to the curvature. In doing so, some semiclassical results, such as the expectation value of the stress-energy tensor to linear order in the metric perturbations and particle creation effects, are obtained. We then solve the equations and compute the two-point correlation functions for the linearized Einstein tensor and for the metric perturbations. In the conformal field case, explicit results are obtained. These results hint that gravitational fluctuations in stochastic semiclassical gravity have a ''non-perturbative'' behavior in some characteristic correlation lengths.

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I. INTRODUCTION

It has been pointed out that the semiclassical theory of gravity [1–5] cannot provide a correct description of the dynamics of the gravitational field in situations where the quantum stress-energy fluctuations are important [1,2,4,6–8]. In such situations, these fluctuations may have relevant back-reaction effects in the form of induced gravitational fluctuations [6] which, in a certain regime, are expected to be described as classical stochastic fluctuations. A generalization of the semiclassical theory is thus necessary to account for these effects. In two previous papers, Refs. [9] and [10], we have shown how a stochastic semiclassical theory of gravity can be formulated to improve the description of the gravitational field when stress-energy fluctuations are relevant.

In Ref. [9], we adopted an axiomatic approach to construct a perturbative generalization of semiclassical gravity which incorporates the back reaction of the lowest order stress-energy fluctuations in the form of a stochastic correction. We started noting that, for a given solution of semiclassical gravity, the lowest order matter stress-energy fluctuations can be associated with a classical stochastic tensor. We then sought a consistent equation in which this stochastic tensor was the source of linear perturbations of the semiclassical metric. The equation obtained is the so-called semiclassical Einstein-Langevin equation.

In Ref. [10], we followed the idea, first proposed by Hu [11] in the context of back reaction in semiclassical gravity, of viewing the metric field as the ''system'' of interest and the matter fields (modeled in that paper by a single scalar field) as being part of its ''environment.'' We then showed that the semiclassical Einstein-Langevin equation introduced in Ref. [10] can be formally derived by a method based on the influence functional of Feynman and Vernon [12] (see also Ref. [13]). That derivation shed light on the physical meaning of the semiclassical Langevin-type equations around specific backgrounds previously obtained with the same functional approach [14–23], since the stochastic source term was shown to be closely linked to the matter stress-energy fluctuations. We also developed a method to compute the semiclassical Einstein-Langevin equation using dimensional regularization, which provides an alternative and more direct way of computing this equation with respect to previous calculations.

This paper is intended to be a first application of the full stochastic semiclassical theory of gravity, where we evaluate the stochastic gravitational fluctuations in a Minkowski background. In order to do so, we first use the method developed in Ref. [10] to derive the semiclassical Einstein-Langevin equation around a class of trivial solutions of semiclassical gravity consisting of Minkowski spacetime and a linear real scalar field in its vacuum state, which may be considered the ground state of semiclassical gravity. Although the Minkowski vacuum is an eigenstate of the total four-momentum operator of a field in Minkowski spacetime, it is not an eigenstate of the stress-energy operator. Hence, even for these solutions of semiclassical gravity, for which the expectation value of the stress-energy operator can always be chosen to be zero, the fluctuations of this operator are non-vanishing. This fact leads to consider the stochastic corrections to these solutions described by the semiclassical Einstein-Langevin equation.

We then solve the Einstein-Langevin equation for the linearized Einstein tensor and compute the associated two-point correlation functions. Even though, in this case, we expect to have negligibly small values for these correlation functions at the domain of validity of the theory, i.e., for points separated by lengths larger than the Planck length, there are several reasons why we think that it is worth carrying out this calculation.

On the one hand, these are, to our knowledge, the first solutions obtained to the full semiclassical Einstein-Langevin equation. We are only aware of analogous solutions to a ''reduced'' version of this equation inspired in a ''mini-superspace'' model [20]. There is also a previous attempt to obtain a solution to the Einstein-Langevin equation in Ref. [17], but, there, the non-local terms in the Einstein-Langevin equation were neglected.

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The Einstein-Langevin equations computed in this paper are simple enough to be explicitly solved and, at least for the case of a conformal field, the expressions obtained for the correlation functions can be explicitly evaluated in terms of elementary functions. Thus, our calculation can serve as a testing ground for the solutions of the Einstein-Langevin equation in more complex situations of physical interest (for instance, for a Robertson-Walker background and a field in a thermal state).

On the other hand, the results of this calculation, which confirm our expectations that gravitational fluctuations are negligible at length scales larger than the Planck length, can be considered as a first check that stochastic semiclassical gravity predicts reasonable results.

In addition, we can extract conclusions on the possible qualitative behavior of the solutions to the Einstein-Langevin equation. Thus, it is interesting to note that the correlation functions are characterized by correlation lengths of the order of the Planck length; furthermore, such correlation lengths enter in a non-analytic way in the correlation functions. This kind of non-analytic behavior is actually quite common in the solutions to Langevin-type equations with dissipative terms and hints at the possibility that correlation functions for other solutions to the Einstein-Langevin equation are also non-analytic in their characteristic correlation lengths.

The plan of the paper is the following. In Sec. II, we give a brief overview of the method developed in Ref. [10] to compute the semiclassical Einstein-Langevin equation. We then consider the background solutions of semiclassical gravity consisting of a Minkowski spacetime and a real scalar field in the Minkowski vacuum. In Sec. III, we compute the kernels which appear in the Einstein-Langevin equation. In Sec. IV, we derive the Einstein-Langevin equation for metric perturbations around Minkowski spacetime. As a side result, we obtain some semiclassical results, which include the expectation value of the stress-energy tensor of a scalar field with arbitrary mass and arbitrary coupling parameter to linear order in the metric perturbations, and also some results concerning the production of particles by metric perturbations: the probability of particle creation and the number and energy of created particles. In Sec. V, we solve this equation for the components of the linearized Einstein tensor and compute the corresponding two-point correlation functions. For the case of a conformal field and spacelike separated points, explicit calculations show that the correlation functions are characterized by correlation lengths of the order of the Planck length. We conclude in Sec. VI with a discussion of our results. We also include some Appendixes with technical details used in the calculations.

Throughout this paper we use the $(+++)$ sign conventions and the abstract index notation of Ref. [24], and we work with units in which $c=\hbar=1$.

II. OVERVIEW

In this section, we give a very brief summary of the main results of Refs. [9] and [10] which are relevant for the computations in the present paper. One starts with a solution of semiclassical gravity consisting of a globally hyperbolic spacetime $(\mathcal{M},g_{ab})$, a linear real scalar field quantized on it and some physically reasonable state for this field (we work in the Heisenberg picture). According to the stochastic semiclassical theory of gravity [9,10], quantum fluctuations in the stress-energy tensor of matter induce stochastic linear perturbations h_{ab} to the semiclassical metric g_{ab}. The dynamics of these perturbations is described by a stochastic equation called the semiclassical Einstein-Langevin equation.

Assuming that our semiclassical gravity solution allows the use of dimensional analytic continuation to define regularized matrix elements of the stress-energy ''operator,'' we shall write the equations in dimensional regularization, that is, assuming an arbitrary dimension n of the spacetime. Using this regularization method, we use a notation in which a subindex n is attached to those quantities that have different physical dimensions from the corresponding physical quantities. The n-dimensional spacetime $(\mathcal{M},g_{ab})$ has to be a solution of the semiclassical Einstein equation in dimensional regularization:

$$\frac{1}{8\pi G_B}(G^{ab}[g]+\Lambda_B g^{ab})-\left(\frac{4}{3}\alpha_B D^{ab}+2\beta_B B^{ab}\right)[g] =\mu^{-(n-4)}\langle\hat{T}_n^{ab}\rangle[g], \tag{2.1}$$

where G_B, Λ_B, α_B and β_B are bare coupling constants and G_{ab} is the Einstein tensor. The tensors D^{ab} and B^{ab} are obtained by functional derivation with respect to the metric of the action terms corresponding to the Lagrangian densities $R_{abcd}R^{abcd}-R_{ab}R^{ab}$ and R^2, respectively, where R_{abcd} is the Riemann tensor, R_{ab} is the Ricci tensor and R is the scalar curvature (see Ref. [10] for the explicit expressions for the tensors D^{ab} and B^{ab}). In the last equation, $\hat{T}_n^{ab}$ is the stress-energy ''operator'' in dimensional regularization and the expectation value is taken in some state for the scalar field in the n-dimensional spacetime. Writing the bare coupling constants in Eq. (2.1) as renormalized coupling constants plus some counterterms which absorb the ultraviolet divergencies of the right hand side, one can take the limit $n\to 4$, which leads to the physical semiclassical Einstein equation.

Assuming that g_{ab} is a solution of Eq. (2.1), the semiclassical Einstein-Langevin equation can be similarly written in dimensional regularization as

$$\frac{1}{8\pi G_B}(G^{ab}[g+h]+\Lambda_B(g^{ab}-h^{ab})) -\left(\frac{4}{3}\alpha_B D^{ab}+2\beta_B B^{ab}\right)[g+h] =\mu^{-(n-4)}\langle\hat{T}_n^{ab}\rangle[g+h]+2\mu^{-(n-4)}\xi_n^{ab}, \tag{2.2}$$

where h_{ab} is a linear stochastic perturbation to g_{ab}, and $h^{ab}\equiv g^{ac}g^{bd}h_{cd}$. In this last equation, ξ_n^{ab} is a Gaussian stochastic tensor characterized by the correlators

$$\langle\xi_n^{ab}(x)\rangle_c=0, \quad \langle\xi_n^{ab}(x)\xi_n^{cd}(y)\rangle_c=N_n^{abcd}(x,y), \tag{2.3}$$

where $8N_n^{abcd}(x,y)\equiv\langle\{\hat{t}_n^{ab}(x),\hat{t}_n^{cd}(y)\}\rangle[g]$, with $\hat{t}_n^{ab}\equiv\hat{T}_n^{ab}-\langle\hat{T}_n^{ab}\rangle$; here, $\langle\ \rangle_c$ means statistical average and $\{\ ,\ \}$ denotes an anticommutator. As we pointed out in Ref. [10], the noise kernel $N_n^{abcd}(x,y)$ is free of ultraviolet divergencies in the limit $n\to 4$. Therefore, in the semiclassical Einstein-Langevin equation (2.2), one can perform exactly the same renormalization procedure as the one for the semiclassical Einstein equation (2.1), and Eq. (2.2) yields the physical semiclassical Einstein-Langevin equation in four spacetime dimensions.

In Ref. [10], we used a method based on the closed time path (CTP) functional technique applied to a system-environment interaction, more specifically, on the influence action formalism of Feynman and Vernon, to obtain an explicit expression for the expansion of $\langle\hat{T}_n^{ab}\rangle[g+h]$ up to first order in h_{cd}. In this way, we can write the Einstein-Langevin equation (2.2) in a more explicit form. This expansion involves the kernel $H_n^{abcd}(x,y)\equiv H_{\mathrm{S}_n}^{abcd}(x,y)+H_{\mathrm{A}_n}^{abcd}(x,y)$, with

$$H_{\mathrm{S}_n}^{abcd}(x,y)\equiv\frac{1}{4}\mathrm{Im}\langle \mathrm{T}^*(\hat{T}_n^{ab}(x)\hat{T}_n^{cd}(y))\rangle[g],$$
$$H_{\mathrm{A}_n}^{abcd}(x,y)\equiv-\frac{i}{8}\langle[\hat{T}_n^{ab}(x),\hat{T}_n^{cd}(y)]\rangle[g], \tag{2.4}$$

where $[\ ,\ ]$ means a commutator, and we use the symbol T* to denote that we have to time order the field operators $\hat{\Phi}_n$ first and then to apply the derivative operators which appear in each term of the product $T^{ab}(x)T^{cd}(y)$, where T^{ab} is the classical stress-energy tensor; see Ref. [10] for more details. In Eq. (2.2), all the ultraviolet divergencies in the limit $n\to 4$, which shall be removed by renormalization of the coupling constants, are in some terms containing $\langle\hat{\Phi}_n^2(x)\rangle$ and in $H_{\mathrm{S}_n}^{abcd}(x,y)$, whereas the kernels $N_n^{abcd}(x,y)$ and $H_{\mathrm{A}_n}^{abcd}(x,y)$ are free of ultraviolet divergencies. These two last kernels can be related to the real and imaginary parts of $\langle\hat{t}_n^{ab}(x)\hat{t}_n^{cd}(y)\rangle$ by

$$N_n^{abcd}(x,y)=\frac{1}{4}\mathrm{Re}\langle\hat{t}_n^{ab}(x)\hat{t}_n^{cd}(y)\rangle,$$
$$H_{\mathrm{A}_n}^{abcd}(x,y)=\frac{1}{4}\mathrm{Im}\langle\hat{t}_n^{ab}(x)\hat{t}_n^{cd}(y)\rangle. \tag{2.5}$$

We now consider the case in which we start with a vacuum state $|0\rangle$ for the field quantized in spacetime $(\mathcal{M},g_{ab})$. In this case, it was shown in Ref. [10] that all the expectation values entering the Einstein-Langevin equation (2.2) can be written in terms of the Wightman and Feynman functions, defined as

$$G_n^{+}(x,y)\equiv\langle 0|\hat{\Phi}_n(x)\hat{\Phi}_n(y)|0\rangle[g],$$
$$iG_{F_n}(x,y)\equiv\langle 0|\mathrm{T}(\hat{\Phi}_n(x)\hat{\Phi}_n(y))|0\rangle[g]. \tag{2.6}$$

For instance, we can write $\langle\hat{\Phi}_n^2(x)\rangle=iG_{F_n}(x,x)=G_n^{+}(x,x)$. The expressions for the kernels, which shall be used in our calculations, can be found in Appendix A.

Perturbations around Minkowski spacetime

An interesting case to be analyzed in the framework of the semiclassical stochastic theory of gravity is that of a Minkowski spacetime solution of semiclassical gravity. The flat metric η_{ab} in a manifold $\mathcal{M}\equiv\mathbb{R}^4$ (topologically) and the usual Minkowski vacuum, denoted as $|0\rangle$, give the class of simplest solutions to the semiclassical Einstein equation [note that each possible value of the parameters (m^2,ξ) leads to a different solution], the so called trivial solutions of semiclassical gravity [25]. In fact, we can always choose a renormalization scheme in which the renormalized expectation value $\langle 0|\hat{T}_R^{ab}|0\rangle[\eta]=0$. Thus, Minkowski spacetime $(\mathbb{R}^4,\eta_{ab})$ and the vacuum state $|0\rangle$ are a solution to the semiclassical Einstein equation with renormalized cosmological constant $\Lambda=0$. The fact that the vacuum expectation value of the renormalized stress-energy operator in Minkowski spacetime should vanish was originally proposed by Wald [2] and it may be understood as a renormalization convention [3,5]. There are other possible renormalization prescriptions (see, for instance, Ref. [26]) in which such vacuum expectation value is proportional to η^{ab}, and this would determine the value of the cosmological constant Λ in the semiclassical equation. Of course, all these renormalization schemes give physically equivalent results: the total effective cosmological constant, i.e., the constant of proportionality in the sum of all the terms proportional to the metric in the semiclassical Einstein and Einstein-Langevin equations, has to be zero.

Although the vacuum $|0\rangle$ is an eigenstate of the total four-momentum operator in Minkowski spacetime, this state is not an eigenstate of $\hat{T}_{ab}^R[\eta]$. Hence, even in these trivial solutions of semiclassical gravity, there are quantum fluctuations in the stress-energy tensor of matter and, as a result, the noise kernel does not vanish. This fact leads to consider the stochastic corrections to this class of trivial solutions of semiclassical gravity. Since, in this case, the Wightman and Feynman functions (2.6), their values in the two-point coincidence limit, and the products of derivatives of two of such functions appearing in expressions (A1) and (A3) (Appendix A) are known in dimensional regularization, we can compute the semiclassical Einstein-Langevin equation using the method outlined above.

In order to perform the calculations, it is convenient to work in a global inertial coordinate system $\{x^\mu\}$ and in the associated basis, in which the components of the flat metric are simply $\eta_{\mu\nu}=\mathrm{diag}(-1,1,\ldots,1)$. In Minkowski spacetime, the components of the classical stress-energy tensor functional reduce to

$$T^{\mu\nu}[\eta,\Phi]=\partial^\mu\Phi\,\partial^\nu\Phi-\frac{1}{2}\eta^{\mu\nu}\partial^\rho\Phi\,\partial_\rho\Phi-\frac{1}{2}\eta^{\mu\nu}m^2\Phi^2+\xi(\eta^{\mu\nu}\Box-\partial^\mu\partial^\nu)\Phi^2, \tag{2.7}$$

where $\Box \equiv \partial_\mu \partial^\mu$, and the formal expression for the components of the corresponding ''operator'' in dimensional regularization is

$$\hat{T}^{\mu\nu}_n[\eta] = \frac{1}{2}\{\partial^\mu \hat{\Phi}_n, \partial^\nu \hat{\Phi}_n\} + \mathcal{D}^{\mu\nu}\hat{\Phi}^2_n, \tag{2.8}$$

where $\mathcal{D}^{\mu\nu}$ are the differential operators $\mathcal{D}^{\mu\nu}_x \equiv (\xi - 1/4)\eta^{\mu\nu}\Box_x - \xi \partial^\mu_x \partial^\nu_x$ and $\hat{\Phi}_n(x)$ is the field operator in the Heisenberg picture in an n-dimensional Minkowski spacetime, which satisfies the Klein-Gordon equation $(\Box - m^2)\hat{\Phi}_n = 0$.

Notice, from Eq. (2.8), that the stress-energy tensor depends on the coupling parameter ξ of the scalar field to the scalar curvature even in the limit of a flat spacetime. Therefore, that tensor differs in general from the canonical stress-energy tensor in flat spacetime, which corresponds to the value $\xi = 0$. Nevertheless, it is easy to see [10] that the n-momentum density components $\hat{T}^{0\mu}_{n\;(\xi)}[\eta]$ (we temporarily use this notation to indicate the dependence on the parameter ξ) and $\hat{T}^{0\mu}_{n\;(\xi=0)}[\eta]$ differ in a space divergence and, hence, dropping surface terms, they both yield the same n-momentum operator:

$$\hat{P}^\mu \equiv \int d^{n-1}\mathbf{x}\, {:}\hat{T}^{0\mu}_{n\;(\xi)}[\eta]{:} = \int d^{n-1}\mathbf{x}\, {:}\hat{T}^{0\mu}_{n\;(\xi=0)}[\eta]{:}, \tag{2.9}$$

where the integration is on a hypersurface $x^0 = \text{constant}$ ($\hat{P}^\mu$ is actually independent of the value of x^0) and we use the notation for coordinates $x^\mu \equiv (x^0, \mathbf{x})$, i.e., $\mathbf{x}$ are space coordinates on each of the hypersurfaces $x^0 = \text{constant}$. The symbol $:\;:$ in Eq. (2.9) means normal ordering of the creation and annihilation operators on the Fock space built on the Minkowski vacuum $|0\rangle$ (in n spacetime dimensions), which is an eigenstate with zero eigenvalue of the operators (2.9).

The Wightman and Feynman functions (2.6) in Minkowski spacetime are well known:

$$G^+_n(x,y) \equiv \langle 0|\hat{\Phi}_n(x)\hat{\Phi}_n(y)|0\rangle[\eta] = i\Delta^+_n(x-y),$$

$$G_{F_n}(x,y) \equiv -i\langle 0|\mathrm{T}(\hat{\Phi}_n(x)\hat{\Phi}_n(y))|0\rangle[\eta] = \Delta_{F_n}(x-y), \tag{2.10}$$

with

$$\Delta^+_n(x) = -2\pi i \int \frac{d^n k}{(2\pi)^n} e^{ikx}\,\delta(k^2+m^2)\,\theta(k^0),$$

$$\Delta_{F_n}(x) = -\int \frac{d^n k}{(2\pi)^n} \frac{e^{ikx}}{k^2+m^2-i\epsilon}, \quad \epsilon \to 0^+, \tag{2.11}$$

where $k^2 \equiv \eta_{\mu\nu}k^\mu k^\nu$ and $kx \equiv \eta_{\mu\nu}k^\mu x^\nu$. Note that the derivatives of these functions satisfy $\partial^x_\mu \Delta^+_n(x-y) = \partial_\mu \Delta^+_n(x-y)$ and $\partial^y_\mu \Delta^+_n(x-y) = -\partial_\mu \Delta^+_n(x-y)$, and similarly for the Feynman propagator $\Delta_{F_n}(x-y)$.

To write down the semiclassical Einstein equation (2.1) for this case, we need to compute the vacuum expectation value of the stress-energy operator components (2.8). Since, from Eq. (2.10), we have that $\langle 0|\hat{\Phi}^2_n(x)|0\rangle = i\Delta_{F_n}(0) = i\Delta^+_n(0)$, which is a constant (independent of x), we have simply

$$\begin{aligned}\langle 0|\hat{T}^{\mu\nu}_n|0\rangle[\eta] &= \frac{1}{2}\langle 0|\{\partial^\mu\hat{\Phi}_n, \partial^\nu\hat{\Phi}_n\}|0\rangle[\eta] \\ &= -i(\partial^\mu\partial^\nu\Delta_{F_n})(0) \\ &= -i\int \frac{d^n k}{(2\pi)^n}\frac{k^\mu k^\nu}{k^2+m^2-i\epsilon} \\ &= \frac{\eta^{\mu\nu}}{2}\left(\frac{m^2}{4\pi}\right)^{n/2}\Gamma\left(-\frac{n}{2}\right),\end{aligned} \tag{2.12}$$

where the integrals in dimensional regularization have been computed in the standard way (see Appendix B) and where $\Gamma(z)$ is the Euler's gamma function. The semiclassical Einstein equation (2.1), which now reduces to

$$\frac{\Lambda_B}{8\pi G_B}\eta^{\mu\nu} = \mu^{-(n-4)}\langle 0|\hat{T}^{\mu\nu}_n|0\rangle[\eta], \tag{2.13}$$

simply sets the value of the bare coupling constant Λ_B/G_B. Note, from Eq. (2.12), that in order to have $\langle 0|\hat{T}^{ab}_R|0\rangle[\eta] = 0$, the renormalized (and regularized) stress-energy tensor ''operator'' for a scalar field in Minkowski spacetime has to be defined as

$$\hat{T}^{ab}_R[\eta] = \mu^{-(n-4)}\hat{T}^{ab}_n[\eta] - \frac{\eta^{ab}}{2}\frac{m^4}{(4\pi)^2} \times \left(\frac{m^2}{4\pi\mu^2}\right)^{(n-4)/2}\Gamma\left(-\frac{n}{2}\right), \tag{2.14}$$

which corresponds to a renormalization of the cosmological constant

$$\frac{\Lambda_B}{G_B} = \frac{\Lambda}{G} - \frac{2}{\pi}\frac{m^4}{n(n-2)}\kappa_n + O(n-4), \tag{2.15}$$

where

$$\kappa_n \equiv \frac{1}{(n-4)}\left(\frac{e^\gamma m^2}{4\pi\mu^2}\right)^{(n-4)/2} = \frac{1}{n-4} + \frac{1}{2}\ln\left(\frac{e^\gamma m^2}{4\pi\mu^2}\right) + O(n-4), \tag{2.16}$$

being γ the Euler's constant. In the case of a massless scalar field, $m^2 = 0$, one simply has $\Lambda_B/G_B = \Lambda/G$. Introducing this renormalized coupling constant into Eq. (2.13), we can take the limit $n \to 4$. We find again that, for $(\mathbb{R}^4, \eta_{ab}, |0\rangle)$ to satisfy the semiclassical Einstein equation, we must take $\Lambda = 0$.

We are now in the position to write down the Einstein-Langevin equations for the components $h_{\mu\nu}$ of the stochastic metric perturbation in dimensional regularization. In our case, using $\langle 0|\hat{\Phi}_n^2(x)|0\rangle = i\Delta_{F_n}(0)$ and the explicit expression for Eq. (2.2) found in Ref. [10], we obtain that this equation reduces to

$$\frac{1}{8\pi G_B}\left[G^{(1)\mu\nu}+\Lambda_B\left(h^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}h\right)\right](x)-\frac{4}{3}\alpha_B D^{(1)\mu\nu}(x)$$
$$-2\beta_B B^{(1)\mu\nu}(x)-\xi G^{(1)\mu\nu}(x)\mu^{-(n-4)}i\Delta_{F_n}(0)$$
$$+2\int d^n y\,\mu^{-(n-4)}H_n^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y)=2\xi^{\mu\nu}(x), \tag{2.17}$$

where $\xi^{\mu\nu}$ are the components of a Gaussian stochastic tensor of zero average and

$$\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\rangle_c=\mu^{-2(n-4)}N_n^{\mu\nu\alpha\beta}(x,y), \tag{2.18}$$

and where indices are raised in $h_{\mu\nu}$ with the flat metric and $h\equiv h^{\rho}_{\rho}$. We use a superindex (1) to denote the components of a tensor linearized around the flat metric. In the last expressions, $N_n^{\mu\nu\alpha\beta}(x,y)$ and $H_n^{\mu\nu\alpha\beta}(x,y)$ are the components of the kernels defined above. In Eq. (2.17), we have made use of the explicit expression for $G^{(1)\mu\nu}$. This expression and those for $D^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ are given in Appendix E; the last two can also be written as

$$D^{(1)\mu\nu}(x)=\frac{1}{2}(3\mathcal{F}_x^{\mu\alpha}\mathcal{F}_x^{\nu\beta}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})h_{\alpha\beta}(x),$$
$$B^{(1)\mu\nu}(x)=2\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}h_{\alpha\beta}(x), \tag{2.19}$$

where $\mathcal{F}_x^{\mu\nu}$ is the differential operator $\mathcal{F}_x^{\mu\nu}\equiv\eta^{\mu\nu}\Box_x-\partial_x^{\mu}\partial_x^{\nu}$.

III. THE KERNELS FOR A MINKOWSKI BACKGROUND

The kernels $N_n^{\mu\nu\alpha\beta}(x,y)$ and $H_n^{\mu\nu\alpha\beta}(x,y)=H_{S_n}^{\mu\nu\alpha\beta}(x,y)+H_{A_n}^{\mu\nu\alpha\beta}(x,y)$ can now be computed using Eq. (2.5) and the expressions (A1) and (A3). In Ref. [10], we have shown that the kernel $H_{A_n}^{\mu\nu\alpha\beta}(x,y)$ plays the role of a dissipation kernel, since it is related to the noise kernel, $N_n^{\mu\nu\alpha\beta}(x,y)$, by a fluctuation-dissipation relation. From the definitions (2.4) and the fact that the Minkowski vacuum $|0\rangle$ is an eigenstate of the operator $\hat{P}^{\mu}$, given by Eq. (2.9), these kernels satisfy

$$\int d^{n-1}\mathbf{x}N_n^{0\mu\alpha\beta}(x,y)=\int d^{n-1}\mathbf{x}H_{A_n}^{0\mu\alpha\beta}(x,y)=0. \tag{3.1}$$

A. The noise and dissipation kernels

Since the two kernels (2.5) are free of ultraviolet divergencies in the limit $n\to 4$, we can deal directly with

$$M^{\mu\nu\alpha\beta}(x-y)\equiv\lim_{n\to 4}\mu^{-2(n-4)}\langle 0|\hat{t}_n^{\mu\nu}(x)\hat{t}_n^{\alpha\beta}(y)|0\rangle[\eta]. \tag{3.2}$$

The kernels $4N^{\mu\nu\alpha\beta}(x,y)=\mathrm{Re}\,M^{\mu\nu\alpha\beta}(x-y)$ and $4H_{\mathrm{A}}^{\mu\nu\alpha\beta}(x,y)=\mathrm{Im}\,M^{\mu\nu\alpha\beta}(x-y)$ are actually the components of the ''physical'' noise and dissipation kernels that will appear in the Einstein-Langevin equations once the renormalization procedure has been carried out. Note that, in the renormalization scheme in which $\hat{T}_R^{ab}[\eta]$ is given by Eq. (2.14), we can write

$$M^{\mu\nu\alpha\beta}(x-y)=\langle 0|\hat{T}_R^{\mu\nu}(x)\hat{T}_R^{\alpha\beta}(y)|0\rangle[\eta],$$

where the limit $n\to 4$ is understood. This kernel can be expressed in terms of the Wightman function in four spacetime dimensions,

$$\Delta^{+}(x)=-2\pi i\int\frac{d^4k}{(2\pi)^4}e^{ikx}\delta(k^2+m^2)\,\theta(k^0), \tag{3.3}$$

in the following way:

$$M^{\mu\nu\alpha\beta}(x)=-2[\partial^{\mu}\partial^{(\alpha}\Delta^{+}(x)\partial^{\beta)}\partial^{\nu}\Delta^{+}(x)$$
$$+\mathcal{D}^{\mu\nu}(\partial^{\alpha}\Delta^{+}(x)\partial^{\beta}\Delta^{+}(x))$$
$$+\mathcal{D}^{\alpha\beta}(\partial^{\mu}\Delta^{+}(x)\partial^{\nu}\Delta^{+}(x))$$
$$+\mathcal{D}^{\mu\nu}\mathcal{D}^{\alpha\beta}(\Delta^{+2}(x))]. \tag{3.4}$$

The different terms in Eq. (3.4) can be easily computed using the integrals

$$I(p)\equiv\int\frac{d^4k}{(2\pi)^4}\delta(k^2+m^2)\,\theta(-k^0)\times\delta[(k-p)^2+m^2]\,\theta(k^0-p^0),$$

$$I^{\mu_1\cdots\mu_r}(p)\equiv\int\frac{d^4k}{(2\pi)^4}k^{\mu_1}\cdots k^{\mu_r}\delta(k^2+m^2)\,\theta(-k^0)\times\delta[(k-p)^2+m^2]\,\theta(k^0-p^0), \tag{3.5}$$

with $r=1,2,3,4$, given in Appendix B; all of them can be expressed in terms of $I(p)$. We obtain expressions (C1)–(C3). It is convenient to separate $I(p)$ in its even and odd parts with respect to the variables p^{μ} as

$$I(p)=I_{\mathrm{S}}(p)+I_{\mathrm{A}}(p), \tag{3.6}$$

where $I_{\mathrm{S}}(-p)=I_{\mathrm{S}}(p)$ and $I_{\mathrm{A}}(-p)=-I_{\mathrm{A}}(p)$. These two functions are explicitly given by

$$I_{\rm S}(p)=\frac{1}{8(2\pi)^3}\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}},$$

$$I_{\rm A}(p)=\frac{-1}{8(2\pi)^3}{\rm sgn}\,p^0\;\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}. \tag{3.7}$$

Using the results of Appendix B, we obtain expressions (C4)–(C6) and, after some calculations, we find

$$M^{\mu\nu\alpha\beta}(x)=\frac{\pi^2}{45}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})\int\frac{d^4p}{(2\pi)^4}\times e^{-ipx}\left(1+4\frac{m^2}{p^2}\right)^2 I(p)+\frac{8\pi^2}{9}\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\times\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2 I(p), \tag{3.8}$$

where $\Delta\xi\equiv\xi-1/6$. The real and imaginary parts of the last expression, which yield the noise and dissipation kernels, are easily recognized as the terms containing $I_{\rm S}(p)$ and $I_{\rm A}(p)$, respectively. To write them explicitly, it is useful to introduce the new kernels

$$N_{\rm A}(x;m^2)\equiv\frac{1}{1920\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\theta(-p^2-4m^2)\times\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2,$$

$$N_{\rm B}(x;m^2,\Delta\xi)\equiv\frac{1}{288\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\theta(-p^2-4m^2)\times\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2, \tag{3.9}$$

$$D_{\rm A}(x;m^2)\equiv\frac{-i}{1920\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\,{\rm sgn}\,p^0\,\theta(-p^2-4m^2)\times\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2,$$

$$D_{\rm B}(x;m^2,\Delta\xi)\equiv\frac{-i}{288\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\,{\rm sgn}\,p^0\,\theta(-p^2-4m^2)\times\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2,$$

and we finally get

$$N^{\mu\nu\alpha\beta}(x,y)=\frac{1}{6}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})N_{\rm A}(x-y;m^2)+\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}N_{\rm B}(x-y;m^2,\Delta\xi), \tag{3.10}$$

$$H_{\rm A}^{\mu\nu\alpha\beta}(x,y)=\frac{1}{6}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})D_{\rm A}(x-y;m^2)+\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}D_{\rm B}(x-y;m^2,\Delta\xi).$$

Notice that the noise and dissipation kernels defined in Eq. (3.9) are actually real because, for the noise kernels, only the $\cos px$ terms of the exponentials e^{ipx} contribute to the integrals, and, for the dissipation kernels, the only contribution of such exponentials comes from the $i\sin px$ terms.

We can now evaluate the contribution of the dissipation kernel components $H_{\rm A}^{\mu\nu\alpha\beta}(x,y)$ to the Einstein-Langevin equations (2.17) [after taking the limit $n\to 4$]. From Eq. (3.10), integrating by parts, and using Eq. (2.19) and the fact that, in four spacetime dimensions, $D^{(1)\mu\nu}(x)=(3/2)A^{(1)\mu\nu}(x)$ (the tensor A^{ab} is obtained from the derivative with respect to the metric of an action term corresponding to the Lagrangian density $C_{abcd}C^{abcd}$, where C_{abcd} is the Weyl tensor, see Ref. [10] for details), it is easy to see that

$$2\int d^4y\,H_{\rm A}^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y)=\int d^4y[D_{\rm A}(x-y;m^2)A^{(1)\mu\nu}(y)+D_{\rm B}(x-y;m^2,\Delta\xi)B^{(1)\mu\nu}(y)]. \tag{3.11}$$

These non-local terms in the semiclassical Einstein-Langevin equations can actually be identified as being part of $\langle\hat{T}_R^{\mu\nu}\rangle[\eta+h]$.

B. The kernel $H_{{\rm S}_n}^{\mu\nu\alpha\beta}(x,y)$

The evaluation of the kernel components $H_{{\rm S}_n}^{\mu\nu\alpha\beta}(x,y)$ is a much more cumbersome task. Since these quantities contain divergencies in the limit $n\to 4$, we shall compute them using dimensional regularization. Using Eq. (A3), these components can be written in terms of the Feynman propagator (2.11) as

$$\mu^{-(n-4)}H_{{\rm S}_n}^{\mu\nu\alpha\beta}(x,y)=\frac{1}{4}{\rm Im}\,K^{\mu\nu\alpha\beta}(x-y), \tag{3.12}$$

where

$$K^{\mu\nu\alpha\beta}(x)\equiv-\mu^{-(n-4)}\Big\{2\partial^{\mu}\partial^{(\alpha}\Delta_{F_n}(x)\partial^{\beta)}\partial^{\nu}\Delta_{F_n}(x)+2\mathcal{D}^{\mu\nu}(\partial^{\alpha}\Delta_{F_n}(x)\partial^{\beta}\Delta_{F_n}(x))$$
$$+2\mathcal{D}^{\alpha\beta}(\partial^{\mu}\Delta_{F_n}(x)\partial^{\nu}\Delta_{F_n}(x))+2\mathcal{D}^{\mu\nu}\mathcal{D}^{\alpha\beta}(\Delta^2_{F_n}(x))+\Big[\eta^{\mu\nu}\partial^{(\alpha}\Delta_{F_n}(x)\partial^{\beta)}+\eta^{\alpha\beta}\partial^{(\mu}\Delta_{F_n}(x)\partial^{\nu)}$$
$$+\Delta_{F_n}(0)(\eta^{\mu\nu}\mathcal{D}^{\alpha\beta}+\eta^{\alpha\beta}\mathcal{D}^{\mu\nu})+\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}(\Delta_{F_n}(x)\Box-m^2\Delta_{F_n}(0))\Big]\delta^n(x)\Big\}. \tag{3.13}$$

Let us define the integrals

$$J_n(p)\equiv\mu^{-(n-4)}\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+m^2-i\epsilon)[(k-p)^2+m^2-i\epsilon]},$$
$$J_n^{\mu_1\cdots\mu_r}(p)\equiv\mu^{-(n-4)}\int\frac{d^nk}{(2\pi)^n}\frac{k^{\mu_1}\cdots k^{\mu_r}}{(k^2+m^2-i\epsilon)[(k-p)^2+m^2-i\epsilon]}, \tag{3.14}$$

with $r=1,2,3,4$, and

$$I_{0_n}\equiv\mu^{-(n-4)}\int\frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+m^2-i\epsilon)},$$
$$I_{0_n}^{\mu_1\cdots\mu_r}\equiv\mu^{-(n-4)}\int\frac{d^nk}{(2\pi)^n}\frac{k^{\mu_1}\cdots k^{\mu_r}}{(k^2+m^2-i\epsilon)}, \tag{3.15}$$

with $r=1,2$, where a limit $\epsilon\to0^+$ is understood in all these expressions. Then, the different terms in Eq. (3.13) can be computed using Eqs. (D1)–(D6). The results for the expansions of the integrals (3.14) and (3.15) around $n=4$ are given in Appendix B. In fact, $I_{0_n}^{\mu}=0$ and the remaining integrals can be written in terms of I_{0_n} and $J_n(p)$ given in Eqs. (B1) and (B4). Using the results of Appendix B, we obtain Eqs. (D7) and (D8) and, from Eqs. (D4)–(D6), we get

$$\mu^{-(n-4)}[\eta^{\mu\nu}\partial^{(\alpha}\Delta_{F_n}(x)\partial^{\beta)}+\eta^{\alpha\beta}\partial^{(\mu}\Delta_{F_n}(x)\partial^{\nu)}]\delta^n(x)=2\eta^{\mu\nu}\eta^{\alpha\beta}\frac{m^2}{n}I_{0_n}\delta^n(x),$$
$$\mu^{-(n-4)}(\Delta_{F_n}(x)\Box-m^2\Delta_{F_n}(0))\delta^n(x)=-I_{0_n}\Box\delta^n(x). \tag{3.16}$$

We are now in the position to work out the explicit expression for $K^{\mu\nu\alpha\beta}(x)$, defined in Eq. (3.13). We use Eqs. (3.16), the results (D1), (D4), (D7) and (D8), the identities $\delta^n(x)=(2\pi)^{-n}\int d^np\,e^{ipx}$, $\mathcal{F}_x^{\mu\nu}\int d^np\,e^{ipx}f(p)=-\int d^np\,e^{ipx}f(p)p^2P^{\mu\nu}$ and $\partial_x^{\mu}\partial_x^{\nu}\int d^np\,e^{ipx}f(p)=-\int d^np\,e^{ipx}f(p)p^{\mu}p^{\nu}$, where $f(p)$ is an arbitrary function of p^{μ} and $P^{\mu\nu}$ is the projector orthogonal to p^{μ} defined as $p^2P^{\mu\nu}\equiv\eta^{\mu\nu}p^2-p^{\mu}p^{\nu}$, and the expansions in Eqs. (B1) and (B4) for $J_n(p)$ and I_{0_n}. After a rather long but straightforward calculation, we get, expanding around $n=4$,

$$K^{\mu\nu\alpha\beta}(x)=\frac{i}{(4\pi)^2}\Big\{\kappa_n\Big[\frac{1}{90}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})\delta^n(x)+4\Delta\xi^2\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\delta^n(x)+\frac{2}{3}\frac{m^2}{(n-2)}(\eta^{\mu\nu}\eta^{\alpha\beta}\Box_x-\eta^{\mu(\alpha}\eta^{\beta)\nu}\Box_x$$
$$+\eta^{\mu(\alpha}\partial_x^{\beta)}\partial_x^{\nu}+\eta^{\nu(\alpha}\partial_x^{\beta)}\partial_x^{\mu}-\eta^{\mu\nu}\partial_x^{\alpha}\partial_x^{\beta}-\eta^{\alpha\beta}\partial_x^{\mu}\partial_x^{\nu})\delta^n(x)+\frac{4m^4}{n(n-2)}(2\eta^{\mu(\alpha}\eta^{\beta)\nu}-\eta^{\mu\nu}\eta^{\alpha\beta})\delta^n(x)\Big]$$
$$+\frac{1}{180}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})\int\frac{d^np}{(2\pi)^n}e^{ipx}\left(1+4\frac{m^2}{p^2}\right)^2\phi(p^2)+\frac{2}{9}\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}$$
$$\times\int\frac{d^np}{(2\pi)^n}e^{ipx}\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2\phi(p^2)-\Big[\frac{4}{675}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})+\frac{1}{270}(60\xi-11)\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\Big]\delta^n(x)$$
$$-m^2\Big[\frac{2}{135}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})+\frac{1}{27}\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\Big]\Delta_n(x)\Big\}+O(n-4), \tag{3.17}$$

where κ_n and $\phi(p^2)$ have been defined in Eqs. (2.16) and (B9), and $\Delta_n(x)$ is given by

$$\Delta_n(x) \equiv \int \frac{d^n p}{(2\pi)^n} e^{ipx} \frac{1}{p^2}. \tag{3.18}$$

The imaginary part of Eq. (3.17) [which, using Eq. (3.12), gives the kernel components $\mu^{-(n-4)} H_{\mathrm{S}_n}^{\mu\nu\alpha\beta}(x,y)$] can be easily obtained multiplying this expression by $-i$ and retaining only the real part, $\varphi(p^2)$, of the function $\phi(p^2)$. Making use of this result, it is easy to compute the contribution of these kernel components to the Einstein-Langevin equations (2.17). Integrating by parts, using Eqs. (E1)–(E5) and Eq. (2.19), and taking into account that, from Eqs. (2.12) and (2.13),

$$\frac{\Lambda_B}{8\pi G_B} = -\frac{1}{4\pi^2} \frac{m^4}{n(n-2)} \kappa_n + O(n-4), \tag{3.19}$$

we finally find

$$\begin{aligned}
2\int d^n y\, \mu^{-(n-4)} H_{\mathrm{S}_n}^{\mu\nu\alpha\beta}(x,y) h_{\alpha\beta}(y) = {} & -\frac{\Lambda_B}{8\pi G_B}\left[h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h\right](x) + \frac{\kappa_n}{(4\pi)^2}\left[\frac{2}{3}\frac{m^2}{(n-2)} G^{(1)\mu\nu} + \frac{1}{90} D^{(1)\mu\nu} + \Delta\xi^2 B^{(1)\mu\nu}\right](x) \\
& + \frac{1}{2880\pi^2}\left\{-\frac{16}{15} D^{(1)\mu\nu}(x) + \left(\frac{1}{6} - 10\Delta\xi\right) B^{(1)\mu\nu}(x) + \int d^n y \int \frac{d^n p}{(2\pi)^n} e^{ip(x-y)} \varphi(p^2)\right. \\
& \times\left[\left(1 + 4\frac{m^2}{p^2}\right)^2 D^{(1)\mu\nu}(y) + 10\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 B^{(1)\mu\nu}(y)\right] \\
& \left. - \frac{m^2}{3}\int d^n y\, \Delta_n(x-y)\big(8D^{(1)\mu\nu}(y) + 5B^{(1)\mu\nu}(y)\big)\right\} + O(n-4).
\end{aligned} \tag{3.20}$$

C. Fluctuation-dissipation relation

From expressions (3.10) and (3.9) it is easy to check that there exists a relation between the noise and dissipation kernels in the form of a fluctuation-dissipation relation which was derived in Ref. [10] in a more general context. Introducing the Fourier transforms in the time coordinates of these kernels as

$$N^{\mu\nu\alpha\beta}(x,y) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} e^{-ip^0(x^0-y^0)} \overline{N}^{\mu\nu\alpha\beta}(p^0;\mathbf{x},\mathbf{y}), \tag{3.21}$$

and similarly for the dissipation kernel, this relation can be written as

$$\overline{H}_{\mathrm{A}}^{\mu\nu\alpha\beta}(p^0;\mathbf{x},\mathbf{y}) = -i \operatorname{sgn} p^0\, \overline{N}^{\mu\nu\alpha\beta}(p^0;\mathbf{x},\mathbf{y}), \tag{3.22}$$

or, equivalently, as

$$H_{\mathrm{A}}^{\mu\nu\alpha\beta}(x^0,\mathbf{x};y^0,\mathbf{y}) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dz^0\, \mathrm{P}\left(\frac{1}{x^0 - z^0}\right) N^{\mu\nu\alpha\beta}(z^0,\mathbf{x};y^0,\mathbf{y}), \tag{3.23}$$

where $\mathrm{P}(1/x^0)$ denotes the principal value distribution.

From Eq. (3.1), taking the limit $n \to 4$, we see that the noise and dissipation kernels must satisfy

$$\int d^3\mathbf{x} N^{0\mu\alpha\beta}(x,y) = \int d^3\mathbf{x} H_{\mathrm{A}}^{0\mu\alpha\beta}(x,y) = 0. \tag{3.24}$$

In order to check the last relations, it is useful to write the $\mathcal{F}_x^{\mu\nu}$ derivatives in expressions (3.10) using $\mathcal{F}_x^{\mu\nu}\int d^4p\, e^{ip(x-y)} f(p) = -\int d^4p\, e^{ip(x-y)} f(p) p^2 P^{\mu\nu}$, where $f(p)$ is any function of p^μ and $P^{\mu\nu}$ is the projector orthogonal to p^μ defined above. The identities (3.24) follow by noting that $p^2 P^{00} = -p^i p_i$ and $p^2 P^{0i} = -p^0 p^i$, where we use the index $i = 1,2,3$ to denote the space components, and that $\int d^3\mathbf{x} \exp(ip_i x^i) = (2\pi)^3 \Pi_{i=1}^3 \delta(p^i)$. It is also easy to check that the noise kernel satisfies $\partial_\mu^x N^{\mu\nu\alpha\beta}(x,y) = 0$ and, hence, the stochastic source in the Einstein-Langevin equations will be conserved up to first order in perturbation theory.

IV. THE SEMICLASSICAL EINSTEIN-LANGEVIN EQUATIONS

The results of the previous section are now ready to be introduced into the Einstein-Langevin equations (2.17). In fact, substituting expression (3.20) in such equations, and using Eqs. (D4) and (B1) for the $\mu^{-(n-4)}\Delta_{F_n}(0)$ term, we get

$$
\begin{aligned}
&\frac{1}{8\pi G_B}G^{(1)\mu\nu}(x)-\frac{4}{3}\alpha_B D^{(1)\mu\nu}(x)-2\beta_B B^{(1)\mu\nu}(x)+\frac{\kappa_n}{(4\pi)^2}\left[-4\Delta\xi\frac{m^2}{(n-2)}G^{(1)\mu\nu}+\frac{1}{90}D^{(1)\mu\nu}+\Delta\xi^2 B^{(1)\mu\nu}\right](x)\\
&\quad+\frac{1}{2880\pi^2}\left\{-\frac{16}{15}D^{(1)\mu\nu}(x)+\left(\frac{1}{6}-10\Delta\xi\right)B^{(1)\mu\nu}(x)+\int d^ny\int\frac{d^np}{(2\pi)^n}e^{ip(x-y)}\varphi(p^2)\right.\\
&\quad\left.\times\left[\left(1+4\frac{m^2}{p^2}\right)^2 D^{(1)\mu\nu}(y)+10\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2 B^{(1)\mu\nu}(y)\right]-\frac{m^2}{3}\int d^ny\,\Delta_n(x-y)(8D^{(1)\mu\nu}+5B^{(1)\mu\nu})(y)\right\}\\
&\quad+2\int d^ny\,\mu^{-(n-4)}H_{\mathrm{A}_n}^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y)+O(n-4)\\
&=2\xi^{\mu\nu}(x).
\end{aligned}
\tag{4.1}
$$

Notice that the terms containing the bare cosmological constant have canceled. These equations can now be renormalized, that is, we can now write the bare coupling constants as renormalized coupling constants plus some suitably chosen counterterms and take the limit $n\to4$. In order to carry out such a procedure, it is convenient to distinguish between massive and massless scalar fields. We shall evaluate these two cases in different subsections.

A. Massive field $(m\neq0)$

In the case of a scalar field with mass $m\neq0$, we can use, as we have done in Eq. (2.15) for the cosmological constant, a renormalization scheme consisting on the subtraction of terms proportional to κ_n. More specifically, we may introduce the renormalized coupling constants $1/G$, α and β as

$$
\begin{aligned}
\frac{1}{G_B}&=\frac{1}{G}+\frac{2}{\pi}\Delta\xi\frac{m^2}{(n-2)}\kappa_n+O(n-4),\\
\alpha_B&=\alpha+\frac{1}{(4\pi)^2}\frac{1}{120}\kappa_n+O(n-4),\\
\beta_B&=\beta+\frac{\Delta\xi^2}{32\pi^2}\kappa_n+O(n-4).
\end{aligned}
\tag{4.2}
$$

Note that for conformal coupling, $\Delta\xi=0$, one has $1/G_B=1/G$ and $\beta_B=\beta$, that is, only the coupling constant α and the cosmological constant need renormalization. Substituting the above expressions into Eq. (4.1), we can now take the limit $n\to4$, using Eqs. (3.18),(3.11) and the fact that, for $n=4$, $D^{(1)\mu\nu}(x)=(3/2)A^{(1)\mu\nu}(x)$. We obtain the semiclassical Einstein-Langevin equations for the physical stochastic perturbations $h_{\mu\nu}$ in the four-dimensional manifold $\mathcal{M}\equiv\mathbb{R}^4$. Introducing the two new kernels

$$
\begin{aligned}
H_{\mathrm{A}}(x;m^2)&\equiv\frac{1}{1920\pi^2}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\left\{\left(1+4\frac{m^2}{p^2}\right)^2\left[-i\pi\,\mathrm{sgn}\,p^0\,\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}+\varphi(p^2)\right]-\frac{8}{3}\frac{m^2}{p^2}\right\},\\
H_{\mathrm{B}}(x;m^2,\Delta\xi)&\equiv\frac{1}{288\pi^2}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\left\{\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2\left[-i\pi\,\mathrm{sgn}\,p^0\,\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}+\varphi(p^2)\right]-\frac{1}{6}\frac{m^2}{p^2}\right\},
\end{aligned}
\tag{4.3}
$$

where $\varphi(p^2)$ is given by the restriction to $n=4$ of expression (B10), these Einstein-Langevin equations can be written as

$$
\begin{aligned}
&\frac{1}{8\pi G}G^{(1)\mu\nu}(x)-2(\alpha A^{(1)\mu\nu}(x)+\beta B^{(1)\mu\nu}(x))+\frac{1}{2880\pi^2}\left[-\frac{8}{5}A^{(1)\mu\nu}(x)+\left(\frac{1}{6}-10\Delta\xi\right)B^{(1)\mu\nu}(x)\right]\\
&\quad+\int d^4y[H_{\mathrm{A}}(x-y;m^2)A^{(1)\mu\nu}(y)+H_{\mathrm{B}}(x-y;m^2,\Delta\xi)B^{(1)\mu\nu}(y)]=2\xi^{\mu\nu}(x),
\end{aligned}
\tag{4.4}
$$

where $\xi^{\mu\nu}$ are the components of a Gaussian stochastic tensor of vanishing mean value and two-point correlation function $\langle\xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\rangle_c=N^{\mu\nu\alpha\beta}(x,y)$, given in Eq. (3.10). Note that the two kernels defined in Eq. (4.3) are real and can be split into an even part and an odd part with respect to the variables x^μ, with the odd terms being the dissipation kernels $D_{\mathrm{A}}(x;m^2)$ and $D_{\mathrm{B}}(x;m^2,\Delta\xi)$ defined in Eq. (3.9). In spite of appearances, one can show that the Fourier transforms of the even parts of these kernels are finite in the limit $p^2\to0$ and, hence, the kernels H_{A} and H_{B} are well defined distributions.

We should mention that, in a previous work in Ref. [18], the same Einstein-Langevin equations were calculated using rather different methods. The way in which the result is written makes difficult a direct comparison with our equations (4.4). For instance, it is not obvious that in those previously derived equations there is some analog of the dissipation kernels related to the noise kernels by a fluctuation-dissipation relation of the form (3.22) or (3.23).

B. Massless field $(m=0)$

In this subsection, we consider the limit $m\to 0$ of equations (4.1). The renormalization scheme used in the previous subsection becomes singular in the massless limit because the expressions (4.2) for α_B and β_B diverge when $m\to 0$. Therefore, a different renormalization scheme is needed in this case. First, note that we may separate κ_n in Eq. (2.16) as $\kappa_n=\tilde{\kappa}_n+\frac{1}{2}\ln(m^2/\mu^2)+O(n-4)$, where

$$\tilde{\kappa}_n\equiv\frac{1}{(n-4)}\left(\frac{e^{\gamma}}{4\pi}\right)^{(n-4)/2}=\frac{1}{n-4}+\frac{1}{2}\ln\left(\frac{e^{\gamma}}{4\pi}\right)+O(n-4), \tag{4.5}$$

and that [see Eq. (B10)]

$$\lim_{m^2\to 0}\left[\varphi(p^2)+\ln(m^2/\mu^2)\right]=-2+\ln\left|\frac{p^2}{\mu^2}\right|. \tag{4.6}$$

Hence, in the massless limit, Eqs. (4.1) reduce to

$$\begin{aligned}&\frac{1}{8\pi G_B}G^{(1)\mu\nu}(x)-\frac{4}{3}\alpha_B D^{(1)\mu\nu}(x)-2\beta_B B^{(1)\mu\nu}(x)+\frac{1}{(4\pi)^2}(\tilde{\kappa}_n-1)\left[\frac{1}{90}D^{(1)\mu\nu}+\Delta\xi^2 B^{(1)\mu\nu}\right](x)\\&\quad+\frac{1}{2880\pi^2}\left\{-\frac{16}{15}D^{(1)\mu\nu}(x)+\left(\frac{1}{6}-10\Delta\xi\right)B^{(1)\mu\nu}(x)+\int d^ny\int\frac{d^np}{(2\pi)^n}e^{ip(x-y)}\ln\left|\frac{p^2}{\mu^2}\right|\left[D^{(1)\mu\nu}(y)+90\Delta\xi^2 B^{(1)\mu\nu}(y)\right]\right\}\\&\quad+\lim_{m^2\to 0}2\int d^ny\,\mu^{-(n-4)}H_{\mathrm{A}_n}^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y)+O(n-4)=2\xi^{\mu\nu}(x).\end{aligned} \tag{4.7}$$

These equations can be renormalized by introducing the renormalized coupling constants $1/G$, α and β as

$$\begin{aligned}&\frac{1}{G_B}=\frac{1}{G},\quad \alpha_B=\alpha+\frac{1}{(4\pi)^2}\frac{1}{120}(\tilde{\kappa}_n-1)+O(n-4),\\&\beta_B=\beta+\frac{\Delta\xi^2}{32\pi^2}(\tilde{\kappa}_n-1)+O(n-4).\end{aligned} \tag{4.8}$$

Thus, in the massless limit, the Newtonian gravitational constant is not renormalized and, in the conformal coupling case, $\Delta\xi=0$, we have again that $\beta_B=\beta$. Introducing the last expressions into Eq. (4.7), we can take the limit $n\to 4$. Note that, by making $m=0$ in Eq. (3.9), the noise and dissipation kernels can be written as

$$\begin{aligned}&N_{\mathrm{A}}(x;m^2=0)=N(x),\quad N_{\mathrm{B}}(x;m^2=0,\Delta\xi)=60\Delta\xi^2N(x),\\&D_{\mathrm{A}}(x;m^2=0)=D(x),\quad D_{\mathrm{B}}(x;m^2=0,\Delta\xi)=60\Delta\xi^2D(x),\end{aligned} \tag{4.9}$$

where

$$N(x)\equiv\frac{1}{1920\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\theta(-p^2), \tag{4.10}$$

$$D(x)\equiv\frac{-i}{1920\pi}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\,\mathrm{sgn}\,p^0\,\theta(-p^2).$$

It is now convenient to introduce the new kernel

$$\begin{aligned}H(x;\mu^2)&\equiv\frac{1}{1920\pi^2}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\\&\quad\times\left[\ln\left|\frac{p^2}{\mu^2}\right|-i\pi\,\mathrm{sgn}\,p^0\,\theta(-p^2)\right]\\&=\frac{1}{1920\pi^2}\lim_{\epsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}e^{ipx}\\&\quad\times\ln\left(\frac{-(p^0+i\epsilon)^2+p^ip_i}{\mu^2}\right).\end{aligned} \tag{4.11}$$

Again, this kernel is real and can be written as the sum of an even part and an odd part in the variables x^{μ}, where the odd part is the dissipation kernel $D(x)$. The Fourier transforms (4.10) and (4.11) can actually be computed and, thus, in this case, we have explicit expressions for the kernels in position space. For $N(x)$ and $D(x)$, we get (see, for instance, Ref. [27])

$$N(x)=\frac{1}{1920\pi}\left[\frac{1}{\pi^3}\mathcal{P}f\left(\frac{1}{(x^2)^2}\right)+\delta^4(x)\right],$$
$$D(x)=\frac{1}{1920\pi^3}\mathrm{sgn}\,x^0\frac{d}{d(x^2)}\,\delta(x^2), \tag{4.12}$$

where $\mathcal{P}f$ denotes a distribution generated by the Hadamard finite part of a divergent integral (see Refs. [28] for the definition of these distributions). The expression for the kernel $H(x;\mu^2)$ can be found in Refs. [29,30] and it is given by

$$\begin{aligned}H(x;\mu^2)&=\frac{1}{960\pi^2}\Bigg\{\mathcal{P}f\left(\frac{1}{\pi}\theta(x^0)\frac{d}{d(x^2)}\,\delta(x^2)\right)\\&\quad+(1-\gamma-\ln\mu)\,\delta^4(x)\Bigg\}\\&=\frac{1}{960\pi^2}\lim_{\lambda\to0^+}\Bigg\{\frac{1}{\pi}\theta(x^0)\,\theta(|\mathbf{x}|-\lambda)\,\frac{d}{d(x^2)}\,\delta(x^2)\\&\quad+[1-\gamma-\ln(\mu\lambda)]\,\delta^4(x)\Bigg\}.\end{aligned} \tag{4.13}$$

See Ref. [29] for the details on how this last distribution acts on a test function. Finally, the semiclassical Einstein-Langevin equations for the physical stochastic perturbations $h_{\mu\nu}$ in the massless case are

$$\begin{aligned}&\frac{1}{8\pi G}G^{(1)\mu\nu}(x)-2(\alpha A^{(1)\mu\nu}(x)+\beta B^{(1)\mu\nu}(x))\\&\quad+\frac{1}{2880\pi^2}\left[-\frac{8}{5}A^{(1)\mu\nu}(x)+\left(\frac{1}{6}-10\Delta\xi\right)B^{(1)\mu\nu}(x)\right]\\&\quad+\int d^4y\,H(x-y;\mu^2)[A^{(1)\mu\nu}(y)+60\Delta\xi^2B^{(1)\mu\nu}(y)]\\&=2\xi^{\mu\nu}(x),\end{aligned} \tag{4.14}$$

where the Gaussian stochastic source components $\xi^{\mu\nu}$ have zero mean value and

$$\begin{aligned}\langle\xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\rangle_c&=\lim_{m\to0}N^{\mu\nu\alpha\beta}(x,y)\\&=\Bigg[\frac{1}{6}(3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu}-\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta})\\&\quad+60\Delta\xi^2\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\Bigg]N(x-y).\end{aligned} \tag{4.15}$$

It is interesting to consider the conformally coupled scalar field, i.e., the case $\Delta\xi=0$, of particular interest because of its similarities with the electromagnetic field. It was shown in Refs. [9,10] that, for this field, the stochastic source tensor must be "traceless" (up to first order in perturbation theory around semiclassical gravity), in the sense that the stochastic variable $\xi^\mu_\mu\equiv\eta_{\mu\nu}\xi^{\mu\nu}$ behaves deterministically as a vanishing scalar field. This can be easily checked by noticing, from Eq. (4.15), that, when $\Delta\xi=0$, one has $\langle\xi^\mu_\mu(x)\xi^{\alpha\beta}(y)\rangle_c=0$, since $\mathcal{F}^\mu_\mu=3\Box$ and $\mathcal{F}^{\mu\alpha}\mathcal{F}^\beta_\mu=\Box\mathcal{F}^{\alpha\beta}$. The semiclassical Einstein-Langevin equations for this particular case [and generalized to a spatially flat Robertson-Walker (RW) background] were first obtained in Ref. [17] (in this reference, the coupling constant β was set to zero). In order to compare with this previous result, it is worth noticing that the description of the stochastic source in terms of a symmetric and "traceless" tensor, with nine independent components $\xi^{\mu\nu}$, is equivalent to a description in terms of a Gaussian stochastic tensor with the same symmetry properties as the Weyl tensor, with components $\xi_c^{\mu\nu\alpha\beta}$, defined as $\xi^{\mu\nu}=-2\partial_\alpha\partial_\beta\xi_c^{\mu\alpha\nu\beta}$; this tensor is used in Ref. [17]. The symmetry properties of the $\xi_c^{\mu\nu\alpha\beta}$ ensure that there are also nine independent components in $-2\partial_\alpha\partial_\beta\xi_c^{\mu\alpha\nu\beta}$. It is easy to show that, for this combination to satisfy the correlation relation (4.15) with $\Delta\xi=0$, the relevant correlators for the new stochastic tensor must be

$$\langle\xi_c^{\mu\nu\alpha\beta}(x)\,\xi_c^{\rho\sigma\lambda\theta}(y)\rangle_{\xi_c}=T^{\mu\nu\alpha\beta\rho\sigma\lambda\theta}N(x-y), \tag{4.16}$$

where $T^{\mu\nu\alpha\beta\rho\sigma\lambda\theta}$ is a linear combination of terms like $\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\alpha\lambda}\eta^{\beta\theta}$ in such a way that it has the same symmetries as the product of two Weyl tensor components $C^{\mu\nu\alpha\beta}C^{\rho\sigma\lambda\theta}$, its explicit expression is given in Ref. [17]. Thus, after a redefinition of the arbitrary mass scale μ in Eq. (4.14) to absorb the constants of proportionality of the local terms with $A^{(1)\mu\nu}(x)$, one can see that the resulting equations for the $\Delta\xi=0$ case are actually equivalent to those found in Ref. [17].

C. Expectation value of the stress-energy tensor

From the above equations one may extract the expectation value of the renormalized stress-energy tensor for a scalar field in a spacetime $(\mathbb{R}^4,\eta_{ab}+h_{ab})$, computed up to first order in perturbation theory around the trivial solution of semiclassical gravity. Such an expectation value can be obtained by identification of Eqs. (4.4) and (4.14) with the components of the physical Einstein-Langevin equation, which in our particular case simply reads

$$\frac{1}{8\pi G}G^{(1)\mu\nu}-2(\alpha A^{(1)\mu\nu}+\beta B^{(1)\mu\nu})=\langle\hat{T}_R^{\mu\nu}\rangle[\eta+h]+2\xi^{\mu\nu}. \tag{4.17}$$

By comparison of Eqs. (4.4) and (4.14) with the last equation, we can identify

$$\begin{aligned}\langle\hat{T}_R^{\mu\nu}(x)\rangle[\eta+h]&=\frac{1}{2880\pi^2}\Bigg[\frac{8}{5}A^{(1)\mu\nu}(x)-\left(\frac{1}{6}-10\Delta\xi\right)\\&\quad\times B^{(1)\mu\nu}(x)\Bigg]-\int d^4y[H_{\mathrm{A}}(x-y;m^2)\\&\quad\times A^{(1)\mu\nu}(y)+H_{\mathrm{B}}(x-y;m^2,\Delta\xi)\\&\quad\times B^{(1)\mu\nu}(y)]+O(h^2),\end{aligned} \tag{4.18}$$

for a massive scalar field, $m\neq 0$, and

$$\langle \hat{T}^{\mu\nu}_{R}(x)\rangle[\eta+h]=\frac{1}{2880\pi^2}\left[\frac{8}{5}A^{(1)\mu\nu}(x)-\left(\frac{1}{6}-10\Delta\xi\right)\times B^{(1)\mu\nu}(x)\right]-\int d^4y H(x-y;\mu^2)\times[A^{(1)\mu\nu}(y)+60\Delta\xi^2 B^{(1)\mu\nu}(y)]+O(h^2), \tag{4.19}$$

for a massless scalar field, $m=0$. Notice that in the massive case we have chosen, as usual, a renormalization scheme such that the expectation value of the renormalized stress-energy tensor does not have local terms proportional to the metric and the Einstein tensor [4]. The result (4.19) agrees with the general form found by Horowitz [30,31] using an axiomatic approach and coincides with that given in Ref. [25]. The particular cases of conformal coupling, $\Delta\xi=0$, and minimal coupling, $\Delta\xi=-1/6$, are also in agreement with the results for this cases given in Refs. [30–34] (modulo local terms proportional to $A^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ due to different choices of the renormalization scheme). For the case of a massive minimally coupled scalar field, $\Delta\xi=-1/6$, our result (4.18) is equivalent to that of Ref. [35].

As it was pointed out above, in the case of conformal coupling, both for massive and massless scalar fields, one has $\beta_B=\beta$. This means that, in these cases, the terms proportional to $B^{(1)\mu\nu}$ in the above expectation values of the stress-energy tensor are actually independent of the renormalization scheme chosen. Due to the conformal invariance of $\int d^4x\sqrt{-g}C_{cabd}C^{cabd}$, the tensor A^{ab} is traceless and we have $A^{(1)\,\mu}_{\;\;\;\;\mu}=0$. Therefore, the terms with $B^{(1)\mu\nu}$ are precisely those which give trace to the expectation value of the stress-energy tensor in Eqs. (4.18) and (4.19). In the massless conformally coupled case, $m=0$ and $\Delta\xi=0$, such terms give the trace anomaly [4] up to first order in $h_{\mu\nu}$:

$$\langle \hat{T}_{R}{}^{\mu}_{\mu}(x)\rangle[\eta+h]=-\frac{1}{2880\pi^2}\frac{1}{6}B^{(1)\,\mu}_{\;\;\;\;\mu}+O(h^2)=\frac{1}{2880\pi^2}\Box R^{(1)}+O(h^2), \tag{4.20}$$

where we have used expression (E3) for $B^{(1)\mu\nu}$.

D. Particle creation

We can also use the result (3.10) for the noise kernel to evaluate the total probability of particle creation and the number of created particles for a real scalar field in a spacetime $(\mathbb{R}^4,\eta_{ab}+h_{ab})$. The metric perturbation h_{ab} (here an arbitrary perturbation) is assumed to vanish, either in an exact way or ''asymptotically,'' in the ''remote past'' and in the ''far future,'' so that the scalar field has well defined ''in'' and ''out'' many particle states. In that case, the absolute value of the logarithm of the vacuum persistence probability $|\langle 0,\mathrm{out}|0,\mathrm{in}\rangle|^2$, where $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ are, respectively, the ''in'' and ''out'' vacua in the Heisenberg picture, gives a measure of the total probability of particle creation. On the other hand, the number of created particles can be defined as the expectation value in the ''in'' vacuum of the number operator for ''out'' particles. As it was shown in Ref. [10], the total probability of particle creation and one half of the number of created particles coincide to lowest non-trivial order in the metric perturbation, these are

$$P[h]=\int d^4x\, d^4y\, h_{\mu\nu}(x)N^{\mu\nu\alpha\beta}(x,y)h_{\alpha\beta}(y)+0(h^3), \tag{4.21}$$

where $N^{\mu\nu\alpha\beta}(x,y)$ is the noise kernel given in Eq. (3.10), which in the massless case reduces to Eq. (4.15). The above expression for the total probability of pair creation by metric perturbations about Minkowski spacetime was first derived in Ref. [36]. Using Eq. (3.10), we can write $P[h]=P_{\rm A}[h]+P_{\rm B}[h]+0(h^3)$, where

$$P_{\rm A}[h]\equiv\frac{1}{6}\int d^4x\, d^4y(3\mathcal{F}^{\mu\alpha}_{x}\mathcal{F}^{\nu\beta}_{x}-\mathcal{F}^{\mu\nu}_{x}\mathcal{F}^{\alpha\beta}_{x})\times N_{\rm A}(x-y;m^2)h_{\mu\nu}(x)h_{\alpha\beta}(y),$$

$$P_{\rm B}[h]\equiv\int d^4x\, d^4y\mathcal{F}^{\mu\nu}_{x}\mathcal{F}^{\alpha\beta}_{x}N_{\rm B}(x-y;m^2,\Delta\xi)\times h_{\mu\nu}(x)h_{\alpha\beta}(y). \tag{4.22}$$

Integrating by parts (we always neglect surface terms), using expression (E5) for $R^{(1)}$, which can also be written as $R^{(1)}=-\mathcal{F}^{\mu\nu}h_{\mu\nu}$, we find

$$P_{\rm B}[h]=\int d^4x\, d^4y\, R^{(1)}(x)N_{\rm B}(x-y;m^2,\Delta\xi)R^{(1)}(y). \tag{4.23}$$

In order to work out $P_{\rm A}[h]$, it is useful to take into account that, using the symmetry properties of the Weyl and Riemann tensors and the expression (E6) for $R^{(1)\rho\sigma\lambda\tau}$, one can write

$$C^{(1)}_{\rho\sigma\lambda\tau}(x)C^{(1)\rho\sigma\lambda\tau}(y)=C^{(1)}_{\rho\sigma\lambda\tau}(x)R^{(1)\rho\sigma\lambda\tau}(y)=-2C^{(1)\rho\sigma\lambda\tau}\delta^{\alpha}_{\rho}\delta^{\beta}_{\lambda}\partial_{\sigma}\partial_{\tau}h_{\alpha\beta}(y). \tag{4.24}$$

Using the last identity, the expression (E7) for $C^{(1)\rho\sigma\lambda\tau}$ and integrating by parts the first expression in Eq. (4.22) we get

$$P_{\rm A}[h]=\int d^4x\, d^4y\, C^{(1)}_{\mu\nu\alpha\beta}(x)N_{\rm A}(x-y;m^2)C^{(1)\mu\nu\alpha\beta}(y). \tag{4.25}$$

Thus, $P_{\rm A}[h]$ and $P_{\rm B}[h]$ depend, respectively, on the Weyl tensor and the scalar curvature to first order in the metric perturbation. The result for the massless case, $m=0$, can be easily obtained from the above expressions, using Eqs. (4.9). If, in addition, we make $\Delta\xi=0$, i.e., conformal coupling, we have $P_{\rm B}[h]=0$. Hence, for a conformal scalar field, particle creation is due to the breaking of conformal flatness in the spacetime, which implies a non-zero Weyl tensor.

In order to compare with previously obtained results, it is useful to introduce the Fourier transform of a field $f(x)$ as $\tilde{f}(p) \equiv \int d^4x\, e^{-ipx} f(x)$. Note that, if $f(x)$ is real, then $\tilde{f}(-p) = \tilde{f}^*(p)$. Using the expressions (3.9) for the kernels $N_{\rm A}$ and $N_{\rm B}$, the above result for the total probability of particle creation and the number of particles created can also be written as

$$P[h] = \frac{1}{1920\pi} \int \frac{d^4p}{(2\pi)^4} \theta(-p^2 - 4m^2) \sqrt{1 + 4\frac{m^2}{p^2}} \times \left[\tilde{C}^{(1)}_{\mu\nu\alpha\beta}(p) \tilde{C}^{(1)*\mu\nu\alpha\beta}(p) \left(1 + 4\frac{m^2}{p^2} \right)^2 + \frac{20}{3} |\tilde{R}^{(1)}(p)|^2 \left(3\Delta\xi + \frac{m^2}{p^2} \right)^2 \right] + O(h^3), \quad (4.26)$$

in agreement with the results of Ref. [37] (except for a sign in the coefficient of the term with $|\tilde{R}^{(1)}(p)|^2$). It is also easy to see that the above result is equivalent to that found in Ref. [38] if we take into account that, for integrals of the form $I \equiv \int d^4p \tilde{f}_{a_1 \cdots a_r}(p) G(p^2) \tilde{f}^{*a_1 \cdots a_r}(p)$, where $f_{a_1 \cdots a_r}(x)$ is any real tensor field in Minkowski spacetime and $G(p^2)$ is any scalar function of p^2, one has that

$$I = 2 \int d^4p\, \theta(p^0) \tilde{f}_{a_1 \cdots a_r}(p) G(p^2) \tilde{f}^{*a_1 \cdots a_r}(p) = 2 \int d^4p\, \theta(-p^0) \tilde{f}_{a_1 \cdots a_r}(p) G(p^2) \tilde{f}^{*a_1 \cdots a_r}(p). \quad (4.27)$$

In the massless conformally coupled case, $m=0$ and $\Delta\xi = 0$, the result (4.26) reduces to that found in Ref. [39].

The energy of the created particles, $E[h]$, defined as the expectation value of the ''out'' energy operator in the ''in'' vacuum can be computed using the expressions derived in Ref. [10]. We find that this energy is given by an expression like Eq. (4.26), but with a factor $2p^0\theta(p^0)$ inserted in the integrand [37,10]. Since the kernels $N_{\rm A}$ and $D_{\rm A}$ are related by the fluctuation-dissipation relation (3.22), and the same holds for $N_{\rm B}$ and $D_{\rm B}$, it is easy to see [similarly to Eq. (4.27)] that

$$E[h] = i \int \frac{d^4p}{(2\pi)^4} p^0 [\tilde{C}^{(1)}_{\mu\nu\alpha\beta}(p) \tilde{C}^{(1)*\mu\nu\alpha\beta}(p) \tilde{D}_{\rm A}(p) + |\tilde{R}^{(1)}(p)|^2 \tilde{D}_{\rm B}(p)] + O(h^3), \quad (4.28)$$

where $\tilde{D}_{\rm A}(p)$ and $\tilde{D}_{\rm B}(p)$ are the Fourier transforms of the dissipation kernels defined in Eq. (3.9). For perturbations of a spatially flat RW spacetime [i.e., $h_{\mu\nu} = 2\Delta a(\eta) \eta_{\mu\nu}$, where $x^0 \equiv \eta$ is the conformal time and $\Delta a(\eta)$ is the perturbation of the scale factor], this last expression agrees with that of Ref. [14], see also Ref. [40].

So far in this subsection the metric perturbations are arbitrary. We may also be interested in the particles created by the back reaction on the metric due to the stress-energy fluctuations. Then we would have to use the solutions of the Einstein-Langevin equations (4.4) and (4.14) in the above results. However, to be consistent, one should look for solutions whose moments vanish asymptotically in the ''remote past'' and in the ''far future.'' These conditions are generally too strong, since they would break the time translation invariance in the correlation functions. In fact, the solutions that we find in the next section do not satisfy these conditions.

V. CORRELATION FUNCTIONS FOR GRAVITATIONAL PERTURBATIONS

In this section, we solve the semiclassical Einstein-Langevin equations (4.4) and (4.14) for the components $G^{(1)\mu\nu}$ of the linearized Einstein tensor. In Sec. V A we use these solutions to compute the corresponding two-point correlation functions, which give a measure of the gravitational fluctuations predicted by the stochastic semiclassical theory of gravity in the present case. Since the linearized Einstein tensor is invariant under gauge transformations of the metric perturbations, these two-point correlation functions are also gauge invariant. Once we have computed the two-point correlation functions for the linearized Einstein tensor, we find solutions for the metric perturbations in Sec. V C and we show how the associated two-point correlation functions can be computed. This procedure to solve the Einstein-Langevin equations is similar to the one used by Horowitz [30], see also Ref. [25], to analyze the stability of Minkowski spacetime in semiclassical gravity.

From expressions (E2) and (E3) restricted to $n=4$, it is easy to see that $A^{(1)\mu\nu}$ and $B^{(1)\mu\nu}$ can be written in terms of $G^{(1)\mu\nu}$ as

$$A^{(1)\mu\nu} = \frac{2}{3} (\mathcal{F}^{\mu\nu} G^{(1)\,\alpha}_{\ \ \ \ \alpha} - \mathcal{F}^{\alpha}_{\alpha} G^{(1)\mu\nu}),$$

$$B^{(1)\mu\nu} = 2\mathcal{F}^{\mu\nu} G^{(1)\,\alpha}_{\ \ \ \ \alpha}, \quad (5.1)$$

where we have used that $3\Box = \mathcal{F}^{\alpha}_{\alpha}$. Therefore, the Einstein-Langevin equations (4.4) and (4.14) can be seen as linear integro-differential stochastic equations for the components $G^{(1)\mu\nu}$. Such equations can be written in both cases, $m \neq 0$ and $m=0$, as

$$\frac{1}{8\pi G} G^{(1)\mu\nu}(x) - 2(\bar{\alpha} A^{(1)\mu\nu}(x) + \bar{\beta} B^{(1)\mu\nu}(x)) + \int d^4y [H_{\rm A}(x-y) A^{(1)\mu\nu}(y) + H_{\rm B}(x-y) B^{(1)\mu\nu}(y)] = 2\xi^{\mu\nu}(x), \quad (5.2)$$

where the new constants $\bar{\alpha}$ and $\bar{\beta}$, and the kernels $H_{\rm A}(x)$ and $H_{\rm B}(x)$ can be identified in each case by comparison of this last equation with Eqs. (4.4) and (4.14). For instance, when $m=0$, we have $H_{\rm A}(x) = H(x;\mu^2)$ and $H_{\rm B}(x)$

$=60\Delta\xi^2 H(x;\mu^2)$. In this case, we can use the arbitrariness of the mass scale μ to eliminate one of the parameters $\overline{\alpha}$ or $\overline{\beta}$.

In order to find solutions to these equations, it is convenient to Fourier transform them. Introducing Fourier transforms as in Sec. IV D, one finds, from Eq. (5.1),

$$\tilde{A}^{(1)\mu\nu}(p)=2p^2\tilde{G}^{(1)\mu\nu}(p)-\frac{2}{3}p^2 P^{\mu\nu}\tilde{G}^{(1)\,\alpha}{}_{\alpha}(p),$$
$$\tilde{B}^{(1)\mu\nu}(p)=-2p^2 P^{\mu\nu}\tilde{G}^{(1)\,\alpha}{}_{\alpha}(p). \tag{5.3}$$

Using these relations, the Fourier transform of Eq. (5.2) reads

$$F^{\mu\nu}{}_{\alpha\beta}(p)\tilde{G}^{(1)\alpha\beta}(p)=16\pi G\tilde{\xi}^{\mu\nu}(p), \tag{5.4}$$

where

$$F^{\mu\nu}{}_{\alpha\beta}(p)\equiv F_1(p)\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}+F_2(p)p^2P^{\mu\nu}\eta_{\alpha\beta}, \tag{5.5}$$

with

$$F_1(p)\equiv 1+16\pi G\,p^2[\tilde{H}_{\rm A}(p)-2\overline{\alpha}],$$
$$F_2(p)\equiv -\frac{16}{3}\pi G[\tilde{H}_{\rm A}(p)+3\tilde{H}_{\rm B}(p)-2\overline{\alpha}-6\overline{\beta}]. \tag{5.6}$$

In Eq. (5.4), $\tilde{\xi}^{\mu\nu}(p)$, the Fourier transform of $\xi^{\mu\nu}(x)$, is a Gaussian stochastic source of zero average and

$$\langle\tilde{\xi}^{\mu\nu}(p)\tilde{\xi}^{\alpha\beta}(p')\rangle_c=(2\pi)^4\delta^4(p+p')\tilde{N}^{\mu\nu\alpha\beta}(p), \tag{5.7}$$

where we have introduced the Fourier transform of the noise kernel. The explicit expression for $\tilde{N}^{\mu\nu\alpha\beta}(p)$ is found from Eqs. (3.10) and (3.9) to be

$$\tilde{N}^{\mu\nu\alpha\beta}(p)=\frac{1}{2880\pi}\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\Bigg[\frac{1}{4}\left(1+4\frac{m^2}{p^2}\right)^2 \times(p^2)^2(3P^{\mu(\alpha}P^{\beta)\nu}-P^{\mu\nu}P^{\alpha\beta}) +10\left(3\Delta\xi+\frac{m^2}{p^2}\right)^2(p^2)^2P^{\mu\nu}P^{\alpha\beta}\Bigg], \tag{5.8}$$

which in the massless case reduces to

$$\lim_{m\to 0}\tilde{N}^{\mu\nu\alpha\beta}(p)=\frac{1}{1920\pi}\theta(-p^2)\times\left[\frac{1}{6}(p^2)^2(3P^{\mu(\alpha}P^{\beta)\nu}-P^{\mu\nu}P^{\alpha\beta})+60\Delta\xi^2(p^2)^2P^{\mu\nu}P^{\alpha\beta}\right]. \tag{5.9}$$

A. Correlation functions for the linearized Einstein tensor

In general, we can write $G^{(1)\mu\nu}=\langle G^{(1)\mu\nu}\rangle_c+G_{\rm f}^{(1)\mu\nu}$, where $G_{\rm f}^{(1)\mu\nu}$ is a solution to Eq. (5.2) [or, in the Fourier transformed version, Eq. (5.4)] with zero average. The averages $\langle G^{(1)\mu\nu}\rangle_c$ must be a solution of the linearized semiclassical Einstein equations obtained by averaging Eq. (5.2) [or Eq. (5.4)]. Solutions to these equations (specially in the massless case, $m=0$) have been studied by several authors [30,41,31,42,43,34,25], particularly in connection with the issue of the stability of the trivial solutions of semiclassical gravity. The two-point correlation functions for the linearized Einstein tensor are given by

$$\mathcal{G}^{\mu\nu\alpha\beta}(x,x')\equiv\langle G^{(1)\mu\nu}(x)G^{(1)\alpha\beta}(x')\rangle_c-\langle G^{(1)\mu\nu}(x)\rangle_c\langle G^{(1)\alpha\beta}(x')\rangle_c=\langle G_{\rm f}^{(1)\mu\nu}(x)G_{\rm f}^{(1)\alpha\beta}(x')\rangle_c. \tag{5.10}$$

Next, we shall seek the family of solutions to the Einstein-Langevin equations which can be written as a linear functional of the stochastic source and whose Fourier transform, $\tilde{G}^{(1)\mu\nu}(p)$, depends locally on $\tilde{\xi}^{\alpha\beta}(p)$. Each of such solutions is a Gaussian stochastic field and, thus, it can be completely characterized by the averages $\langle G^{(1)\mu\nu}\rangle_c$ and the two-point correlation functions (5.10). For such a family of solutions, $\tilde{G}_{\rm f}^{(1)\mu\nu}(p)$ is the most general solution to Eq. (5.4) which is linear, homogeneous and local in $\tilde{\xi}^{\alpha\beta}(p)$. It can be written as

$$\tilde{G}_{\rm f}^{(1)\mu\nu}(p)=16\pi G\,D^{\mu\nu}{}_{\alpha\beta}(p)\tilde{\xi}^{\alpha\beta}(p), \tag{5.11}$$

where $D^{\mu\nu}{}_{\alpha\beta}(p)$ are the components of a Lorentz invariant tensor field distribution in Minkowski spacetime (by ''Lorentz invariant'' we mean invariant under the transformations of the orthochronous Lorentz subgroup; see Ref. [30] for more details on the definition and properties of these tensor distributions), symmetric under the interchanges $\alpha\leftrightarrow\beta$ and $\mu\leftrightarrow\nu$, which is the most general solution of

$$F^{\mu\nu}{}_{\rho\sigma}(p)D^{\rho\sigma}{}_{\alpha\beta}(p)=\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}. \tag{5.12}$$

In addition, we must impose the conservation condition to the solutions: $p_\nu\tilde{G}_{\rm f}^{(1)\mu\nu}(p)=0$, where this zero must be understood as a stochastic variable which behaves deterministically as a zero vector field. We can write $D^{\mu\nu}{}_{\alpha\beta}(p)=D_p^{\mu\nu}{}_{\alpha\beta}(p)+D_h^{\mu\nu}{}_{\alpha\beta}(p)$, where $D_p^{\mu\nu}{}_{\alpha\beta}(p)$ is a particular solution to Eq. (5.12) and $D_h^{\mu\nu}{}_{\alpha\beta}(p)$ is the most general solution to the corresponding homogeneous equation. Correspondingly [see Eq. (5.11)], we can write $\tilde{G}_{\rm f}^{(1)\mu\nu}(p)=\tilde{G}_p^{(1)\mu\nu}(p)+\tilde{G}_h^{(1)\mu\nu}(p)$. To find the particular solution, we try an ansatz of the form

$$D_p^{\mu\nu}{}_{\alpha\beta}(p)=d_1(p)\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}+d_2(p)p^2P^{\mu\nu}\eta_{\alpha\beta}. \tag{5.13}$$

Substituting this ansatz into Eqs. (5.12), it is easy to see that it solves these equations if we take

$$d_1(p)=\left[\frac{1}{F_1(p)}\right]_r,\quad d_2(p)=-\left[\frac{F_2(p)}{F_1(p)F_3(p)}\right]_r, \tag{5.14}$$

with

$$F_3(p) \equiv F_1(p) + 3p^2 F_2(p) = 1 - 48\pi G\, p^2 [\tilde{H}_{\rm B}(p) - 2\bar{\beta}], \tag{5.15}$$

and where the notation $[\ ]_r$ means that the zeros of the denominators are regulated with appropriate prescriptions in such a way that $d_1(p)$ and $d_2(p)$ are well defined Lorentz invariant scalar distributions. This yields a particular solution to the Einstein-Langevin equations:

$$\tilde{G}_p^{(1)\mu\nu}(p) = 16\pi G\, D_p^{\mu\nu}{}_{\alpha\beta}(p)\, \tilde{\xi}^{\alpha\beta}(p), \tag{5.16}$$

which, since the stochastic source is conserved, satisfies the conservation condition. Note that, in the case of a massless scalar field, $m=0$, the above solution has a functional form analogous to that of the solutions of linearized semiclassical gravity found in the Appendix of Ref. [25]. Notice also that, for a massless conformally coupled field, $m=0$ and $\Delta\xi=0$, the second term in the right hand side of Eq. (5.13) will not contribute in the correlation functions (5.10), since, as we have pointed out in Sec. IV B, in this case the stochastic source is ''traceless.''

Next, we can work out the general form for $D_h^{\mu\nu}{}_{\alpha\beta}(p)$, which is a linear combination of terms consisting of a Lorentz invariant scalar distribution times one of the products $\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}$, $p^2 P^{\mu\nu}\eta_{\alpha\beta}$, $\eta^{\mu\nu}\eta_{\alpha\beta}$, $\eta^{\mu\nu}p^2 P_{\alpha\beta}$, $\delta^{(\mu}_{(\alpha}p^2 P^{\nu)}_{\beta)}$ and $p^2 P^{\mu\nu} p^2 P_{\alpha\beta}$. However, taking into account that the stochastic source is conserved, we can omit some terms in $D_h^{\mu\nu}{}_{\alpha\beta}(p)$ and simply write

$$\tilde{G}_h^{(1)\mu\nu}(p) = 16\pi G\, D_h^{\mu\nu}{}_{\alpha\beta}(p)\, \tilde{\xi}^{\alpha\beta}(p), \tag{5.17}$$

with

$$D_h^{\mu\nu}{}_{\alpha\beta}(p) = h_1(p)\, \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + h_2(p) p^2 P^{\mu\nu}\eta_{\alpha\beta} + h_3(p)\, \eta^{\mu\nu}\eta_{\alpha\beta}, \tag{5.18}$$

where $h_1(p)$, $h_2(p)$ and $h_3(p)$ are Lorentz invariant scalar distributions. From the fact that $D_h^{\mu\nu}{}_{\alpha\beta}(p)$ must satisfy the homogeneous equation corresponding to Eq. (5.12), we find that $h_1(p)$ and $h_3(p)$ have support on the set of points $\{p^\mu\}$ for which $F_1(p)=0$, and that $h_2(p)$ has support on the set of points $\{p^\mu\}$ for which $F_1(p)=0$ or $F_3(p)=0$. Moreover, the conservation condition for $\tilde{G}_h^{(1)\mu\nu}(p)$ implies that the term with $h_3(p)$ is only allowed in the case of a massless conformally coupled field, $m=0$ and $\Delta\xi=0$. From Eq. (5.7), we get

$$\langle \tilde{G}_h^{(1)\mu\nu}(p)\tilde{\xi}^{\alpha\beta}(p')\rangle_c = (2\pi)^4 16\pi G\, \delta^4(p+p') D_h^{\mu\nu}{}_{\rho\sigma}(p) \tilde{N}^{\rho\sigma\alpha\beta}(p). \tag{5.19}$$

Note, from expressions (5.8) and (5.9), that the support of $\tilde{N}^{\mu\nu\alpha\beta}(p)$ is on the set of points $\{p^\mu\}$ for which $-p^2 \geqslant 0$ when $m=0$, and for which $-p^2-4m^2>0$ when $m\neq 0$. At such points, using expressions (5.6), (5.15), (4.11) and (4.3), it is easy to see that $F_1(p)$ is always different from zero, and that $F_3(p)$ is also always different from zero, except for some particular values of $\Delta\xi$ and $\bar{\beta}$:

(a) when $m=0$, $\Delta\xi=0$ and $\bar{\beta}>0$;

(b) when $m\neq 0$, $0<\Delta\xi<(1/12)$ and

$$\bar{\beta} = (\Delta\xi/32\pi^2)[\pi/(Gm^2) + 1/36].$$

In case (a), $F_3(p)=0$ for the set of points $\{p^\mu\}$ satisfying $-p^2 = 1/(96\pi G\bar{\beta})$; in case (b), $F_3(p)=0$ for $\{p^\mu\}$ such that $-p^2 = m^2/(3\Delta\xi)$. Hence, except for the above cases (a) and (b), the intersection of the supports of $\tilde{N}^{\mu\nu\alpha\beta}(p)$ and $D^{\rho\sigma}_{h\lambda\gamma}(p)$ is an empty set and, thus, the correlation function (5.19) is zero. In cases (a) and (b), we can have a contribution to Eq. (5.19) coming from the term with $h_2(p)$ in Eq. (5.18) of the form $D_h^{\mu\nu}{}_{\rho\sigma}(p)\tilde{N}^{\rho\sigma\alpha\beta}(p) = H_3(p;\{C\}) p^2 P^{\mu\nu} \tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$, where $H_3(p;\{C\})$ is the most general Lorentz invariant distribution satisfying $F_3(p)H_3(p;\{C\})=0$, which depends on a set of arbitrary parameters represented as $\{C\}$. However, from Eq. (5.8), we see that $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$ is proportional to $\theta(-p^2-4m^2)(1+4m^2/p^2)^{1/2}(3\Delta\xi + m^2/p^2)^2$. Thus, in case (a), we have $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)=0$ and, in case (b), the intersection of the supports of $\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p)$ and of $H_3(p;\{C\})$ is an empty set. Therefore, from the above analysis, we conclude that $\tilde{G}_h^{(1)\mu\nu}(p)$ gives no contribution to the correlation functions (5.10), since $\langle \tilde{G}_h^{(1)\mu\nu}(p)\tilde{\xi}^{\alpha\beta}(p')\rangle_c = 0$, and we have simply $\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = \langle G_p^{(1)\mu\nu}(x) G_p^{(1)\alpha\beta}(x')\rangle_c$, where $G_p^{(1)\mu\nu}(x)$ is the inverse Fourier transform of Eq. (5.16).

The correlation functions (5.10) can then be computed from

$$\langle \tilde{G}_p^{(1)\mu\nu}(p)\tilde{G}_p^{(1)\alpha\beta}(p')\rangle_c = 64(2\pi)^6 G^2 \delta^4(p+p') D_p^{\mu\nu}{}_{\rho\sigma}(p) \times D_p^{\alpha\beta}{}_{\lambda\gamma}(-p) \tilde{N}^{\rho\sigma\lambda\gamma}(p). \tag{5.20}$$

It is easy to see from the above analysis that the prescriptions $[\ ]_r$ in the factors D_p are irrelevant in the last expression and, thus, they can be suppressed. Taking into account that $F_l(-p) = F_l^*(p)$, with $l=1,2,3$, we get from Eqs. (5.13) and (5.14)

$$\langle \tilde{G}_p^{(1)\mu\nu}(p)\tilde{G}_p^{(1)\alpha\beta}(p')\rangle_c = 64(2\pi)^6 G^2 \frac{\delta^4(p+p')}{|F_1(p)|^2}\left[\tilde{N}^{\mu\nu\alpha\beta}(p) - \frac{F_2(p)}{F_3(p)}p^2 P^{\mu\nu}\tilde{N}^{\alpha\beta\rho}{}_{\rho}(p) - \frac{F_2^*(p)}{F_3^*(p)}p^2 P^{\alpha\beta}\tilde{N}^{\mu\nu\rho}{}_{\rho}(p) + \frac{|F_2(p)|^2}{|F_3(p)|^2}p^2 P^{\mu\nu}p^2 P^{\alpha\beta}\tilde{N}^{\rho}{}_{\rho}{}^{\sigma}{}_{\sigma}(p)\right]. \tag{5.21}$$

This last expression is well defined as a bi-distribution and can be easily evaluated using Eq. (5.8). We find

$$\langle \tilde{G}_p^{(1)\mu\nu}(p)\tilde{G}_p^{(1)\alpha\beta}(p')\rangle_c = \frac{2}{45}(2\pi)^5 G^2 \frac{\delta^4(p+p')}{|F_1(p)|^2}\,\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left[\frac{1}{4}\left(1+4\frac{m^2}{p^2}\right)^2 (p^2)^2(3P^{\mu(\alpha}P^{\beta)\nu} - P^{\mu\nu}P^{\alpha\beta}) + 10\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 (p^2)^2 P^{\mu\nu}P^{\alpha\beta}\left|1-3p^2\frac{F_2(p)}{F_3(p)}\right|^2\right]. \tag{5.22}$$

To derive the correlation functions (5.10), we have to take the inverse Fourier transform of the above result. We finally obtain

$$\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = \frac{\pi}{45}G^2\mathcal{F}_x^{\mu\nu\alpha\beta}\mathcal{G}_{\rm A}(x-x') + \frac{8\pi}{9}G^2\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\mathcal{G}_{\rm B}(x-x'), \tag{5.23}$$

with

$$\tilde{\mathcal{G}}_{\rm A}(p) \equiv \theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(1+4\frac{m^2}{p^2}\right)^2\frac{1}{|F_1(p)|^2},$$

$$\tilde{\mathcal{G}}_{\rm B}(p) \equiv \theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}\left(3\Delta\xi + \frac{m^2}{p^2}\right)^2 \times \frac{1}{|F_1(p)|^2}\left|1-3p^2\frac{F_2(p)}{F_3(p)}\right|^2, \tag{5.24}$$

and $\mathcal{F}_x^{\mu\nu\alpha\beta} \equiv 3\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu} - \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}$, and where $F_l(p)$, $l=1,2,3$, are given in Eqs. (5.6) and (5.15). Notice that, for a massless field ($m=0$), we have

$$F_1(p) = 1 + 16\pi G p^2 \tilde{H}(p;\bar{\mu}^2),$$

$$F_2(p) = -\frac{16}{3}\pi G[(1+180\Delta\xi^2)\tilde{H}(p;\bar{\mu}^2) - 6\Upsilon], \tag{5.25}$$

$$F_3(p) = 1 - 48\pi G p^2[60\Delta\xi^2\tilde{H}(p;\bar{\mu}^2) - 2\Upsilon],$$

with $\bar{\mu} \equiv \mu\exp(1920\pi^2\bar{\alpha})$ and $\Upsilon \equiv \bar{\beta} - 60\Delta\xi^2\bar{\alpha}$, and where $\tilde{H}(p;\mu^2)$ is the Fourier transform of $H(x;\mu^2)$ given in Eq. (4.11).

B. Conformal field case

The above correlation functions become simpler when the scalar field is massless and conformally coupled, i.e., when $m=0$ and $\Delta\xi=0$, since in this case $\mathcal{G}_{\rm B}(x)=0$ and $\tilde{\mathcal{G}}_{\rm A}(p)$ reduces to $\tilde{\mathcal{G}}_{\rm A}(p) = \theta(-p^2)|F_1(p)|^{-2}$. Introducing the function $\varphi(\chi;\lambda) \equiv [1-\chi\ln(\lambda\chi/e)]^2 + \pi^2\chi^2$, with $\chi \geq 0$ and $\lambda > 0$, $\mathcal{G}_{\rm A}(x)$ can be written as

$$\mathcal{G}_{\rm A}(x) = \frac{(120\pi)^{3/2}}{2\pi^3 L_P^3}\frac{1}{|\mathbf{x}|}\int_0^\infty d|\mathbf{q}||\mathbf{q}|\sin\left[\frac{\sqrt{120\pi}}{L_P}|\mathbf{x}||\mathbf{q}|\right] \times \int_0^\infty dq^0\cos\left[\frac{\sqrt{120\pi}}{L_P}x^0 q^0\right]\frac{\theta(-q^2)}{\varphi(-q^2;\lambda)}, \tag{5.26}$$

where $L_P \equiv \sqrt{G}$ is the Planck length, $\lambda \equiv 120\pi e/(L_P^2\bar{\mu}^2)$, and we use the notation $x^\mu = (x^0,\mathbf{x})$ and $q^\mu = (q^0,\mathbf{q})$. Notice that, if we assume that $\bar{\mu} \leq L_P^{-1}$, then $\lambda \gtrsim 10^3$. For those values of the parameter λ (and also for smaller values), the function $\varphi(\chi;\lambda)$ has a minimum at some value of χ that we denote as $\chi_0(\lambda)$. This can be found by solving the equation $\pi^2\chi_0 = [1-\chi_0\ln(\lambda\chi_0/e)][1+\ln(\lambda\chi_0/e)]$ numerically [discarding a solution $\chi_M(\lambda) < \chi_0(\lambda)$, at which the function $\varphi(\chi;\lambda)$ has a maximum]. Since the main contribution to the integral (5.26) come from the values of $-q^2$ around $-q^2 = \chi_0(\lambda)$, $\varphi(\chi;\lambda)$ can be approximately replaced in this integral by

$$\varphi_{\rm ap}(\chi;\lambda) \equiv [1-\kappa(\lambda)\chi]^2 + \pi^2\chi^2 = [\kappa^2(\lambda)+\pi^2]\chi^2 - 2\kappa(\lambda)\chi + 1,$$

with $\kappa(\lambda) \equiv \ln(\lambda\chi_0(\lambda)/e)$. For $(\lambda/5) \sim 10^3 - 10^7$, we have $\kappa \sim 10$.

Let the spacetime points x and x' be different and spacelike separated. In this case, we can choose an inertial coordinate system for which $(x-x')^\mu = (0,\mathbf{x}-\mathbf{x}')$ and $\mathcal{G}^{\mu\nu\alpha\beta}(x,x')$ will be a function of $\mathbf{x}-\mathbf{x}'$ only that can be written as

$$\mathcal{G}^{\mu\nu\alpha\beta}(\mathbf{x}-\mathbf{x}') = \mathcal{G}_1^{\mu\nu\alpha\beta}(\mathbf{x}-\mathbf{x}') + \mathcal{G}_2^{\mu\nu\alpha\beta}(\mathbf{x}-\mathbf{x}') + \mathcal{G}_3^{\mu\nu\alpha\beta}(\mathbf{x}-\mathbf{x}'), \tag{5.27}$$

with

$$\mathcal{G}_a^{\mu\nu\alpha\beta}(\mathbf{x}) \equiv \frac{\pi}{45} G^2 \mathcal{F}_{a_{\mathbf{x}}}^{\mu\nu\alpha\beta} I_a(\mathbf{x}), \tag{5.28}$$

$a=1,2,3$, where $I_1(\mathbf{x}) \equiv \mathcal{G}_{\rm A}(x)|_{x^\mu=(0,\mathbf{x})}$, $I_2(\mathbf{x}) \equiv (\partial_x^0)^2 \mathcal{G}_{\rm A}(x)|_{x^\mu=(0,\mathbf{x})}$, $I_3(\mathbf{x}) \equiv (\partial_x^0)^4 \mathcal{G}_{\rm A}(x)|_{x^\mu=(0,\mathbf{x})}$, and $\mathcal{F}_{a_{\mathbf{x}}}^{\mu\nu\alpha\beta}$ are some differential operators. Note that the terms containing an odd number of ∂_x^0 derivatives are zero. The differential operators $\mathcal{F}_{1_{\mathbf{x}}}^{\mu\nu\alpha\beta}$ are given by $\mathcal{F}_1^{\mu\nu\alpha\beta} = 3\mathcal{D}^{\mu(\alpha}\mathcal{D}^{\beta)\nu} - \mathcal{D}^{\mu\nu}\mathcal{D}^{\alpha\beta}$, with $\mathcal{D}^{\mu\nu} \equiv (\eta^{\mu\nu}\delta^{ij} - \delta^{\mu i}\delta^{\nu j})\partial_i\partial_j$. The non-null components of the remaining operators are $\mathcal{F}_2^{00ij} = 3\partial^i\partial^j - \delta^{ij}\triangle$, $\mathcal{F}_2^{0i0j} = \frac{1}{2}(\partial^i\partial^j + 3\delta^{ij}\triangle)$, $\mathcal{F}_3^{ijkl} = -\delta^{ij}\delta^{kl} + 3\delta^{i(k}\delta^{l)j}$, $\mathcal{F}_2^{ijkl} = 2(\delta^{ij}\delta^{kl} - 3\delta^{i(k}\delta^{l)j})\triangle - \delta^{ij}\partial^k\partial^l - \delta^{kl}\partial^i\partial^j + 3(\delta^{i(k}\partial^{l)}\partial^j + \delta^{j(k}\partial^{l)}\partial^i)$, where $\triangle \equiv \delta^{ij}\partial_i\partial_j$ is the usual (Euclidean space) Laplace operator. From the above expressions, we can see that $\mathcal{G}^{000i}(\mathbf{x}-\mathbf{x}') = \mathcal{G}^{0ijk}(\mathbf{x}-\mathbf{x}') = 0$, but the remaining correlation functions $\mathcal{G}^{\mu\nu\alpha\beta}(\mathbf{x}-\mathbf{x}')$ are in principle non-null.

With the approximation described above, the integrals $I_a(\mathbf{x})$ can be written as

$$I_a(\mathbf{x}) \simeq \frac{(-1)^{a+1}}{2\pi^3}\left(\frac{120\pi}{L_P^2}\right)^{a+1/2} \frac{1}{|\mathbf{x}|}\int_0^\infty d|\mathbf{q}| \times \sin\left[\frac{\sqrt{120\pi}}{L_P}|\mathbf{x}||\mathbf{q}|\right] |\mathbf{q}| J_a(|\mathbf{q}|), \tag{5.29}$$

where

$$J_1(|\mathbf{q}|) \equiv \int_{|\mathbf{q}|}^\infty dq^0 \frac{1}{\varphi_{\rm ap}(-q^2;\lambda)}, \quad J_2(|\mathbf{q}|) \equiv \int_{|\mathbf{q}|}^\infty dq^0 \frac{(q^0)^2}{\varphi_{\rm ap}(-q^2;\lambda)},$$
$$J_3(|\mathbf{q}|) \equiv \frac{-|\mathbf{q}|}{\kappa^2(\lambda)+\pi^2} + \int_{|\mathbf{q}|}^\infty dq^0 \left[\frac{(q^0)^4}{\varphi_{\rm ap}(-q^2;\lambda)} - \frac{1}{[\kappa^2(\lambda)+\pi^2]}\right]. \tag{5.30}$$

Noting that $\varphi_{\rm ap}(-q^2;\lambda)$ has four zeros in the complex q^0 plane at $\pm p(|\mathbf{q}|)$, $\pm p^*(|\mathbf{q}|)$, where $p(s)$ (we make $s \equiv |\mathbf{q}|$) is the complex function with

$$\left.\begin{matrix}\operatorname{Re} p(s)\\ \operatorname{Im} p(s)\end{matrix}\right\} = \left[\frac{\sqrt{[(\kappa^2+\pi^2)s^2+\kappa]^2+\pi^2} \pm (\kappa^2+\pi^2)s^2 \pm \kappa}{2(\kappa^2+\pi^2)}\right]^{1/2}, \tag{5.31}$$

we can decompose

$$\frac{1}{\varphi_{\rm ap}(-q^2;\lambda)} = \frac{1}{4(\kappa^2+\pi^2)}\,\frac{1}{|p|^2 \operatorname{Re} p}\left[\frac{q^0+2\operatorname{Re} p}{(q^0)^2+2\operatorname{Re} p\, q^0+|p|^2} - \frac{(q^0-2\operatorname{Re} p)}{(q^0)^2-2\operatorname{Re} p\, q^0+|p|^2}\right], \tag{5.32}$$

and then we can perform the integrals $J_a(s)$, $a=1,2,3$. The results for these integrals can be found in Appendix F.

Next, to carry on with the calculation, we need to introduce some suitable approximations for the functions $J_a(s)$ in the integrals (5.29). In order to do so, we study the behavior of these functions for small and large values of s. For $s\,J_1(s)$, we find that it can be well approximated by an arctan function. In fact, on the one hand, $s\,J_1(s)$ tends very quickly to a constant limiting value $\lim_{s\to\infty} s\,J_1(s) = a/4$, where $a \equiv 1+(2/\pi)\arctan(\kappa/\pi)$. On the other hand, for small values of s, we have $s\,J_1(s) \simeq [\sqrt{120\pi}a/(2\pi b)]s + O(s^2)$, with $b \equiv (4a/\pi^2)[15\pi(\sqrt{\kappa^2+\pi^2}-\kappa)]^{1/2}$. Hence, we can approximate

$$s\,J_1(s) \simeq \frac{a}{2\pi}\arctan\left(\frac{\sqrt{120\pi}}{b}s\right). \tag{5.33}$$

Performing the integral $I_1(\mathbf{x})$ [see Eq. (5.29)] with this approximation, we get, for $|\mathbf{x}| \neq 0$,

$$I_1(\mathbf{x}) \simeq \frac{15}{\pi^2}\frac{a}{L_P^2}\frac{1}{|\mathbf{x}|^2} e^{-b|\mathbf{x}|/L_P}. \tag{5.34}$$

The function $J_2(s)$ behaves as $J_2(s) \simeq (a/4)s + O(s^{-1}\ln s)$ for large values of s, and as $J_2(s) \simeq (a/4)(120\pi)^{-1/2}\gamma + O(s^2)$, with $\gamma \equiv 240(\kappa^2+\pi^2)^{-1/2}b^{-1}$, for small values of s. This function can be well approximated by

$$J_2(s) \simeq \frac{a}{4}\left[s^2 + \frac{\gamma^2}{120\pi}\right]^{1/2}, \tag{5.35}$$

and, substituting the last expression in the integral $I_2(\mathbf{x})$ [see Eq. (5.29)], we obtain, for $|\mathbf{x}| \neq 0$,

$$I_2(\mathbf{x}) \simeq \frac{15}{\pi^2}\frac{a}{L_P^4}\frac{\gamma^2}{|\mathbf{x}|^2} K_2(\gamma|\mathbf{x}|/L_P), \tag{5.36}$$

where $K_\nu(z)$ denote the modified Bessel functions of the second kind. For $J_3(s)$, we find that $J_3(s) \simeq (a/4)s^3 + O(s\ln s)$ for large values of s, and that

$$J_3(s) \simeq (a/4)(120\pi)^{-3/2}\delta^3 + O(s),$$

with

$$\delta \equiv 4(\kappa^2+\pi^2)^{-1/2}[450\pi\, b^{-1}(2\kappa - \sqrt{\kappa^2+\pi^2})]^{1/3},$$

for s small. With the approximation

$$J_3(s) \simeq \frac{a}{4}\left[s^2 + \frac{\delta^2}{120\pi}\right]^{3/2}, \tag{5.37}$$

we can compute the integral $I_3(\mathbf{x})$ [see Eq. (5.29)] for $|\mathbf{x}| \neq 0$, and we find

$$I_3(\mathbf{x}) \simeq \frac{45}{\pi^2}\frac{a}{L_P^5}\frac{\delta^3}{|\mathbf{x}|^3} K_3(\delta|\mathbf{x}|/L_P). \tag{5.38}$$

Numerical calculations confirm that the above approximations are reasonable. For $\kappa \sim 10$, we have $a, b, \delta \sim 1$ and $\gamma \sim 10$.

The results (5.34), (5.36) and (5.38) are now ready to be substituted into Eq. (5.28), from where we can compute the different contributions to the correlation functions (5.27). Using the relation $(d/dz)[z^{-\nu}K_\nu(z)] = -z^{-\nu}K_{\nu+1}(z)$, and defining $\sigma_b \equiv b|\mathbf{x}|/L_P$, $\sigma_\gamma \equiv \gamma|\mathbf{x}|/L_P$, $\sigma_\delta \equiv \delta|\mathbf{x}|/L_P$, we get, after a rather long but straightforward calculation, the following results for the non-zero components of $\mathcal{G}_a^{\mu\nu\alpha\beta}(\mathbf{x})$ [with $|\mathbf{x}| \neq 0$]:

$$\mathcal{G}_1^{0000}(\mathbf{x}) \simeq \frac{2}{3\pi}\frac{ab^6}{L_P^4}\frac{e^{-\sigma_b}}{\sigma_b^2}\left[1+\frac{4}{\sigma_b}+\frac{12}{\sigma_b^2}+\frac{24}{\sigma_b^3}+\frac{24}{\sigma_b^4}\right],$$

$$\mathcal{G}_1^{00ij}(\mathbf{x}) \simeq \frac{1}{3\pi}\frac{ab^6}{L_P^4}\frac{e^{-\sigma_b}}{\sigma_b^2}\left[\delta^{ij}\left(1+\frac{5}{\sigma_b}+\frac{16}{\sigma_b^2}+\frac{32}{\sigma_b^3}+\frac{32}{\sigma_b^4}\right) - \frac{x^i x^j}{|\mathbf{x}|^2}\left(1+\frac{7}{\sigma_b}+\frac{24}{\sigma_b^2}+\frac{48}{\sigma_b^3}+\frac{48}{\sigma_b^4}\right)\right],$$

$$\mathcal{G}_1^{0i0j}(\mathbf{x}) = -\frac{3}{2}\mathcal{G}_1^{00ij}(\mathbf{x}),$$

$$\begin{aligned}\mathcal{G}_1^{ijkl}(\mathbf{x}) \simeq \frac{1}{3\pi}\frac{ab^6}{L_P^4}\frac{e^{-\sigma_b}}{\sigma_b^2}\Bigg[&-(\delta^{ij}\delta^{kl} - 3\delta^{i(k}\delta^{l)j})\left(1+\frac{6}{\sigma_b}+\frac{18}{\sigma_b^2}+\frac{30}{\sigma_b^3}+\frac{24}{\sigma_b^4}\right) + 10\delta^{i(k}\delta^{l)j}\left(\frac{1}{\sigma_b^2}+\frac{5}{\sigma_b^3}+\frac{8}{\sigma_b^4}\right)\\ &+\frac{1}{|\mathbf{x}|^2}(\delta^{ij}x^k x^l + \delta^{kl}x^i x^j - 3\delta^{i(k}x^{l)}x^j - 3\delta^{j(k}x^{l)}x^i)\left(1+\frac{5}{\sigma_b}+\frac{6}{\sigma_b^2}-\frac{18}{\sigma_b^3}-\frac{48}{\sigma_b^4}\right)\\ &-\frac{10}{|\mathbf{x}|^2}(\delta^{i(k}x^{l)}x^j + \delta^{j(k}x^{l)}x^i)\left(\frac{1}{\sigma_b}+\frac{9}{\sigma_b^2}+\frac{33}{\sigma_b^3}+\frac{48}{\sigma_b^4}\right) + \frac{2}{|\mathbf{x}|^4}x^i x^j x^k x^l\left(1+\frac{14}{\sigma_b}+\frac{87}{\sigma_b^2}+\frac{279}{\sigma_b^3}+\frac{384}{\sigma_b^4}\right)\Bigg],\end{aligned} \tag{5.39}$$

$$\mathcal{G}_2^{00ij}(\mathbf{x}) \simeq \frac{1}{3\pi}\frac{a\gamma^6}{L_P^4}\frac{K_4(\sigma_\gamma)}{\sigma_\gamma^2}\left(3\frac{x^i x^j}{|\mathbf{x}|^2} - \delta^{ij}\right),$$

$$\mathcal{G}_2^{0i0j}(\mathbf{x}) \simeq \frac{1}{6\pi}\frac{a\gamma^6}{L_P^4}\frac{K_4(\sigma_\gamma)}{\sigma_\gamma^2}\left(\frac{x^i x^j}{|\mathbf{x}|^2} + 3\delta^{ij}\right) - \frac{5}{3\pi}\frac{a\gamma^6}{L_P^4}\frac{K_3(\sigma_\gamma)}{\sigma_\gamma^3}\delta^{ij},$$

$$\begin{aligned}\mathcal{G}_2^{ijkl}(\mathbf{x}) \simeq \frac{1}{3\pi}\frac{a\gamma^6}{L_P^4}\frac{K_4(\sigma_\gamma)}{\sigma_\gamma^2}&\left[2(\delta^{ij}\delta^{kl} - 3\delta^{i(k}\delta^{l)j}) - \frac{1}{|\mathbf{x}|^2}(\delta^{ij}x^k x^l + \delta^{kl}x^i x^j - 3\delta^{i(k}x^{l)}x^j - 3\delta^{j(k}x^{l)}x^i)\right]\\ &-\frac{4}{3\pi}\frac{a\gamma^6}{L_P^4}\frac{K_3(\sigma_\gamma)}{\sigma_\gamma^3}(\delta^{ij}\delta^{kl} - 3\delta^{i(k}\delta^{l)j}),\end{aligned}$$

$$\mathcal{G}_3^{ijkl}(\mathbf{x}) \simeq -\frac{1}{\pi}\frac{a\delta^6}{L_P^4}\frac{K_3(\sigma_\delta)}{\sigma_\delta^3}(\delta^{ij}\delta^{kl} - 3\delta^{i(k}\delta^{l)j}).$$

Note that, for $\sigma \gg 1$, we have the following asymptotic expansions for the modified Bessel functions in the above expressions:

$$K_4(\sigma) \simeq \sqrt{\frac{\pi}{2\sigma}} e^{-\sigma}\left[1+\frac{63}{8}\frac{1}{\sigma}+O\left(\frac{1}{\sigma^2}\right)\right], \qquad (5.40)$$

$$K_3(\sigma) \simeq \sqrt{\frac{\pi}{2\sigma}} e^{-\sigma}\left[1+\frac{35}{8}\frac{1}{\sigma}+O\left(\frac{1}{\sigma^2}\right)\right].$$

C. Correlation functions for the metric perturbations

Starting from the solutions found for the linearized Einstein tensor, which are characterized by the two-point correlation functions (5.23) [or, in terms of Fourier transforms, Eq. (5.22)], we can now solve the equations for the metric perturbations. Working in the harmonic gauge, $\partial_\nu \bar{h}^{\mu\nu}=0$ (this zero must be understood in the same statistical sense as above), where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu}-(1/2)\,\eta_{\mu\nu}h$, and using Eqs. (2.19) and (E1), these equations reduce to $\Box \bar{h}^{\mu\nu}(x) = -2G^{(1)\mu\nu}(x)$, or, in terms of Fourier transforms, $p^2 \tilde{\bar{h}}^{\mu\nu}(p) = 2\tilde{G}^{(1)\mu\nu}(p)$. As above, we can write $\bar{h}^{\mu\nu} = \langle \bar{h}^{\mu\nu}\rangle_c + \bar{h}^{\mu\nu}_{\rm f}$, where $\bar{h}^{\mu\nu}_{\rm f}$ is a solution to these equations with zero average, and the two-point correlation functions are given by

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') \equiv \langle \bar{h}^{\mu\nu}(x)\bar{h}^{\alpha\beta}(x')\rangle_c - \langle \bar{h}^{\mu\nu}(x)\rangle_c \langle \bar{h}^{\alpha\beta}(x')\rangle_c = \langle \bar{h}^{\mu\nu}_{\rm f}(x)\bar{h}^{\alpha\beta}_{\rm f}(x')\rangle_c . \qquad (5.41)$$

We can now seek solutions of the form $\tilde{\bar{h}}^{\mu\nu}_{\rm f}(p) = 2D(p)\tilde{G}^{(1)\mu\nu}_{\rm f}(p)$, where $D(p)$ is a Lorentz invariant scalar distribution in Minkowski spacetime, which is the most general solution of $p^2 D(p)=1$. Note that, since the linearized Einstein tensor is conserved, solutions of this form automatically satisfy the harmonic gauge condition. As above, we can write $D(p)=[1/p^2]_r + D_h(p)$, where $D_h(p)$ is the most general solution to the associated homogeneous equation and, correspondingly, we have $\tilde{\bar{h}}^{\mu\nu}_{\rm f}(p) = \tilde{\bar{h}}^{\mu\nu}_{p}(p) + \tilde{\bar{h}}^{\mu\nu}_{h}(p)$. However, since $D_h(p)$ has support on the set of points for which $p^2=0$, it is easy to see from Eq. (5.22) [from the factor $\theta(-p^2-4m^2)$] that $\langle \tilde{\bar{h}}^{\mu\nu}_{h}(p)\tilde{G}^{(1)\alpha\beta}_{\rm f}(p')\rangle_c = 0$ and, thus, the two-point correlation functions (5.41) can be computed from $\langle \tilde{\bar{h}}^{\mu\nu}_{\rm f}(p)\tilde{\bar{h}}^{\alpha\beta}_{\rm f}(p')\rangle_c = \langle \tilde{\bar{h}}^{\mu\nu}_{p}(p)\tilde{\bar{h}}^{\alpha\beta}_{p}(p')\rangle_c$. From Eq. (5.22) and due to the factor $\theta(-p^2-4m^2)$, it is also easy to see that the prescription $[\ ]_r$ is irrelevant in this correlation function and we obtain

$$\langle \tilde{\bar{h}}^{\mu\nu}_{p}(p)\tilde{\bar{h}}^{\alpha\beta}_{p}(p')\rangle_c = \frac{4}{(p^2)^2}\langle \tilde{G}^{(1)\mu\nu}_{p}(p)\tilde{G}^{(1)\alpha\beta}_{p}(p')\rangle_c , \qquad (5.42)$$

where $\langle \tilde{G}^{(1)\mu\nu}_{p}(p)\tilde{G}^{(1)\alpha\beta}_{p}(p')\rangle_c$ is given in Eq. (5.22). The right hand side of this equation is a well defined bi-distribution, at least for $m\neq 0$ (the θ function provides the suitable cutoff). In the massless field case, since the noise kernel is obtained as the limit $m\to 0$ of the noise kernel for a massive field, it seems that the natural prescription to avoid the divergencies on the lightcone $p^2=0$ is a Hadamard finite part (see Refs. [28] for its definition). Taking this prescription, we also get a well defined bi-distribution for the massless limit of the last expression. Finally, we find the result

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') = \frac{4\pi}{45}G^2 \mathcal{F}^{\mu\nu\alpha\beta}_x \mathcal{H}_{\rm A}(x-x') + \frac{32\pi}{9}G^2 \mathcal{F}^{\mu\nu}_x \mathcal{F}^{\alpha\beta}_x \mathcal{H}_{\rm B}(x-x'), \qquad (5.43)$$

where $\tilde{\mathcal{H}}_{\rm A}(p) \equiv [1/(p^2)^2]\tilde{\mathcal{G}}_{\rm A}(p)$ and $\tilde{\mathcal{H}}_{\rm B}(p) \equiv [1/(p^2)^2]\tilde{\mathcal{G}}_{\rm B}(p)$, with $\tilde{\mathcal{G}}_{\rm A}(p)$ and $\tilde{\mathcal{G}}_{\rm B}(p)$ given by Eq. (5.24). The two-point correlation functions for the metric perturbations can be easily obtained using $h_{\mu\nu} = \bar{h}_{\mu\nu} - (1/2)\,\eta_{\mu\nu}\bar{h}^{\alpha}_{\alpha}$.

VI. DISCUSSION

Our main results for the correlation functions are Eqs. (5.23) and (5.43). In the case of a conformal field, the correlation functions of the linearized Einstein tensor have been explicitly evaluated and the results are given in Eq. (5.39). From the exponential factors $e^{-\sigma}$ in these results, we see that the correlation functions of the linearized Einstein tensor are in this case characterized by correlation lengths of the order of the Planck length. A similar behavior is expected for the correlation functions of the metric perturbations. Hence, as expected in this case, the correlation functions are negligibly small for points separated by distances large compared to the Planck length. At such scales, the dynamics of gravitational perturbations of Minkowski spacetime can be simply described by semiclassical gravity [30,41,31,42,43,34,25]. Deviations from semiclassical gravity are only important for points separated by Planckian or sub-Planckian scales. However, for such scales, our results (5.39) are not reliable, since we expect that gravitational fluctuations of genuine quantum nature to be relevant and, thus, the classical description breaks down. It is interesting to note, however, that these results for correlation functions are non-analytic in their characteristic correlation lengths. This kind of non-analytic behavior is actually quite typical of the solutions of Langevin-type equations with dissipative terms. An example in the context of a reduced version of the semiclassical Einstein-Langevin equation is given in Ref. [20].

For background solutions of semiclassical gravity with other scales present apart from the Planck scales (for instance, for matter fields in a thermal state), stress-energy fluctuations may be important at larger scales. For such backgrounds, stochastic semiclassical gravity might predict correlation functions with characteristic correlation lengths much larger than the Planck scales, so as to be relevant and reliable on a certain range of scales. It seems quite plausible, nevertheless, that these correlation functions would remain non-analytic in their characteristic correlation lengths. This would imply that these correlation functions could not be obtained from a calculation involving a perturbative expansion in the characteristic correlation lengths. In particular, if these correlation lengths are proportional to the Planck constant $\hbar$, the gravitational correlation functions could not be obtained from an expansion in $\hbar$. Hence, stochastic semiclassical gravity might predict a behavior for gravitational correlation functions different from that of the analogous functions in perturbative quantum gravity [44]. This is not necessarily inconsistent with having neglected action terms of higher order in $\hbar$ when considering semiclassical gravity as an effective theory [25].

We conclude this section with some comments about a technical point on the obtained solutions of stochastic semiclassical gravity. It concerns the issue that the Einstein-Langevin equations, as well as the semiclassical Einstein equations, contain derivatives of order higher than two. Because of this fact, these equations can have some ''pathological'' solutions (e.g., ''runaway'' solutions) which are presumably unphysical [45,43,46,25]. Thus, one needs to apply some criterion to discern the ''physical'' from the unphysical solutions. However, as it is discussed in Ref. [25] (see also Refs. [47]), even in the context of ''pure'' (non-stochastic) semiclassical gravity, this is still an open problem. Two main proposals, both based in the works by Simon [45,43,46], have been made concerning this issue: the ''perturbative expandability'' (in $\hbar$) criterion [45,43,46] and the ''reduction of order'' procedure [25].

The first proposal consists in identifying a subclass of ''physical'' solutions which are analytic in the Planck constant $\hbar$. This proposal has been successful in eliminating the instability of Minkowski spacetime found by Horowitz [30,31]. However, on the one hand, this proposal seems to be too restrictive since, as it has been pointed out in Ref. [25], one could not describe effects such as the continuous mass loss of a black hole due to Hawking radiation. On the other hand, there can be situations in which the formal series obtained when seeking approximate perturbative solutions (to a finite order in $\hbar$) does not converge to a solution to the semiclassical equations [25]. In our case, if we had tried to find solutions to Eq. (5.2) as a Taylor expansion in $\hbar$, we would have obtained a series for $\tilde{G}^{(1)}_{\mu\nu}(p)$ which, as the above solutions, would be linear and local in $\tilde{\xi}_{\alpha\beta}(p)$, but whose corresponding two-point correlation functions for the conformal field case would not converge to Eq. (5.23).

The ''reduction of order'' procedure provides in some cases a reasonable way to modify the semiclassical equations in order to eliminate spurious solutions. But, as it has been emphasized in Ref. [25], it is not clear at all whether a reduction of order procedure can always be applied to the semiclassical Einstein equation (and how this procedure should be applied). For the Einstein-Langevin equation, this issue has not been, to our knowledge, properly addressed. A naive application of the prescription to Eq. (5.2) seems to downplay the role of the dissipative terms with respect to the noise source. In fact, to lowest order, we obtain $G^{(1)\mu\nu} = 16\pi G\xi^{\mu\nu}$, where there is no contribution of the dissipation kernel. From this equation, we get the well-known result $\langle G^{(1)\mu\nu}\rangle_c = 0$ [25,43], and also $\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = (16\pi)^2 L_P^4 N^{\mu\nu\alpha\beta}(x,x')$. For a massless field, using Eqs. (3.10), (4.9) and (4.12), this gives

$$\mathcal{G}^{\mu\nu\alpha\beta}(x,x') = (2/15)(L_P^4/\pi^2) \times[(1/6)\mathcal{F}_x^{\mu\nu\alpha\beta} + 60\Delta\xi^2 \mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}] \times[\mathcal{P}f[1/((x-x')^2)^2] + \pi^3\delta^4(x-x')].$$

For the two-point correlation functions (5.41), we get, in the harmonic gauge,

$$\mathcal{H}^{\mu\nu\alpha\beta}(x,x') = (4\pi/45)L_P^4\mathcal{F}_x^{\mu\nu\alpha\beta}\mathcal{I}_{\rm A}(x-x') + (32\pi/9)L_P^4\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}\mathcal{I}_{\rm B}(x-x'),$$

with

$$\tilde{\mathcal{I}}_{\rm A}(p) \equiv \theta(-p^2-4m^2)(p^2)^{-2} \times\sqrt{1+4m^2/p^2}(1+4m^2/p^2)^2$$

and

$$\tilde{\mathcal{I}}_{\rm B}(p) \equiv \theta(-p^2-4m^2)(p^2)^{-2}\sqrt{1+4m^2/p^2}(3\Delta\xi + m^2/p^2)^2.$$

Comparing the last results for the massless case with the ones obtained in Sec. V, we note that the main qualitative feature is the absence of the exponential factors $e^{-\sigma}$, which make the two-point correlation functions to decay much more slowly with the distance, i.e., like a power instead of an exponential law. This fact is due to the lack of dissipative terms in the reduced order equations. The conclusion is that one should probably implement a more sophisticated version of the reduction of order procedure so as to keep some contribution of the dissipation kernel in the reduced order equations.

For these reasons, in our work we have not attempted any of these procedures and we have simply sought some solutions to the full equations (5.2). Our solutions for the conformal field case have the physically reasonable feature of having negligible two-point functions for points separated by scales larger than the Planck length.

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APPENDIX A: THE KERNELS FOR A VACUUM STATE

The kernels for a vacuum state can be computed in terms of the Wightman and Feynman functions defined in Eq. (2.6) using

$$\langle 0|\hat{t}_n^{ab}(x)\hat{t}_n^{cd}(y)|0\rangle = 4(N_n^{abcd}(x,y)+iH_{\mathrm{A}_n}^{abcd}(x,y)) = \nabla_x^a\nabla_y^c G_n^+(x,y)\nabla_x^b\nabla_y^d G_n^+(x,y)+\nabla_x^a\nabla_y^d G_n^+(x,y)\nabla_x^b\nabla_y^c G_n^+(x,y)$$
$$+2\mathcal{D}_x^{ab}(\nabla_y^c G_n^+(x,y)\nabla_y^d G_n^+(x,y))+2\mathcal{D}_y^{cd}(\nabla_x^a G_n^+(x,y)\nabla_x^b G_n^+(x,y))+2\mathcal{D}_x^{ab}\mathcal{D}_y^{cd}(G_n^{+2}(x,y)), \quad \text{(A1)}$$

where $\mathcal{D}^{ab}$ is the differential operator

$$\mathcal{D}_x^{ab} \equiv \left(\xi-\frac{1}{4}\right)g^{ab}(x)\Box_x+\xi(R^{ab}(x)-\nabla_x^a\nabla_x^b), \quad \text{(A2)}$$

and

$$H_{\mathrm{S}_n}^{abcd}(x,y) = -\frac{1}{4}\mathrm{Im}\Bigg[\nabla_x^a\nabla_y^c G_{F_n}(x,y)\nabla_x^b\nabla_y^d G_{F_n}(x,y)+\nabla_x^a\nabla_y^d G_{F_n}(x,y)\nabla_x^b\nabla_y^c G_{F_n}(x,y)+2\mathcal{D}_x^{ab}(\nabla_y^c G_{F_n}(x,y)\nabla_y^d G_{F_n}(x,y))$$
$$+2\mathcal{D}_y^{cd}(\nabla_x^a G_{F_n}(x,y)\nabla_x^b G_{F_n}(x,y))+2\mathcal{D}_x^{ab}\mathcal{D}_y^{cd}(G_{F_n}^2(x,y))+\frac{1}{2}[g^{ab}(x)(\nabla_y^c G_{F_n}(x,y)\nabla_y^d+\nabla_y^d G_{F_n}(x,y)\nabla_y^c)$$
$$+g^{cd}(y)(\nabla_x^a G_{F_n}(x,y)\nabla_x^b+\nabla_x^b G_{F_n}(x,y)\nabla_x^a)]\frac{\delta^n(x-y)}{\sqrt{-g(x)}}+(g^{ab}(x)\mathcal{D}_y^{cd}+g^{cd}(y)\mathcal{D}_x^{ab})\left(\frac{\delta^n(x-y)}{\sqrt{-g(x)}}G_{F_n}(x,y)\right)$$
$$+\frac{1}{4}g^{ab}(x)g^{cd}(y)G_{F_n}(x,y)(\Box_x-m^2-\xi R(x))\frac{\delta^n(x-y)}{\sqrt{-g(x)}}\Bigg]. \quad \text{(A3)}$$

APPENDIX B: MOMENTUM INTEGRALS

Some useful expressions for the momentum integrals in dimensional regularization defined in Eqs. (3.14) and (3.15) are

$$I_{0_n} = \frac{i}{(4\pi)^2}m^2\left(\frac{m^2}{4\pi\mu^2}\right)^{(n-4)/2}\Gamma\left(1-\frac{n}{2}\right) = \frac{i}{(4\pi)^2}\frac{4m^2}{(n-2)}\kappa_n+O(n-4), \quad \text{(B1)}$$

$$I_{0_n}^{\mu} = 0, \quad \text{(B2)}$$

$$I_{0_n}^{\mu\nu} = -m^2\eta^{\mu\nu}\frac{I_{0_n}}{n}, \quad \text{(B3)}$$

$$J_n(p) = \frac{-i}{(4\pi)^2}[2\kappa_n+\phi(p^2)+O(n-4)], \quad \text{(B4)}$$

$$J_n^{\mu}(p) = \frac{J_n(p)}{2}p^{\mu}, \quad \text{(B5)}$$

$$J_n^{\mu\nu}(p) = \frac{J_n(p)}{4}\left[p^{\mu}p^{\nu}-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2P^{\mu\nu}}{(n-1)}\right]+\frac{I_{0_n}}{2}\frac{1}{p^2}\left[p^{\mu}p^{\nu}+\frac{p^2P^{\mu\nu}}{n-1}\right], \quad \text{(B6)}$$

$$J_n^{\mu\nu\alpha}(p) = \frac{J_n(p)}{8}\left[p^{\mu}p^{\nu}p^{\alpha}-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2}{(n-1)}(P^{\mu\nu}p^{\alpha}+P^{\mu\alpha}p^{\nu}+P^{\alpha\nu}p^{\mu})\right]$$
$$+\frac{I_{0_n}}{4}\frac{1}{p^2}\left[3p^{\mu}p^{\nu}p^{\alpha}+\frac{p^2}{(n-1)}(P^{\mu\nu}p^{\alpha}+P^{\mu\alpha}p^{\nu}+P^{\alpha\nu}p^{\mu})\right], \quad \text{(B7)}$$

$$J_n^{\mu\nu\alpha\beta}(p)=\frac{J_n(p)}{16}\bigg[p^\mu p^\nu p^\alpha p^\beta-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2}{(n-1)}(P^{\mu\nu}p^\alpha p^\beta+P^{\nu\alpha}p^\mu p^\beta+P^{\nu\beta}p^\mu p^\alpha+P^{\mu\alpha}p^\nu p^\beta+P^{\mu\beta}p^\nu p^\alpha+P^{\alpha\beta}p^\mu p^\nu)$$
$$+\left(1+4\frac{m^2}{p^2}\right)^2\frac{(p^2)^2}{(n^2-1)}(P^{\mu\nu}P^{\alpha\beta}+P^{\mu\alpha}P^{\nu\beta}+P^{\mu\beta}P^{\nu\alpha})\bigg]+\frac{I_{0_n}}{8}\frac{1}{p^2}\bigg[\left(7-\frac{12}{n}\frac{m^2}{p^2}\right)p^\mu p^\nu p^\alpha p^\beta+\left(\frac{1}{n-1}-\frac{4}{n}\frac{m^2}{p^2}\right)$$
$$\times p^2(P^{\mu\nu}p^\alpha p^\beta+P^{\nu\alpha}p^\mu p^\beta+P^{\nu\beta}p^\mu p^\alpha+P^{\mu\alpha}p^\nu p^\beta+P^{\mu\beta}p^\nu p^\alpha+P^{\alpha\beta}p^\mu p^\nu)-\left(\frac{1}{n^2-1}-\frac{4(2n-1)}{n(n^2-1)}\frac{m^2}{p^2}\right)$$
$$\times(p^2)^2(P^{\mu\nu}P^{\alpha\beta}+P^{\mu\alpha}P^{\nu\beta}+P^{\mu\beta}P^{\nu\alpha})\bigg], \tag{B8}$$

where $p^2P^{\mu\nu}\equiv\eta^{\mu\nu}p^2-p^\mu p^\nu$, κ_n is defined in Eq. (2.16),

$$\phi(p^2)\equiv\int_0^1 d\alpha\,\ln\left(1+\frac{p^2}{m^2}\alpha(1-\alpha)-i\epsilon\right)=-i\pi\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}+\varphi(p^2), \tag{B9}$$

with $\epsilon\to0^+$, and

$$\varphi(p^2)\equiv\int_0^1 d\alpha\,\ln\left|1+\frac{p^2}{m^2}\alpha(1-\alpha)\right|$$
$$=-2+\sqrt{1+4\frac{m^2}{p^2}}\ln\left|\frac{\sqrt{1+4\frac{m^2}{p^2}}+1}{\sqrt{1+4\frac{m^2}{p^2}}-1}\right|\theta\left(1+4\frac{m^2}{p^2}\right)+2\sqrt{-1-4\frac{m^2}{p^2}}$$
$$\times\operatorname{arccotan}\left(\sqrt{-1-4\frac{m^2}{p^2}}\right)\theta\left(-1-4\frac{m^2}{p^2}\right). \tag{B10}$$

We can also write $\phi(p^2)$ in a more compact way as

$$\phi(p^2)=-2+\sqrt{1+4\frac{m^2}{p^2}}\ln\left(\frac{\sqrt{1+4(m^2-i\epsilon)/p^2}+1}{\sqrt{1+4(m^2-i\epsilon)/p^2}-1}\right). \tag{B11}$$

Other useful integrals in momentum space defined in Eq. (3.5) are

$$I(p)=\frac{1}{4(2\pi)^3}\theta(-p^0)\theta(-p^2-4m^2)\sqrt{1+4\frac{m^2}{p^2}}, \tag{B12}$$

$$I^\mu(p)=\frac{I(p)}{2}p^\mu, \tag{B13}$$

$$I^{\mu\nu}(p)=\frac{I(p)}{4}\left[p^\mu p^\nu-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2P^{\mu\nu}}{3}\right], \tag{B14}$$

$$I^{\mu\nu\alpha}(p)=\frac{I(p)}{8}\left[p^\mu p^\nu p^\alpha-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2}{3}(P^{\mu\nu}p^\alpha+P^{\mu\alpha}p^\nu+P^{\alpha\nu}p^\mu)\right], \tag{B15}$$

$$I^{\mu\nu\alpha\beta}(p)=\frac{I(p)}{16}\bigg[p^\mu p^\nu p^\alpha p^\beta-\left(1+4\frac{m^2}{p^2}\right)\frac{p^2}{3}(P^{\mu\nu}p^\alpha p^\beta+P^{\nu\alpha}p^\mu p^\beta+P^{\nu\beta}p^\mu p^\alpha+P^{\mu\alpha}p^\nu p^\beta+P^{\mu\beta}p^\nu p^\alpha+P^{\alpha\beta}p^\mu p^\nu)$$
$$+\left(1+4\frac{m^2}{p^2}\right)^2\frac{(p^2)^2}{15}(P^{\mu\nu}P^{\alpha\beta}+P^{\mu\alpha}P^{\nu\beta}+P^{\mu\beta}P^{\nu\alpha})\bigg]. \tag{B16}$$

APPENDIX C: PRODUCTS OF WIGHTMAN FUNCTIONS

For the products of derivatives of Wightman functions involved in the calculations of Sec. III A, we obtain the following expressions:

$$\Delta^{+2}(x)=-(2\pi)^2\int\frac{d^4p}{(2\pi)^4}e^{-ipx}I(p), \tag{C1}$$

$$\partial^{\mu}\Delta^{+}(x)\partial^{\nu}\Delta^{+}(x)=(2\pi)^2\int\frac{d^4p}{(2\pi)^4}e^{-ipx}[I^{\mu}(p)p^{\nu}-I^{\mu\nu}(p)], \tag{C2}$$

$$\partial^{\mu}\partial^{\nu}\Delta^{+}(x)\partial^{\alpha}\partial^{\beta}\Delta^{+}(x)=-(2\pi)^2\int\frac{d^4p}{(2\pi)^4}e^{-ipx}[I^{\mu\nu}(p)p^{\alpha}p^{\beta}-2I^{\mu\nu(\alpha}(p)p^{\beta)}+I^{\mu\nu\alpha\beta}(p)], \tag{C3}$$

with $I(p)$, $I^{\mu}(p)$, $I^{\mu\nu}(p)$, $I^{\mu\nu\alpha}(p)$ and $I^{\mu\nu\alpha\beta}(p)$ given by Eqs. (B12)–(B16). From these expressions, using the results of Appendix B, we obtain

$$\partial^{\mu}\Delta^{+}(x)\partial^{\nu}\Delta^{+}(x)=-\pi^2\partial_x^{\mu}\partial_x^{\nu}\int\frac{d^4p}{(2\pi)^4}e^{-ipx}I(p)-\frac{\pi^2}{3}\mathcal{F}_x^{\mu\nu}\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(1+4\frac{m^2}{p^2}\right)I(p), \tag{C4}$$

$$\partial^{\mu}\partial^{(\alpha}\Delta^{+}(x)\partial^{\beta)}\partial^{\nu}\Delta^{+}(x)=-\frac{\pi^2}{4}\partial_x^{\mu}\partial_x^{\nu}\partial_x^{\alpha}\partial_x^{\beta}\int\frac{d^4p}{(2\pi)^4}e^{-ipx}I(p) \tag{C5}$$

$$-\frac{\pi^2}{12}(\mathcal{F}_x^{\mu\nu}\partial_x^{\alpha}\partial_x^{\beta}+\mathcal{F}_x^{\alpha\beta}\partial_x^{\mu}\partial_x^{\nu})\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(1+4\frac{m^2}{p^2}\right)I(p)-\frac{\pi^2}{60}(\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}+2\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu})$$
$$\times\int\frac{d^4p}{(2\pi)^4}e^{-ipx}\left(1+4\frac{m^2}{p^2}\right)^2 I(p). \tag{C6}$$

APPENDIX D: PRODUCTS OF FEYNMAN FUNCTIONS

For the products of derivatives of Feynman functions that we need for the calculations of Sec. III B, we obtain the following results:

$$\mu^{-(n-4)}\Delta_{F_n}^2(x)=\int\frac{d^np}{(2\pi)^n}e^{ipx}J_n(p), \tag{D1}$$

$$\mu^{-(n-4)}\partial^{\mu}\Delta_{F_n}(x)\partial^{\nu}\Delta_{F_n}(x)=-\int\frac{d^np}{(2\pi)^n}e^{ipx}[J_n^{\mu}(p)p^{\nu}-J_n^{\mu\nu}(p)], \tag{D2}$$

$$\mu^{-(n-4)}\partial^{\mu}\partial^{\nu}\Delta_{F_n}(x)\partial^{\alpha}\partial^{\beta}\Delta_{F_n}(x)=\int\frac{d^np}{(2\pi)^n}e^{ipx}[J_n^{\mu\nu}(p)p^{\alpha}p^{\beta}-2J_n^{\mu\nu(\alpha}(p)p^{\beta)}+J_n^{\mu\nu\alpha\beta}(p)], \tag{D3}$$

$$\mu^{-(n-4)}\Delta_{F_n}(0)=-I_{0_n}, \tag{D4}$$

$$\mu^{-(n-4)}\partial^{\mu}\Delta_{F_n}(x)\partial^{\nu}\delta^n(x)=\int\frac{d^np}{(2\pi)^n}e^{ipx}(I_{0_n}^{\mu}p^{\nu}-I_{0_n}^{\mu\nu}), \tag{D5}$$

$$\mu^{-(n-4)}\Delta_{F_n}(x)\Box\delta^n(x)=\int\frac{d^np}{(2\pi)^n}e^{ipx}(p^2I_{0_n}+2p_{\mu}I_{0_n}^{\mu}+I_{0_n\,\mu}^{\mu}). \tag{D6}$$

Using the results of Appendix B, we find from the above expressions

$$\mu^{-(n-4)}\partial^{\mu}\Delta_{F_n}(x)\partial^{\nu}\Delta_{F_n}(x)=\frac{1}{4}\partial_x^{\mu}\partial_x^{\nu}\int\frac{d^np}{(2\pi)^n}e^{ipx}J_n(p)+\frac{1}{12}\mathcal{F}_x^{\mu\nu}\int\frac{d^np}{(2\pi)^n}e^{ipx}\left(1+4\frac{m^2}{p^2}\right)J_n(p)$$
$$+\frac{1}{2}\int\frac{d^np}{(2\pi)^n}e^{ipx}\left[I_{0_n}\left(\frac{p^{\mu}p^{\nu}}{p^2}+\frac{1}{3}P^{\mu\nu}\right)-\frac{i}{(4\pi)^2}\frac{1}{9}(p^2+6m^2)P^{\mu\nu}\right]+O(n-4), \tag{D7}$$

$$\mu^{-(n-4)}\partial^{\mu}\partial^{(\alpha}\Delta_{F_n}(x)\partial^{\beta)}\partial^{\nu}\Delta_{F_n}(x)=\frac{1}{16}\partial_x^{\mu}\partial_x^{\nu}\partial_x^{\alpha}\partial_x^{\beta}\int\frac{d^np}{(2\pi)^n}e^{ipx}J_n(p)+\frac{1}{48}(\mathcal{F}_x^{\mu\nu}\partial_x^{\alpha}\partial_x^{\beta}+\mathcal{F}_x^{\alpha\beta}\partial_x^{\mu}\partial_x^{\nu})\int\frac{d^np}{(2\pi)^n}e^{ipx}\left(1+4\frac{m^2}{p^2}\right)$$
$$\times J_n(p)+\frac{1}{240}(\mathcal{F}_x^{\mu\nu}\mathcal{F}_x^{\alpha\beta}+2\mathcal{F}_x^{\mu(\alpha}\mathcal{F}_x^{\beta)\nu})\int\frac{d^np}{(2\pi)^n}e^{ipx}\left(1+4\frac{m^2}{p^2}\right)^2J_n(p)-\frac{1}{8}$$
$$\times\int\frac{d^np}{(2\pi)^n}e^{ipx}\left\{I_{0_n}\left[\frac{1}{p^2}\left(1+\frac{12}{n}\frac{m^2}{p^2}\right)p^{\mu}p^{\nu}p^{\alpha}p^{\beta}+\frac{1}{3}(P^{\mu\nu}p^{\alpha}p^{\beta}+P^{\alpha\beta}p^{\mu}p^{\nu})\right.\right.$$
$$+\frac{4}{n}\frac{m^2}{p^2}(P^{\mu\nu}p^{\alpha}p^{\beta}+P^{\alpha\beta}p^{\mu}p^{\nu}+2P^{\mu(\alpha}p^{\beta)}p^{\nu}+2P^{\nu(\alpha}p^{\beta)}p^{\mu})+\frac{1}{15}\left(p^2+\frac{28}{n}m^2\right)$$
$$\left.\times(P^{\mu\nu}P^{\alpha\beta}+2P^{\mu(\alpha}P^{\beta)\nu})\right]-\frac{i}{(4\pi)^2}\frac{1}{9}(p^2+6m^2)(P^{\mu\nu}p^{\alpha}p^{\beta}+P^{\alpha\beta}p^{\mu}p^{\nu})$$
$$\left.-\frac{i}{4\pi^2}\frac{1}{225}(2(p^2)^2+20m^2p^2+45m^4)(P^{\mu\nu}P^{\alpha\beta}+2P^{\mu(\alpha}P^{\beta)\nu})\right\}+O(n-4),\qquad\text{(D8)}$$

where $P^{\mu\nu}$ is the projector orthogonal to p^{μ} defined above.

APPENDIX E: LINEARIZED TENSORS AROUND FLAT SPACETIME

Some curvature tensors linearized around flat spacetime are given by the following expressions:

$$G^{(1)\mu\nu}=R^{(1)\mu\nu}-\frac{1}{2}\eta^{\mu\nu}R^{(1)},\qquad\text{(E1)}$$

$$D^{(1)\mu\nu}=\partial^{\mu}\partial^{\nu}R^{(1)}+\frac{1}{2}\eta^{\mu\nu}\Box R^{(1)}-3\Box R^{(1)\mu\nu},\qquad\text{(E2)}$$

$$B^{(1)\mu\nu}=2(\partial^{\mu}\partial^{\nu}R^{(1)}-\eta^{\mu\nu}\Box R^{(1)}),\qquad\text{(E3)}$$

with

$$R^{(1)\mu\nu}=\frac{1}{2}(\partial_{\alpha}\partial^{\mu}h^{\nu\alpha}+\partial_{\alpha}\partial^{\nu}h^{\mu\alpha}-\Box h^{\mu\nu}-\partial^{\mu}\partial^{\nu}h),\qquad\text{(E4)}$$

$$R^{(1)}=\eta_{\alpha\beta}R^{(1)\alpha\beta}=\partial^{\alpha}\partial^{\beta}h_{\alpha\beta}-\Box h,\qquad\text{(E5)}$$

and

$$R^{(1)\mu\nu\alpha\beta}=\frac{1}{2}(\partial^{\mu}\partial^{\beta}h^{\nu\alpha}+\partial^{\nu}\partial^{\alpha}h^{\mu\beta}-\partial^{\mu}\partial^{\alpha}h^{\nu\beta}-\partial^{\nu}\partial^{\beta}h^{\mu\alpha}).\qquad\text{(E6)}$$

In four spacetime dimensions, the linearized Weyl tensor is given by

$$C^{(1)\mu\nu\alpha\beta}=\frac{1}{12}[6(\eta^{\nu\rho}\eta^{\alpha\sigma}\partial^{\mu}\partial^{\beta}+\eta^{\mu\rho}\eta^{\beta\sigma}\partial^{\nu}\partial^{\alpha}-\eta^{\nu\rho}\eta^{\beta\sigma}\partial^{\mu}\partial^{\alpha}-\eta^{\mu\rho}\eta^{\alpha\sigma}\partial^{\nu}\partial^{\beta})+3(\eta^{\mu\alpha}\eta^{\rho\sigma}\partial^{\nu}\partial^{\beta}$$
$$+\eta^{\mu\alpha}\eta^{\nu\rho}\eta^{\beta\sigma}\Box-\eta^{\mu\alpha}\eta^{\nu\rho}\partial^{\beta}\partial^{\sigma}-\eta^{\mu\alpha}\eta^{\beta\sigma}\partial^{\nu}\partial^{\rho}+\eta^{\nu\beta}\eta^{\rho\sigma}\partial^{\mu}\partial^{\alpha}+\eta^{\nu\beta}\eta^{\mu\rho}\eta^{\alpha\sigma}\Box$$
$$-\eta^{\nu\beta}\eta^{\mu\rho}\partial^{\alpha}\partial^{\sigma}-\eta^{\nu\beta}\eta^{\alpha\sigma}\partial^{\mu}\partial^{\rho}-\eta^{\nu\alpha}\eta^{\rho\sigma}\partial^{\mu}\partial^{\beta}-\eta^{\nu\alpha}\eta^{\mu\rho}\eta^{\beta\sigma}\Box+\eta^{\nu\alpha}\eta^{\mu\rho}\partial^{\beta}\partial^{\sigma}+\eta^{\nu\alpha}\eta^{\beta\sigma}\partial^{\mu}\partial^{\rho}$$
$$-\eta^{\mu\beta}\eta^{\rho\sigma}\partial^{\nu}\partial^{\alpha}-\eta^{\mu\beta}\eta^{\nu\rho}\eta^{\alpha\sigma}\Box+\eta^{\mu\beta}\eta^{\nu\rho}\partial^{\alpha}\partial^{\sigma}+\eta^{\mu\beta}\eta^{\alpha\sigma}\partial^{\nu}\partial^{\rho})+2(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\nu\alpha}\eta^{\mu\beta})(\partial^{\rho}\partial^{\sigma}-\eta^{\rho\sigma}\Box)]h_{\rho\sigma}.\qquad\text{(E7)}$$

APPENDIX F: THE INTEGRALS $J_a(S)$

For the integrals $J_a(s)$, $a=1,2,3$, defined in Eq. (5.30), we find the following results:

$$J_1(s)=\frac{1}{4(\kappa^2+\pi^2)|p|^2}\left\{\frac{1}{2\,\mathrm{Re}\,p}\ln\left[\frac{s^2-2\,\mathrm{Re}\,p\,s+|p|^2}{s^2+2\,\mathrm{Re}\,p\,s+|p|^2}\right]+\frac{1}{\mathrm{Im}\,p}\left[\pi-\arctan\left(\frac{s+\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)-\arctan\left(\frac{s-\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)\right]\right\}, \tag{F1}$$

$$J_2(s)=\frac{1}{4(\kappa^2+\pi^2)}\left\{\frac{1}{2\,\mathrm{Re}\,p}\ln\left[\frac{s^2+2\,\mathrm{Re}\,p\,s+|p|^2}{s^2-2\,\mathrm{Re}\,p\,s+|p|^2}\right]+\frac{1}{\mathrm{Im}\,p}\left[\pi-\arctan\left(\frac{s+\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)-\arctan\left(\frac{s-\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)\right]\right\}, \tag{F2}$$

$$\begin{aligned}J_3(s)=\frac{1}{4(\kappa^2+\pi^2)}\Bigg\{&-4s+\frac{1}{2\,\mathrm{Re}\,p}[3(\mathrm{Re}\,p)^2-(\mathrm{Im}\,p)^2]\ln\left[\frac{s^2+2\,\mathrm{Re}\,p\,s+|p|^2}{s^2-2\,\mathrm{Re}\,p\,s+|p|^2}\right]\\&+\frac{1}{\mathrm{Im}\,p}[(\mathrm{Re}\,p)^2-3(\mathrm{Im}\,p)^2]\left[\pi-\arctan\left(\frac{s+\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)-\arctan\left(\frac{s-\mathrm{Re}\,p}{\mathrm{Im}\,p}\right)\right]\Bigg\},\end{aligned} \tag{F3}$$

where p is a function of s given by expressions (5.31), which give $|p|^2=[[(\kappa^2+\pi^2)s^2+\kappa]^2+\pi^2]^{1/2}/(\kappa^2+\pi^2)$.

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