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Stochastic background of gravitational waves

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A continuous stochastic background of gravitational waves (GWs) for burst sources is produced if the mean time interval between the occurrence of bursts is smaller than the average time duration of a single burst at the emission, i.e., the so-called duty cycle must be greater than one. To evaluate the background of GWs produced by an ensemble of sources, during their formation, for example, one needs to know the average energy flux emitted during the formation of a single object and the formation rate of such objects as well. In many cases the energy flux emitted during an event of production of GWs is not known in detail; only characteristic values for the dimensionless amplitude and frequencies are known. Here we present a shortcut to calculate stochastic backgrounds of GWs produced from cosmological sources. For this approach it is not necessary to know in detail the energy flux emitted at each frequency. Knowing the characteristic values for the ''lumped'' dimensionless amplitude and frequency we show that it is possible to calculate the stochastic background of GWs produced by an ensemble of sources.

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I. INTRODUCTION

The detection of gravitational radiation will probably mark a new revolution in the history of astronomy. It is worth mentioning that the detection of gravitational waves (GWs) will directly verify the predictions of the general relativity theory concerning the existence or not of such waves, as well as other theories of gravity [1].

The realm of astrophysics is the place where one finds sources of GWs detectable by the GW observatories. There is a host of possible astrophysical sources of GWs: namely, supernovas, the collapse of a star or star cluster to form a black hole, inspiral and coalescence of compact binaries, the fall of stars and black holes into supermassive black holes, rotating neutron stars, ordinary binary stars, relics of the big bang, vibrating or colliding of monopoles, cosmic strings, etc., among others [1]. Nowadays there is a great effort to study, from the theoretical point of view, which are the most promising sources of GWs to be detected, in particular, their wave forms, characteristic frequencies, and the number of sources a year that one expects to observe.

In a few years, instead of building models trying to understand how the sources of GWs work, it will be possible, starting from the observations (wave forms, amplitudes, polarizations, etc.), to really understand how the GW emission takes place.

Because of the fact the GWs are produced by a large variety of astrophysical sources and cosmological phenomena it is quite probable that the Universe is pervaded by a background of such waves. Binary stars of a variety of stars (ordinary, compact, or combinations of them), population III stars, phase transitions in the early Universe, and cosmic strings are examples of sources that could generate such putative background of GWs.

As the GWs possess a very weak interaction with matter passing through it with impunity, relic radiation (spectral properties, for example) once detected can provide information on the physical conditions from the era in which the GWs were produced. In principle it will be possible, for example, to get information from the epoch when the galaxies and stars started to form and evolve.

Here we present, in particular, a shortcut to the calculation of stochastic background of GWs. For this approach it is not necessary to know in detail the energy flux of the GWs produced in a given burst event. If the characteristic values for the dimensionless amplitude and frequency are known and the event rate is given it is possible to calculate the stochastic background of GWs produced by an ensemble of sources of the same kind.

This paper is organized as follows. In Sec. II we show how to calculate the stochastic background of GWs starting from characteristic values for the dimensionless amplitude and frequency as well as the burst event rate. In Sec. III we apply the idea presented in Sec. II to the calculation of a stochastic background of GWs from a cosmological population of black holes, and finally in Sec. IV we present the conclusions.

II. A SHORTCUT TO THE CALCULATION OF STOCHASTIC BACKGROUND OF GWs

The GWs can be characterized by their dimensionless amplitude h, and frequency ν. The spectral energy density, the flux of GWs, received on Earth, F_ν, in erg $\mathrm{cm^{-2}\ s^{-1}\ Hz^{-1}}$, is (see, e.g., Refs. [2,3])

$$F_\nu = \frac{c^3 s_{\mathrm{h}} \omega_{\mathrm{obs}}^2}{16\pi G}, \tag{1}$$

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where $\omega_{obs}=2\pi\nu_{obs}$ with ν_{obs} the GW frequency observed on Earth (in Hz), c is the speed of light, G is the gravitational constant, and $\sqrt{s_h}$ is the strain amplitude of the GW (in $Hz^{-1/2}$). For $\omega \geqslant 0$, Eq. (1) must be multiplied by a factor of 2 in order to account for the folding of negative frequencies into positive (see, e.g., Ref. [4]). The stochastic background produced by an ensemble of sources, of the same kind, would have a spectral density of the flux and strain amplitude also related to the above equation. The strain amplitude at a given frequency at the present time could be, for example, a contribution of sources of the same kind but with different masses producing GWs at different redshifts. Thus, the ensemble of sources produces a background whose characteristic amplitude at the present time is $\sqrt{s_h}$.

On the other hand, the spectral density of the flux can be written as (see, e.g., Refs. [4,5])

$$F_\nu=\int f_\nu(\nu_{obs})dR, \tag{2}$$

where $f_\nu(\nu_{obs})$ is the energy flux per unit of frequency (in $\mathrm{erg\,cm^{-2}\,Hz^{-1}}$) produced by a unique source and dR is the differential rate of production of GWs by the source.

The energy flux per unit frequency $f_\nu(\nu_{obs})$ can be written as follows (see, e.g., Ref. [6])

$$f_\nu(\nu_{obs})=\frac{\pi c^3}{2G}h^2_{single}, \tag{3}$$

where h_{single} is the dimensionless amplitude produced by an event that generates a signal with observed frequency ν_{obs}.

Then, the resulting equation for the spectral density of the flux is

$$F_\nu=\frac{\pi c^3}{2G}\int h^2_{single}dR. \tag{4}$$

From the above equations we obtain for the strain

$$s_h=\frac{1}{\nu^2_{obs}}\int h^2_{single}dR. \tag{5}$$

Thus, the dimensionless amplitude reads

$$h^2_{BG}=\frac{1}{\nu_{obs}}\int h^2_{single}dR. \tag{6}$$

With the above equations one finds, for example, the dimensionless amplitude of the GWs produced by an ensemble of sources of the same kind that generates a signal observed at frequency ν_{obs}. Note that in this formulation it is not necessary to know in detail the energy flux of GWs at each frequency. Knowing the characteristic amplitude for a given source, h_{single}, associated to an event burst of GWs, and the rate of production of GWs, it is possible to obtain the stochastic background of an ensemble of these sources.

It is worth mentioning that if the collective effect of bursts of GWs really form a continuous background the quantity called duty cycle must be greater than one. In other words, the mean time interval between the occurrence of bursts must be smaller than the typical duration of each burst. The duty cycle is defined as follows:

$$D(z)=\int dR\Delta\overline{\tau}_{GW}(1+z), \tag{7}$$

where $\Delta\overline{\tau}_{GW}$ is the average time duration of single bursts at the emission (see, e.g., Ref. [4]).

In the present study we are using the relationship between h_{BG} and h_{single} [Eq. (6)], as Ferrari *et al.* [4,5], in either case of duty cycle, large or small. In the next section we apply the technique for a case in which the duty cycle is small; in another study, to appear elsewhere, we apply it to a case where the duty cycle is large (see Ref. [7]).

The energy density of GWs is usually written in terms of the closure energy density of GWs per logarithmic frequency interval, which is given by

$$\Omega_{GW}=\frac{1}{\rho_c}\frac{d\rho_{GW}}{d\log\nu_{obs}}, \tag{8}$$

where ρ_c is the critical density ($\rho_c=3H^2/8\pi G$). The above can be written as

$$\Omega_{GW}=\frac{\nu_{obs}}{c^3\rho_c}F_\nu=\frac{4\pi^2}{3H^2}\nu^2_{obs}h^2_{obs}. \tag{9}$$

III. APPLICATION: STOCHASTIC BACKGROUND OF GWs FROM A COSMOLOGICAL POPULATION OF STELLAR BLACK HOLES

In this section we apply the formulation presented in the preceding section to calculate the background of GWs from a cosmological population of stellar black holes.

From Eq. (6) one sees that it is necessary to know (a) h_{single}, here named h_{BH}, the characteristic amplitude of the burst of GWs produced during the black hole formation; (b) dR, the differential rate of production of GWs, here named dR_{BH}, the differential rate of black hole formation. It is worth noting that we are implicitly assuming that during the formation of each black hole there is a production of a burst of GWs.

To proceed it is necessary to know the star formation history of the Universe, which we adopted from a study performed by Madau *et al.* [8], which holds for the redshift range $0<z<5$. It is also necessary to know (a) the initial mass function (IMF), which we assume to be the Salpeter IMF, and (b) the smallest progenitor mass which is expected to lead to black holes (see Refs. [9,10]).

A. The rate of stellar black holes formation

The differential rate of black hole formation can be written as

$$dR_{BH}=\dot{\rho}_*(z)\frac{dV}{dz}\phi(m)dmdz, \tag{10}$$

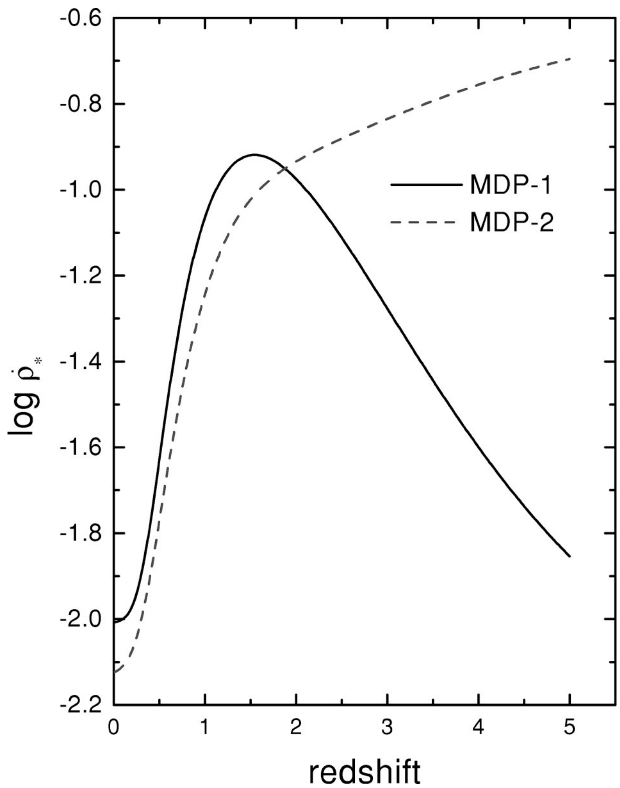

FIG. 1. Evolution of the log of the SFR density ($M_\odot\,\mathrm{yr}^{-1}\,\mathrm{Mpc}^{-3}$) for $\Omega_0=1$, $(\Lambda=0)$, $h=0.5$, and a Salpeter IMF. The solid line represents the SFR density evolution given by Eq. (11), MDP-1, beyond the dotted line corresponds to the SFR density given by Eq. (12), MDP-2.

where $\dot{\rho}_*(z)$ is the star formation rate (SFR) density (in $M_\odot\,\mathrm{yr}^{-1}\,\mathrm{Mpc}^{-3}$), dV is the comoving volume element, and $\phi(m)$ the IMF (see Refs. [4,5,7]).

The SFR density can be derived from observations. In particular, our present view of the Universe at redshifts $z \lesssim 4-5$ has been extended by recent data obtained with the Hubble Space Telescope (HST) and other large telescopes (see, e.g., Refs. [11–13]).

It has been shown that, in general, the measured comoving luminosity density is proportional to the SFR density. Thus, the star formation evolution can be derived from recent UV-optical observations of star forming galaxies out to redshifts $\sim 4-5$ [14]. Figure 1 shows the SFR density obtained by Madau *et al.* [14].

In particular, there are two different fits to the SFR density presented by these authors. The first fit for the SFR density (here after referred to as MDP-1) is given by

$$\dot{\rho}_*(z)=0.049[t_9^5 e^{-t_9/0.64}+0.2(1-e^{-t_9/0.64})] \times M_\odot\,\mathrm{yr}^{-1}\,\mathrm{Mpc}^{-3}, \quad (11)$$

where t_9 is the Hubble time in Gyr $[t_9=13/(1+z)^{3/2}]$.

The second fit for the SFR density (hereafter referred to as MDP-2) is given by

$$\dot{\rho}_*(z)=0.336\,e^{-t_9/1.6}+0.0074(1-e^{-t_9/0.64}) +0.0197\,t_9^5 e^{-t_9/0.64} M_\odot\,\mathrm{yr}^{-1}\,\mathrm{Mpc}^{-3}. \quad (12)$$

In the above fits, Eq. (11) and (12), Madau and collaborators considered an Einstein–de Sitter cosmology $(\Omega_0=1)$ with Hubble constant $H_0=50\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and cosmological constant $\Lambda=0$. Note that for a different cosmological scenario it is necessary to rescale the SFR density.

The fit given by Eq. (11) traces the rise, peak, and sharp drop of the observed UV emissivity at redshifts $z \lesssim 2$, while the fit given by Eq. (12) considers that half of the present-day stars, the fraction contained in spheroidal systems [15], were formed at $z>2.5$ and were enshrouded by dust. This fact produces an increase in the SFR density at redshifts $z>2.5$ (see Fig. 1) contrary to the sharp drop described in Eq. (11).

The consistency of $\dot{\rho}_*(z)$ given by Eq. (12) with the Hubble Deep Field (HDF) analysis is obtained assuming a dust extinction that increases with redshift. This fact is consistent with the evolution of the luminosity density, but overpredicts the metal mass density at high redshifts as derived from quasar absorbers (see Ref. [8]).

Despite this fact, it is interesting also to analyze the GW production with $\dot{\rho}_*(z)$ given by MDP-2 [Eq. (12)] because this SFR density produces a large number of supernovas at $z>2.5$, when compared to the SFR history described by MDP-1.

Concerning the IMF here we consider Salpeter's, as already mentioned. Thus,

$$\phi(m)=Am^{-(1+x)}, \quad (13)$$

where A is a normalization constant and $x=1.35$ the Salpeter exponent.

The IMF is defined in such a way that $\phi(m)dm$ represents the number of stars in the mass interval $[m,m+dm]$. The normalization of the IMF is obtained through the relation

$$\int_{\mathrm{m}_l}^{\mathrm{m_u}} m\,\phi(m)\,dm=1, \quad (14)$$

with $\mathrm{m}_l=0.1M_\odot$ and $\mathrm{m_u}=125M_\odot$. Using this normalization of the mass spectrum, we obtain $A=0.17(M_\odot)^{0.35}$.

In the present work we follow Timmes, Woosley and Wheaver [9] (see also Ref. [10]), who obtain from stellar evolution calculations, the minimal progenitor mass to form black holes, namely, $18M_\odot$ to $30M_\odot$ depending on the iron core masses. Then, we assume that the minimum mass able to form a remnant black hole is $m_{\min}=25M_\odot$. For the remnant mass, M_r, we take $M_\mathrm{r}=\alpha m$, where m is the mass of the progenitor star and $\alpha=0.1$ (see, e.g., Refs. [4,5]).

In Fig. 2 we show the evolution of the rate of black hole formation $R_{\mathrm{BH}}(z)$, i.e., the number of black holes formed per unit time within the comoving volume out to redshift z, for MDP-1 and MDP-2 for a cosmological scenario with $\Omega_0=1.0$ and $h_0=0.5$. Note that MDP-1 and MDP-2 are similar for $z<2.5$, and for $z>2.5$ they are quite different.

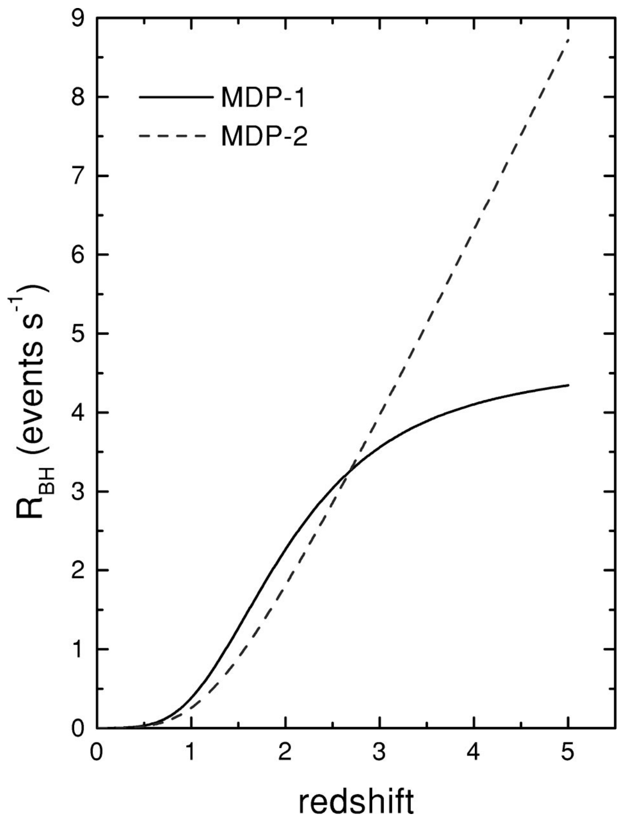

FIG. 2. Evolution of the rate of black hole formation occurring per unit time within the comoving volume out to redshift z for $\Omega_0 = 1$, ($\Lambda = 0$), $h = 0.5$, and a Salpeter IMF. The solid line represents the rate of black hole formation when we used the Eq. (11) for the SFR density, MDP-1, beyond the dotted line corresponds to the rate of black hole formation when we used the SFR density given by Eq. (12), MDP-2 (see Fig. 1).

B. The gravitational wave production

To obtain the stochastic background, besides knowing the differential rate of black holes formation presented in Sec. II, one needs to know $h_{\rm BH}$, the characteristic dimensionless amplitude generated during the black hole formation. Following Thorne [1], $h_{\rm BH}$ reads

$$h_{\rm BH} = \left(\frac{15}{2\pi}\varepsilon\right)^{1/2} \frac{G}{c^2}\frac{M_{\rm r}}{r_{\rm z}} \simeq 7.4\times 10^{-20} \varepsilon^{1/2} \left(\frac{M_{\rm r}}{M_\odot}\right)\left(\frac{r_{\rm z}}{1\ {\rm Mpc}}\right)^{-1}, \tag{15}$$

where ε is the efficiency of generation of GWs and $r_{\rm z}$ is the distance to the source.

The collapse to a black hole produces a signal with frequency (see, e.g., Ref. [1])

$$\nu_{\rm obs} = \frac{1}{5\pi M_{\rm r}}\frac{c^3}{G}(1+z)^{-1} \simeq 1.3\times 10^4 {\rm Hz}\left(\frac{M_\odot}{M_{\rm r}}\right)(1+z)^{-1}, \tag{16}$$

where the factor $(1+z)^{-1}$ takes into account the redshift effect on the emission frequency, that is, a signal emitted at frequency $\nu_{\rm e}$ at redshift z is observed at frequency $\nu_{\rm obs} = \nu_{\rm e}(1+z)^{-1}$.

From Eqs. (6), (10), and (15) we obtain, for the dimensionless amplitude (for $\alpha = 0.1$),

$$h_{\rm BG}^2 = \frac{(7.4\times 10^{-21})^2 \varepsilon}{\nu_{\rm obs}} \left[\int_{z_{c_f}}^{z_{c_i}} \int_{m_{\rm min}}^{m_{\rm u}} \left(\frac{m}{M_\odot}\right)^2 \left(\frac{d_L}{1\ {\rm Mpc}}\right)^{-2} \times \dot{\rho}_*(z)\frac{dV}{dz}\phi(m)\,dm\,dz \right], \tag{17}$$

where in the above equation $d_{\rm L}$ is the luminosity distance to the source.

The comoving volume element is given by

$$dV = 4\pi\left(\frac{c}{H_0}\right) r_{\rm z}^2 \frac{dz}{(1+z)}, \tag{18}$$

and the comoving distance, $r_{\rm z}$, is

$$r_{\rm z} = \frac{2c[1-(1+z)^{-1/2}]}{H_0}. \tag{19}$$

In the above equation the density parameter is considered $\Omega_0 = 1$ and H_0 is the present value of the Hubble parameter.

The comoving distance is related to the luminosity distance by

$$d_{\rm L} = r_{\rm z}(1+z). \tag{20}$$

With the above equations we can calculate the dimensionless amplitude produced by an ensemble of black holes that generates a signal observed at frequency $\nu_{\rm obs}$.

It is worth mentioning that the formulation used here is similar to that used by Ferrari *et al.* [4,5], but instead of using an average energy flux taken from Stark and Piran [16], who simulated the axisymmetric collapse of a rotating polytropic star to a black hole, we use Eq. (15) to obtain the energy flux, which takes into account the most relevant quasinormal modes of a rotating black hole and represents a kind of average over the rotational parameter. Both formulations present similar results, since in the end the most important contributions to the energy flux come from the quasinormal modes of the black hole formed, which account for most of the gravitational radiation produced during the collapse process. In a paper to appear elsewhere [17] we present a detailed comparison between our formulation and results with those by Ferrari *et al.* [4].

C. Numerical results

Figure 3 presents the amplitude of GWs as a function of the observed frequency obtained from Eq. (17) for the two SFR densities present in Sec. II. We obtained that the star formation rate given by Eq. (12), MDP-2, produces a maximum amplitude $h_{\rm BG}$ lower than the MDP-1 SFR density. This seems to be a contradiction since $R_{\rm BH}$ is higher for MDP-2. Note, however, that for $z < 2.5$, $R_{\rm BH}$ is higher for MDP-1 and due to this fact the maximum amplitude peak is higher for MDP-1. The SFR density described by MDP-2 produces a higher $R_{\rm BH}$ for $z > 2.5$, but the contribution of these events does not contribute to enhance the $h_{\rm BG}$ peak, but

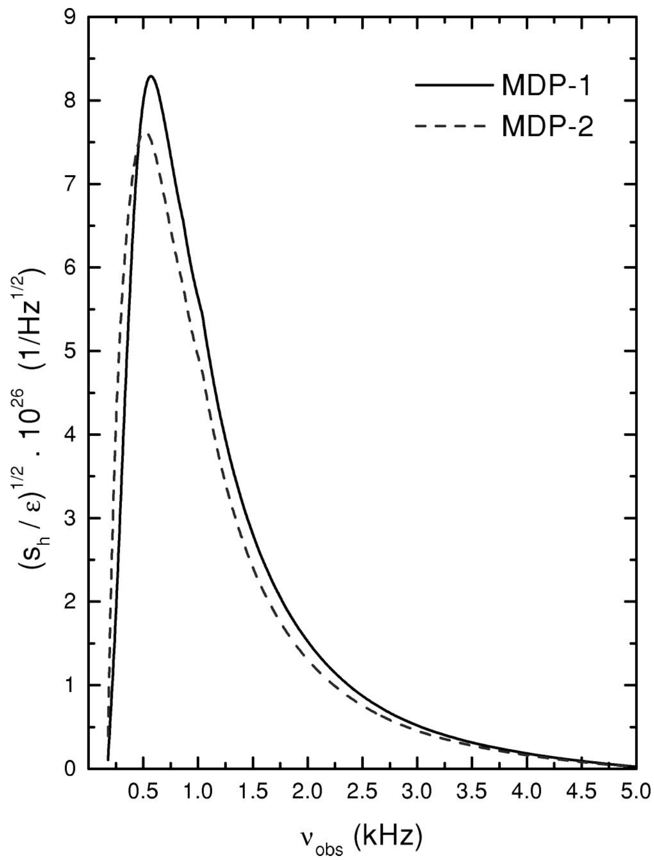

FIG. 3. The background amplitude of the GWs as a function of ν_{obs}. The solid line (dashed line) represents $(s_h/\varepsilon)^{1/2}$ for the MDP-1 SFR density (MDP-2 SFR density).

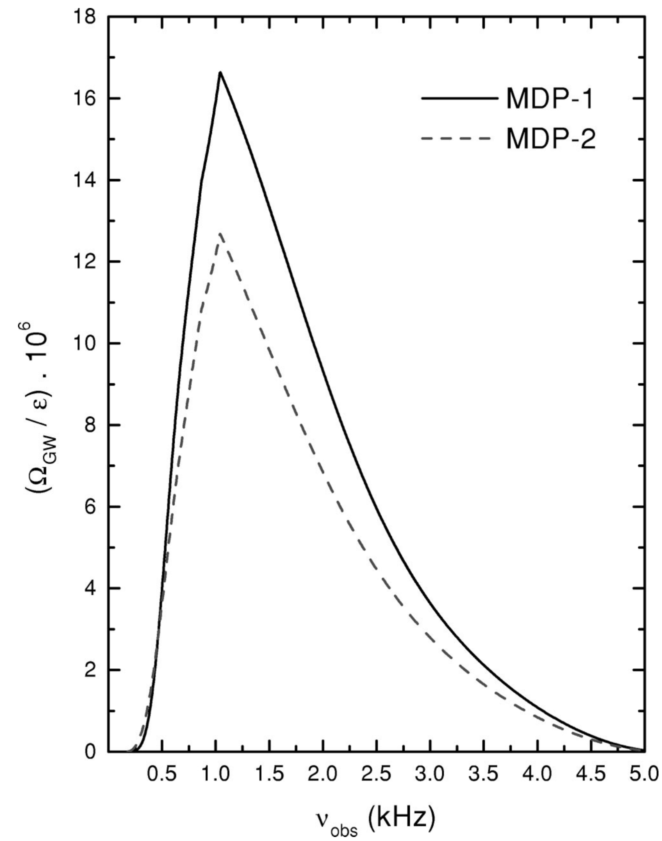

FIG. 4. The closure energy density of GWs as a function of ν_{obs}. The solid line (dashed line) represents Ω_{GW}/ε for the MDP-1 SFR density (MDP-2 SFR density).

instead contributes to enhance h_{BG} at the lowest frequencies due to the redshift effect (see also Fig. 2).

A comparison of our results with those of Ferrari *et al.* [4] shows that the formulation used here presents similar results. It is worth mentioning that in the comparison we have adopted $\varepsilon \sim 10^{-4}$. Note that in the present study, instead of using the average energy flux emitted during the axisymmeric collapse of a rotating polytropic star to a black hole with different values for the rotational parameter, we use Eq. (15). In this equation there is no explicit dependence of h_{BH} on the rotational parameter, it represents a kind of characteristic value for the amplitude of GWs during the black hole formation. The characteristic frequency given by Eq. (16) has to do with the frequency of the lowest $m=0$ quasinormal mode of a black hole, which is believed to be excited during its formation.

One could argue that it is surprising that our results agree so well with those of Ferrari *et al.*, particularly those of Fig. 3. The reason for this good agreement is related to the fact that the main contribution to the strain amplitude in the Ferrari *et al.* calculations comes from the lowest quasinormal mode of the black hole formed, which is also the main contribution present in the Eq. (15) we use in our study. A close comparison shows, however, that the peak of the curve in the Fig. 3 occurs for a frequency higher than that of Ferrari et al. Although most of the energy comes from the quasinormal mode, there is a contribution from the lower frequencies of the energy flux (see Ref. [4]), which moves the peak of the strain amplitude of Ferrari *et al.* to the left as compared to ours.

We show in Fig. 4 Ω_{GW} as a function of the observed frequency. Note that the MDP-1 SFR density presents, due to the higher contribution to R_{BH} for $z<2.5$, a more relevant contribution to the closure energy density of GWs for almost all frequencies, than the MDP-2 SFR density does.

Comparing Fig. 4 with the corresponding figures of Ferrari *et al.*, one notes that their curves are broader than ours. This occurs due to the fact the closure energy density is directly proportional to the energy flux, and therefore more sensitive to their frequency dependence. The Ferrari *et al.* energy flux as a function of frequency is broader than we use here, this is why their closure energy density as function of frequency curves are broader than ours.

Certainly, modifying the exponent x in Eq. (13) to $x > 1.35$, we will obtain a more steeply falling IMF, corresponding to a lower number of massive stars than that obtained using Salpeter's IMF. This produces, as a result, a lower rate of black holes formation than that obtained here, narrowing the curves present in Figs. 3 and 4. However, the agreement with the study performed by Ferrari *et al.* will be still good, since their study presents the same dependence on the IMF. Thus both results (and models) will be modified in the same way.

IV. CONCLUSIONS

Here we present a shortcut to the calculation of stochastic background of GWs. For this approach it is not necessary to know in detail the energy flux at each frequency of the GWs produced in a given burst event; if the characteristic values for the ''lumped'' dimensionless amplitude and frequency are known, and the event rate is given, it is possible to calculate the stochastic background of GWs produced by an ensemble of sources of the same kind.

Since one knows the dominant processes of GWs emission one can calculate the stochastic background of an ensemble of black holes. We argue that the same holds for other processes of GWs production, particularly those involving cosmological sources, since the number of sources could be large enough to produce stochastic backgrounds.

We apply this formulation to the study of a stochastic background of GWs produced during the formation of a cosmological population of stellar black holes. We compare the results obtained here with a study by Ferrari *et al.* [4], who take into account in their calculations an average energy flux for the GWs emitted during the formation of black holes obtained from simulations by Stark and Piran [16]. Our results are in good agreement.

For most sources of GWs only characteristic values for the dimensionless amplitude and frequency are known; if these sources are numerous, a stochastic background of GWs could be produced. We argue that the formulation presented here could be applied to other calculations of stochastic backgrounds as well.

ACKNOWLEDGMENTS

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