Approximate binary-black-hole metric

Kashif Alvi

Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125

An approximate solution to Einstein's equations representing two widely separated non-rotating black holes in a circular orbit is constructed by matching a post-Newtonian metric to two perturbed Schwarzschild metrics. The spacetime metric is presented in a single coordinate system valid up to the apparent horizons of the black holes. This metric could be useful in numerical simulations of binary black holes. Initial data extracted from this metric have the advantages of being linked to the early inspiral phase of the binary system, and of not containing spurious gravitational waves.

PACS number(s): 04.25.Nx, 04.30.Db, 04.70.−s

I. INTRODUCTION

One of the outstanding issues in gravitational wave research is calculating the wave output from the last stages of inspiral of binary black holes. This intermediate binary black hole problem has been discussed by Brady, Creighton, and Thorne [1]. The purpose of this paper is to provide an approximate four-dimensional binary-black-hole metric from which initial data can be extracted and evolved numerically into and through the intermediate binary black hole region.

The approach I take is based on the work of Manasse [2] and D'Eath [3,4]. I consider two widely separated non-rotating black holes in a circular orbit. The black holes' mass ratio is not restricted—they can have comparable masses. However, the masses are assumed to be much smaller than the distance between them.[1] As a result spacetime can be divided into four regions, each with its own approximation scheme to solve Einstein's equations. There is a strong-gravity region near each of the black holes which is described by the Schwarzschild solution plus a perturbation due to the companion's tidal field. This perturbation is constrained to satisfy the linearized Einstein equations about the Schwarzschild metric. The companion black hole's electric-type and magnetic-type tidal fields are both taken into account in calculating the perturbation.

Outside the strong-gravity regions but within the near zone, the metric can be approximated by a post-Newtonian expansion. Further out is the radiation zone which contains outgoing gravitational waves and can be described by a post-Minkowski expansion of the metric.

There are overlap zones in this spacetime where the regions described above intersect in pairs. In the overlap zones, two different approximation schemes—one from each of the two intersecting regions—are both valid. The perturbative expansions produced by the two approximation schemes are matched in the overlap zones using the framework of matched asymptotic expansions. The post-Newtonian near-zone metric—taken from [5]—and the radiation-zone metric—taken from [6]—already match in their overlap re-

gion. In this paper, the post-Newtonian near-zone metric is matched to a perturbed Schwarzschild metric in the matching or buffer zone surrounding each black hole. This yields information on the asymptotic behavior of the Schwarzschild perturbation at large distances from the horizon, and on the coordinate transformation between the two buffer-zone coordinate systems.

The Schwarzschild perturbation and coordinate transformation are not uniquely determined. However, a different choice of transformation—and hence different form of Schwarzschild perturbation—should still represent the same physical situation. In other words, different perturbations that match to the post-Newtonian near-zone metric are expected to be related via gauge transformations. For the purposes of this paper, it is sufficient to find one transformation and one Schwarzschild perturbation associated with each black hole that result in a match between the post-Newtonian near-zone metric and the distorted-black-hole metrics.

An approximate spacetime metric is put together by joining the regional metrics at some specific 3-surfaces in the matching zones. The final 4-metric is written in a single coordinate system valid up to (but not inside) the apparent horizons of the black holes. This metric is useful not only as a source of initial data for numerical evolution, but also as a check on the early stages of such an evolution.

It has been suggested that numerical simulation of binary black holes (BBHs) should be performed in corotating coordinates [1]. For this reason the metric in final form is given in corotating coordinates. The BBH spacetime can be sliced and spatial coordinates chosen in any convenient way when extracting initial data from the metric. (Asymptotically inertial coordinates can be used, for example.)

Initial data generated by the method presented in this paper have the advantage of being connected to the early inspiral phase of the BBHs. Detailed gravitational waveforms from this early inspiral phase have already been calculated using post-Newtonian expansions. These waveforms will be easily linked to the waveforms obtained by evolving initial data extracted from the metric presented here.

Initial data from this metric have the additional advantages of not containing spurious gravitational waves and of reliably describing the physical situation of coalescing BBHs. The accuracy of this description can be improved by taking the calculation in this paper to higher orders.

---

[1]Throughout this paper I use geometrized units in which $G=c=1$.

In Sec. II, the near-zone and radiation-zone metrics are written down. In Sec. III, the first black hole's tidal deformation is calculated. In Sec. IV, the buffer-zone coordinate transformations are determined, and the distorted-black-hole metrics are written in corotating post-Newtonian coordinates. The full spacetime metric is summarized in Sec. V.

II. NEAR-ZONE AND RADIATION-ZONE METRICS

Blanchet and collaborators ([7] and references therein) and Will and Wiseman [6] have calculated in detail the near-zone and radiation-zone gravitational fields of compact binary systems. The approach taken by Will and Wiseman is particularly useful here because they use a single coordinate system—harmonic coordinates—to cover both the near zone and the radiation zone. As a result, expressions for the radiation-zone metric components taken from [6] automatically match (to some finite order) the harmonic-coordinate, post-Newtonian, near-zone metric components calculated in [5]. For this reason I work initially in harmonic coordinates (t',x',y',z') with the origin of the spatial coordinates placed at the binary system's center of mass. I use only the first post-Newtonian (1PN) metric, not the full 2.5PN metric given in [5].[2] Consistently with this, I put the black holes on Newtonian trajectories: they are taken to be in circular orbits with Keplerian orbital angular velocities. Moreover, I use the post-Newtonian metric for point-like particles; in the near zone, I ignore the black holes' internal structure. The near-zone gravitational effects of the black holes' multipole moments can in principle be computed by matching out to the near zone the tidally-distorted Schwarzschild metrics obtained in this paper. However, these effects are too small to be included in this paper; this is discussed further in Sec. IV B.

A. Binary-system parameters

Label the black holes BH1 and BH2, and let m_1 and m_2 be their respective masses. Define

$$m = m_1 + m_2, \quad \delta m = m_1 - m_2, \quad \mu = \frac{m_1 m_2}{m}. \quad (2.1)$$

Denote the harmonic-coordinate trajectories of the black holes by $x_A^j(t')$ for $A = 1,2$ and $j = 1,2,3$. In other words, $x_A^j(t')$ are the spatial coordinates at time t' of the center of attraction of the gravitational field of black hole A.

In this section, boldface letters are used to denote spatial coordinates. For example $\mathbf{x}_A = (x_A^1, x_A^2, x_A^3) = (x_A, y_A, z_A)$. The notation $\mathbf{a} \cdot \mathbf{b}$ is used for the quantity $\delta_{jk} a^j b^k$, and $|\mathbf{a}|$ is by definition $(\mathbf{a} \cdot \mathbf{a})^{1/2}$.

Denote the black holes' separation $|\mathbf{x}_1 - \mathbf{x}_2|$ by b. The circular, Newtonian trajectories of the black holes are

$$\mathbf{x}_1(t') = \frac{m_2}{m} \mathbf{b}(t'), \quad \mathbf{x}_2(t') = -\frac{m_1}{m} \mathbf{b}(t') \quad (2.2)$$

where

$$\mathbf{b}(t') = \mathbf{x}_1(t') - \mathbf{x}_2(t') = b(\cos \omega t', \sin \omega t', 0) \quad (2.3)$$

and

$$\omega = \sqrt{\frac{m}{b^3}} \quad (2.4)$$

is the Keplerian orbital angular velocity. Define

$$\epsilon = \sqrt{\frac{m}{b}}, \quad r = (x'^2 + y'^2 + z'^2)^{1/2},$$

$$r_A = |\mathbf{x}' - \mathbf{x}_A|, \quad \mathbf{n}_A = \frac{\mathbf{x}' - \mathbf{x}_A}{r_A},$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.5)$$

$$\mathbf{v}_A = \frac{d\mathbf{x}_A}{dt'}, \quad v_A = |\mathbf{v}_A|,$$

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \epsilon(-\sin \omega t', \cos \omega t', 0),$$

for $A = 1,2$. By assumption, $\epsilon \ll 1$.

B. Demarcation of four regions in the BBH spacetime

Let us first fix precisely four regions in this binary-black-hole spacetime; each of these regions will receive a metric calculated as an approximate solution to the Einstein equations. With such a partition of spacetime in mind, define the inner limits $r_1^{in} = \sqrt{m_1 b}$ and $r_2^{in} = \sqrt{m_2 b}$. These are just convenient choices for the inner limits. The important property r_1^{in} has is that both $r_1^{in}/b \to 0$ and $m_1/r_1^{in} \to 0$ as $m_1/b \to 0$. Similarly $r_2^{in}/b \to 0$ and $m_2/r_2^{in} \to 0$ as $m_2/b \to 0$. Also define the outer limit $r^{out} = \lambda_c/2\pi = b/2\epsilon$ where $\lambda_c = \pi/\omega$ is the characteristic wavelength of gravitational radiation emitted by the binary system.

Divide spacetime into four regions that are bounded by the black holes' apparent horizons and the surfaces $r_1 = r_1^{in}$, $r_2 = r_2^{in}$, and $r = r^{out}$: (i) the region $r_1 < r_1^{in}$ (but outside the apparent horizon of BH1), labeled region I; (ii) the region $r_2 < r_2^{in}$ (but outside the apparent horizon of BH2), labeled region II; (iii) the subset of the near zone specified by $r_1 > r_1^{in}$, $r_2 > r_2^{in}$, and $r < r^{out}$, labeled region III; and (iv) the region $r > r^{out}$, labeled region IV. The near zone contains region III and overlaps with regions I and II; the radiation zone corresponds to region IV. The buffer zone around black hole A contains $r_A = r_A^{in}$ and satisfies $m_A \ll r_A \ll b$. These regions of spacetime are illustrated in Fig. 1.

C. Near-zone metric in harmonic coordinates

In the near zone, the 1PN harmonic-coordinate metric with two point-like particles representing the black holes is [5]
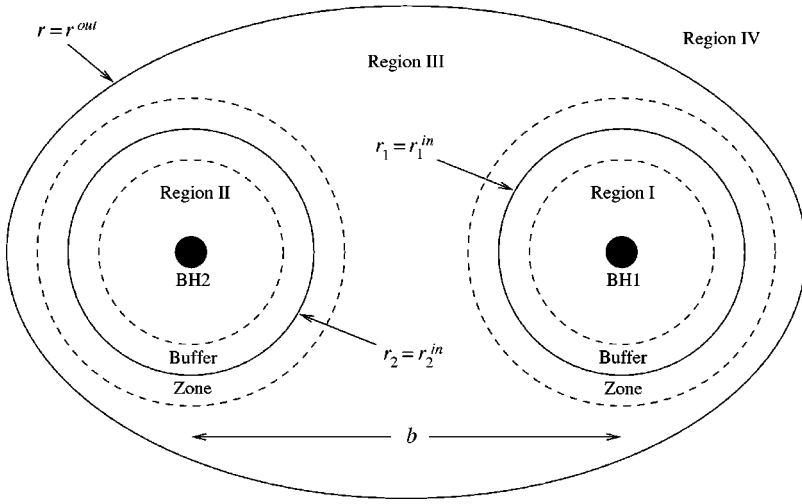

FIG. 1. Schematic illustration of the various regions in the binary-black-hole spacetime. Regions I, II, III, and IV are demarcated by solid lines; the buffer zones are bounded by dashed lines.

$$g_{0'0'} = -1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} - 2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)^2$$

$$+ \frac{m_1}{r_1}[4v_1^2 - (\mathbf{n}_1 \cdot \mathbf{v}_1)^2] + \frac{m_2}{r_2}[4v_2^2 - (\mathbf{n}_2 \cdot \mathbf{v}_2)^2]$$

$$- 2\frac{m_1 m_2}{b}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{m_1 m_2}{b^3}\mathbf{b}\cdot(\mathbf{n}_1 - \mathbf{n}_2),$$

$$g_{0'i'} = -4\left(\frac{m_1}{r_1}v_1^i + \frac{m_2}{r_2}v_2^i\right), \qquad (2.6)$$

$$g_{i'j'} = \delta_{ij}\left(1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right).$$

This metric presumably differs in the near zone by a small amount from an exact solution to the Einstein equations representing binary black holes. I take the neglected terms in the 2.5PN metric [5] to be an estimate of the errors in the 1PN metric (2.6).

The largest neglected terms in $g_{0'0'}$ are of the form $m^3/b^2 r_A$, $m^3/b r_A^2$, m_A^3/r_A^3, m^3/b^3, and $\epsilon m^3 r^2/b^5$. (The last term represents a radiation reaction potential.) Let us compute the orders of magnitude of these terms at various places in region III. If $r_A \gtrsim r_A^{in} \sim b\epsilon$ (here and henceforth "$\sim$" means "is of the order of" and $a \gtrsim b$ means $a > b$ and $a \sim b$) for $A = 1$ or 2, then the error in $g_{0'0'}$ (denoted $\delta g_{0'0'}$) is of $O(\epsilon^3)$ and comes from neglecting a term of the form

m_A^3/r_A^3. If both $r_1 \sim b$ and $r_2 \sim b$, then $\delta g_{0'0'} \sim \epsilon^6$. Finally, if $r \lesssim r^{out} \sim b/\epsilon$ (so that $r_A \sim b/\epsilon$ for $A = 1$ and 2), then the error $\delta g_{0'0'} \sim \epsilon^5$ arises from neglecting the radiation reaction potential. Note that it is reasonable to consider the "absolute" errors $\delta g_{\mu'\nu'}$ in the metric components since the coordinate system being used is asymptotically inertial and the errors are only calculated in regions of weak gravity where deviations from a flat metric are small.

A similar analysis for $g_{0'i'}$ yields $\delta g_{0'i'} \sim \epsilon^3$ if $r_A \gtrsim r_A^{in}$ for $A = 1$ or 2, $\delta g_{0'i'} \sim \epsilon^5$ if both $r_1 \sim b$ and $r_2 \sim b$, and $\delta g_{0'i'} \sim \epsilon^5$ if $r \lesssim r^{out}$. Lastly, $\delta g_{i'j'} \sim \epsilon^2$ if $r_A \gtrsim r_A^{in}$ for $A = 1$ or 2 (this comes from neglecting a term of the form m_A^2/r_A^2 in $g_{i'j'}$), $\delta g_{i'j'} \sim \epsilon^4$ if both $r_1 \sim b$ and $r_2 \sim b$, and $\delta g_{i'j'} \sim \epsilon^5$ if $r \lesssim r^{out}$.

D. Near-zone metric in corotating coordinates

The metric (2.6) is transformed to corotating coordinates (t, x, y, z) defined by

$$t' = t, \quad x' = x\cos\omega t - y\sin\omega t,$$
$$y' = x\sin\omega t + y\cos\omega t, \quad z' = z. \qquad (2.7)$$

In terms of the new coordinates,

$$r = (x^2 + y^2 + z^2)^{1/2}. \qquad (2.8)$$

Putting the expressions (2.2)–(2.5) in Eq. (2.6) and transforming to corotating coordinates gives

$$ds^2 = dt^2\left[-1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} - 2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right)^2 + \frac{3\mu}{b}\left(\frac{m_2}{r_1} + \frac{m_1}{r_2}\right) - \frac{\mu}{b}\left(\frac{m_2}{r_1^3} + \frac{m_1}{r_2^3}\right)y^2 - 2\mu\epsilon^2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) - 7\mu\epsilon^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\frac{x}{b}\right.$$

$$\left. + \omega^2\left(1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right)(x^2 + y^2)\right] + 2\omega\left(1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right)dt(xdy - ydx) - 8\mu\epsilon\left(\frac{1}{r_1} - \frac{1}{r_2}\right)dtdy + \left(1 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2}\right)$$

$$\times(dx^2 + dy^2 + dz^2), \qquad (2.9)$$

where, in terms of the new coordinates, the quantities r_A are

$$r_1 = [(x - m_2 b/m)^2 + y^2 + z^2]^{1/2},$$

$$r_2 = [(x + m_1 b/m)^2 + y^2 + z^2]^{1/2}. \tag{2.10}$$

This is the final form of the metric in region III. (Note, however, that this metric is valid throughout the near zone, which includes the buffer zones around the black holes.) It remains to specify the metric in regions I, II, and IV. I postpone until Sec. II F discussion of the errors $\delta g_{\mu\nu}$ in the metric components in the new, rotating coordinate system (t,x,y,z).

E. Radiation-zone metric in harmonic coordinates

The radiation-zone metric can be extracted from [6]. In that paper, a potential $h^{\mu'\nu'}$ is defined by

$$h^{\mu'\nu'} = \eta^{\mu'\nu'} - (-g')^{1/2} g^{\mu'\nu'}, \tag{2.11}$$

where $\eta^{\mu'\nu'} = \text{diag}(-1,1,1,1)$, $g_{\mu'\nu'}$ is the spacetime metric, and $g' = \det(g_{\mu'\nu'})$. Equation (5.5) of [6] gives $h^{\mu'\nu'}$ in the radiation zone in harmonic coordinates (t',x',y',z') for a system of several bodies. After correcting a typo in that equation,[3] I specialize to a system of two bodies of masses m_1 and m_2 in a circular orbit specified by Eqs. (2.2)–(2.5). This yields[4]

$$h^{0'0'}(t',x',y',z') = \frac{4\tilde{m}}{r} + \frac{7m^2}{r^2} + 2\left[\frac{1}{r}Q^{ij}(u')\right]_{,ij} - \frac{2}{3}\left[\frac{1}{r}Q^{ijk}(u')\right]_{,ijk},$$

$$h^{0'i'}(t',x',y',z') = -2\left\{\frac{1}{r}[\dot{Q}^{ij}(u') - \epsilon^{ijl}J^l(u')]\right\}_{,j} + \frac{2}{3}\left\{\frac{1}{r}[\dot{Q}^{ijk}(u') - 2\epsilon^{ikl}J^{lj}(u')]\right\}_{,jk}, \tag{2.12}$$

$$h^{i'j'}(t',x',y',z') = \frac{m^2}{r^2}n'^i n'^j + \frac{2}{r}\ddot{Q}^{ij}(u') - \frac{2}{3}\left\{\frac{1}{r}[\ddot{Q}^{ijk}(u') - 4\epsilon^{(i|kl|}J^{l|j)}(u')]\right\}_{,k},$$

where

$$\tilde{m} = m(1 - \mu/2b), \quad u' = t' - r, \quad \mathbf{n}' = \mathbf{x}'/r \tag{2.13}$$

and

$$Q^{ij} = \sum_{A=1}^{2} m_A x_A^i x_A^j = \mu b^i b^j, \quad Q^{ijk} = \sum_{A=1}^{2} m_A x_A^i x_A^j x_A^k = -\mu(\delta m/m)b^i b^j b^k,$$

$$J^i = \sum_{A=1}^{2} m_A \epsilon^{ilm} x_A^l v_A^m = \mu \epsilon^{ilm} b^l v^m, \quad J^{ij} = \sum_{A=1}^{2} m_A \epsilon^{ilm} x_A^l v_A^m x_A^j = -\mu(\delta m/m)\epsilon^{ilm} b^l v^m b^j. \tag{2.14}$$

Putting the expressions (2.14) in Eq. (2.12) and using Eqs. (2.2)–(2.5) gives

$$h^{0'0'} = \frac{4\tilde{m}}{r} + \frac{7m^2}{r^2} + \frac{2\mu}{r}\left\{2(\mathbf{n}'\cdot\mathbf{v})^2 - \frac{2m}{b^3}(\mathbf{n}'\cdot\mathbf{b})^2 + \frac{6}{r}(\mathbf{n}'\cdot\mathbf{b})(\mathbf{n}'\cdot\mathbf{v}) + \frac{1}{r^2}[3(\mathbf{n}'\cdot\mathbf{b})^2 - b^2]\right\} + \frac{2\mu}{r}\frac{\delta m}{m}\left\{\frac{7m}{b^3}(\mathbf{n}'\cdot\mathbf{b})^2(\mathbf{n}'\cdot\mathbf{v})\right.$$

$$\left. -2(\mathbf{n}'\cdot\mathbf{v})^3 + \frac{1}{r}(\mathbf{n}'\cdot\mathbf{b})\left[\frac{6m}{b^3}(\mathbf{n}'\cdot\mathbf{b})^2 - 12(\mathbf{n}'\cdot\mathbf{v})^2 - \frac{m}{b}\right] + \frac{3}{r^2}(\mathbf{n}'\cdot\mathbf{v})[b^2 - 5(\mathbf{n}'\cdot\mathbf{b})^2] + \frac{1}{r^3}(\mathbf{n}'\cdot\mathbf{b})[3b^2 - 5(\mathbf{n}'\cdot\mathbf{b})^2]\right\},$$

$$h^{0'i'} = \frac{4\mu}{r}\left\{\left[(\mathbf{n}'\cdot\mathbf{v}) + \frac{1}{r}(\mathbf{n}'\cdot\mathbf{b})\right]v^i - \frac{m}{b^3}(\mathbf{n}'\cdot\mathbf{b})b^i\right\} - \frac{2\mu}{r}\frac{\delta m}{m}\left\{-\frac{m}{b^3}(\mathbf{n}'\cdot\mathbf{b})[3(\mathbf{n}'\cdot\mathbf{b})v^i + 4(\mathbf{n}'\cdot\mathbf{v})b^i] + 2(\mathbf{n}'\cdot\mathbf{v})^2 v^i\right.$$

$$\left. + \frac{1}{r}\left[6(\mathbf{n}'\cdot\mathbf{b})(\mathbf{n}'\cdot\mathbf{v})v^i - \frac{3m}{b^3}(\mathbf{n}'\cdot\mathbf{b})^2 b^i + \frac{m}{b}b^i\right] + \frac{1}{r^2}[3(\mathbf{n}'\cdot\mathbf{b})^2 - b^2]v^i\right\}, \tag{2.15}$$

$$h^{i'j'} = \frac{m^2}{r^2}n'^i n'^j + \frac{4\mu}{r}\left[v^i v^j - \frac{m}{b^3}b^i b^j\right] + \frac{2\mu}{r}\frac{\delta m}{m}\left\{\frac{6m}{b^3}(\mathbf{n}'\cdot\mathbf{b})v^{(i}b^{j)} + \left[(\mathbf{n}'\cdot\mathbf{v}) + \frac{1}{r}(\mathbf{n}'\cdot\mathbf{b})\right]\left(\frac{m}{b^3}b^i b^j - 2v^i v^j\right)\right\},$$

---

[3]The term $4m/r'$ in the expression for h^{00} should instead be $4\tilde{m}/r'$.
[4]Note that I have replaced r' in Eq. (5.5) of [6] by r.

where $\mathbf{v}$ and $\mathbf{b}$ are evaluated at the retarded time $u' = t' - r$.

The metric $g_{\mu'\nu'}$ can be gotten from Eq. (2.15) as follows: from Eq. (2.11) we have

$$g^{\mu'\nu'} = (-g')^{-1/2}(\eta^{\mu'\nu'} - h^{\mu'\nu'}). \tag{2.16}$$

Take the determinant of both sides of Eq. (2.16); this yields $g' = \det(\eta^{\mu'\nu'} - h^{\mu'\nu'})$. So g' can be calculated once $h^{\mu'\nu'}$ is known, and then $g^{\mu'\nu'}$ can be gotten from Eq. (2.16). Inverting the matrix $g^{\mu'\nu'}$ gives the spacetime metric $g_{\mu'\nu'}$.

When performing these calculations, I keep all terms of the form $m^{3-p/2}b^{-3(1-p/2)}r^{-p}$ for integer $p > 0$. I also keep—at each order in r—all terms that are of lower order in m/b than this, and throw away terms of higher order in m/b. This means in particular that no terms of $O(r^{-5})$ are kept. This scheme of organizing terms is consistent with the ordering of terms in Eq. (5.5) of [6].

The result of these calculations is the following radiation-zone metric in harmonic coordinates:

$$g_{0'0'} = -1 + \frac{2\widetilde{m}}{r} - \frac{2m^2}{r^2} + \frac{\mu}{r}\left\{2(\mathbf{n'}\cdot\mathbf{v})^2 - \frac{2m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 + \frac{6}{r}(\mathbf{n'}\cdot\mathbf{b})(\mathbf{n'}\cdot\mathbf{v}) + \frac{1}{r^2}[3(\mathbf{n'}\cdot\mathbf{b})^2 - b^2]\right\}$$

$$+ \frac{\mu}{r}\frac{\delta m}{m}\left\{(\mathbf{n'}\cdot\mathbf{v})\left[\frac{7m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 - 2(\mathbf{n'}\cdot\mathbf{v})^2 - \frac{m}{b}\right] + \frac{2}{r}(\mathbf{n'}\cdot\mathbf{b})\left[\frac{3m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 - 6(\mathbf{n'}\cdot\mathbf{v})^2 - \frac{m}{b}\right]\right.$$

$$\left. + \frac{3}{r^2}(\mathbf{n'}\cdot\mathbf{v})[b^2 - 5(\mathbf{n'}\cdot\mathbf{b})^2] + \frac{1}{r^3}(\mathbf{n'}\cdot\mathbf{b})[3b^2 - 5(\mathbf{n'}\cdot\mathbf{b})^2]\right\},$$

$$g_{0'i'} = -\frac{4\mu}{r}\left\{\left[(\mathbf{n'}\cdot\mathbf{v}) + \frac{1}{r}(\mathbf{n'}\cdot\mathbf{b})\right]v^i - \frac{m}{b^3}(\mathbf{n'}\cdot\mathbf{b})b^i\right\} + \frac{2\mu}{r}\frac{\delta m}{m}\left(\left\{2(\mathbf{n'}\cdot\mathbf{v})^2 - \frac{3m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 + \frac{6}{r}(\mathbf{n'}\cdot\mathbf{b})(\mathbf{n'}\cdot\mathbf{v})\right.\right.$$

$$\left.\left. + \frac{1}{r^2}[3(\mathbf{n'}\cdot\mathbf{b})^2 - b^2]\right\}v^i + \left\{-\frac{4m}{b^3}(\mathbf{n'}\cdot\mathbf{b})(\mathbf{n'}\cdot\mathbf{v}) + \frac{m}{rb}\left[1 - \frac{3}{b^2}(\mathbf{n'}\cdot\mathbf{b})^2\right]\right\}b^i\right),$$

$$g_{i'j'} = \delta_{ij}\left(1 + \frac{2\widetilde{m}}{r} + \frac{m^2}{r^2} + \frac{\mu}{r}\left\{2(\mathbf{n'}\cdot\mathbf{v})^2 - \frac{2m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 + \frac{6}{r}(\mathbf{n'}\cdot\mathbf{b})(\mathbf{n'}\cdot\mathbf{v}) + \frac{1}{r^2}[3(\mathbf{n'}\cdot\mathbf{b})^2 - b^2]\right\}\right.$$

$$+ \frac{\mu}{r}\frac{\delta m}{m}\left\{(\mathbf{n'}\cdot\mathbf{v})\left[\frac{7m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 - 2(\mathbf{n'}\cdot\mathbf{v})^2 + \frac{m}{b}\right] + \frac{6}{r}(\mathbf{n'}\cdot\mathbf{b})\left[\frac{m}{b^3}(\mathbf{n'}\cdot\mathbf{b})^2 - 2(\mathbf{n'}\cdot\mathbf{v})^2\right] + \frac{3}{r^2}(\mathbf{n'}\cdot\mathbf{v})[b^2 - 5(\mathbf{n'}\cdot\mathbf{b})^2]\right.$$

$$\left.\left. + \frac{1}{r^3}(\mathbf{n'}\cdot\mathbf{b})[3b^2 - 5(\mathbf{n'}\cdot\mathbf{b})^2]\right\}\right) + \frac{m^2}{r^2}n'^i n'^j + \frac{4\mu}{r}\left(v^i v^j - \frac{m}{b^3}b^i b^j\right) + \frac{2\mu}{r}\frac{\delta m}{m}\left\{\frac{6m}{b^3}(\mathbf{n'}\cdot\mathbf{b})v^{(i}b^{j)}\right.$$

$$\left. + \left[(\mathbf{n'}\cdot\mathbf{v}) + \frac{1}{r}(\mathbf{n'}\cdot\mathbf{b})\right]\left(\frac{m}{b^3}b^i b^j - 2v^i v^j\right)\right\}, \tag{2.17}$$

where $\mathbf{v}$ and $\mathbf{b}$ are evaluated at the retarded time $u' = t' - r$.

The errors $\delta g_{\mu'\nu'}$ in these metric components in region IV can be estimated by computing the orders of magnitude of neglected terms, which are of the form $m^{3-p/2}b^{-3(1-p/2)}r^{-p}(m/b)^{n/2}$ for integers $p > 0$ and $n > 0$. These terms include $m^3/b^2 r$, $\epsilon m^2/r^2$, $m^2 b/r^3$, etc. This gives $\delta g_{\mu'\nu'} \sim \epsilon^7$ for $r \gtrsim r^{out} \sim b/\epsilon$, $\delta g_{\mu'\nu'} \sim \epsilon^8$ for $r \sim b/\epsilon^2 \gg r^{out}$, and $\delta g_{\mu'\nu'} \ll \epsilon^8$ for $r \gg b/\epsilon^2$.

F. Radiation-zone metric in corotating coordinates

Substituting the expressions (2.2)–(2.5) into the metric (2.17) and transforming this metric to corotating coordinates (t,x,y,z) [defined in Eq. (2.7)] gives

$$ds^2 = dt^2\left[-1 + \frac{2m}{r}\left(1 - \frac{\mu}{2b}\right) - \frac{2m^2}{r^2} + A + \frac{2\,\epsilon^2}{b}(x\cos\omega r - y\sin\omega r)B - 2\,\epsilon(x\sin\omega r + y\cos\omega r)D + \omega^2(x^2 + y^2)E\right.$$

$$\left. + \frac{\epsilon^4}{b^2}(x\cos\omega r - y\sin\omega r)^2 N + \epsilon^2(x\sin\omega r + y\cos\omega r)^2 S - \frac{12\mu\epsilon^5}{r^2 b^2}\frac{\delta m}{m}(x\cos\omega r - y\sin\omega r)^2(x\sin\omega r + y\cos\omega r)\right]$$

$$+ 2dt\left(\epsilon[\sin(\omega r)dx + \cos(\omega r)dy]B + b[\cos(\omega r)dx - \sin(\omega r)dy]D + \omega(xdy - ydx)E + \frac{\epsilon^3}{b}(x\cos\omega r - y\sin\omega r)\right.$$

$$\times[\sin(\omega r)dx + \cos(\omega r)dy]N - \epsilon b(x\sin\omega r + y\cos\omega r)[\cos(\omega r)dx - \sin(\omega r)dy]S + \frac{6\mu\epsilon^4}{r^2 b}\frac{\delta m}{m}(x\cos\omega r - y\sin\omega r)$$

$$\times\left[(x\cos 2\omega r - y\sin 2\omega r)dx - (x\sin 2\omega r + y\cos 2\omega r)dy]\right) + E(dx^2 + dy^2 + dz^2) + \frac{m^2}{r^4}(xdx + ydy + zdz)^2$$

$$+ \epsilon^2[\sin(\omega r)dx + \cos(\omega r)dy]^2 N + b^2[\cos(\omega r)dx - \sin(\omega r)dy]^2 S$$

$$+ \frac{12\mu\epsilon^3}{r^2}\frac{\delta m}{m}(x\cos\omega r - y\sin\omega r)[\cos(\omega r)dx - \sin(\omega r)dy][\sin(\omega r)dx + \cos(\omega r)dy] \tag{2.18}$$

where

$$A = \frac{\mu}{r^3}\left\{2\epsilon^2[(x\sin\omega r + y\cos\omega r)^2 - (x\cos\omega r - y\sin\omega r)^2] + \frac{6b\epsilon}{r}(x\sin\omega r + y\cos\omega r)(x\cos\omega r - y\sin\omega r)\right.$$

$$+ \frac{b^2}{r^2}[3(x\cos\omega r - y\sin\omega r)^2 - r^2]\right\} + \frac{\mu}{r^4}\frac{\delta m}{m}\left\{\frac{b^3}{r^3}(x\cos\omega r - y\sin\omega r)[3r^2 - 5(x\cos\omega r - y\sin\omega r)^2] + \frac{3b^2\epsilon}{r^2}\right.$$

$$\times(x\sin\omega r + y\cos\omega r)[r^2 - 5(x\cos\omega r - y\sin\omega r)^2] + \frac{2m}{r}(x\cos\omega r - y\sin\omega r)[3(x\cos\omega r - y\sin\omega r)^2$$

$$\left. - 6(x\sin\omega r + y\cos\omega r)^2 - r^2] + \epsilon^3(x\sin\omega r + y\cos\omega r)[7(x\cos\omega r - y\sin\omega r)^2 - 2(x\sin\omega r + y\cos\omega r)^2 - r^2]\right\},$$

$$B = -\frac{4\mu}{r^2}\left[\epsilon(x\sin\omega r + y\cos\omega r) + \frac{b}{r}(x\cos\omega r - y\sin\omega r)\right] + \frac{2\mu}{r^3}\frac{\delta m}{m}\left\{\epsilon^2[2(x\sin\omega r + y\cos\omega r)^2\right.$$

$$\left. - 3(x\cos\omega r - y\sin\omega r)^2] + \frac{6b\epsilon}{r}(x\sin\omega r + y\cos\omega r)(x\cos\omega r - y\sin\omega r) + \frac{b^2}{r^2}[3(x\cos\omega r - y\sin\omega r)^2 - r^2]\right\},$$

$$D = \frac{2\mu\epsilon^2}{r^2}\left(\frac{2}{b}(x\cos\omega r - y\sin\omega r) - \frac{1}{r}\frac{\delta m}{m}\left\{\frac{4\epsilon}{b}(x\cos\omega r - y\sin\omega r)(x\sin\omega r + y\cos\omega r)\right.\right.$$

$$\left.\left. + \frac{1}{r}[3(x\cos\omega r - y\sin\omega r)^2 - r^2]\right\}\right),$$

$$E = 1 + \frac{2m}{r}\left(1 - \frac{\mu}{2b}\right) + \frac{m^2}{r^2} + A + \frac{2\mu\epsilon^2}{r^2}\frac{\delta m}{m}\left[\frac{b}{r}(x\cos\omega r - y\sin\omega r) + \epsilon(x\sin\omega r + y\cos\omega r)\right],$$

$$N = \frac{4\mu}{r}\left\{1 - \frac{\delta m}{m}\left[\frac{\epsilon}{r}(x\sin\omega r + y\cos\omega r) + \frac{b}{r^2}(x\cos\omega r - y\sin\omega r)\right]\right\},$$

$$S = \frac{2\mu\epsilon^2}{rb}\left\{-\frac{2}{b} + \frac{1}{r}\frac{\delta m}{m}\left[\frac{\epsilon}{b}(x\sin\omega r + y\cos\omega r) + \frac{1}{r}(x\cos\omega r - y\sin\omega r)\right]\right\}, \tag{2.19}$$

and $\epsilon=(m/b)^{1/2}$, $\omega=(m/b^3)^{1/2}$. This is the final form of the metric in region IV.

It is now necessary to evaluate the errors $\delta g_{\mu\nu}$ in the metric components in the corotating coordinate system (t,x,y,z). Since this coordinate system is not asymptotically inertial, it no longer makes sense to compute absolute errors. The rotation of the coordinates introduces terms of $O(\omega r)$ and $O[(\omega r)^2]$ in g_{00} and terms of $O(\omega r)$ in g_{0i}. For this reason, I define the ''normalized'' errors $\Delta g_{00}=\delta g_{00}/(\omega r)^2$, $\Delta g_{0i}=\delta g_{0i}/\omega r$, and $\Delta g_{ij}=\delta g_{ij}$ in region IV. It follows that $\Delta g_{\mu\nu}\sim\epsilon^7$ for $r\gtrsim r^{out}\sim b/\epsilon$, $\Delta g_{\mu\nu}\sim\epsilon^8$ for $r\sim b/\epsilon^2\gg r^{out}$, and $\Delta g_{\mu\nu}\ll\epsilon^8$ for $r\gg b/\epsilon^2$.

In region III, ωr is less than 1 (and in the buffer zones, $\omega r\ll1$). So rotation of the coordinates is not important in analyzing errors in the metric (2.9) in region III. I continue to use absolute errors in that region. The errors $\delta g_{\mu\nu}$ in the metric components (2.9) in corotating coordinates in region III are the same as the errors in harmonic coordinates (see Sec. II C): (i) $\delta g_{00}\sim\epsilon^3$, $\delta g_{0i}\sim\epsilon^3$, and $\delta g_{ij}\sim\epsilon^2$ if $r_A\gtrsim r_A^{in}$ for $A=1$ or 2; (ii) $\delta g_{00}\sim\epsilon^6$, $\delta g_{0i}\sim\epsilon^5$, and $\delta g_{ij}\sim\epsilon^4$ if both $r_1\sim b$ and $r_2\sim b$; and (iii) $\delta g_{\mu\nu}\sim\epsilon^5$ if $r\lesssim r^{out}$.

Since the analysis by Will and Wiseman [6] of compact binary systems uses a single coordinate chart to cover both the near and radiation zones, the near-zone metric (2.9) automatically matches (to some finite order; see below) the radiation-zone metric (2.18) at $r=r^{out}$. The match is not perfect because I have truncated the relevant perturbative expansions at finite order. As a result, there are discontinuities in the metric components at $r=r^{out}$. The orders of magnitude of these discontinuities can be estimated as follows: first expand r_A^{-1} in powers of b/r for $r>b$ and substitute this expansion in Eq. (2.9); then expand Eq. (2.18) in powers of ωr for $r<r^{out}$; finally, compare the two. The result is that the discontinuities in $g_{\mu\nu}$, denoted $[g_{\mu\nu}]$, are $[g_{\mu\nu}]\sim\epsilon^5$ at $r=r^{out}$.

III. TIDAL DEFORMATION OF THE FIRST BLACK HOLE

The metric (2.9) is valid not only in region III but also in the buffer zones around the black holes. The next step is to match this metric to a tidally distorted black-hole metric in the buffer zone around BH1. There are two coordinate systems which overlap in the buffer zone. The first is the corotating post-Newtonian coordinate system (t,x,y,z) defined in Eq. (2.7). The second—to be called the internal coordinate system—covers the strong-gravity region near the first black hole and is valid from the black hole's apparent horizon up into (and through) the buffer zone. The internal coordinates are chosen to be isotropic coordinates (T,X,Y,Z) in which the unperturbed Schwarzschild metric is

$$ds^2=-\left(\frac{1-m_1/2R}{1+m_1/2R}\right)^2dT^2+\left(1+\frac{m_1}{2R}\right)^4(dX^2+dY^2+dZ^2),$$

(3.1)

where

$$R=(X^2+Y^2+Z^2)^{1/2}.$$

(3.2)

The region these coordinates cover will be called the internal region; it contains region I in particular.

The first step in matching the near-zone metric (2.9) to the internal metric [which is the Schwarzschild metric (3.1) plus tidal perturbations] is to write the metric (2.9) in internal coordinates. Then the near-zone metric and the internal metric are both expanded in positive powers of m_1/R and R/b in the buffer zone. Finally, corresponding terms in the two asymptotic expansions are equated. The near-zone metric determines in this way the asymptotic form of the tidal perturbations on BH1. These perturbations are further constrained to solve the linearized Einstein equations about the Schwarzschild metric (3.1) and to be finite at the horizon $R=m_1/2$.

The asymptotic form of the Schwarzschild perturbations in internal coordinates can be determined—independently of the matching procedure described above—by calculating the electric- and magnetic-type tidal fields of BH2 in the buffer zone surrounding BH1. Once this asymptotic form is known, the matching procedure can be used to constrain the coordinate transformation taking corotating coordinates (t,x,y,z) to internal coordinates (T,X,Y,Z). This is the approach to matching taken in this paper. In the next two sections, I calculate the second black hole's tidal fields and the perturbations they induce on the first black hole.

A. Tidal fields of the companion black hole

Thorne and Hartle [8] have analyzed the motion of an isolated black hole in an arbitrary surrounding spacetime. They define and discuss the black hole's local asymptotic rest frame (LARF). In the LARF of BH1, the metric can be expanded in powers of the black hole's mass m_1 as follows:[5]

$$g=g^{(0)}+m_1g^{(1)}+m_1^2g^{(2)}+\cdots.$$

(3.3)

Here the metric $g^{(0)}$ represents the external universe without BH1; the rest of the terms represent the black hole's internal gravitational field and the nonlinear interaction between internal and external fields. In this section, I will focus on the external metric $g^{(0)}$ and use it to constrain internal perturbations. Throughout this section and in the rest of this paper, boldface letters denote spacetime tensors of all ranks (including 4-vectors).

In the case of binary black holes, the external metric is simply that of a single black hole; it is the metric of the companion black hole BH2 of mass m_2. This metric must be expressed in LARF coordinates. With this goal in mind, consider first a freely-falling observer in a circular, equatorial orbit around a Kerr black hole of mass m_2. (I will later specialize to a non-rotating black hole.) The Kerr black hole represents BH2 while the observer's local Lorentz frame (same as proper reference frame) represents the LARF of BH1.

---

[5]This expansion is written in Eq. (2.5) of [8]; I have substituted m_1 for M in that equation.

The metric near the observer's world line is determined by the Kerr black hole's electric-type and magnetic-type tidal fields as seen in the observer's local Lorentz frame. These tidal fields can be evaluated by taking components of the Kerr spacetime's Weyl tensor $\mathbf{C}$ in a parallel-propagated orthonormal tetrad along the observer's world line. The vectors in this tetrad form the coordinate basis of the local Lorentz frame at the location of the geodesic orbit.

The electric-type tidal field as seen by such an observer has been calculated by Fishbone [9] and Marck [10]. Marck has computed a parallel-propagated orthonormal tetrad $(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ along arbitrary geodesics of the Kerr spacetime, with $\boldsymbol{\lambda}_0$ equal to the 4-velocity of the geodesic. He obtains the electric-type tidal field by evaluating

$$R_{0i0j} = \mathbf{C}(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_0, \boldsymbol{\lambda}_j). \tag{3.4}$$

I specialize his tetrad to circular, equatorial geodesics; I also label the tetrad vectors (and hence coordinate axes) differently. Initially (that is, at proper time $\mathcal{T}=0$), I choose $\boldsymbol{\lambda}_1$ to be radially outward (in Boyer-Lindquist coordinates); $\boldsymbol{\lambda}_2$ is chosen so that the projections of $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_2$ on a constant-Boyer-Lindquist-time-t surface are parallel; and $\boldsymbol{\lambda}_3$ is then chosen to give $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3)$ positive (i.e., right-handed) orientation. With this choice of tetrad, I obtain the magnetic-type tidal field using Marck's work by evaluating

$$R_{0ijk} = \mathbf{C}(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_k). \tag{3.5}$$

The results of the calculations (3.4) and (3.5) with the above choice of tetrad are

$$R_{0101} = \frac{m_2}{d^3}\left[1 - 3\left(1 + \frac{\mathcal{W}^2}{d^2}\right)\cos^2\bar{\Omega}\mathcal{T}\right],$$

$$R_{0202} = \frac{m_2}{d^3}\left[1 - 3\left(1 + \frac{\mathcal{W}^2}{d^2}\right)\sin^2\bar{\Omega}\mathcal{T}\right],$$

$$R_{0303} = \frac{m_2}{d^3}\left(1 + \frac{3\mathcal{W}^2}{d^2}\right),$$

$$R_{0102} = R_{0201} = -\frac{3m_2}{d^3}\left(1 + \frac{\mathcal{W}^2}{d^2}\right)\cos\bar{\Omega}\mathcal{T}\sin\bar{\Omega}\mathcal{T},$$

$$R_{0112} = -R_{0121} = R_{0323} = -R_{0332}$$
$$= \frac{3m_2\mathcal{W}}{d^4}\left(1 + \frac{\mathcal{W}^2}{d^2}\right)^{1/2}\cos\bar{\Omega}\mathcal{T},$$

$$R_{0212} = -R_{0221} = R_{0331} = -R_{0313}$$
$$= \frac{3m_2\mathcal{W}}{d^4}\left(1 + \frac{\mathcal{W}^2}{d^2}\right)^{1/2}\sin\bar{\Omega}\mathcal{T}, \tag{3.6}$$

and the rest of the Weyl-tensor components are zero. Here $\mathcal{T}$ is proper time along the geodesic, m_2 is the mass of the Kerr black hole, d is the Boyer-Lindquist radial coordinate of the circular, equatorial orbit, and

$$\bar{\Omega} = \sqrt{\frac{m_2}{d^3}} \tag{3.7}$$

is the (exact) rotation rate of the black hole's tidal field as seen in the local Lorentz frame. The quantity $\mathcal{W}$ is given by

$$\frac{\mathcal{W}}{d} = \frac{\sqrt{m_2/d} \pm a/d}{(1 - 3m_2/d \mp 2a\sqrt{m_2/d^3})^{1/2}}, \tag{3.8}$$

where $m_2 a$ is the black hole's angular momentum. The upper sign in Eq. (3.8) is for a retrograde orbit while the lower one is for a direct or prograde orbit.

Notice that for $d \gg m_2$, the electric- and magnetic-type tidal field components at $\mathcal{T}=0$ are simply related via a Lorentz boost with low velocity $(m_2/d)^{1/2}$. For example,

$$R_{0112}|_{\mathcal{T}=0} = -(m_2/d)^{1/2}[R_{0101} + R_{1212}]_{\mathcal{T}=0}$$
$$= -(m_2/d)^{1/2}[2R_{0101} + R_{0202}]_{\mathcal{T}=0} \tag{3.9}$$

to lowest order in m_2/d. This fact will be used later in this section.

In the local Lorentz frame, the spacetime metric can be written as an expansion in powers of distance $\mathcal{R}$ from the observer's geodesic world line [11]. The two types of tidal field (3.4) and (3.5) determine the metric up to and including terms of $O(\mathcal{R}^2)$. After exploiting some gauge freedom (see Sec. V A 2 of [12] and Eq. (2.7) of [8]), the metric can be written in local coordinates $(\mathcal{T}, \mathcal{X}, \mathcal{Y}, \mathcal{Z})$ as

$$g_{00} = -1 - R_{0i0j}(\mathcal{T})\mathcal{X}^i\mathcal{X}^j + O(\mathcal{R}^3),$$

$$g_{0i} = -\frac{2}{3}R_{0jik}(\mathcal{T})\mathcal{X}^j\mathcal{X}^k + O(\mathcal{R}^3), \tag{3.10}$$

$$g_{ij} = \delta_{ij}[1 - R_{0k0m}(\mathcal{T})\mathcal{X}^k\mathcal{X}^m] + O(\mathcal{R}^3),$$

where $\mathcal{R} = (\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2)^{1/2}$. Substituting the expressions (3.6) in Eq. (3.10) gives

$$g_{00} = -1 + \frac{m_2}{d^3}\left[3\left(1 + \frac{\mathcal{W}^2}{d^2}\right)(\mathcal{X}\cos\bar{\Omega}\mathcal{T} + \mathcal{Y}\sin\bar{\Omega}\mathcal{T})^2 - \mathcal{R}^2\right.$$
$$\left. - \frac{3\mathcal{W}^2}{d^2}\mathcal{Z}^2\right],$$

$$g_{0\mathcal{X}} = \frac{2m_2\mathcal{W}}{d^4}\left(1 + \frac{\mathcal{W}^2}{d^2}\right)^{1/2}[(\mathcal{Z}^2 - \mathcal{Y}^2)\sin\bar{\Omega}\mathcal{T}$$
$$- \mathcal{X}\mathcal{Y}\cos\bar{\Omega}\mathcal{T}],$$

$$g_{0\mathcal{Y}} = \frac{2m_2\mathcal{W}}{d^4}\left(1 + \frac{\mathcal{W}^2}{d^2}\right)^{1/2}[(\mathcal{X}^2 - \mathcal{Z}^2)\cos\bar{\Omega}\mathcal{T}$$
$$+ \mathcal{X}\mathcal{Y}\sin\bar{\Omega}\mathcal{T}],$$

$$g_{0\mathcal{Z}} = \frac{2m_2\mathcal{W}}{d^4}\left(1+\frac{\mathcal{W}^2}{d^2}\right)^{1/2}(\mathcal{Y}\cos\bar{\Omega}\mathcal{T}-\mathcal{X}\sin\bar{\Omega}\mathcal{T})\mathcal{Z},$$

$$g_{ij}=\delta_{ij}\left\{1+\frac{m_2}{d^3}\left[3\left(1+\frac{\mathcal{W}^2}{d^2}\right)(\mathcal{X}\cos\bar{\Omega}\mathcal{T}+\mathcal{Y}\sin\bar{\Omega}\mathcal{T})^2\right.\right.$$
$$\left.\left.-\mathcal{R}^2-\frac{3\mathcal{W}^2}{d^2}\mathcal{Z}^2\right]\right\} \quad (3.11)$$

up to and including terms of $O(\mathcal{R}^2)$.

The rotation rate $\bar{\Omega}$ is only correct for test-particle orbits and is exact in that case. The correct rotation rate Ω of the second black hole's tidal field—measured in a local inertial frame in the first black hole's LARF—is actually determined by the post-Newtonian metric (2.9) and by the requirements that (i) this metric match the LARF metric [given in Eq. (3.24) in Sec. III B below]; and (ii) the LARF coordinate system be non-rotating relative to local inertial frames. The rotation rate Ω is calculated in Sec. IV A by transforming the metric (2.9) to internal coordinates and requiring a match to the LARF metric (3.24). There it will be seen that the rotation rate is[6]

$$\Omega=\omega\left[1-\frac{\mu}{b}+O(\epsilon^3)\right]. \quad (3.12)$$

Note that post-Newtonian corrections to the orbital angular velocity ω of $O(\epsilon^2\omega)$ have not been included in this paper.

The metric (3.11) is valid for all radii d which allow a circular, equatorial, geodesic orbit. To apply Eq. (3.11) to the situation of widely-separated, non-rotating, binary black holes, I specialize to a Schwarzschild black hole by setting $a=0$ and take the limit of small m_2/d, keeping only lowest-order terms in m_2/d. [In particular, I replace $\mathcal{W}/d$ with $(m_2/d)^{1/2}$.] I then replace d with b, $\bar{\Omega}$ with Ω, and local coordinates $(\mathcal{T},\mathcal{X},\mathcal{Y},\mathcal{Z})$ with internal coordinates (T,X,Y,Z) [which are described above Eq. (3.1)]. The result is

$$g_{00}=-1+\frac{m_2}{b^3}[3(X\cos\Omega T+Y\sin\Omega T)^2-R^2],$$

$$g_{0X}=\frac{2m_2}{b^3}\sqrt{\frac{m_2}{b}}[(Z^2-Y^2)\sin\Omega T-XY\cos\Omega T],$$

$$g_{0Y}=\frac{2m_2}{b^3}\sqrt{\frac{m_2}{b}}[(X^2-Z^2)\cos\Omega T+XY\sin\Omega T],$$

$$g_{0Z}=\frac{2m_2}{b^3}\sqrt{\frac{m_2}{b}}(Y\cos\Omega T-X\sin\Omega T)Z,$$

$$g_{ij}=\delta_{ij}\left\{1+\frac{m_2}{b^3}[3(X\cos\Omega T+Y\sin\Omega T)^2-R^2]\right\}, \quad (3.13)$$

where $R=(X^2+Y^2+Z^2)^{1/2}$ as defined in Eq. (3.2).

The metric (3.13) is still not applicable to binary black holes since the observer was taken to be massless [there are no factors of m_1 in Eq. (3.13)]. This can be fixed easily. As mentioned above, the factors of $(m_2/b)^{1/2}$ in g_{0i} in Eq. (3.13) arise from a Lorentz boost with low velocity $(m_2/b)^{1/2}$. But the correct (Newtonian) relative velocity between the black holes is $\epsilon=[(m_1+m_2)/b]^{1/2}$. So I replace the factors of $(m_2/b)^{1/2}$ in Eq. (3.13) by $(m/b)^{1/2}$. The resulting metric includes the second black hole's tidal fields but does not include the first black hole's gravitational field:

$$g_{00}=-1+\frac{m_2}{b^3}[3(X\cos\Omega T+Y\sin\Omega T)^2-R^2],$$

$$g_{0X}=\frac{2m_2}{b^3}\sqrt{\frac{m}{b}}[(Z^2-Y^2)\sin\Omega T-XY\cos\Omega T],$$

$$g_{0Y}=\frac{2m_2}{b^3}\sqrt{\frac{m}{b}}[(X^2-Z^2)\cos\Omega T+XY\sin\Omega T],$$

$$g_{0Z}=\frac{2m_2}{b^3}\sqrt{\frac{m}{b}}(Y\cos\Omega T-X\sin\Omega T)Z,$$

$$g_{ij}=\delta_{ij}\left\{1+\frac{m_2}{b^3}[3(X\cos\Omega T+Y\sin\Omega T)^2-R^2]\right\}. \quad (3.14)$$

In the buffer zone around BH1, this metric provides the asymptotic form of the perturbation on BH1.

B. Schwarzschild perturbation

The next stage is to solve the linearized Einstein equations (LEEs) about the Schwarzschild metric for a perturbation which is finite at the horizon $R=m_1/2$ and asymptotes to the form (3.14) as $R/m_1\to\infty$. For ease in dealing with the LEEs, I transform to spherical, isotropic, internal coordinates (T,R,θ,ϕ) by letting

$$X=R\sin\theta\cos\phi, \quad Y=R\sin\theta\sin\phi, \quad Z=R\cos\theta. \quad (3.15)$$

The unperturbed Schwarzschild metric in these coordinates is

$$ds^2=-\left(\frac{1-m_1/2R}{1+m_1/2R}\right)^2 dT^2+\left(1+\frac{m_1}{2R}\right)^4\left[dR^2+R^2\right.$$
$$\left.\times(d\theta^2+\sin^2\theta d\phi^2)\right]. \quad (3.16)$$

The metric (3.14) in these coordinates is

---

[6]This rotation rate can also be calculated by looking at geodetic precession of parallel-propagated vectors in the LARF [13].

$$ds^2 = -dT^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$-4\frac{m_2 R^3}{b^3}\sqrt{\frac{m}{b}}dT[\cos\theta\sin(\phi-\Omega T)d\theta$$

$$+\sin\theta\cos(2\theta)\cos(\phi-\Omega T)d\phi]$$

$$+\frac{m_2 R^2}{b^3}[3\sin^2\theta\cos^2(\phi-\Omega T)-1][dT^2+dR^2$$

$$+R^2(d\theta^2+\sin^2\theta d\phi^2)]. \tag{3.17}$$

The linearity of the LEEs allows me to look separately for solutions corresponding to the electric-type and magnetic-type tidal fields. First I look for a perturbation $\mathbf{h}_1 = \mathbf{g} - \mathbf{g}_s$ (where $\mathbf{g}_s$ is the unperturbed Schwarzschild metric and $\mathbf{g}$ is the full metric including the perturbation) of BH1 which corresponds to the electric-type tidal field of BH2 and is of the form

$$\mathbf{h}_1 = \frac{m_2 R^2}{b^3}[3\sin^2\theta\cos^2(\phi-\Omega T)-1][f_1(R)dT^2$$

$$+f_2(R)dR^2+f_3(R)R^2(d\theta^2+\sin^2\theta d\phi^2)], \quad (3.18)$$

as suggested by Eq. (3.17). In this notation, dT, dR, $d\theta$, and $d\phi$ are coordinate one-forms and dT^2 denotes the tensor product $dT \otimes dT$. The functions f_1, f_2, and f_3 are to be determined by solving the LEEs with the following boundary conditions: (i) $f_1(R)$, $f_2(R)$, and $f_3(R)$ are required to approach 1 as $R/m_1 \to \infty$ so that the perturbation (3.18) matches the electric-type tidal field in Eq. (3.17); and (ii) $\mathbf{h}_1$ is required to be finite at $R = m_1/2$.

Consider solving the LEEs order by order in $\epsilon = (m/b)^{1/2}$. Time derivatives of the components of $\mathbf{h}_1$ produce factors of $m_1\Omega \sim \epsilon^3$ in the LEEs and can thus be neglected. A solution for $\mathbf{h}_1$ can then be found using the Regge-Wheeler formalism [14] for analysis of stationary Schwarzschild perturbations. Regge and Wheeler decompose perturbations into even- and odd-parity modes and analyze them in a particular gauge chosen to simplify computations. In their classification $\mathbf{h}_1$ is a superposition of static[7] even-parity modes with angular numbers $l=2$ and $m=-2,0,2$. The general solution of the LEEs for static even-parity modes with $l \geq 2$ is well-known in Schwarzschild coordinates and is given in Sec. IV of [15], for example. A particular solution with $l=2$ that is finite at the black hole's horizon and contains an arbitrary multiplicative constant is easily obtained from the general solution, and is given in Eqs. (6.5) and (6.7) of [3], for example.[8] After transforming this solution to isotropic coordinates, the multiplicative constant is determined by imposing the boundary condition (i) (given at

the end of the previous paragraph). This yields the following solution for the radial factors $f_1(R)$, $f_2(R)$, and $f_3(R)$ in isotropic coordinates:

$$f_1(R) = \left(1-\frac{m_1}{2R}\right)^4,$$

$$f_2(R) = \left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^6, \tag{3.19}$$

$$f_3(R) = \left(1+\frac{m_1}{2R}\right)^4\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}\right].$$

Next I look for a perturbation $\mathbf{h}_2 = \mathbf{g} - \mathbf{g}_s$ of BH1 corresponding to the magnetic-type tidal field of BH2 and of the form

$$\mathbf{h}_2 = -\frac{4m_2}{b^3}\sqrt{\frac{m}{b}}R^3 F(R)dT[\cos\theta\sin(\phi-\Omega T)d\theta$$

$$+\sin\theta\cos 2\theta\cos(\phi-\Omega T)d\phi], \tag{3.20}$$

as suggested by Eq. (3.17). The function F is to be determined by solving the LEEs with the following boundary conditions: (i) $F(R) \to 1$ as $R/m_1 \to \infty$ so that the perturbation (3.20) matches the magnetic-type tidal field in Eq. (3.17); and (ii) $\mathbf{h}_2$ finite at $R = m_1/2$. As was done for $\mathbf{h}_1$, time dependence is ignored in $\mathbf{h}_2$ since time derivatives produce higher-order terms. In the Regge-Wheeler classification, $\mathbf{h}_2$ is a superposition of stationary odd-parity modes with angular numbers $l=2$ and $m=-1,1$. The general solution of the LEEs for stationary odd-parity modes that are finite at the horizon and have $l \geq 2$ is given in Schwarzschild coordinates in Eq. (38) of [14]. This solution is only determined up to a multiplicative constant. The particular case $l=2$ is easily obtained from the general solution, and is given in Eq. (6.10) of [3], for example.[8] After transforming this solution to isotropic coordinates, the multiplicative constant is determined by imposing the boundary condition (i) [given below Eq. (3.20)]. This yields the following solution for the radial factor $F(R)$ in isotropic coordinates:

$$F(R) = \left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4. \tag{3.21}$$

The metric in the internal region near BH1 is now complete. It is given by the Schwarzschild metric (3.16) plus the perturbations (3.18) and (3.20) with radial factors given in Eqs. (3.19) and (3.21); in other words, $\mathbf{g} = \mathbf{g}_s + \mathbf{h}_1 + \mathbf{h}_2$. In spherical isotropic coordinates (T,R,θ,ϕ), this internal metric is

---

[7]Time dependence in Eq. (3.18) is to be ignored, as explained above.

[8]The notation in Sec. VI of [3] may be confusing: R there denotes a dimensionless quantity obtained from the Schwarzschild radial coordinate r_s by $R = r_s/M$ where M—in my notation m_1—is the mass of the black hole being perturbed.

$$ds^2 = -\left(\frac{1-m_1/2R}{1+m_1/2R}\right)^2 dT^2 + \left(1+\frac{m_1}{2R}\right)^4 [dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)] - \frac{4m_2}{b^3}\sqrt{\frac{m}{b}}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4 R^3 dT$$

$$\times [\cos\theta\sin(\phi-\Omega T)d\theta + \sin\theta\cos(2\theta)\cos(\phi-\Omega T)d\phi] + \frac{m_2 R^2}{b^3}[3\sin^2\theta\cos^2(\phi-\Omega T)-1]\left\{\left(1-\frac{m_1}{2R}\right)^4 dT^2\right.$$

$$\left.+\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^6 dR^2 + \left(1+\frac{m_1}{2R}\right)^4\left[\left(1+\frac{m_1}{2R}\right)^4 - \frac{2m_1^2}{R^2}\right]R^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\}. \tag{3.22}$$

In isotropic coordinates (T,X,Y,Z), this metric is

$$g_{00} = -\left(\frac{1-m_1/2R}{1+m_1/2R}\right)^2 + \frac{m_2}{b^3}\left(1-\frac{m_1}{2R}\right)^4 [3(X\cos\Omega T + Y\sin\Omega T)^2 - R^2],$$

$$g_{0X} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4 [(Z^2-Y^2)\sin\Omega T - XY\cos\Omega T],$$

$$g_{0Y} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4 [(X^2-Z^2)\cos\Omega T + XY\sin\Omega T], \tag{3.23}$$

$$g_{0Z} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4 (Y\cos\Omega T - X\sin\Omega T)Z,$$

$$g_{ij} = \left(1+\frac{m_1}{2R}\right)^4\left(\delta_{ij} + \frac{m_2}{b^3}[3(X\cos\Omega T + Y\sin\Omega T)^2 - R^2]\left\{\left[\left(1+\frac{m_1}{2R}\right)^4 - \frac{2m_1^2}{R^2}\right]\delta_{ij}\right.\right.$$

$$\left.\left.- \frac{2m_1}{R}\left(1+\frac{m_1^2}{4R^2}\right)\frac{X^i X^j}{R^2}\right\}\right).$$

Expanding the components (3.23) in positive powers of m_1/R and R/b in the buffer zone $m_1 \ll R \ll b$ and keeping only lowest-order terms yields the local-asymptotic-rest-frame metric:

$$g_{00} = -1 + \frac{2m_1}{R} + \frac{m_2}{b^3}[3(X\cos\Omega T + Y\sin\Omega T)^2 - R^2],$$

$$g_{0X} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}[(Z^2-Y^2)\sin\Omega T - XY\cos\Omega T],$$

$$g_{0Y} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}[(X^2-Z^2)\cos\Omega T + XY\sin\Omega T], \tag{3.24}$$

$$g_{0Z} = \frac{2m_2}{b^3}\sqrt{\frac{m}{b}}(Y\cos\Omega T - X\sin\Omega T)Z,$$

$$g_{ij} = \delta_{ij}\left\{1 + \frac{2m_1}{R} + \frac{m_2}{b^3}[3(X\cos\Omega T + Y\sin\Omega T)^2 - R^2]\right\}.$$

This metric includes the first black hole's (weak) gravitational field as well as the second black hole's tidal fields.

IV. DISTORTED-BLACK-HOLE METRICS IN COROTATING COORDINATES

The post-Newtonian metric (2.9), when expressed in internal coordinates (T,X,Y,Z) in the buffer zone around BH1, must take the form (3.24). The next step is to find explicitly the coordinate transformation in the buffer zone taking corotating post-Newtonian coordinates to these internal coordinates. Applying the inverse of this transformation to the internal metric (3.23) will put that metric in corotating coordinates (t,x,y,z). An identical procedure will then be followed to obtain the metric near BH2 in corotating coordinates.

A. Buffer-zone coordinate transformation

In this section, a series of coordinate transformations are performed on the metric (2.9) in the buffer zone of BH1 to bring it to the form (3.24). Composing these transformations gives the final transformation from corotating to internal coordinates. Throughout this process terms of $O(m^2)$ are dropped; justification for this will be given at the end of the section.

Begin with the near-zone metric (2.9) with terms of $O(m^2)$ removed. Restrict attention to the buffer zone $m_1 \ll r_1 \ll b$ since this is where the corotating coordinate system and internal coordinate system overlap. Center the coordi-

nate grid on BH1 by shifting the origin to (x,y,z) $=(m_2b/m,0,0)$. This is done by defining a new coordinate

$$\xi = x - \frac{m_2 b}{m}. \tag{4.1}$$

Next expand the metric in powers of the distance

$$r_1 = (\xi^2 + y^2 + z^2)^{1/2} \tag{4.2}$$

to the new origin. The expansion for r_2^{-1} is

$$\frac{1}{r_2} = [(b+\xi)^2 + y^2 + z^2]^{-1/2} = \frac{1}{b} - \frac{\xi}{b^2} + \frac{2\xi^2 - y^2 - z^2}{2b^3} + \cdots. \tag{4.3}$$

Positive powers of r_1 in the metric components come in the form $(r_1/b)^p$ with integer $p>0$. Since $r_1 \ll b$ in the buffer zone, discard terms of $O[(r_1/b)^3]$ or higher. This results in the following metric:

$$\begin{aligned}
ds^2 = dt^2[&-1 + 2m_1/r_1 + (2m_2/b)(1+m_2/2m) + (m_2/b^3)(2\xi^2 - y^2 - z^2) + (m/b^3)(\xi^2+y^2)] - 2\omega y\,dt\,d\xi[1+2m_2/b \\
&+ 2m_1/r_1 - 2m_2\xi/b^2] + 2\omega dt\,dy\{(m_2/m)(b+4m_1+2m_2) - 2\mu b/r_1 + \xi(1+2m_1/r_1 - 2\mu/b) + (m_2/b^2)[2m_1\xi^2/m \\
&- (1+m_1/m)(y^2+z^2)]\} + (d\xi^2 + dy^2 + dz^2)[1 + 2m_1/r_1 + 2m_2/b - 2m_2\xi/b^2 + (m_2/b^3)(2\xi^2 - y^2 - z^2)]. \tag{4.4}
\end{aligned}$$

Now renormalize the time-coordinate by defining

$$t = \grave{t}[1 + (m_2/b)(1+m_2/2m)], \tag{4.5}$$

and then perform a partial Lorentz transformation by setting

$$\begin{aligned}
\grave{t} &= \tilde{t} + (m_2\omega/m)(b+4m_1+3m_2+m_2^2/2m)\tilde{y}, \\
\xi &= \tilde{x}, \quad y = \tilde{y}, \quad z = \tilde{z}.
\end{aligned} \tag{4.6}$$

In the new coordinates $(\tilde{t},\tilde{x},\tilde{y},\tilde{z})$, the metric (4.4) is

$$\begin{aligned}
ds^2 = d\tilde{t}^2[&-1 + 2m_1/\tilde{r} + (m_2/b^3)(3\tilde{x}^2 - \tilde{r}^2) + (m/b^3)(\tilde{x}^2+\tilde{y}^2)] - 2\omega d\tilde{t}\,d\tilde{x}\{\tilde{y}[1+(m_2/2mb)(6m_1+7m_2)+2m_1/\tilde{r}] \\
&- 2m_2\tilde{x}\tilde{y}/b^2\} + 2\omega d\tilde{t}\,d\tilde{y}\{\tilde{x}[1+(m_2/2mb)(-2m_1+3m_2)+2m_1/\tilde{r}] + (m_2/b^2)(3\tilde{x}^2 - \tilde{y}^2 - 2\tilde{z}^2)\} + (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2) \\
&\times [1 + 2m_1/\tilde{r} + 2m_2/b - 2m_2\tilde{x}/b^2 + (m_2/b^3)(3\tilde{x}^2 - \tilde{r}^2)] + (m_2/b)d\tilde{y}[(m_2/m + 2\tilde{x}/b)d\tilde{y} - (2\tilde{y}/b)d\tilde{x}] \tag{4.7}
\end{aligned}$$

where $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{1/2}$ and terms of $O(m^2)$ have been dropped, as is done throughout this section.

Next clean up the spatial part of the metric by putting

$$\begin{aligned}
\tilde{t} &= \hat{t}, \quad \tilde{x} = \hat{x}(1-m_2/b) + (m_2/2b^2)(\hat{x}^2 + \hat{y}^2 - \hat{z}^2), \\
\tilde{y} &= \hat{y}[1 - (m_2/2mb)(2m_1+3m_2)], \quad \tilde{z} = \hat{z}(1-m_2/b) + (m_2/b^2)\hat{x}\hat{z}. \tag{4.8}
\end{aligned}$$

Transforming the metric (4.7) using Eq. (4.8) results in

$$\begin{aligned}
ds^2 = d\hat{t}^2[&-1 + 2m_1/\hat{r} + (m_2/b^3)(3\hat{x}^2 - \hat{r}^2) + (m/b^3)(\hat{x}^2+\hat{y}^2)] - 2\omega d\hat{t}\,d\hat{x}[\hat{y}(1+m_2/b+2m_1/\hat{r}) - m_2\hat{x}\hat{y}/b^2] \\
&+ 2\omega d\hat{t}\,d\hat{y}\{\hat{x}[1-(m_2/mb)(3m_1+m_2)+2m_1/\hat{r}] + (m_2/2b^2)(7\hat{x}^2 - 3\hat{y}^2 - 5\hat{z}^2)\} + 2(m_2\omega/b^2)\hat{y}\hat{z}d\hat{t}\,d\hat{z} \\
&+ (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)[1 + 2m_1/\hat{r} + (m_2/b^3)(3\hat{x}^2 - \hat{r}^2)] \tag{4.9}
\end{aligned}$$

where $\hat{r} = (\hat{x}^2 + \hat{y}^2 + \hat{z}^2)^{1/2}$.

Focus attention on the terms

$$\begin{aligned}
2\omega d\hat{t}\{\hat{x}d\hat{y}[1 - (m_2/mb)(3m_1+m_2) \\
+ 2m_1/\hat{r}] - \hat{y}d\hat{x}(1+m_2/b+2m_1/\hat{r})\}
\end{aligned}$$

in Eq. (4.9). These terms contain information about the rotation of the coordinate axes. However, they are not yet in the form of the rotation terms $2\Omega(1+2m_1/\hat{r})d\hat{t}(\hat{x}d\hat{y} - \hat{y}d\hat{x})$ that result from rotating—at a constant rate Ω and in an active sense, i.e., using a pull-back map—the metric $ds^2 = d\hat{t}^2(-1+2m_1/\hat{r}) + (d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2)(1+2m_1/\hat{r})$, which

is a fragment of Eq. (4.9). An additional coordinate transformation is required to bring the former terms into the latter form. With this goal in mind, look first for a gauge transformation taking the perturbation

$$\gamma = 2\omega d\hat{t}\{\hat{x}d\hat{y}[1-(m_2/mb)(3m_1+m_2)+2m_1/\hat{r}]$$
$$-\hat{y}d\hat{x}(1+m_2/b+2m_1/\hat{r})\} \tag{4.10}$$

on a flat background metric $ds^2=-d\hat{t}^2+d\hat{x}^2+d\hat{y}^2+d\hat{z}^2$ to the perturbation

$$\check{\gamma}=2\Omega(1+2m_1/\hat{r})d\hat{t}(\hat{x}d\hat{y}-\hat{y}d\hat{x}). \tag{4.11}$$

In other words, look for a vector field $\boldsymbol{\eta}$ such that

$$\check{\gamma}_{\hat{\mu}\hat{\nu}}=\gamma_{\hat{\mu}\hat{\nu}}-2\partial_{(\hat{\mu}}\eta_{\hat{\nu})}. \tag{4.12}$$

In order to solve Eq. (4.12), it suffices to consider $\boldsymbol{\eta}$ with only one nonzero component $\eta^{\hat{t}}=\eta^{\hat{t}}(\hat{x},\hat{y},\hat{z})$. The perturbations (4.10) and (4.11) when put in Eq. (4.12) yield

$$-\Omega\hat{y}(1+2m_1/\hat{r})=\check{\gamma}_{\hat{t}\hat{x}}=\gamma_{\hat{t}\hat{x}}+\partial\eta^{\hat{t}}/\partial\hat{x}$$
$$=-\omega\hat{y}(1+m_2/b+2m_1/\hat{r})+\partial\eta^{\hat{t}}/\partial\hat{x}, \tag{4.13}$$

$$\Omega\hat{x}(1+2m_1/\hat{r})=\check{\gamma}_{\hat{t}\hat{y}}=\gamma_{\hat{t}\hat{y}}+\partial\eta^{\hat{t}}/\partial\hat{y}$$
$$=\omega\hat{x}[1-(m_2/mb)(3m_1+m_2)$$
$$+2m_1/\hat{r}]+\partial\eta^{\hat{t}}/\partial\hat{y}. \tag{4.14}$$

These two equations determine the rotation rate Ω as follows: the function $\eta^{\hat{t}}(\hat{x},\hat{y},\hat{z})$ must satisfy $\partial^2\eta^{\hat{t}}/\partial\hat{x}\partial\hat{y}=\partial^2\eta^{\hat{t}}/\partial\hat{y}\partial\hat{x}$. Taking $\partial/\partial\hat{y}$ of Eq. (4.13) and $\partial/\partial\hat{x}$ of Eq. (4.14), equating the mixed partials of $\eta^{\hat{t}}$, and ignoring terms of $O(m^2)$ yields the following equation for Ω:

$$\Omega-\omega[1-(m_2/b)(1+2m_1/m)]=\omega(1+m_2/b)-\Omega, \tag{4.15}$$

which has solution $\Omega=\omega[1-\mu/b+O(\epsilon^3)]$. This is the rotation rate of the second black hole's tidal field as seen in the first black hole's local asymptotic rest frame (LARF); this value confirms the claim in Sec. III A [see Eq. (3.12)]. With Ω in hand, Eqs. (4.13) and (4.14) now yield

$$\eta^{\hat{t}}=(m_2\omega/b)(1+m_1/m)\hat{x}\hat{y}. \tag{4.16}$$

Gauge transformations can also be thought of as resulting from infinitesimal coordinate transformations. The coordi-

nate transformation corresponding to the gauge transformation given in Eqs. (4.12) and (4.16) is

$$\hat{t}=\check{t}-(m_2\omega/b)(1+m_1/m)\check{x}\check{y}, \tag{4.17}$$

$$\hat{x}=\check{x}, \quad \hat{y}=\check{y}, \quad \hat{z}=\check{z}.$$

The metric (4.9) expressed in the new coordinates $(\check{t},\check{x},\check{y},\check{z})$ is

$$ds^2=d\check{t}^2[-1+2m_1/\check{r}+(m_2/b^3)(3\check{x}^2-\check{r}^2)+(m/b^3)$$
$$\times(\check{x}^2+\check{y}^2)]+2\Omega(1+2m_1/\check{r})d\check{t}(\check{x}d\check{y}-\check{y}d\check{x})$$
$$+(m_2\omega/b^2)d\check{t}[2\check{x}\check{y}d\check{x}+(7\check{x}^2-3\check{y}^2-5\check{z}^2)d\check{y}$$
$$+2\check{y}\check{z}d\check{z}]+(d\check{x}^2+d\check{y}^2+d\check{z}^2)[1+2m_1/\check{r}$$
$$+(m_2/b^3)(3\check{x}^2-\check{r}^2)], \tag{4.18}$$

where $\check{r}=(\check{x}^2+\check{y}^2+\check{z}^2)^{1/2}$.

The next step is to undo the rotation of the coordinate system. But first some fine-adjustment of coordinates is needed in order to obtain the LARF metric (3.24). To find out what is required, the metric (3.24) can be put in coordinates rotating with angular velocity Ω. It turns out that the fine-adjustment needed is

$$\check{t}=\bar{t}+(m_2\omega/2b^2)(3\bar{x}^2-\bar{y}^2-\bar{z}^2)\bar{y}, \tag{4.19}$$

$$\check{x}=\bar{x}, \quad \check{y}=\bar{y}, \quad \check{z}=\bar{z}.$$

In the new coordinates $(\bar{t},\bar{x},\bar{y},\bar{z})$, the metric (4.18) is

$$ds^2=d\bar{t}^2[-1+2m_1/\bar{r}+(m_2/b^3)(3\bar{x}^2-\bar{r}^2)$$
$$+(m/b^3)(\bar{x}^2+\bar{y}^2)]+2\Omega(1+2m_1/\bar{r})d\bar{t}(\bar{x}d\bar{y}-\bar{y}d\bar{x})$$
$$+4(m_2\omega/b^2)d\bar{t}[-\bar{x}\bar{y}d\bar{x}+(\bar{x}^2-\bar{z}^2)d\bar{y}+\bar{y}\bar{z}d\bar{z}]$$
$$+(d\bar{x}^2+d\bar{y}^2+d\bar{z}^2)[1+2m_1/\bar{r}+(m_2/b^3)(3\bar{x}^2-\bar{r}^2)], \tag{4.20}$$

where $\bar{r}=(\bar{x}^2+\bar{y}^2+\bar{z}^2)^{1/2}$. Now eliminate the rotation of coordinates by defining

$$\bar{t}=T, \quad \bar{x}=X\cos\Omega T+Y\sin\Omega T, \tag{4.21}$$

$$\bar{y}=-X\sin\Omega T+Y\cos\Omega T, \quad \bar{z}=Z.$$

Transforming the metric (4.20) using Eq. (4.21) results in the LARF metric (3.24).

The transformation from corotating post-Newtonian coordinates (t,x,y,z) to isotropic internal coordinates (T,X,Y,Z) can now be obtained by composing the transformations (4.1), (4.5), (4.6), (4.8), (4.17), (4.19), and (4.21). Inverting this composite map gives the following transformation from internal to corotating coordinates:

$$T=t\left[1-\frac{m_2}{b}\left(1+\frac{m_2}{2m}\right)\right]$$
$$-y\left[\frac{m_2}{\sqrt{mb}}+\frac{m_2}{b}\sqrt{\frac{m}{b}}\left(3+\frac{m_1}{m}+\frac{m_2^2}{2m^2}\right)\right]$$
$$+\frac{m_2 y}{b^2}\sqrt{\frac{m}{b}}\left[\left(1+\frac{m_1}{m}\right)\xi-\frac{1}{2b}(3\xi^2-y^2-z^2)\right],$$

$$X=\Gamma\cos\Omega T-\Lambda\sin\Omega T,$$

$$Y=\Gamma\sin\Omega T+\Lambda\cos\Omega T,$$

$$(4.22)$$

$$Z=z\left(1+\frac{m_2}{b}-\frac{m_2\xi}{b^2}\right),$$

where

$$\xi=x-\frac{m_2 b}{m},\quad \Gamma=\xi\left(1+\frac{m_2}{b}\right)-\frac{m_2}{2b^2}(\xi^2+y^2-z^2),$$

$$(4.23)$$

$$\Lambda=y\left[1+\frac{m_2}{b}\left(1+\frac{m_2}{2m}\right)\right],\quad \Omega=\omega\left(1-\frac{\mu}{b}\right),$$

and terms of $O(m^2)$ have been dropped. In terms of the coordinates (x,y,z),

$$R=(\Gamma^2+\Lambda^2+Z^2)^{1/2} \qquad (4.24)$$

[cf. Eq. (3.2)].

There are two reasons why terms of $O(m^2)$ were dropped from the metric (2.9) at the beginning of this section. First, suppose that such terms were kept and were used to calculate higher-order deformation of the black hole. Since internal metric components are coupled to each other via the Einstein equations (in particular, the components of a black hole perturbation are coupled via the linearized Einstein equations), to be fully consistent, terms of $O(m^2)$ would have to be included in the spatial part g_{ij} of the metric (2.9). But these terms are of higher order than first post-Newtonian, and so have not been included in this paper.

Second, black-hole perturbations with asymptotic form $m^2 r_1^p/b^{p+2}$ $(p\geq2)$ in the buffer zone, which come from terms of $O(m^2)$ in g_{00} in Eq. (2.9), are actually smaller in the internal region than the perturbation with asymptotic form $m_2 r_1^3/b^4$ in the buffer zone; the latter perturbation has been ignored in this paper. Once terms of $O(m^2)$ were dropped in Eq. (2.9), all terms of $O(m^2)$ were consistently discarded in this section.

B. Internal metric in corotating coordinates

In this section, the transformation (4.22) is applied to the internal metric (3.23) throughout region I (not just in the buffer zone). This puts the internal metric in corotating post-Newtonian coordinates (t,x,y,z). In order to preserve finiteness of the perturbations (3.18), (3.19) and (3.20), (3.21) at the horizon of BH1, all terms must be kept when performing the transformation. The rotation in Eq. (4.22) can easily be performed on the metric (3.22) by first defining $\varphi=\phi-\Omega T$ and then setting $\Gamma=R\sin\theta\cos\varphi$, $\Lambda=R\sin\theta\sin\varphi$, and $Z=R\cos\theta$. To complete the transformation (4.22), define the functions $P_{\alpha\beta}(x,y,z)$ for $\alpha,\beta=0,\ldots,3$ to be components of the internal metric in coordinates (T,Γ,Λ,Z); write the components as functions of (x,y,z) using Eqs. (4.22) and (4.23). Explicitly, the functions $P_{\alpha\beta}$ are

$$P_{00}=-\left(\frac{1-m_1/2R}{1+m_1/2R}\right)^2+\frac{m_2}{b^3}\left(1-\frac{m_1}{2R}\right)^4(3\Gamma^2-R^2)-\frac{4\epsilon m_2}{b^3}\Omega\Gamma\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4(2Z^2-R^2)$$
$$+\Omega^2\left(1+\frac{m_1}{2R}\right)^4(\Gamma^2+\Lambda^2)\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}\right]\right\}, \qquad (4.25)$$

$$P_{01}=P_{10}=-\frac{2\epsilon m_2}{b^3}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4\Gamma\Lambda-\Omega\Lambda\left(1+\frac{m_1}{2R}\right)^4\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}\right]\right\},$$

$$P_{02}=P_{20}=\frac{2\epsilon m_2}{b^3}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4(\Gamma^2-Z^2)+\Omega\Gamma\left(1+\frac{m_1}{2R}\right)^4\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}\right]\right\},$$

$$P_{03}=P_{30}=\frac{2\,\epsilon m_2}{b^3}\left(1-\frac{m_1}{2R}\right)^2\left(1+\frac{m_1}{2R}\right)^4\Lambda Z,$$

$$P_{11}=\left(1+\frac{m_1}{2R}\right)^4\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}-\frac{2m_1}{R^3}\left(1+\frac{m_1^2}{4R^2}\right)\Gamma^2\right]\right\},$$

$$P_{22}=\left(1+\frac{m_1}{2R}\right)^4\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}-\frac{2m_1}{R^3}\left(1+\frac{m_1^2}{4R^2}\right)\Lambda^2\right]\right\},$$

$$P_{33}=\left(1+\frac{m_1}{2R}\right)^4\left\{1+\frac{m_2}{b^3}(3\Gamma^2-R^2)\left[\left(1+\frac{m_1}{2R}\right)^4-\frac{2m_1^2}{R^2}-\frac{2m_1}{R^3}\left(1+\frac{m_1^2}{4R^2}\right)Z^2\right]\right\},$$

$$P_{12}=P_{21}=-\frac{2m_1m_2}{R^3b^3}\left(1+\frac{m_1}{2R}\right)^4\left(1+\frac{m_1^2}{4R^2}\right)(3\Gamma^2-R^2)\Gamma\Lambda,$$

$$P_{13}=P_{31}=-\frac{2m_1m_2}{R^3b^3}\left(1+\frac{m_1}{2R}\right)^4\left(1+\frac{m_1^2}{4R^2}\right)(3\Gamma^2-R^2)\Gamma Z,$$

$$P_{23}=P_{32}=-\frac{2m_1m_2}{R^3b^3}\left(1+\frac{m_1}{2R}\right)^4\left(1+\frac{m_1^2}{4R^2}\right)(3\Gamma^2-R^2)\Lambda Z,$$

where $\epsilon=(m/b)^{1/2}$, $\Omega=\omega(1-\mu/b)$, $R=(\Gamma^2+\Lambda^2+Z^2)^{1/2}$, and Γ, Λ, and Z are given in terms of (x,y,z) in Eqs. (4.22) and (4.23).

Next define the functions $K_\rho^\sigma(x,y,z)$ for $\rho,\sigma=0,\dots,3$ by $K_\rho^0=\partial T/\partial x^\rho$, $K_\rho^1=\partial\Gamma/\partial x^\rho$, $K_\rho^2=\partial\Lambda/\partial x^\rho$, and $K_\rho^3=\partial Z/\partial x^\rho$, where (T,Γ,Λ,Z) are to be expressed in terms of (t,x,y,z) using Eqs. (4.22) and (4.23). Explicitly, the functions K_ρ^σ are

$$K_0^0=1-\frac{m_2}{b}\left(1+\frac{m_2}{2m}\right),\quad K_1^0=\frac{\epsilon m_2 y}{b^2}\left(1+\frac{m_1}{m}-\frac{3\xi}{b}\right),\tag{4.26}$$

$$K_2^0=-\epsilon\left[\frac{m_2}{m}+\frac{m_2}{b}\left(3+\frac{m_1}{m}+\frac{m_2^2}{2m^2}\right)\right]+\frac{\epsilon m_2}{b^2}\left[\left(1+\frac{m_1}{m}\right)\xi-\frac{1}{2b}(3\xi^2-3y^2-z^2)\right],$$

$$K_3^0=\frac{\epsilon m_2}{b^3}yz,$$

$$K_0^1=0,\quad K_1^1=1+\frac{m_2}{b}-\frac{m_2\xi}{b^2},\quad K_2^1=-\frac{m_2 y}{b^2},\quad K_3^1=\frac{m_2 z}{b^2},$$

$$K_0^2 = 0, \quad K_1^2 = 0, \quad K_2^2 = 1 + \frac{m_2}{b}\left(1 + \frac{m_2}{2m}\right), \quad K_3^2 = 0,$$

$$K_0^3 = 0, \quad K_1^3 = -\frac{m_2 z}{b^2}, \quad K_2^3 = 0, \quad K_3^3 = 1 + \frac{m_2}{b} - \frac{m_2 \xi}{b^2},$$

where $\epsilon = (m/b)^{1/2}$ and $\xi = x - m_2 b/m$.

The metric in region I can now be written in terms of the functions $P_{\alpha\beta}$ and K_ρ^σ. It is given in corotating coordinates (t,x,y,z) by

$$g_{\mu\nu}(x,y,z) = \sum_{\alpha,\sigma=0}^{3} P_{\alpha\sigma}(x,y,z) K_\mu^\alpha(x,y,z) K_\nu^\sigma(x,y,z) \tag{4.27}$$

with $P_{\alpha\beta}$ and K_ρ^σ as defined in Eqs. (4.25) and (4.26). Note that the metric components are explicitly independent of time t. This metric is valid throughout region I (up to the first black hole's apparent horizon) and matches (to some finite order; see below) the post-Newtonian metric (2.9) at $r_1 = r_1^{in}$.

Errors in the internal metric (4.27) will only be analyzed in the weak-gravity buffer zone $m_1 \ll r_1 \ll b$. The largest errors come from inaccuracies in the coordinate transformation (4.22). Terms of the form $(m^2/b^2)(r_1/b)^p$ for integer $p \geq 1$ have been ignored in Eq. (4.22). This leads to errors $\delta g_{\mu\nu} \sim \epsilon^4$ for $r_1 \lesssim r_1^{in}$.

The match between the internal and post-Newtonian metrics at $r_1 = r_1^{in}$ is not perfect; there are discontinuities $[g_{\mu\nu}]$ in the metric components on that 3-surface. A term of the form m_1^3/r_1^3 in the internal metric component g_{00} [as given in Eq. (4.27)] is not matched in the post-Newtonian metric component g_{00} in Eq. (2.9); as a result, $[g_{00}] \sim \epsilon^3$ at $r_1 = r_1^{in}$. Similarly, a term of the form m_1^2/r_1^2 is not matched in g_{ij}, so $[g_{ij}] \sim \epsilon^2$. Lastly, a term of the form m_1^2/R^2 in the internal-coordinate metric component g_{00} in Eq. (3.23) gives via a (partial) Lorentz boost an unmatched term of the form $\epsilon m_1^2/r_1^2$ in the internal-metric component g_{0i} in corotating coordinates [given in Eq. (4.27)]; so $[g_{0i}] \sim \epsilon^3$.

The internal metric (4.27) contains terms of the form $(m_1/r_1)^p(m_2 r_1^2/b^3)$, $p \geq 1$, in the buffer zone. These terms represent the first black hole's multipole moments and the nonlinear interaction of internal and external gravitational fields. They are of $O(\epsilon^{p+4})$, $p \geq 1$, in the buffer zone and have not been matched to the post-Newtonian near-zone metric (2.9). At the level of accuracy achieved in this paper, the metric (2.9) need not be modified to include the near-zone gravitational effects of the black holes' deformation.

C. Metric near the second black hole

An identical procedure can now be followed to obtain the metric in corotating post-Newtonian coordinates in region II.

However, it is not necessary to repeat all the steps. This metric can simply be gotten as follows: exchange $m_1 \leftrightarrow m_2$ in the internal metric (3.23) and in the transformation (4.22); take $x \to -x$ and $y \to -y$ in Eq. (4.22); then transform. In other words, the metric components in region II [denoted $g_{\mu\nu}^{II}(t,x,y,z)$] are related to those in region I [$g_{\mu\nu}^{I}(t,x,y,z)$] by $g_{\mu\nu}^{II}(t,x,y,z) = (-1)^p g_{\mu\nu}^{I}(t,-x,-y,z)$ (with $m_1 \leftrightarrow m_2$), where p is the number of the indices μ and ν that are equal to 1 or 2.

Define $\bar{P}_{\alpha\beta}$ to be $P_{\alpha\beta}$ with m_1 and m_2 exchanged, and similarly $\bar{K}_\rho^\sigma$ to be K_ρ^σ with $m_1 \leftrightarrow m_2$. Then the metric in region II is given in corotating coordinates (t,x,y,z) by

$$g_{\mu\nu}(x,y,z) = (-1)^p \sum_{\alpha,\sigma=0}^{3} \bar{P}_{\alpha\sigma}(-x,-y,z)$$
$$\times \bar{K}_\mu^\alpha(-x,-y,z) \bar{K}_\nu^\sigma(-x,-y,z) \tag{4.28}$$

where p is, as above, the number of the indices μ and ν that are equal to 1 or 2. Again, the metric components are explicitly independent of time t. This metric is valid up to the second black hole's apparent horizon and matches (to a finite order) the post-Newtonian metric (2.9) at $r_2 = r_2^{in}$. Error analysis for this metric is identical to the analysis above for the metric in region I.

V. RESULTS AND DISCUSSION

The result of this calculation is an approximate solution to Einstein's equations representing two widely-separated nonrotating black holes in a circular orbit. The metric has been expressed in a single set of coordinates valid up to the black holes' apparent horizons; the coordinate system chosen is corotating coordinates (t,x,y,z). In these coordinates, the metric components are explicitly independent of time t. The metric is specified in region I by Eq. (4.27), in region II by Eq. (4.28), in region III by Eq. (2.9), and in region IV by Eq. (2.18). At the boundaries $r_1 = r_1^{in}$, $r_2 = r_2^{in}$, and $r = r^{out}$ of these regions, there are discontinuities in the metric components that result from truncation of perturbative expansions and finite-order matching. The magnitudes of these discontinuities can be reduced by taking this calculation to higher orders.

The full 4-metric is summarized below:

$$g_{00} = \begin{cases} \displaystyle\sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z)K_0^\mu(x,y,z)K_0^\nu(x,y,z) & \text{in region I,} \\[2mm] \displaystyle\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z)\bar{K}_0^\mu(-x,-y,z)\bar{K}_0^\nu(-x,-y,z) & \text{in region II,} \\[2mm] \begin{aligned} &-1+\frac{2m_1}{r_1}+\frac{2m_2}{r_2}-2\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)^2+\frac{3\mu}{b}\left(\frac{m_2}{r_1}+\frac{m_1}{r_2}\right)-\frac{\mu}{b}\left(\frac{m_2}{r_1^3}+\frac{m_1}{r_2^3}\right)y^2 \\ &-2\mu\epsilon^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)-7\mu\epsilon^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{x}{b}+\omega^2\left(1+\frac{2m_1}{r_1}+\frac{2m_2}{r_2}\right)(x^2+y^2) \end{aligned} & \text{in region III,} \\[2mm] \begin{aligned} &-1+\frac{2m}{r}\left(1-\frac{\mu}{2b}\right)-\frac{2m^2}{r^2}+A+\frac{2\epsilon^2}{b}(x\cos\omega r-y\sin\omega r)B \\ &-2\epsilon(x\sin\omega r+y\cos\omega r)D+\omega^2(x^2+y^2)E \\ &+\frac{\epsilon^4}{b^2}(x\cos\omega r-y\sin\omega r)^2N+\epsilon^2(x\sin\omega r+y\cos\omega r)^2 S \\ &-\frac{12\mu\epsilon^5}{r^2b^2}\frac{\delta m}{m}(x\cos\omega r-y\sin\omega r)^2(x\sin\omega r+y\cos\omega r) \end{aligned} & \text{in region IV,} \end{cases} \tag{5.1}$$

$$g_{0x} = \begin{cases} \displaystyle\sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z)K_0^\mu(x,y,z)K_1^\nu(x,y,z) & \text{in region I,} \\[2mm] -\displaystyle\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z)\bar{K}_0^\mu(-x,-y,z)\bar{K}_1^\nu(-x,-y,z) & \text{in region II,} \\[2mm] -\omega y\left(1+\frac{2m_1}{r_1}+\frac{2m_2}{r_2}\right) & \text{in region III,} \\[2mm] \begin{aligned} &\epsilon B\sin\omega r+bD\cos\omega r-\omega Ey+\frac{\epsilon^3}{b}(x\cos\omega r-y\sin\omega r)N\sin\omega r \\ &-\epsilon b(x\sin\omega r+y\cos\omega r)S\cos\omega r \\ &+\frac{6\mu\epsilon^4}{r^2b}\frac{\delta m}{m}(x\cos\omega r-y\sin\omega r)(x\cos2\omega r-y\sin2\omega r) \end{aligned} & \text{in region IV,} \end{cases} \tag{5.2}$$

$$g_{0y} = \begin{cases} \displaystyle\sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z)K_0^\mu(x,y,z)K_2^\nu(x,y,z) & \text{in region I ,} \\[2mm] -\displaystyle\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z)\bar{K}_0^\mu(-x,-y,z)\bar{K}_2^\nu(-x,-y,z) & \text{in region II,} \\[2mm] \omega x\left(1+\frac{2m_1}{r_1}+\frac{2m_2}{r_2}\right)-4\mu\epsilon\left(\frac{1}{r_1}-\frac{1}{r_2}\right) & \text{in region III,} \\[2mm] \begin{aligned} &\epsilon B\cos\omega r-bD\sin\omega r+\omega Ex+\frac{\epsilon^3}{b}(x\cos\omega r-y\sin\omega r)N\cos\omega r \\ &+\epsilon b(x\sin\omega r+y\cos\omega r)S\sin\omega r \\ &-\frac{6\mu\epsilon^4}{r^2b}\frac{\delta m}{m}(x\cos\omega r-y\sin\omega r)(x\sin2\omega r+y\cos2\omega r) \end{aligned} & \text{in region IV,} \end{cases} \tag{5.3}$$

$$g_{0z} = \begin{cases} \displaystyle\sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z)K_0^\mu(x,y,z)K_3^\nu(x,y,z) & \text{in region I,} \\[2mm] \displaystyle\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z)\bar{K}_0^\mu(-x,-y,z)\bar{K}_3^\nu(-x,-y,z) & \text{in region II,} \\[2mm] 0 & \text{in regions III and IV,} \end{cases} \tag{5.4}$$

$$g_{xx} = \begin{cases} \sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_1^{\mu}(x,y,z) K_1^{\nu}(x,y,z) & \text{in region I,} \\[2ex] \sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_1^{\mu}(-x,-y,z) \bar{K}_1^{\nu}(-x,-y,z) & \text{in region II,} \\[2ex] 1 + \dfrac{2m_1}{r_1} + \dfrac{2m_2}{r_2} & \text{in region III,} \\[2ex] E + \dfrac{m^2 x^2}{r^4} + \epsilon^2 N \sin^2 \omega r + b^2 S \cos^2 \omega r + \dfrac{6\mu\epsilon^3}{r^2} \dfrac{\delta m}{m} (x\cos\omega r - y\sin\omega r)\sin 2\omega r & \text{in region IV,} \end{cases} \tag{5.5}$$

$$g_{yy} = \begin{cases} \sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_2^{\mu}(x,y,z) K_2^{\nu}(x,y,z) & \text{in region I,} \\[2ex] \sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_2^{\mu}(-x,-y,z) \bar{K}_2^{\nu}(-x,-y,z) & \text{in region II,} \\[2ex] 1 + \dfrac{2m_1}{r_1} + \dfrac{2m_2}{r_2} & \text{in region III,} \\[2ex] E + \dfrac{m^2 y^2}{r^4} + \epsilon^2 N \cos^2 \omega r + b^2 S \sin^2 \omega r - \dfrac{6\mu\epsilon^3}{r^2} \dfrac{\delta m}{m} (x\cos\omega r - y\sin\omega r)\sin 2\omega r & \text{in region IV,} \end{cases} \tag{5.6}$$

$$g_{zz} = \begin{cases} \sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_3^{\mu}(x,y,z) K_3^{\nu}(x,y,z) & \text{in region I,} \\[2ex] \sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_3^{\mu}(-x,-y,z) \bar{K}_3^{\nu}(-x,-y,z) & \text{in region II,} \\[2ex] 1 + \dfrac{2m_1}{r_1} + \dfrac{2m_2}{r_2} & \text{in region III,} \\[2ex] E + \dfrac{m^2 z^2}{r^4} & \text{in region IV,} \end{cases} \tag{5.7}$$

$$g_{xy} = \begin{cases} \sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_1^{\mu}(x,y,z) K_2^{\nu}(x,y,z) & \text{in region I,} \\[2ex] \sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_1^{\mu}(-x,-y,z) \bar{K}_2^{\nu}(-x,-y,z) & \text{in region II,} \\[2ex] 0 & \text{in region III,} \\[2ex] \dfrac{m^2}{r^4} xy + \dfrac{1}{2}(\epsilon^2 N - b^2 S)\sin 2\omega r \\[2ex] + \dfrac{6\mu\epsilon^3}{r^2} \dfrac{\delta m}{m}(x\cos\omega r - y\sin\omega r)\cos 2\omega r & \text{in region IV,} \end{cases} \tag{5.8}$$

$$g_{xz} = \begin{cases} \sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_1^{\mu}(x,y,z) K_3^{\nu}(x,y,z) & \text{in region I,} \\[2ex] -\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_1^{\mu}(-x,-y,z) \bar{K}_3^{\nu}(-x,-y,z) & \text{in region II,} \\[2ex] 0 & \text{in region III,} \\[2ex] \dfrac{m^2}{r^4} xz & \text{in region IV,} \end{cases} \tag{5.9}$$

$$
g_{yz} = \begin{cases}
\displaystyle\sum_{\mu,\nu=0}^{3} P_{\mu\nu}(x,y,z) K_2^\mu(x,y,z) K_3^\nu(x,y,z) & \text{in region I,} \\[3ex]
\displaystyle-\sum_{\mu,\nu=0}^{3} \bar{P}_{\mu\nu}(-x,-y,z) \bar{K}_2^\mu(-x,-y,z) \bar{K}_3^\nu(-x,-y,z) & \text{in region II,} \\[3ex]
0 & \text{in region III,} \\[2ex]
\dfrac{m^2}{r^4} yz & \text{in region IV.}
\end{cases}
\tag{5.10}
$$

In the expressions above, $m = m_1 + m_2$, $\mu = m_1 m_2/m$, $\delta m = m_1 - m_2$, $\epsilon = (m/b)^{1/2}$, $\omega = (m/b^3)^{1/2}$, $r_1 = [(x - m_2 b/m)^2 + y^2 + z^2]^{1/2}$, and $r_2 = [(x + m_1 b/m)^2 + y^2 + z^2]^{1/2}$. Region I is specified by $r_1 < (m_1 b)^{1/2}$ and region II by $r_2 < (m_2 b)^{1/2}$ (but these regions do not extend inside the black holes' apparent horizons). Region III is specified by $r_1 > (m_1 b)^{1/2}$, $r_2 > (m_2 b)^{1/2}$, and $r < b/2\epsilon$; and region IV by $r > b/2\epsilon$. The functions A, B, D, E, N, and S are defined in Eq. (2.19). The functions $P_{\alpha\beta}$ and K_ρ^σ are defined in Eqs. (4.25) and (4.26). The functions $\bar{P}_{\alpha\beta}$ and $\bar{K}_\rho^\sigma$ are obtained from $P_{\alpha\beta}$ and K_ρ^σ respectively by exchanging m_1 and m_2.

TABLE I. Errors and discontinuities in the metric components in corotating coordinates. Numbers denote orders in $\epsilon = (m/b)^{1/2}$; e.g., 4 denotes $O(\epsilon^4)$. The last two columns contain normalized errors.

	Region I or II	disc. at	Region III				disc. at	Region IV	
	$r_1 \lesssim r_1^{in}$ or $r_2 \lesssim r_2^{in}$	$r_1 = r_1^{in}$ or $r_2 = r_2^{in}$	$r_1 \gtrsim r_1^{in}$ or $r_2 \gtrsim r_2^{in}$	$r_1 \sim b$ and $r_2 \sim b$	$r \lesssim r^{out}$	$r = r^{out}$	$r \gtrsim r^{out}$	$r \sim b/\epsilon^2$
g_{00}	4	3	3	6	5	5	7	8
g_{0i}	4	3	3	5	5	5	7	8
g_{ij}	4	2	2	4	5	5	7	8

The errors and discontinuities in the metric components are summarized in Table I. The discontinuities should be smoothed out before initial data are extracted from the metric. In addition, initial data taken should be relaxed numerically to approach more closely an exact solution of the constraint equations. It is expected that higher-order versions of this calculation will differ by smaller amounts from an exact solution of the Einstein equations.

[1] P. R. Brady, J. D. E. Creighton, and K. S. Thorne, Phys. Rev. D **58**, 061501 (1998).

[2] F. K. Manasse, J. Math. Phys. **4**, 746 (1963).

[3] P. D. D'Eath, Phys. Rev. D **11**, 1387 (1975).

[4] P. D. D'Eath, Phys. Rev. D **12**, 2183 (1975).

[5] L. Blanchet, G. Faye, and B. Ponsot, Phys. Rev. D **58**, 124002 (1998).

[6] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).

[7] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).

[8] K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).

[9] L. G. Fishbone, Astrophys. J. **185**, 43 (1973).

[10] J.-A. Marck, Proc. R. Soc. London **A385**, 431 (1983).

[11] F. K. Manasse and C. W. Misner, J. Math. Phys. **4**, 735 (1963).

[12] *Black Holes: The Membrane Paradigm*, edited by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale University Press, New Haven, 1986).

[13] K. S. Thorne (private communication).

[14] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[15] J. B. Hartle, Phys. Rev. D **8**, 1010 (1973).