

Thermodynamics of large N noncommutative super Yang-Mills theory

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(Received 12 October 1999; revised manuscript received 10 February 2000; published 22 May 2000)

We study the thermodynamics of the large N noncommutative super Yang-Mills theory in the strong 't Hooft coupling limit in the spirit of the AdS-CFT correspondence. It has already been noticed that some thermodynamic quantities of near-extremal D3-branes with NS B fields, which are dual gravity configurations of the noncommutative $\mathcal{N}=4$ super Yang-Mills theory, are the same as those without B fields. In this paper, (1) we examine the $\alpha'^3 R^4$ corrections to the free energy and find that the part of the tree-level contribution remains unchanged, but the one-loop and the nonperturbative D-instanton corrections are suppressed, compared to the ordinary case. (2) We consider the thermodynamics of a bound state probe consisting of D3-branes and D-strings in the near-extremal D3-brane background with B fields, and find that the thermodynamics of the probe is the same as that of a D3-brane probe in the D3-brane background without B fields. (3) The stress-energy tensor of the noncommutative super Yang-Mills theory is calculated via the AdS-CFT correspondence. It is found that the tensor is not isotropic and its trace does not vanish, which confirms that the super Yang-Mills theory is not conformal even in four dimensions due to the noncommutative nature of space. Our results render further evidence for the argument that the large N noncommutative and ordinary super Yang-Mills theories are equivalent not only in the weak coupling limit, but also in the strong coupling limit.

PACS number(s): 04.70.Dy, 11.15.Pg, 11.27.+d

I. INTRODUCTION

The super Yang-Mills (SYM) theory on noncommutative spaces is a natural generalization of the SYM theory on ordinary commutative spaces. Such a noncommutative SYM theory has been found to arise naturally in a certain limit of string theory with Neveu-Schwarz (NS) B fields [1–6]. The spirit of the AdS conformal field theory (CFT) correspondence [7–10] leads one to try to find out the supergravity dual of the noncommutative SYM theory. Recently Hashimoto and Itzhaki [11] and Maldacena and Russo [12] constructed independently the supergravity dual configurations of the noncommutative SYM theories, which are the decoupling limits of D-brane solutions with NS B fields. Some of the latter have been also constructed in [13–15] before.¹ The supergravity dual of the noncommutative SYM theory can also be constructed by using the relationship between the open string moduli and closed string moduli [17]. In this construction, the only input is a simple form of the running string tension as a function of energy.

In the AdS-CFT correspondence, of particular interest is the D3-brane solution. Its decoupling limit has the structure $\text{AdS}_5 \times S^5$, and the type IIB string theory on this background is supposed to be dual to the four-dimensional $\mathcal{N}=4$ SYM theory in the large N and strong 't Hooft coupling limit. At finite temperature the theory is described by the near-extremal D3-brane configuration [7,18]. According to the AdS-CFT correspondence, the decoupling limit of D3-brane solutions with B fields is supposed to be the dual gravity description of the noncommutative SYM theory in four di-

mensions [11,12]. An interesting question then arises: Are the total numbers of degrees of freedom the same for the noncommutative and ordinary SYM theories at any given scale? On the weak 't Hooft coupling side, according to the analysis of planar diagrams [19], the large N noncommutative and ordinary SYM theories are equivalent; the planar diagrams depends on the noncommutativity parameter only through the external momenta and noncommutative effects can be seen in the nonplanar diagrams. Explicit perturbative calculations [20] provide evidence to this assertion. On the strong 't Hooft coupling side, Maldacena and Russo [12] have discussed the thermodynamics of near-extremal D3-branes with B fields and found that the entropy and other thermodynamic quantities are the same as those of the corresponding D3-branes without B fields.² On this basis, they argued that the total number of physical degrees of freedom of the noncommutative SYM theory at any given scale coincides with the ordinary case.

In the present paper we would like to investigate further aspects of thermodynamics of the noncommutative SYM theory from the supergravity side and to compare them with the ordinary SYM cases. In Sec. II, we introduce the black D3-brane solutions with NS B fields and calculate some of their thermodynamic quantities. Most of the results are known, but these are needed for our discussions. This also serves to establish our notation. In Sec. III we calculate the corrections from the higher derivative terms ($\alpha'^3 R^4$) to the free energy of the noncommutative SYM theory. To compare the results with the ordinary case, we use a T -duality transformation to transform the D3-brane solution with B fields and a varying dilaton to that with constant dilaton and B

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¹These and more general solutions are also discussed in [16] in type IIA, type IIB and $d=11$ supergravities.

²This conclusion also holds for other D-branes with B fields. For related discussions see Refs. [21–23].

fields. In the latter configuration, we find that the contribution coming from the tree-level term remains the same as that in the ordinary case, but the contributions from the one-loop and nonperturbative D-instanton terms are suppressed. This result is consistent with that in the weak coupling limit [19].

In Sec. IV we consider the thermodynamics of a static bound state probe consisting of D3-branes and D-strings in the background produced by near-extremal D3-branes with B fields. According to the interpretation of the D-brane action, the supergravity interaction potential between the probe and the source D-branes can be interpreted as the contribution of massive states to the free energy of SYM theory when the SYM theory is in the Higgs phase, and the distance between the probe and the source can be regarded as a mass scale of the SYM theory. We find that the thermodynamics of the bound state probe again remains the same as that of a D3-brane probe in the near-extremal D3-brane background without B fields. In Sec. V, we compute the stress-energy tensor of the noncommutative SYM theory on the supergravity side. As is already known, the thermal excitations of D3-branes without B fields are of the form of an ideal gas in four dimensions. The entropy of near-extremal D3-branes can be accounted for by the ideal gas model [24]; its stress-energy tensor is isotropic and its trace vanishes [25], which confirms that the SYM theory is conformally invariant in four dimensions. Our result shows that the stress-energy tensor of the noncommutative SYM theory is not isotropic and its trace does not vanish, which reflects the fact that the noncommutative SYM theory is not conformal even in four dimensions. Section VI is devoted to a summary of our results and a discussion.

II. BLACK D3-BRANE SOLUTION WITH B FIELDS AND ITS THERMODYNAMICS

The supergravity solution corresponding to D3-branes with a nonvanishing NS B field has been constructed in [13] and [15]. The simplest way to get the solution is to start with a D3-brane solution without B fields. First make T duality along x_3 (the world-volume coordinates are x_0, x_1, x_2 , and x_3), which gives a D2-brane solution with a smeared coordinate x_3 , perform a rotation with an angle θ in the x_2 - x_3 plane, and then T -dualize back on x_3 . This procedure yields the desired solution with a nonvanishing B field along the x_2 and x_3 directions [12]. The prescription is also applicable to the black D3-brane solutions. The black D3-brane solution with B fields along the x_2 and x_3 directions can be written in the string metric as

$$ds^2 = H^{-1/2}[-fdx_0^2 + dx_1^2 + h(dx_2^2 + dx_3^2)] + H^{1/2}[f^{-1}dr^2 + r^2d\Omega_5^2], \quad (2.1)$$

where

$$H = 1 + \frac{r_0^4 \sinh^2 \alpha}{r^4}, \quad f = 1 - \frac{r_0^4}{r^4}, \quad h^{-1} = H^{-1} \sin^2 \theta + \cos^2 \theta,$$

$$B_{23}^{(1)} = \frac{\sin \theta}{\cos \theta} H^{-1} h, \quad e^{2\phi} = g^2 h,$$

$$B_{01}^{(2)} = (1 - H^{-1}) \sin \theta \coth \alpha / g, \quad (2.2)$$

$$C_{0123} = (1 - H^{-1}) h \cos \theta \coth \alpha / g.$$

The D3-brane charge satisfies $R'^4 \cos \theta = 4\pi g \alpha'^2 N_3$. Here $R'^4 = r_0^4 \sinh \alpha \cosh \alpha$, N_3 is the number of coincident D3-branes, and $g = g_\infty$ is the asymptotic value of the coupling constant. The solution interpolates between the black D-string solution ($\theta = \pi/2$) with the smeared coordinates x_2 and x_3 and the black D3-brane solution without B fields ($\theta = 0$). In fact the solution describes a nonthreshold bound state consisting of D3-branes and D-strings due to the presence of the nonzero B field [15].

Taking the decoupling limit [12]

$$\alpha' \rightarrow 0: \quad \tan \theta = \frac{\tilde{b}}{\alpha'}, \quad x_{0,1} = \tilde{x}_{0,1}, \quad x_{2,3} = \frac{\alpha'}{\tilde{b}} \tilde{x}_{2,3},$$

$$r = \alpha' R^2 u, \quad r_0 = \alpha' R^2 u_0, \quad g = \alpha' \tilde{g}, \quad (2.3)$$

where \tilde{b} , u , u_0 , \tilde{g} , and \tilde{x}_μ are kept fixed, the solution (2.1) becomes

$$ds^2 = \alpha' R^2 \left[u^2 (-\tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2) + u^2 \tilde{h} (d\tilde{x}_2^2 + d\tilde{x}_3^2) + \frac{du^2}{u^2 \tilde{f}} + d\Omega_5^2 \right], \quad (2.4)$$

where

$$\tilde{f} = 1 - u_0^4 / u^4, \quad \tilde{h}^{-1} = 1 + a^4 u^4, \quad a^2 = \tilde{b} R^2,$$

$$e^{2\phi} = \hat{g}^2 \tilde{h}, \quad \tilde{B}_{23} = \frac{\alpha'}{\tilde{b}} \frac{a^4 u^4}{1 + a^4 u^4}, \quad (2.5)$$

$\hat{g} = \tilde{g} \tilde{b}$ is the value of the string coupling in the IR, and $R^4 = 4\pi \hat{g} N_3 = 2g_{\text{YM}}^2 N_3 \equiv \lambda$ is the 't Hooft coupling constant of gauge theory.

Let us first discuss the extremal case $\tilde{f} = 1$ in the solution (2.4). The solution (2.4) reduces to the familiar product spacetime $\text{AdS}_5 \times S^5$ for $a = 0$, while it deviates from the anti-de Sitter space for $a \neq 0$. Thus, in the spirit of the AdS-CFT correspondence the solution (2.4) is proposed to be the gravity dual of the noncommutative SYM theory and the parameter a reflects the noncommutative nature of space. When $u \rightarrow 0$, the solution (2.4) approaches the $\text{AdS}_5 \times S^5$, which corresponds to the IR regime of the gauge theory. This is in agreement with the expectation that the noncommutative SYM theory reduces to the ordinary SYM theory at long distances.

Next, for nonextremal solution (2.4), just like the pure black D3-brane case, the thermodynamics of the nonextremal solution (2.4) should be equivalent to that of the non-

commutative SYM theory in the large N and strong 't Hooft coupling limit. However, the solution (2.4) is neither asymptotically flat nor asymptotically anti-de Sitter. Hence it is difficult to calculate the energy excitation of the noncommutative SYM theory directly from the solution (2.4). To discuss the thermodynamics of the noncommutative theory, we rather start with the black D3-brane solution (2.1). For our purposes, it is convenient to rewrite the solution in the Einstein frame, which has the following form:

$$ds_{\text{E}}^2 = h^{-1/4} H^{-1/2} [-fdx_0^2 + dx_1^2 + h(dx_2^2 + dx_3^2)] + h^{-1/4} H^{1/2} [f^{-1} dr^2 + r^2 d\Omega_5^2]. \quad (2.6)$$

We can further make compactification of the D3-brane world volume and then go to the Einstein frame. From the resulting metric, we can easily obtain the Arnowitt-Deser-Misner (ADM) mass M , Hawking temperature T , and entropy S of the solution, which are found to be

$$M = \frac{5\pi^2 r_0^4 V_3}{16g^2 G_{10}} \left(1 + \frac{4}{5} \sinh^2 \alpha \right),$$

$$T = \frac{1}{\pi r_0 \cosh \alpha},$$

$$S = \frac{V_3 \pi^3}{4g^2 G_{10}} r_0^5 \cosh \alpha, \quad (2.7)$$

where $G_{10} = 2^3 \pi^6 \alpha'^4$ is the gravitational constant in ten dimensions and V_3 is the spatial volume of the world volume of the D3-brane. We are interested in comparing these thermodynamic quantities with those of black D3-branes without B fields. We have just found that these quantities are independent of the parameter θ . Thus they are exactly the same as those without B fields [26].³ Furthermore, it is worth pointing out that it is independent of the parameter θ so that these thermodynamic quantities (2.7) are also those of the black D-string solution with two smeared coordinates. For later use, let us note the relation between the numbers of D3-branes and D-strings in the solution (2.1). The charge density of D3-branes is

$$Q_3 = \frac{\pi^2 r_0^4}{4g G_{10}} \cos \theta \sinh \alpha \cosh \alpha, \quad (2.8)$$

while the charge density of D-strings is

$$Q_1 = \frac{\pi^2 r_0^4 V_2}{4g G_{10}} \sin \theta \sinh \alpha \cosh \alpha, \quad (2.9)$$

where V_2 is the area of the rectangular torus spanned by the two smeared coordinates x_2 and x_3 . Using the charge quan-

tization rule we obtain the following relation between the number N_3 of D-branes and the number N_1 of D-strings:

$$\frac{N_1}{N_3} = \frac{V_2}{(2\pi)^2 \alpha'} \tan \theta. \quad (2.10)$$

In the decoupling limit, the relation (2.10) becomes

$$\frac{N_1}{N_3} = \frac{\tilde{V}_2}{(2\pi)^2 \tilde{b}}, \quad (2.11)$$

where $\tilde{V}_2 = \tilde{b}^2 V_2 / \alpha'^2$ is the area of the torus after rescaling. The excitation above the extremality of the black D3-brane corresponds to a thermal state of the corresponding SYM theory. Considering the limit (2.3), from Eq. (2.7) we have the energy E , temperature T , and entropy of the large N noncommutative SYM theory in the strong coupling limit:

$$E = \frac{3\pi^3 \tilde{V}_3 R^8 u_0^4}{(2\pi)^7 \hat{g}^2},$$

$$T = \frac{u_0}{\pi},$$

$$S = \frac{4\pi^4 \tilde{V}_3 R^8 u_0^3}{(2\pi)^7 \hat{g}^2}, \quad (2.12)$$

where $\tilde{V}_3 = \tilde{b}^2 V_3 / \alpha'^2$. Obviously these thermodynamic quantities satisfy the first law of thermodynamics, $dE = TdS$. The free energy F of the gauge theory, defined as $F = E - TS$, can be expressed in terms of the temperature:

$$F = -\frac{\pi^2}{8} \tilde{V}_3 N_3^2 T^4. \quad (2.13)$$

Of course, the free energy is also the same as that of ordinary SYM theory [24,27]. This result is quite interesting, which leads Maldacena and Russo [12] to argue that at any given scale the total number of degrees of freedom of the noncommutative SYM theory coincides with the ordinary case in the large N limit. No doubt it would be of much interest to further investigate this result and try to see if this is modified by any corrections. Motivated by this observation, we are now going to compute the higher-derivative term corrections to the free energy of the noncommutative SYM theory.

III. $\alpha'^3 R^4$ CORRECTIONS TO THE FREE ENERGY

The black configuration (2.1) is an exact solution of type IIB supergravity, which is a low-energy approximation of superstring, keeping only the leading contribution of massless states in the α' expansion. The nonleading contributions from massive string states appear as corrections to this low-energy action in the form of higher-derivative curvature terms. In type IIB supergravity the lowest correction can be symbolically written as $\alpha'^3 R_{\mu\nu\rho\sigma}^4$, where $R_{\mu\nu\rho\sigma}$ represents the Riemann tensor of spacetime. The tree-level contribution

³The results in [26] are for rotating D3-branes. For a comparison, take $l=0$ in the corresponding quantities in [26].

of the four-graviton amplitude to the effective action is [28]

$$S_{R^4}^{\text{tree}} = \frac{\zeta(3)}{3 \times 2^6 \times 16\pi G_{10}} \int d^{10}x \sqrt{-g} \alpha'^3 e^{-2\phi} R^4 \quad (3.1)$$

in the string frame. Exploiting the field redefinition ambiguity [29] and noting that for the D3-branes without B fields, the extremal background $\text{AdS}_5 \times S^5$ is a conformally flat spacetime, the corrections can be written in the Einstein frame as [27,30]

$$I_{R^4}^{\text{tree}} = -\frac{\gamma}{16\pi G_{10}} \int d^{10}x \sqrt{g} e^{-3\phi/2} \left[C^{hmnk} C_{pmnq} C_h{}^{rsp} C^q{}_{rsk} + \frac{1}{2} C^{hkmn} C_{pqmn} C_h{}^{rsp} C^q{}_{rsk} \right], \quad (3.2)$$

where $\gamma = \zeta(3)\alpha'^3/8$ and C_{pqmn} denotes the Weyl tensor. Such corrections to the free energy of the ordinary SYM theory on the three-torus T^3 , on the three-sphere S^3 , and even on a hyperbolic space H^3 have been calculated in [27,31–35]. In particular, for the large three-torus T^3 case the free energy correction is [27]

$$\delta F_{R^4}^{\text{tree}} = -\frac{\pi^2}{8} N_3^2 \tilde{V}_3 T^4 \frac{15}{8} \zeta(3) \lambda^{-3/2}. \quad (3.3)$$

Thus the free energy including the correction is

$$F_1 = F + \delta F_{R^4}^{\text{tree}} = -\frac{\pi^2}{8} N_3^2 \tilde{V}_3 T^4 \left[1 + \frac{15}{8} \zeta(3) \lambda^{-3/2} \right], \quad (3.4)$$

so that the leading correction is positive. If one writes the total free energy as

$$F_{\text{total}} = -f(\lambda) \frac{\pi^2}{6} N_3^2 \tilde{V}_3 T^4, \quad (3.5)$$

for the large 't Hooft coupling λ , one has

$$f(\lambda) = \frac{3}{4} + \frac{45}{32} \zeta(3) \lambda^{-3/2} + \dots \quad (3.6)$$

It is expected that the interpolation function f smoothly approaches 1 in the weak coupling limit ($\lambda \rightarrow 0$) [27].

In fact in type IIB supergravity the one-loop and nonperturbative D-instanton contributions are also of the form R^4 and of the same order (α'^3). If one writes

$$\rho = \rho_1 + i\rho_2 = c^{(0)} + i e^{-\phi}, \quad (3.7)$$

where $c^{(0)}$ is the Ramond-Ramond (RR) pseudoscalar, the effective action of the R^4 part can be expressed as [28,30]

$$S_{R^4}^{\text{IIB}} = \frac{1}{3 \times 2^7 \times 16\pi G_{10}} \int d^{10}x \sqrt{-g} \alpha'^3 e^{-\phi/2} f_4(\rho, \bar{\rho}) R^4, \quad (3.8)$$

where f_4 is given by the nonholomorphic Eisenstein series:

$$f_4(\rho, \bar{\rho}) = \sum_{(m,n) \neq (0,0)} \frac{\rho_2^{3/2}}{|m + \rho n|^3}. \quad (3.9)$$

For the small string coupling which is required for the validity of the supergravity description, the function f_4 can be expanded as [30]

$$e^{-\phi/2} f_4 \approx 2\zeta(3) e^{-2\phi} + \frac{2\pi^2}{3} + (4\pi)^{3/2} e^{-\phi/2} \sum_{M>0} Z_M M^{1/2} (e^{-2\pi M(e^{-\phi} + ic^{(0)})} + e^{-2\pi M(e^{-\phi} - ic^{(0)})}) [1 + \mathcal{O}(e^\phi/M)], \quad (3.10)$$

where M runs over integers. Here the first term gives the tree-level contribution, the second term gives the one-loop contribution, and the remaining denotes the contribution of the nonperturbative D-instantons. The coefficient Z_M is defined as

$$Z_M \equiv \sum_{m|M} \frac{1}{m^2}, \quad (3.11)$$

where $m|M$ denotes that the sum is taken over the divisors of M . Considering the contributions from the one-loop term and from D-instantons, $2\zeta(3)\lambda^{-3/2}$ in Eq. (3.6) is replaced by [27]

$$2\zeta(3)\lambda^{-3/2} \rightarrow 2\zeta(3)\lambda^{-3/2} + \frac{1}{24N_3^2} \lambda^{1/2} + \frac{1}{N_3^{3/2}} \times h(e^{-4\pi^2/g_{\text{YM}}^2}) [1 + \mathcal{O}(g_{\text{YM}}^2)], \quad (3.12)$$

where h represents an infinite series of instanton corrections. In particular, the one-loop contribution to the entropy correction [$\delta S = -\delta(\delta F)/\delta T$] is

$$\delta S_{\text{one}} = \frac{5\pi^2}{256} \lambda^{1/2} \tilde{V}_3 T^3. \quad (3.13)$$

In the effective low-energy action of type IIB supergravity, except for the R^4 terms, in the same order (α'^3) there exist other terms, for instance, an eight-derivative four-dilaton term [36], supersymmetric terms accompanying R^4 terms, and so on (see [27,37] and references therein). For the ordinary SYM theory, however, those terms will not make contributions since the dilaton is a constant and the five-form field strength is the same as that in the extremal background. For the noncommutative SYM theory, namely, the black D3-branes with nonvanishing B field, from Eq. (2.5) we see that the dilaton is no longer a constant and hence its derivative terms and other possible terms involving the derivatives of dilaton and curvature tensors are expected to make a contribution to the free energy correction. Also other terms unknown so far might have potential contributions in this order. Unfortunately, until now there has not been a complete expression of the effective low-energy action to the order

(α'^3), to the best of our knowledge. This makes it difficult to evaluate the free energy correction of noncommutative SYM theory via the supergravity description and to compare with the ordinary SYM case.⁴

To resolve this difficulty, we will adopt the following approach to attain insight into the (α'^3) correction to the free energy of the noncommutative SYM theory, rather than the usual way to evaluate the free energy correction by substituting the unperturbed solution (2.4) into those terms of order (α'^3), all of which are not known exactly. As is well known, the T duality is a perturbative symmetry of full string theories valid loop by loop. This symmetry holds in the low-energy supergravities as well. The low-energy effective action remains unchanged under the T -duality transformation. Therefore, if one can transform the solution (2.4) with a varying dilaton to a solution with a constant dilaton, one may get the free energy correction of the former via the latter. Indeed it has been found that such a T -duality transformation exists.

Following [38], defining

$$\mu = \frac{\tilde{V}_2}{(2\pi)^2 \alpha'} (\tilde{B}_{23} + i\sqrt{G_{22}G_{33}}), \quad (3.14)$$

the relevant T -duality transformation is given by the $SL(2, Z)$ transformation

$$\mu \rightarrow \hat{\mu} = \frac{a\mu + b}{c\mu + d}, \quad (3.15)$$

where $ad - bc = 1$. Applying this transformation to the solution (2.4) yields

$$ds^2 = \alpha' R^2 \left[u^2 (-\tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2 + d\tilde{x}_2^2 + d\tilde{x}_3^2) + \frac{du^2}{u^2 \tilde{f}} + d\Omega_5^2 \right], \quad (3.16)$$

$$e^{2\phi} = \frac{(2\pi)^4 \tilde{g}^2 \tilde{b}^4}{\tilde{V}_2^2}, \quad \hat{B}_{23} = \frac{\alpha'}{\tilde{b}}, \quad (3.17)$$

for $c = -1$ and $d = \tilde{V}_2 / (2\pi)^2 \tilde{b}$ when the latter is an integer. Note from Eq. (2.11) that $d = N_1 / N_3$ must be a rational number. If this is not an integer, after some steps of Morita equivalence transformation as in [38], one can reach a solution like Eq. (3.16). The solution (3.16) is asymptotically of the structure $AdS_5 \times S_5$, completely the same as that describing the ordinary SYM theory at finite temperature. Actually the solution (3.16) describes a twisted ordinary SYM theory due to the presence of a constant NS B field. The ordinary

⁴In the earlier version of this paper, we calculated the free energy correction from the term (3.2) and found that it is always less than the ordinary case. We thank Troels Harmark and Niels Obers for raising a question on the validity of that calculation to compare with the ordinary case.

SYM theory resides on a dual torus with area $\hat{V}_2 = (2\pi)^4 \tilde{b}^2 / \tilde{V}_2$ and its Yang-Mills coupling constant is

$$\hat{g}_{\text{YM}}^2 = \frac{(2\pi)^3 \tilde{g} \tilde{b}^2}{\tilde{V}_2} = g_{\text{YM}}^2 \frac{(2\pi)^2 \tilde{b}}{\tilde{V}_2}. \quad (3.18)$$

This ordinary SYM theory is equivalent to the noncommutative SYM theory described by the solution (2.4) in the sense of the Morita equivalence [38]. Note further that the number of D3-branes in Eq. (3.16) is $\tilde{V}_2 N_3 / (2\pi)^2 \tilde{b} = N_1$ according to Eq. (2.11), rather than N_3 in Eq. (2.4). It is quite interesting to note that the area, the Yang-Mills coupling constant, and the rank of the gauge group of the ordinary SYM theory in Eq. (3.16) are different from those of the noncommutative SYM theory in Eq. (2.4), but that the 't Hooft coupling constants for both theories are the same:

$$\hat{\lambda} = 2 \hat{g}_{\text{YM}}^2 N_3 \frac{\tilde{V}_2}{(2\pi)^2 \tilde{b}} = \lambda. \quad (3.19)$$

Considering that the spatial volume of the world volume in Eq. (3.16) is

$$\hat{V}_3 = \frac{(2\pi)^4 \tilde{b}^2}{\tilde{V}_2^2} \tilde{V}_3, \quad (3.20)$$

we conclude that the thermodynamics of the solutions (3.16) is the same as the one of the solutions (2.4), that is, the thermodynamics of the noncommutative SYM theory because the Hawking temperature is unchanged and $N_1^2 \hat{V}_3 = N_3^2 \tilde{V}_3$. Indeed, the Morita equivalence transformation will not change the thermodynamics of gauge field theory [38]. Thus we expect that the $\alpha'^3 R^4$ correction in Eq. (3.16) gives us the free energy correction of the noncommutative SYM theory in order (α'^3). As just mentioned above, the advantage to consider the solution (3.16), rather than Eq. (2.4) is that one does not have to worry about the contributions from the derivative terms of dilatons and possible other terms since the dilaton and the NS B field are constants here.

Now it is easy to get the free energy correction of noncommutative SYM theory from the (α'^3) terms in the effective low-energy action according to the above consideration. It is obtained from Eqs. (3.5), (3.6), and (3.12), with the replacements of g_{YM}^2 by \hat{g}_{YM}^2 , \tilde{V}_3 by \hat{V}_3 , and N_3 by $N_1 = \tilde{V}_2 N_3 / (2\pi)^2 \tilde{b}$, respectively. Considering the invariance of the 't Hooft coupling constant, we get the correction function (3.6) of the noncommutative SYM theory by replacing

$$2\zeta(3)\lambda^{-3/2} \rightarrow 2\zeta(3)\lambda^{-3/2} + \frac{(2\pi)^4 \tilde{b}^2}{\tilde{V}_2^2} \frac{1}{24N_3^2} \lambda^{1/2} + \left(\frac{(2\pi)^2 \tilde{b}}{\tilde{V}_2} \right)^{3/2} \frac{1}{N_3^{3/2}} h(e^{-\tilde{V}_2 / g_{\text{YM}}^2 \tilde{b}}) \times \left[1 + \mathcal{O} \left(\frac{(2\pi)^2 \tilde{b}}{\tilde{V}_2} g_{\text{YM}}^2 \right) \right]. \quad (3.21)$$

Comparing this with the ordinary case (3.12), we see that the first term is unchanged, but the remaining two terms are suppressed because $(2\pi)^2 \tilde{b}/\tilde{V}_2 \ll 1$. Recall that the first term comes from the tree-level contribution, the second term from the one-loop contribution, and the third is the nonperturbative D-instanton contribution. This indicates that in the strong 't Hooft coupling the large N noncommutative and ordinary SYM theories are equivalent because the first term corresponds to the planar diagrams and the second term to the nonplanar diagrams. This result is also consistent with the argument, which is made in the weak 't Hooft coupling limit, that planar diagrams depend on the noncommutativity parameter only through the external momenta and nonplanar diagrams are generally more convergent than their commutative counterparts [19]. The previous and the present sections provide evidence of the equivalence between the large N noncommutative and ordinary SYM theory. It would be interesting to accumulate further evidence for this equivalence. In the next section we will do it by studying the thermodynamics of a probe brane in the background (2.4).

IV. THERMODYNAMICS OF A PROBE BRANE

We know that the solution (2.1) describes N_3 D3-branes coinciding with each other. The configuration represents the noncommutative SYM theory with gauge group $U(N_3)$ in the Higgs branch, in which the vacuum expectation values (VEVs) of scalar fields are zero. Therefore the thermodynamics given in Sec. II is the one for noncommutative SYM theory in the Higgs branch. We now want to discuss the thermodynamics of the noncommutative SYM theory in the Coulomb branch, in which the VEVs of some scalar fields do not vanish. Corresponding to the Coulomb branch should be a multicenter configuration of D3-brane solutions. One of the simplest cases is that N parallel coinciding D3-branes are separated along a single transverse direction by a distance from a single D3-brane. The gauge symmetry is then broken from $U(N+1)$ to $U(N) \times U(1)$ and the distance can be regarded as a mass scale in the gauge field. However, no stable, multicenter, nonextremal configurations of D-branes have been known.⁵ As an approximation, one may consider the probe method. That is, we put an unexcited probe brane in the background of other nonextremal D-branes and regard this as an approximate multicenter solution. Such a method has been used recently to study the thermodynamics of SYM theory in the Higgs phase [41–44].

Considering that the noncommutative and ordinary SYM theories have the same thermodynamics at a given scale in the Higgs branch, it would be interesting to compare them also in the Coulomb branch. To this aim, in this section, we investigate the thermodynamics of a probe in the nonextremal D3-brane background. According to the interpretation of the D-brane action, the supergravity interaction potential between the probe and the near-extremal D3-branes (as the source) can be interpreted as the contribution of massive

states to the free energy of gauge fields in the large N and strong 't Hooft coupling limit [41]. When a NS B field is present, the dynamics of a probe D3-brane is governed by the following action:

$$S = -T_3 \int d^4x e^{-\phi} \sqrt{-\det(\hat{G} - \hat{B}^{(1)})} - T_3 \int \hat{C} - T_3 \int \hat{B}^{(2)} \wedge \hat{B}^{(1)}, \tag{4.1}$$

where $T_3 = 1/(2\pi)^3 \alpha'^2$ is the tension of the D3-brane. In fact this is a bound state probe consisting of D3-branes and D-strings. Explicit evidence for this is the tension of the probe, $T_3 \sqrt{1 + \tan^2 \theta}$. We will see more evidence below.

Substituting the solution (2.1) into the probe action (4.1), one has

$$S = -\frac{T_3 V_3}{g \cos \theta} \int d\tau H^{-1} [\sqrt{f} - 1 + H_0 - H], \tag{4.2}$$

where we have subtracted a constant potential at spatial infinity and

$$H_0 = 1 + \frac{R'^4}{r^4}. \tag{4.3}$$

In the extremal background where $f=1$, one can see from Eq. (4.2) that the static interaction potential between the probe and the source vanishes. Note that the source is a nonthreshold bound state consisting of D3-branes and D-strings, and the static potential will no longer vanish unless the probe is also the same bound state. In the nonextremal background, of course, the static potential exists always. In the decoupling limit (2.3), we arrive at

$$F_p = \frac{\tilde{V}_3 N_3 u^4}{2\pi^2} \left[\sqrt{1 - \frac{u_0^4}{u^4}} - 1 + \frac{u_0^4}{2u^4} \right], \tag{4.4}$$

which agrees with the result in [41] and [42] for a D3-brane probe in the near-extremal D3-brane background without B field.⁶ When the probe is on the horizon of the source, the free energy of the probe is

$$F_p|_{u=u_0} = -\frac{\tilde{V}_3 N_3 u_0^4}{4\pi^2} = -\frac{\pi^2 \tilde{V}_3 N_3 T^4}{4}. \tag{4.5}$$

Comparing with the free energy of the source (2.13), we find that

⁵There is a small difference between the probe free energies in [41] and [42], which arises as follows. In the decoupling limit, although $H_0 \approx 1/\alpha'^2 R^4 u^4$, and $H \approx 1/\alpha'^2 R^4 u^4$, the difference $H_0 - H$ does not vanish, but gives a finite value $u_0^4/2u^4$. This is just the additional term appearing in [42]. The additional term is important in the interpretation of the probe free energy.

⁶It is possible to have nonextremal configurations for continuously distributed D-branes. For D3-branes, see [39,40], for example.

$$F_p|_{u=u_0} = \frac{dF}{dN_3}. \quad (4.6)$$

The number of D3-branes in the probe is 1, so we may rewrite the above equation as

$$F_p|_{u=u_0} \approx F(N_3+1) - F(N_3), \quad (4.7)$$

for a large N_3 . This implies that, from the point of view of thermodynamics, the nonextremal D-branes reside on the horizon because the probe branes on the horizon can be viewed as a part of source branes.

In the low-temperature or long-distance limit, expanding the free energy (4.4) and using $u_0 = \pi T$, we get⁷

$$F_p = -\frac{\pi^2 \tilde{V}_3 N_3 T^4}{4} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n (n+1)!} \left(\frac{\pi T}{u}\right)^{4n}. \quad (4.8)$$

This is consistent with the expectation that, in the weak coupling and low-temperature limit, the contributions of one and two loops are exponentially suppressed [41,42]. The leading term is a three-loop contribution.

In the high-temperature or short-distance limit, we have to use the isotropic coordinates defined in Eq. (5.3) below.⁸ Defining the mass scale $M = (\sqrt{2}\rho - u_0)$, we obtain

$$F_p = -\frac{\pi^2 \tilde{V}_3 N_3 T^4}{4} \frac{1}{(1 + M/\pi T)^4}. \quad (4.9)$$

Expanding Eq. (4.9) for the small $M/\pi T$ yields

$$F_p = -\frac{1}{4} \pi^2 \tilde{V}_3 N_3 T^4 \left[1 - 4 \left(\frac{M}{\pi T}\right) + 10 \left(\frac{M}{\pi T}\right)^2 - 20 \left(\frac{M}{\pi T}\right)^3 + \dots \right]. \quad (4.10)$$

Let us compare this with the free energy in the weak coupling limit. The one-loop free energy of the $\mathcal{N}=4$ SYM theory in the weak coupling has the following high-temperature expansion [41]:

$$F_M(T \gg M) = -\frac{1}{3} \pi^2 N_3 \tilde{V}_3 T^4 \left[1 - 3 \left(\frac{M}{\pi T}\right)^2 + 4 \left(\frac{M}{\pi T}\right)^3 + \dots \right]. \quad (4.11)$$

⁷Note that there is a difference by a factor of R^2 in the rescaling of r and r_0 from the definitions in [41] and [42].

⁸Note that there is a difference by a factor of $\sqrt{2}$ in the definition of the coordinate ρ between this paper and [41,42]. If we use the definition in [41,42], the metric will not be asymptotically flat as $\rho \rightarrow \infty$.

It is very similar to that for the strong coupling limit (4.10) except that the term $(M/\pi T)$ is absent in the weak coupling. In particular, in the massless approximation, keeping only the leading terms in Eqs. (4.10) and (4.11), one may see that there is also the well-known difference by a factor of $3/4$, which occurs in comparing the supergravity calculation and weak coupling calculation of the entropy for the $\mathcal{N}=4$ SYM theory in the Higgs branch [24,27].

The main result of this section is that the static interaction potential between a D3-brane probe with NS B fields in the background of D3-branes with B fields is the same as that of a D3-brane probe in the corresponding background without B fields. From the point of view of field theory, the static potential comes from planar diagrams [45], which is clearer from the viewpoint of open strings extended between the probe branes and source branes. This further renders evidence that the large N noncommutative and ordinary SYM theories are also equivalent in the strong coupling limit.

V. STRESS-ENERGY TENSOR OF THE NONCOMMUTATIVE SYM THEORY

In Sec. II we have seen that the entropy of the noncommutative SYM theory is the same as that of the SYM theory at a given temperature scale or energy scale. However, from the Einstein frame metric of Eq. (2.4) we see that, when $u \rightarrow \infty$, the area of the torus x_2, x_3 contracts, while the radius of the S^5 expands. The contraction of area of the torus is just compensated for by the expansion of the volume of the S^5 . This seems to imply that there is a redistribution of the degrees of freedom [12]. To compare the distribution of thermal states between the noncommutative SYM and ordinary SYM theories, it is enough to calculate the stress-energy tensor of the noncommutative SYM theory on the supergravity side.

For this purpose, we adopt the method developed by Myers [25] by generalizing the ADM mass density formula of p -branes [46]. The stress-energy tensor for the p -brane world-volume can be expressed as

$$T_{ab} = \frac{1}{16\pi g^2 G_{10}} \int_{r \rightarrow \infty} d\Omega_{8-p} r^{8-p} n^i \times [\eta_{ab}(\partial_i h_c^c + \partial_i h_j^j - \partial_j h_i^i) - \partial_i h_{ab}], \quad (5.1)$$

where n^i is a radial unit in the transverse subspace, while $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the deviation of the (Einstein frame) metric from that for flat space. The labels $a, b = 0, 1, \dots, p$ run over the world-volume directions, while $i, j = 1, 2, \dots, 9-p$ denote the transverse directions. In addition, it should be remembered that the calculations in Eq. (5.1) must be done using asymptotically Cartesian coordinates.

Rewriting the Einstein metric (2.6) in isotropic coordinates, one has

$$ds_E^2 = h^{-1/4} H^{-1/2} [-fdx_0^2 + dx_1^2 + h(dx_2^2 + dx_3^2)] + h^{-1/4} H^{1/2} r^2 \rho^{-2} [d\rho^2 + \rho^2 d\Omega_3^2], \quad (5.2)$$

where

$$r^2 = \rho^2 \left(1 + \frac{r_0^4}{4\rho^4} \right), \quad \rho^2 = \frac{1}{2} (r^2 + \sqrt{r^4 - r_0^4}). \quad (5.3)$$

Substituting Eq. (5.2) into Eq. (5.1) and setting $p=3$ yields

$$T_{ab} = \frac{\pi^3}{16\pi g^2 G_{10}} \text{diag}[5r_0^4 + 4\tilde{R}^4, -r_0^4 - 4\tilde{R}^4, -r_0^4 - 4\tilde{R}^4 \cos^2 \theta, -r_0^4 - 4\tilde{R}^4 \cos^2 \theta], \quad (5.4)$$

where $\tilde{R}^4 = \sqrt{R'^8 + r_0^8/4} - r_0^4/2$. The stress-energy tensor (5.4) includes the contribution from the extremal background, which must be subtracted from it in order to acquire the required quantity. The contribution of the extremal background can be obtained directly from Eq. (5.4) by setting $r_0=0$:

$$(T_{ab})_{\text{ext}} = \frac{\pi^3}{16\pi g^2 G_{10}} \text{diag}[4R'^4, -4R'^4, -4R'^4 \cos^2 \theta, -4R'^4 \cos^2 \theta]. \quad (5.5)$$

Subtracting Eq. (5.5) from Eq. (5.4) and taking the near-extremal limit $\tilde{R}^4 \approx R'^4 - r_0^4/2$, we finally get the stress-energy tensor for the noncommutative SYM theory in the large N and strong coupling limit,

$$(\Delta T)_{ab} = \frac{\pi^3 r_0^4}{16\pi g^2 G_{10}} \text{diag}[3, 1, 2 \cos^2 \theta - 1, 2 \cos^2 \theta - 1], \quad (5.6)$$

and its trace

$$\Delta T = - \frac{4\pi^3 r_0^4}{16\pi g^2 G_{10}} \sin^2 \theta. \quad (5.7)$$

For $\theta=0$, Eq. (5.6) reduces to the result for the ordinary SYM theory, which is of the form of an ideal gas in 3+1 dimensions. In that case, its trace is zero. This is in accordance with the fact that the ordinary SYM theory is conformally invariant in four dimensions. On the other hand, for $\theta \neq 0$, Eq. (5.6) gives the stress-energy tensor for the noncommutative SYM theory. In this case, the tensor is not isotropic and its trace does not vanish. It reflects the fact that the noncommutative SYM theory is not conformal even in four dimensions due to the noncommutativity of space. In addition, we confirm that the T_{00} component of the stress-energy tensor (5.6) in the decoupling limit indeed gives the energy density of the noncommutative SYM theory given in Eq. (2.12).

VI. CONCLUSIONS

To summarize, we have investigated some aspects of thermodynamics for the noncommutative SYM theory in the large N and strong 't Hooft coupling limit on the supergravity side, and compared them with the ordinary case. Although the entropy and other thermodynamic quantities of black D3-branes with NS B fields are the same as those without B fields, the stress-energy tensor of thermal excitations is different. For the ordinary SYM theory the stress-energy tensor is of the form of an ideal gas in four dimensions. It is isotropic and its trace is zero. On the other hand, for the noncommutative SYM theory, the tensor is not isotropic and its trace does not vanish, which confirms that the noncommutative SYM theory is not conformally invariant even in four dimensions due to the noncommutative nature of space. Note that in the solution (2.1) the NS B field has components only in the x_2, x_3 directions. This means that the coordinates x_2 and x_3 are noncommutative, while x_0 and x_1 are the ordinary commutative coordinates. One may consider more general D3-brane solutions with both B_{01} and B_{23} components. We do not expect that the stress-energy tensor will be isotropic in that case either. The result is indeed so, and we have confirmed this by a very similar calculation to that in Sec. V.

We have considered the higher-derivative term corrections in the order (α'^3) to the free energy of the noncommutative SYM theory in Sec. III. Because there has not been a complete expression of the low-energy effective action of type IIB supergravity to order (α'^3) , to make sense of the calculation and to compare the case of the ordinary case which has been investigated in [27], we transformed the near-extremal D3-brane solution with varying dilatons to a solution (3.16) with a constant dilaton by a T -duality transformation. Those two solutions are equivalent in the sense of the Morita equivalence. Using the latter solution, we have found that the tree-level contribution is the same as the ordinary case, but the one-loop and the nonperturbative D-instanton contributions are suppressed, compared to the ordinary case. Note that the tree-level part corresponds to the planar diagrams, and the one-loop part to the nonplanar diagrams in the field theory. This provides evidence that the large N noncommutative and ordinary SYM theories are also equivalent in the strong 't Hooft coupling limit.

We have also studied the thermodynamics of a bound state probe consisting of D3-branes and D-strings in the background produced by the black D3-branes with B fields and compared it with that of a D3-brane probe in the background produced by the black D3-branes without B fields. In accordance with the interpretation of the D-brane action, the free energy of a static probe can be regarded as the contribution of massive states to the free energy of noncommutative SYM theory in the Higgs phase and the distance between the probe and the source can be explained as a mass scale in the gauge theory. From the thermodynamics of the probe we have found that the free energies for the ordinary and noncommutative cases agree. In fact the dynamic inter-

action potential between the probe and the source also agrees with the ordinary case. Because the interaction potential comes from the planar diagrams from the point of view of field theory, the agreement further suggests that the large N noncommutative and ordinary SYM theories are equivalent not only in the weak coupling limit [19], but also in the strong coupling limit.

ACKNOWLEDGMENTS

We would like to thank T. Harmark and N. Obers for their very helpful correspondences. This work was supported in part by the Japan Society for the Promotion of Science and by a grant-in-aid from the Ministry of Education, Science, Sports and Culture No. 99020.

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