Vacuum polarization of scalar fields near Reissner-Nordström black holes and the resonance behavior in field-mass dependence

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We study vacuum polarization of quantized massive scalar fields ϕ in equilibrium at the black-hole temperature in a Reissner-Nordström background. By means of the Euclidean space Green's function we analytically derive the renormalized expression $\langle\phi^2\rangle_H$ at the event horizon with the area $4\pi r_+^2$. It is confirmed that the polarization amplitude $\langle\phi^2\rangle_H$ is free from any divergence due to the infinite redshift effect. Our main purpose is to clarify the dependence of $\langle\phi^2\rangle_H$ on the field mass m in relation to the excitation mechanism. It is shown for small-mass fields with $mr_+\ll 1$ how the excitation of $\langle\phi^2\rangle_H$ caused by a finite black-hole temperature is suppressed as m increases, and it is verified for very massive fields with $mr_+\gg 1$ that $\langle\phi^2\rangle_H$ decreases in proportion to m^{-2} with an amplitude equal to the DeWitt-Schwinger approximation. In particular, we find a resonance behavior with a peak amplitude at $mr_+\simeq 0.38$ in the field-mass dependence of vacuum polarization around nearly extreme (low-temperature) black holes. The difference between Scwarzschild and nearly extreme black holes is discussed in terms of the mass spectrum of quantum fields dominant near the event horizon.

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I. INTRODUCTION

The quantum behavior of matter fields in black hole spacetime has been extensively studied in order to understand the various physical effects. In particular, the existence of a state of quantum fields in equilibrium at a finite temperature near the event horizon has attracted much attention, because it clearly represents the thermodynamic properties of stationary black holes. The problem of vacuum polarization for this Hartle-Hawking state [1] may be described in terms of the Euclidean space Green's function $G_E(x,x')$, which is periodic with respect to the Euclidean time $\tau=it$. If one considers a quantized scalar field ϕ, the vacuum polarization $\langle\phi^2(x)\rangle$ can be calculated by using the equality

$$\langle\phi^2(x)\rangle=\mathrm{Re}\{\lim_{x'\to x}G_E(x,x')\},\qquad(1)$$

in which the renormalized expression is derived through the method of point splitting.

It is well known that the black-hole temperature T defined as the inverse of the period of $G_E(x,x')$ is proportional to the surface gravity κ on the event horizon as follows:

$$T=\kappa/2\pi.\qquad(2)$$

(Throughout this paper we use units such that $G=c=\hbar=k_B=1$.) If the origin of the vacuum polarization $\langle\phi^2(x)\rangle$ is claimed to be purely induced by the finite black-hole temperature, the amplitude should decrease toward zero in the extreme black-hole limit $\kappa\to 0$. In fact, we can see this behavior of $\langle\phi^2\rangle$ by applying the analytical approximation of the renormalized value obtained by Anderson, Hiscock, and Samuel [2] to the Reissner-Nordström background, for which the analytic continuation of the metric into Euclidean space is given by

$$ds^2=f(r)d\tau^2+f^{-1}(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2,\qquad(3)$$

where $f=(r-r_+)(r-r_-)/r^2$, and using mass M and charge Q parameters of the black hole, we have

$$r_\pm=M\pm\sqrt{M^2-Q^2}.\qquad(4)$$

For massless scalar fields the analytical approximation denoted by $\langle\phi^2\rangle_T$ reduces to

$$\langle\phi^2(r)\rangle_T=\frac{\kappa^2}{48\pi^2}\,\frac{(r+r_+)(r^2+r_+^2)}{r^2(r-r_-)}.\qquad(5)$$

Therefore, in nearly extreme Reissner-Nordström spacetime such that $\kappa r_+=(r_+-r_-)/(2r_+)\ll 1$, the vacuum polarization of massless fields is strongly suppressed. (This is also justified by the result of Frolov [3] estimated at the event horizon $r=r_+$.)

Such an excitation of vacuum polarization induced by finite black-hole temperature is an important aspect of quantum matter fields in black-hole backgrounds, and it may remain valid for massive scalar fields too. Then, field mass m will just play the role of suppressing the amplitude of $\langle\phi^2\rangle$ in comparison with massless fields. In this paper, however, we would like to emphasize another remarkable effect due to field mass, which we call mass-induced excitation as a remaining part of $\langle\phi^2\rangle$ in the low-temperature limit $T\to 0$. Note that massive fields can have a characteristic correlation scale corresponding to the Compton wavelength $1/m$. Our purpose is to show that nearly extreme (low-temperature) black holes can enhance the excitation of quantum fields with the Compton wavelength $1/m$ of order of the black-hole radius (i.e., $mr_+\sim 1$). This mass-induced excitation may be

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expected as a result of wave modes in resonance with the potential barrier surrounding a black hole, for which the tail part of $\langle\phi^2\rangle$ in the large-mass limit $mr_+ \gg 1$ is generated with the amplitude decreasing in proportion to $1/m^2$ [4,5] according to the DeWitt-Schwinger approximation developed by Christensen [6].

In this paper our investigation is focused on the Reissner-Nordström background as the simplest example which allows us to consider the low-temperature limit $\kappa r_+ \ll 1$ keeping an arbitrary value of mr_+. (The black-hole temperature and the field mass are measured in units of the inverse of a fixed black-hole radius r_+. In the Schwarzschild background with $\kappa r_+ = 1/2$ we cannot discuss the field-mass dependence of $\langle\phi^2\rangle$ in such a low-temperature limit, and any resonance behavior of the polarization amplitude $\langle\phi^2\rangle$ at $mr_+ \sim 1$ will become obscure by virtue of a contamination of the temperature-induced excitation in the mass range of $mr_+ \ll 1$ [7].) Then, following the analysis given by Anderson and his collaborators [2,5], we compute the vacuum polarization of massive scalar fields, for which we have the analytical approximation of the form

$$\langle\phi^2\rangle_{ap} = \langle\phi^2\rangle_T + \langle\phi^2\rangle_{m^2}. \tag{6}$$

Here the additional contribution from field mass becomes

$$\langle\phi^2\rangle_{m^2} = \frac{m^2}{16\pi^2}\left\{1 - 2\gamma - \ln\left(\frac{m^2 f}{4\kappa^2}\right)\right\}, \tag{7}$$

with Euler's constant γ. Unfortunately, this field-mass term contains a logarithmic divergence at the event horizon $r = r_+$. Therefore, in Sec. II we develop the technique of analytical calculation to cancel such a divergent term, by paying the price that $\langle\phi^2\rangle$ is evaluated only near the event horizon. It is checked in Sec. III that the renormalized value of $\langle\phi^2\rangle$ at the event horizon becomes identical, up to the leading terms of order of $1/m^2 r_+^2$, with the result derived by DeWitt-Schwinger expansion in the large-mass limit. In Sec. IV, using the small-mass approximation $mr_+ \ll 1$, we show the tendency of temperature-induced excitation to be suppressed with increase of field mass. We find in Sec. V the mass-induced enhancement of the polarization amplitude $\langle\phi^2\rangle$, by giving explicitly the dependence on field mass in the low-temperature limit $\kappa r_+ \ll 1$. The final section summarizes the results representing a remarkable difference of field-mass dependence of the polarization amplitude for scalar fields in equilibrium at various black-hole temperatures.

II. CORRECTION TO THE WKB APPROXIMATION

Let us start from a brief introduction of the method to compute the renormalized value of $\langle\phi^2\rangle$ in Reissner-Nordström background (3), which has been developed by Anderson and his collaborators [2,5]. Using Eq. (1) for a massive scalar field ϕ obeying the equation

$$(\Box - m^2)\phi(x) = 0, \tag{8}$$

the unrenormalized expression is given by

$$\langle\phi^2(r)\rangle = \lim_{\epsilon\to 0}\left\{\frac{\kappa}{4\pi^2}\sum_{n=0}^{\infty} c_n \cos(n\kappa\epsilon) A_n(r)\right\}, \tag{9}$$

where $c_0 = 1/2$ and $c_n = 1$ for $n \geq 1$. The separation of two points in $G_E(x,x')$ is chosen to be only in time as $\epsilon \equiv \tau - \tau'$, and the radial part $A_n(r)$ for each quantum number n is given by the sum of radial modes $p_{nl}(r)$ and $q_{nl}(r)$,

$$A_n(r) = \sum_{l=0}^{\infty}\left\{(2l+1)p_{nl}(r)q_{nl}(r) - \frac{1}{r\sqrt{f}}\right\}, \tag{10}$$

where l is the angular-momentum quantum number, and the subtraction term $1/r\sqrt{f}$ is necessary for removing the divergence in the sum over l. The radial mode q_{nl} satisfies the equation

$$\frac{d^2 q_{nl}}{dr^2} + \frac{1}{r^2 f}\frac{d(r^2 f)}{dr}\frac{dq_{nl}}{dr} - \left\{\frac{n^2\kappa^2}{f^2} + \frac{l(l+1) + m^2 r^2}{fr^2}\right\} q_{nl} = 0, \tag{11}$$

and it is chosen to be regular at $r = \infty$ and divergent at $r = r_+$. The same equation is satisfied by p_{nl}, which is chosen to be well behaved at $r = r_+$ and divergent at $r = \infty$.

The WKB approximation for the modes may be used to calculate the mode sums (10), by assuming the forms

$$p_{nl} = \frac{1}{(2r^2 W)^{1/2}}\exp\left(\int (W/f)dr\right), \tag{12}$$

and

$$q_{nl} = \frac{1}{(2r^2 W)^{1/2}}\exp\left(-\int (W/f)dr\right), \tag{13}$$

where the zeroth-order solution is chosen to be

$$W^2 \simeq n^2\kappa^2 + \left\{\left(l + \frac{1}{2}\right)^2 + m^2 r^2\right\}\frac{f}{r^2}. \tag{14}$$

To renormalize $\langle\phi^2\rangle$ in the limit $\epsilon \to 0$ of point splitting, we subtract the counterterms $\langle\phi^2\rangle_{DS}$ generated from the DeWitt-Schwinger expansion of $\langle\phi^2\rangle$,

$$\langle\phi^2\rangle_{DS} = \frac{1}{8\pi^2\sigma} + \frac{m^2}{16\pi^2}\left\{-1 + 2\gamma + \ln\left(\frac{m^2|\sigma|}{2}\right)\right\} + \frac{1}{96\pi^2}R_{ab}\frac{\sigma^a\sigma^b}{\sigma}, \tag{15}$$

where σ is equal to one-half the square of the distance between the two points x and x', and $\sigma^a \equiv \nabla^a\sigma$. Then, for the renormalized value defined by

$$\langle\phi^2\rangle_{ren} = \langle\phi^2\rangle - \langle\phi^2\rangle_{DS}, \tag{16}$$

we can arrive at the analytical approximation (6), if the second-order WKB approximation for W is used in the mode sums for $n\geqslant 1$ [2,5].

Though Eq. (6) can clearly show a spatial distribution of the vacuum polarization, the validity is rather restrictive. For example, in the asymptotically flat region $r\to\infty$ it fails to give the expected dependence on field mass. It is instructive for later discussions to calculate precisely $\langle\phi^2\rangle_{ren}$ of thermal fields in equilibrium at a temperature T in flat background (corresponding to $f=1$), following the method of the Euclidean space Green's function $G_E(x,x')$. Denoting T by $\kappa/2\pi$, we obtain the exact solutions for p_{nl} and g_{nl} in flat background as follows,

$$p_{nl}=\frac{1}{r^{1/2}}I_{l+\frac{1}{2}}(r\sqrt{m^2+n^2\kappa^2}), \tag{17}$$

and

$$q_{nl}=\frac{1}{r^{1/2}}K_{l+\frac{1}{2}}(r\sqrt{m^2+n^2\kappa^2}), \tag{18}$$

and the mode sum over l in A_n results in

$$A_n=-\sqrt{m^2+n^2\kappa^2}. \tag{19}$$

If we use the Plana sum formula for a function $g(k)$

$$\sum_{j=k}^{\infty}g(j)=\frac{1}{2}g(k)+\int_k^{\infty}g(x)dx+i\int_0^{\infty}\frac{dx}{e^{2\pi x}-1}\times[g(k+ix)-g(k-ix)], \tag{20}$$

the unrenormalized value is written by the integral form

$$\langle\phi^2\rangle=\lim_{\epsilon\to 0}\left\{\frac{\kappa}{4\pi^2}\left[-\int_0^{\infty}dn\,\cos(n\kappa\epsilon)\sqrt{m^2+n^2\kappa^2}+\int_{m/\kappa}^{\infty}\frac{2dn}{e^{2\pi n}-1}\sqrt{\kappa^2n^2-m^2}\right]\right\}. \tag{21}$$

The first term in the right-hand side of Eq. (21) is completely canceled by the subtraction of the DeWitt-Schwinger counterterms (15), in which we have $\sigma=-\epsilon^2/2$, while the second term gives the renormalized value $\langle\phi^2\rangle_{ren}$ in flat background, which for massless fields reduces to

$$\langle\phi^2\rangle_{ren}=T^2/12, \tag{22}$$

and becomes equal to Eq. (6) estimated in the asymptotically flat region. However, in the large-mass limit $m\gg\kappa$, we obtain

$$\langle\phi^2\rangle_{ren}=m^{1/2}(T/2\pi)^{3/2}e^{-m/T}, \tag{23}$$

because the second integral over n in Eq. (21) should run from the large lower limit $m/\kappa\gg 1$ to infinity. This leads to a crucial difference from the approximated form (6), for which A_n is expressed in inverse powers of $n\kappa$ such that

$$A_n\simeq-\frac{n\kappa}{f}+\left(\frac{1}{12r^2}-m^2\right)\bigg/\;2n\kappa, \tag{24}$$

as a result of the mode sum over l using the zeroth-order solution (14) for W. It is clear that the sum of such an expansion form of A_n over $n\geqslant 1$ misses the exponential behavior $e^{-2\pi m/\kappa}$ of $\langle\phi^2\rangle_{ren}$ in the asymptotically flat region.

Now let us turn our attention to vacuum polarization at the event horizon $f=0$, which is the main concern in this paper. Fortunately, we can claim that the above-mentioned deviation of Eq. (6) from the precise estimation becomes irrelevant, if we consider the limit $f\to 0$. This is because owing to the redshift factor f in W the expansion (24) remains valid even for a large mass $m\gg\kappa$, by keeping the condition $m\sqrt{f}/\kappa\ll 1$. Then, concerning vacuum polarization of massive fields at the event horizon, we can use Eq. (6) to show the dependence of $\langle\phi^2\rangle_{ren}$ on m. Of course, one may point out another crucial problem, that Eq. (6) contains a logarithmic divergence at $r=r_+$. However, this singular behavior is due to the sum of A_n over the limited range of $n\geqslant 1$. Because the expansion form (24) also breaks down for $n=0$, the contribution of A_0 to $\langle\phi^2\rangle_{ren}$ is omitted in the calculation of Eq. (6). We would like to clarify an important role of the $n=0$ mode to give a regular value at the event horizon for the renormalized vacuum polarization $\langle\phi^2\rangle_{ren}$ (and also for the renormalized stress-energy tensor $\langle T_{ab}\rangle_{ren}$).

To this end we propose the procedure to treat more precisely the mode sum over l in A_n at the event horizon, which is applicable to the lower n modes. Note that near the event horizon the exact solution for q_{nl} should have the expansion form

$$q_{nl}=z^{n/2}\ln z\sum_{s=0}^{\infty}\alpha_s z^s+z^{-n/2}\sum_{s=0}^{\infty}\beta_s z^s, \tag{25}$$

with some coefficients α_s and β_s. The rescaled radial coordinate z is defined by $z\equiv(r-r_+)/r_+\ll 1$. This expansion form is not useful to calculate A_n at the event horizon, because the sums over l should be done without expanding in powers of z for requiring the convergence. Then, the important points to be mentioned here are the existence of the logarithmic term $z^{n/2}\ln z$ and the power-law behavior $z^{-n/2}$ dominant for $n\geqslant 1$ in the limit $z\to 0$ (except for the $n=0$ mode in which the logarithmic term becomes dominant). For the modes p_{nl} regular at the event horizon the dominant power-law behavior is given by $z^{n/2}$, and the WKB forms (13) and (12) for q_{nl} and p_{nl} remain exact up to these dominant power-law terms. Hence, the value of A_n for $n\geqslant 1$ is exactly given by the WKB calculation in the limit $z\to 0$, and we will obtain a precise value of $\langle\phi^2\rangle_{ren}$ at the event horizon by taking account of the additional correction A_0 to Eq. (6).

To resolve the problem of logarithmic divergence, however, it is important to note that the WKB form for q_{nl} fails to give the logarithmic behavior, which should play the role of canceling the logarithmic term contained in the DeWitt-Schwinger renormalization counterterms. (Because the leading logarithmic behavior in A_n would be $z^n\ln z$, the value of $\langle\phi^2\rangle_{ren}$ can become regular at the event horizon only by considering a more precise treatment of the $n=0$ mode beyond the WKB level, while the same analysis for the $n=1$ mode is also necessary to obtain a regular value of $\langle T_a^b\rangle_{ren}$.) Hence, our key approach is to study the modified Bessel forms for the modes instead of the WKB forms as follows:

$$p_{nl}=\left(\frac{\chi}{r^2 w}\right)^{1/2} I_n(\chi), \tag{26}$$

and

$$q_{nl}=\left(\frac{\chi}{r^2 w}\right)^{1/2} K_n(\chi), \tag{27}$$

where we have

$$\chi=\int_{r_+}^{r}(w/f)dr, \tag{28}$$

for which it is easy to check the validity of the Wronskian condition

$$p_{nl}\frac{dq_{nl}}{dr}-q_{nl}\frac{dp_{nl}}{dr}=-\frac{1}{r^2 f}. \tag{29}$$

The ordinary WKB forms are given if we assume p_{nl} and q_{nl} to be proportional to $I_{1/2}$ and $K_{1/2}$, respectively. Now, the function w introduced in place of W should satisfy the equation

$$\frac{w^2}{f^2}\left\{1+\frac{1}{\chi^2}\left(n^2-\frac{1}{4}\right)\right\}=\frac{n^2\kappa^2}{f^2}+\frac{l(l+1)+m^2r^2}{fr^2}+\frac{1}{2w}\frac{d^2w}{dr^2}-\frac{3}{4}\frac{1}{w^2}\left(\frac{dw}{dr}\right)^2+\frac{1}{2r^2fw}\frac{d(r^2w)}{dr}\frac{df}{dr}. \tag{30}$$

If w is rewritten into

$$w\equiv f^{1/2}y/r_+, \tag{31}$$

the solution of Eq. (30) allows the expansion form

$$y=B\left(1+\sum_{s=1}^{\infty}y_s z^s\right). \tag{32}$$

From the well-known behavior of the modified Bessel function $K_n(\chi)$ near $\chi=0$, it is easy to see that q_{nl} has the expected logarithmic behavior near the event horizon.

By substituting Eq. (32) into Eq. (30) with the expansion in powers of z, we obtain the recurrence relation between the coefficients B and y_s. For example, the lowest relation leads to

$$\frac{2\kappa r_+}{3}(n^2-1)\left(y_1-2+\frac{1}{2\kappa r_+}\right)=\nu(\nu+1)+2\kappa r_+-B^2, \tag{33}$$

where $\nu(\nu+1)=l(l+1)+m^2r_+^2$. From the expansion up to the next power of z the relation between y_1 and y_2 turns out to be

$$\frac{2\kappa r_+}{5}(n^2-4)y_2=-\nu(\nu+1)y_1-l(l+1)+U(\kappa r_+,n,y_1), \tag{34}$$

where U is a slightly complicated quadratic function of y_1 which depends on n and κr_+ only. An important point of the expansion form (32) is that we can require y_s to remain finite in the limit $l\to\infty$, for which from Eqs. (33) and (34) the asymptotic values of B and y_1 reduce to

$$B^2=l(l+1)+m^2r_+^2+\frac{1}{3}+n^2\left(2\kappa r_+-\frac{1}{3}\right)+O(l^{-2}), \tag{35}$$

and

$$y_1=-1+O(l^{-2}). \tag{36}$$

This dependence of y_s on l allows us to calculate the mode sum over l in A_n by neglecting the terms with the higher powers of z in Eq. (32), and in the following Eq. (35) will be verified in terms of the cancellation of the logarithmic divergence in $\langle\phi^2\rangle_{ren}$.

We also remark that the amplitude of $\langle\phi^2\rangle_{ren}$ at the event horizon should not be interpreted as a quantity determined only by local geometry. The relations (33) and (34) allow us to give a conjecture that the recurrence relation is truncated within a finite sequence, and for the nth mode the finite set consisting of B, $y_1,\ldots,y_{n-1}$ is completely determined for any value of l. However, the coefficient y_n remains unknown, unless the higher infinite sequence of the recurrence relation is consistently solved for satisfying the boundary condition $y\to(m^2r_+^2+n^2\kappa^2r_+^2)^{1/2}$ at $z\to\infty$ as an eigenvalue problem. In particular, for $n=0$ we cannot give B for lower values of l without a further analysis of Eq. (11). This is the problem to be solved in the subsequent sections, and in this section we use Eq. (35) for $n=0$ to derive the logarithmic term in A_0.

By taking the limit $z\to 0$, we can give the mode sum over l for $n=0$ written by the form

$$A_0=\sum_{l=0}^{\infty}\left\{\frac{2l+1}{\kappa r_+^2}K_0(B\sqrt{2z/\kappa r_+})I_0(B\sqrt{2z/\kappa r_+})-\frac{1}{r_+\sqrt{2\kappa r_+ z}}\right\}. \tag{37}$$

Then, we apply the Plana sum formula (20) to Eq. (37), in which the modified Bessel functions are allowed to reduce to

$$K_0(B\sqrt{2z/\kappa r_+})\simeq-\gamma-\ln(B\sqrt{z/2\kappa r_+}), \tag{38}$$

and

$$I_0(B\sqrt{2z/\kappa r_+})\simeq 1, \tag{39}$$

except for the integral defined by

$$\int_0^\infty dl\left\{\frac{2l+1}{\kappa r_+^2}K_0(B\sqrt{2z/\kappa r_+})I_0(B\sqrt{2z/\kappa r_+})-\frac{1}{r_+\sqrt{2\kappa r_+ z}}\right\}. \tag{40}$$

To calculate the integral (40), let us recall that B is a function of l satisfying

$$2BdB/dl=2l+1+O(l^{-2}) \tag{41}$$

in the large l limit and replace the integral of the modified Bessel functions over l by that over B to use the integral formula

$$\int 2BK_0(Bv)I_0(Bv)dB=B^2\{K_0(Bv)I_0(Bv)+K_1(Bv)I_1(Bv)\} \tag{42}$$

for any variable v. Then, the same approximations with Eqs. (38) and (39) is applicable to the remaining integral given by

$$\int_0^\infty \frac{dl}{\kappa r_+^2}\left(2l+1-2B\frac{dB}{dl}\right)K_0(B\sqrt{2z/\kappa r_+})I_0(B\sqrt{2z/\kappa r_+}), \tag{43}$$

and we arrive at the final result for A_0 in the limit $z\to 0$ such that

$$A_0=\frac{S_0}{\kappa r_+^2}+\frac{m^2}{\kappa}\left\{\gamma+\frac{1}{2}\ln\left(\frac{z}{2\kappa r_+}\right)\right\}, \tag{44}$$

where

$$S_0=\left(B_0^2-\frac{1}{2}\right)\ln B_0-\frac{B_0^2}{2}-\int_0^\infty dl\left(2l+1-2B\frac{dB}{dl}\right)\ln B-\int_0^\infty\frac{idl}{e^{2\pi l}-1}\{(2il+1)\ln B(il)+(2il-1)\ln B(-il)\}, \tag{45}$$

if we denote $B(l=0)$ by B_0. Hence, by adding $\kappa A_0/8\pi^2$ to $\langle\phi^2\rangle_{ap}$, the logarithmic divergence at the event horizon turns out to be canceled, and we obtain the renormalized value denoted by $\langle\phi^2\rangle_H$ as follows:

$$\langle\phi^2\rangle_H=\frac{\kappa}{24\pi^2 r_+}+\frac{m^2}{16\pi^2}\{1-\ln(m^2r_+^2)\}+\frac{S_0}{8\pi^2 r_+^2}. \tag{46}$$

It is interesting to note that the absence of the logarithmic divergence of $\langle\phi^2\rangle_{ren}$ at the event horizon is assured only by giving the asymptotic value (35) of B for the $n=0$ mode with very large l, which is determined through the local analysis near $r=r_+$. Though in general we cannot obtain the renormalized value itself without deriving B for lower l modes, the large-mass limit can be an exceptional case for which the local analysis remains useful, and we calculate $\langle\phi^2\rangle_H$ up to the order of m^{-2} in the next section as a simple application of the procedure presented here.

III. THE LARGE-MASS LIMIT

To calculate the integral of B in S_0 over l under the large-mass limit $mr_+\gg 1$, it is convenient to give the expansion form of B in inverse powers of $\nu(\nu+1)$, by keeping the quantity $\mu\equiv m^2r_+^2/\nu(\nu+1)$ to be of order of unity. [For the first integral present in S_0 we cannot assume $l(l+1)$ to be much smaller than mr_+, while for the second integral the approximation $\mu\simeq 1-l(l+1)(mr_+)^{-2}$ may be allowed.] The expansion of B^2 should be done up to the terms of order of $1/\nu(\nu+1)$ for obtaining the m^{-2} terms of $\langle\phi^2\rangle_H$. Then, the recurrence relation subsequent to Eqs. (33) and (34) becomes necessary, for which the leading terms turn out to be

$$y_2=-\frac{y_1^2}{2}+\frac{3}{2}(1-\mu)+O(m^{-2}). \tag{47}$$

The key point of Eq. (47) is the absence of y_3 in the leading-order relation, from which Eqs. (33) and (34) for $n=0$ can give

$$y_1=-1+\mu+\frac{\kappa r_+}{\nu(\nu+1)}\eta+0(m^{-4}), \tag{48}$$

and

$$B^2=\nu(\nu+1)+\frac{1}{3}(1+2\kappa r_+\mu)+\frac{2\kappa^2r_+^2}{3\nu(\nu+1)}\eta+O(m^{-4}), \tag{49}$$

where

$$\eta=-\frac{1}{60\kappa^2r_+^2}+\left(\frac{4}{5}-\frac{1}{15\kappa r_+}\right)\mu-\frac{37}{15}\mu^2. \tag{50}$$

Now it is easy to calculate the integrals in Eq. (45) up to the terms of order of $(mr_+)^{-2}$, and we can confirm the cancellation of all the terms much larger than $(mr_+)^{-2}$ in the expression (46) for $\langle\phi^2\rangle_H$, giving the result

$$\langle\phi^2\rangle_H=\frac{1}{720\pi^2m^2r_+^4}(16\kappa^2r_+^2-4\kappa r_++1). \tag{51}$$

Note that the well-known m^{-2} term $\langle\phi^2\rangle_{m^{-2}}$ of the DeWitt-Schwinger approximation for $\langle\phi^2\rangle$ can be written by

$$\langle\phi^2\rangle_{m^{-2}}=\frac{1}{2880\pi^2m^2}(R_{abcd}R^{abcd}-R_{ab}R^{ab}) \tag{52}$$

for the Reissner-Nordström background (with vanishing Ricci scalar), where R_{abcd} and R_{ab} are the Riemann and Ricci tensors, respectively. If evaluated at the event horizon $r=r_+$, this DeWitt-Schwinger term is found to be identical with Eq. (51). Hence, for very massive fields with $mr_+\gg 1$ in equilibrium at black-hole temperature $T=\kappa/2\pi$, we can claim the validity of the DeWitt-Schwinger approximation near the event horizon, as was previously shown in numerical calculations [2,5]. Further, if mr_+ is fixed, the tail part (51) in the range $mr_+\gg 1$ becomes minimum at the black-hole temperature corresponding to $\kappa r_+=1/8$, rather than at the low-temperature limit $\kappa r_+\ll 1$. The m-κ coupling can give a slightly complicated change to the amplitude of vacuum polarization. In the next section we see a result of the m-κ coupling as the suppression of temperature-induced excitation in a small-mass range.

IV. THE SMALL-MASS LIMIT

Now we consider scalar fields with very small mass $mr_+\ll 1$, for which the temperature-induced excitation given by Eq. (5) will dominate. To reveal some correction due to the small field mass, let us begin with a brief analysis of purely massless fields. It is easy to see that Eq. (11) for the massless $n=0$ modes becomes equal to Legendre's differential equation, if we use the variable x defined by $x=1+(z/\kappa r_+)$. Then, from the behavior of Legendre functions at $x\to 1$ and $x\to\infty$, the modes q_{0l} and p_{0l} should be chosen to be

$$q_{0l}=Q_l(x),\quad p_{0l}=P_l(x). \tag{53}$$

The mode sum in Eq. (10) for $n=0$ is known to be precisely zero for any x [8], and from Eq. (46) the vacuum polarization at the event horizon reduces to

$$\langle\phi^2\rangle_H=\frac{\kappa}{24\pi^2 r_+}, \tag{54}$$

which should be interpreted to be purely induced by the black-hole temperature. For purpose of extending the result to massive fields, it is useful to check explicitly through the procedure given in the previous sections that S_0 in Eq. (46) vanishes.

Recall that the function $Q_l(x)$ has logarithmic branch point at $x=1$, and the dominant behavior near the point is

$$Q_l\simeq\frac{1}{2}\ln\left(\frac{2}{x-1}\right)-\psi(1+l)-\gamma, \tag{55}$$

where $\psi(s)$ is the logarithmic derivative of the gamma function (i.e., a polygamma function), and we have $\psi(1)=-\gamma$ for Euler's constant γ. By comparing the logarithmic behavior of Q_l with Eq. (38) for the modified Bessel function, we can determine the coefficient B as follows:

$$B=\exp\{\psi(1+l)\}. \tag{56}$$

To calculate the integrals over l in S_0, we use integral representations for the polygamma function. For example, we obtain

$$-\int_0^\infty\frac{idl}{e^{2\pi l}-1}\{(2il+1)\psi(1+il)+(2il-1)\psi(1-il)\}$$
$$=\int_0^\infty dt\left\{\frac{e^{-t}}{6t}-\frac{2t^{-2}+t^{-1}}{e^t-1}+\frac{1}{4}\left(\frac{\cosh(t/2)}{\sinh^3(t/2)}-\coth(t/2)+1\right)\right\}, \tag{57}$$

by virtue of the formula

$$\psi(s)=\int_0^\infty dt\left(\frac{e^{-t}}{t}-\frac{e^{-ts}}{1-e^{-t}}\right). \tag{58}$$

Another useful formula is given by

$$\psi(s)=\ln s-\frac{1}{2s}-\frac{1}{12s^2}-\int_0^\infty dt\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right)e^{-ts}, \tag{59}$$

through which we arrive at the result

$$\int_0^\infty dl\left\{2e^{2\psi(1+l)}\frac{d\psi(1+l)}{dl}-(2l+1)\right\}\psi(1+l)$$
$$=\left(\frac{1}{2}+\gamma\right)e^{-2\gamma}-\frac{1}{3}+\int_0^\infty dt\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}\right)\times\left(\frac{2}{t^2}+\frac{1}{t}\right). \tag{60}$$

Then, it becomes easy to calculate the integral over t for the sum of Eqs. (57) and (60), and we obtain $S_0=0$.

For the massive $n=0$ mode we rewrite Eq. (11) into the form

$$(x^2-1)\frac{d^2q_{0l}}{dx^2}+2x\frac{dq_{0l}}{dx}-\{l(l+1)+m^2r_+^2(\kappa r_+x+1-\kappa r_+)^2\}q_{0l}=0, \tag{61}$$

which can clarify the deviation from Legendre's differential equation. In this section a small-mass field having $mr_+\ll 1$ is assumed, and the solution perturbed by the field mass is given by

$$q_{0l}=Q_{l'}(x)+q_l(x), \tag{62}$$

where $l'-l\equiv\delta=O(m^2r_+^2)$. Because the terms proportional to $m^2r_+^2$ in Eq. (61) are dependent on x, we use the recurrence formula valid for $Q_{l'}$ (and also for $P_{l'}$) such that

$$(l'+1)Q_{l'+1}-(2l'+1)xQ_{l'}+l'Q_{l'-1}=0, \tag{63}$$

and the perturbed part q_l is expanded in terms of Legendre functions as follows:

$$q_l=\sum_{k=1}^{\infty}(c_k^{(l)}Q_{l'+k}+c_{-k}^{(l)}Q_{l'-k}). \tag{64}$$

The coefficients c_k and c_{-k} together with the eigenvalue δ are determined by solving the recurrence relation

$$c_k^{(l)}\{(l'+k)(l'+k+1)-l(l+1)-m^2r_+^2v_{l'+k}^{(0)}\}$$
$$=m^2r_+^2\sum_{j=1}^{2}(v_{l'+k}^{(j)}c_{k+j}^{(l)}+v_{l'+k}^{(-j)}c_{k-j}^{(l)}), \tag{65}$$

where $c_0^{(l)}=1$, and

$$v_i^{(0)}=(1-\kappa r_+)^2+\kappa^2r_+^2\frac{2i(2i+1)-1}{(2i-1)(2i+3)},$$

$$v_i^{(1)}=2\kappa r_+(1-\kappa r_+)\frac{i+1}{2i+3},$$

$$v_i^{(-1)}=2\kappa r_+(1-\kappa r_+)\frac{i}{2i-1}, \tag{66}$$

$$v_i^{(2)}=\kappa^2r_+^2\frac{(i+1)(i+2)}{(2i+3)(2i+5)},$$

$$v_i^{(-2)}=\kappa^2r_+^2\frac{i(i-1)}{(2i-3)(2i-1)}.$$

Then, the first-order perturbation is found to be

$$q_l=\frac{m^2\kappa r_+^3}{2l+1}\left\{(1-\kappa r_+)(Q_{l+1}-Q_{l-1})+\frac{\kappa r_+}{2}\left(\frac{(l+1)(l+2)Q_{l+2}}{(2l+3)^2}-\frac{l(l-1)Q_{l-2}}{(2l-1)^2}\right)\right\}, \tag{67}$$

and

$$\delta=\frac{m^2r_+^2}{2l+1}\left\{(1-\kappa r_+)^2+\kappa^2r_+^2\frac{2l(l+1)-1}{(2l-1)(2l+3)}\right\}, \tag{68}$$

for which the coefficient B is estimated to be

$$B=e^{\psi(l+1)}\left\{1+\delta\frac{d\psi(l+1)}{dl}+m^2r_+^2\left(\frac{\kappa r_+(1-\kappa r_+)}{l(l+1)}+\frac{\kappa^2r_+^2}{(2l-1)(2l+3)}\right)\right\}. \tag{69}$$

Using these equations, one may calculate the polarization amplitude $\langle\phi^2\rangle_H$ at the event horizon. However, for $l=0$ the value of B becomes divergent as a result of the existence of the undefined function Q_{-k} in Eq. (67). This will mean a dominant contribution of the $l=0$ mode in the small-mass limit.

To estimate more precisely $B=B_0$ for $l=0$, the subscript l in the Legendre functions should be replaced by l', taking account of the approximate relation $Q_{\delta-k}\simeq P_{k-1}/\delta$ for $\delta\ll 1$. Then, the term $m^2r_+^2Q_{\delta-1}$ which appears in q_0 should be interpreted to be of order of unity, contradictory to the perturbation scheme. This problem is resolved if we add another independent solution for Eq. (61) written by

$$p_0=d_0^{(0)}P_\delta+\sum_{k=1}^{\infty}(d_k^{(0)}P_{\delta+k}+d_{-k}^{(0)}P_{\delta-k}) \tag{70}$$

to q_0 as follows,

$$q_0=\sum_{k=1}^{\infty}(c_k^{(0)}Q_{\delta+k}+c_{-k}^{(0)}Q_{\delta-k})+p_0, \tag{71}$$

where we require that $\delta^{-1}c_{-1}^{(0)}+d_0^{(0)}\equiv\varepsilon\ll 1$ for $d_0^{(0)}$ of order of unity. Of course, the coefficients $d_k^{(0)}$ should satisfy the same recurrence relation with $c_k^{(0)}$, and we obtain for $k\geqslant 1$

$$d_{2k-1}^{(0)}=O((mr_+)^{2k}),\quad d_{2k}^{(0)}=O((mr_+)^{2k}), \tag{72}$$

in addition to the ratio $d_{-k}^{(0)}/d_{k-1}^{(0)}=O(m^2r_+^2)$. Then, the asymptotic behavior of the $l=0$ mode q_{00} at $x\gg 1$ is approximately given by

$$q_{00}\simeq\frac{1}{x}+\sum_{k=0}^{\infty}\frac{\Gamma(k+(1/2))}{\sqrt{\pi}\Gamma(k+1)}(\delta^{-1}c_{-k-1}^{(0)}+d_k^{(0)})(2x)^k, \tag{73}$$

which should be consistent with the boundary condition

$$q_{00}\simeq\frac{1}{x}\exp(-m\kappa r_+^2x) \tag{74}$$

at a distant region far from the event horizon.

To check the consistency, let us derive the approximate recurrence relation which is valid up to the leading order of $m^2r_+^2$ and reduces to

$$\frac{c_{-1-2k}^{(0)}}{c_{1-2k}^{(0)}}=\frac{d_{2k}^{(0)}}{d_{2k-2}^{(0)}}=m^2\kappa^2r_+^4\frac{2k-1}{(2k+1)(4k-1)(4k-3)}, \tag{75}$$

and

$$\frac{\delta^{-1}c^{(0)}_{-2k-2}+d^{(0)}_{2k+1}}{\delta^{-1}c^{(0)}_{-2k}+d^{(0)}_{2k-1}}=m^2\kappa^2r_+^4\frac{2k}{(2k+2)(4k+1)(4k-1)}. \tag{76}$$

Noting the relations between the lowest coefficients such that

$$\delta^{-1}c^{(0)}_{-1}=2m^2\kappa r_+^3(1-\kappa r_+) \tag{77}$$

and

$$\delta^{-1}c^{(0)}_{-2}+d^{(0)}_{1}=m^2\kappa^2r_+^4/2, \tag{78}$$

we arrive at the result

$$q_{00}\simeq\sum_{k=1}^{\infty}\frac{(m\kappa r_+^2x)^{2k}}{x(2k)!}+\varepsilon\sum_{k=1}^{\infty}\frac{(m\kappa r_+^2x)^{2k-2}}{(2k-1)!}, \tag{79}$$

which can satisfy the boundary condition if $\varepsilon=-m\kappa r_+^2$.

Unfortunately, we cannot determine ε to the order of $m^2r_+^2$, unless the recurrence relation is studied to the higher order. Hence, we only keep the leading correction of order of mr_+ in the $l=0$ mode,

$$q_{00}\simeq Q_0-m\kappa r_+^2, \tag{80}$$

which means that $B_0=e^{-\gamma}(1+m\kappa r_+)$. For the $l\geqslant 1$ modes q_{0l} we must also consider the perturbation with the terms written by the Legendre functions $P_k(x)$. However, it is sure that no perturbation of order of mr_+ does not appear for $l\geqslant 1$, and we obtain

$$S_0\simeq-\ln(1+m\kappa r_+^2)\simeq-m\kappa r_+^2, \tag{81}$$

if we omit the higher-order corrections. Now the vacuum polarization given by Eq. (46) for small-mass fields becomes approximately

$$\langle\phi^2\rangle_H\simeq\frac{\kappa}{24\pi^2r_+}(1-3mr_+), \tag{82}$$

which clearly shows that the temperature-induced excitation is suppressed by field mass. As m becomes larger, the amplitude may monotonously decrease in the whole mass range extending to $mr_+\gg 1$ where the DeWitt-Schwinger approximation $\langle\phi^2\rangle_H\sim(mr_+)^{-2}$ is valid. This simple dependence on m is supported through numerical calculations for several values of mr_+ in Schwarzschild background ($\kappa r_+=1/2$) [7]. In the next section, however, we point out a different dependence on field mass, which is a resonant behavior of $\langle\phi^2\rangle_H$ remarkable in the low-temperature case $\kappa r_+\ll 1$.

V. MASS-INDUCED EXCITATION

Let us turn attention to quantum fields at the event horizon of nearly extreme black holes to show an interesting feature of the mass-induced excitation of vacuum polarization. Then, we do not limit the range of the parameter mr_+, but we solve Eq. (61) under the assumption $\kappa r_+\ll 1$ with the help of the technique of asymptotic matching.

At large values of x Eq. (61) reduces to the form

$$\frac{d^2q_{0l}}{dx^2}+\frac{2}{x}\frac{dq_{0l}}{dx}-\left(\frac{\nu(\nu+1)}{x^2}+\frac{2m^2\kappa r_+^3}{x}+m^2\kappa^2r_+^4\right)q_{0l}=0, \tag{83}$$

in which we cannot neglect the terms depending on κr_+ to require the exponential decrease of q_{0l}. For the approximate differential equation we obtain the solution

$$q_{0l}=W_{-mr_+,\nu+1/2}(2m\kappa r_+^2x)/x, \tag{84}$$

where $W_{a,b}$ denotes the Whittaker function with the asymptotic behavior

$$W_{a,b}(u)\simeq u^a\exp(-u/2) \tag{85}$$

as $u\to\infty$. This asymptotic solution can remain valid in the range

$$1\ll x\ll 1/\kappa r_+, \tag{86}$$

where we obtain the approximate behavior

$$q_{0l}\simeq\frac{\Gamma(-2\nu-1)}{\Gamma(mr_+-\nu)}(2m\kappa r_+^2x)^{\nu+1}x^{-1}+\frac{\Gamma(2\nu+1)}{\Gamma(mr_++\nu+1)}(2m\kappa r_+^2x)^{-\nu}x^{-1}. \tag{87}$$

Note that if $x\ll 1/\kappa r_+$, Eq. (61) becomes approximately equal to Legendre's differential equation, giving the solution

$$q_{0l}=CP_\nu(x)+DQ_\nu(x). \tag{88}$$

The coefficients C and D should be determined by matching with the approximate solution (87), and it is easy to see that the ratio C/D is of order of $(m\kappa r_+^2)^{2\nu+1}$. Hence, we can neglect the term P_ν in q_{0l}, and the asymptotic behavior at $x\to 1$ turns out to be

$$q_{0l}\simeq-D\left\{\frac{1}{2}\ln\left(\frac{x-1}{2}\right)+\gamma+\psi(\nu+1)\right\}, \tag{89}$$

from which we obtain

$$B=e^{\psi(\nu+1)}, \tag{90}$$

for calculating S_0 (and $\langle\phi^2\rangle_H$) through Eq. (45).

A useful expression of S_0 to understand the field-mass dependence is derived if we use the integral formula

$$\psi(\nu+1)=\frac{1}{2}\ln\left(\nu^2+\nu+\frac{1}{4}\right)+\int_0^\infty\frac{2tdt}{(e^{2\pi t}+1)[t^2+\nu^2+\nu+(1/4)]}. \quad (91)$$

In fact, for $F(l)\equiv(-i)\{(2il+1)\ln B(il)+(2il-1)\ln B(-il)\}$ which is one of the integrands in S_0, we obtain

$$F(l)=l\ln\{(l^2-\zeta)^2+l^2\}+\arctan\left(\frac{l}{\zeta-l^2}\right)-\int_0^\infty\frac{8tdt}{e^{2\pi t}+1}\frac{l^2+(1/2)-t^2-\zeta}{(l^2-t^2-\zeta)^2+l^2}, \quad (92)$$

where $\zeta=m^2r_+^2+(1/4)$, and the value of $\arctan(u)$ runs from 0 to π in the range $0\leqslant u\leqslant\infty$. Further, the integral given by

$$\int dl\left(2l+1-2B\frac{dB}{dl}\right)\ln B \quad (93)$$

is rewritten into the form

$$\frac{1}{2}\left\{\nu(\nu+1)+\frac{1}{4}\right\}\left\{\ln\left[\nu(\nu+1)+\frac{1}{4}\right]-1\right\}-e^{2\psi(\nu+1)}\left\{\psi(\nu+1)-\frac{1}{2}\right\}+2\int_0^\infty\frac{tdt}{e^{2\pi t}+1}\ln\left[t^2+\nu(\nu+1)+\frac{1}{4}\right], \quad (94)$$

which is equal to zero as $l\to\infty$. We therefore arrive at the result

$$S_0=\frac{1}{2}\left(\zeta-\frac{1}{2}\right)\ln\zeta-\frac{\zeta}{2}+\int_0^\infty\left\{\frac{tG(t)}{e^{2\pi t}+1}+\frac{H(t)}{e^{2\pi t}-1}\right\}dt, \quad (95)$$

where

$$G(t)=2\ln(t^2+\zeta)-\frac{1}{t^2+\zeta}-8\int_0^\infty\frac{dl}{e^{2\pi l}-1}\frac{l^2+(1/2)-t^2-\zeta}{(l^2-t^2-\zeta)^2+l^2}, \quad (96)$$

and

$$H(t)=t\ln\{(t^2-\zeta)^2+t^2\}+\arctan\left(\frac{t}{\zeta-t^2}\right). \quad (97)$$

Under the low-temperature approximation $\kappa r_+\ll 1$ we neglect the term $\kappa/24\pi^2r_+$ in Eq. (46), and the polarization amplitude at the event horizon is finally given by

$$8\pi^2r_+^2\langle\phi^2\rangle_H=\frac{m^2r_+^2}{2}\ln\left(\frac{\zeta}{m^2r_+^2}\right)-\frac{1}{8}(1+\ln\zeta)+\int_0^\infty\left\{\frac{tG(t)}{e^{2\pi t}+1}+\frac{H(t)}{e^{2\pi t}-1}\right\}dt. \quad (98)$$

Now it is easy to check the value of $\langle\phi^2\rangle_H$ in the large-mass limit $mr_+\gg 1$, and we obtain

$$8\pi^2r_+^2\langle\phi^2\rangle_H\simeq\frac{1}{90m^2r_+^2}, \quad (99)$$

for which we can reconfirm that it is equal to the DeWitt-Schwinger approximation (with $\kappa r_+\to 0$). We can also consider the small-mass limit $mr_+\ll 1$ under the condition $m/\kappa\gg 1$, and the approximate expression of $\langle\phi^2\rangle_H$ becomes

$$8\pi^2r_+^2\langle\phi^2\rangle_H\simeq -m^2r_+^2\left\{\frac{1}{2}+\gamma+\ln(mr_+)\right\}, \quad (100)$$

which can remain positive by virtue of the existence of the logarithmic term $-m^2r_+^2\ln(mr_+)$.

We evaluate numerically the integrals in the expression of $\langle\phi^2\rangle_H$, and the field-mass dependence is shown in Fig. 1. Note that the maximum excitation of $\langle\phi^2\rangle_H$ occurs at $mr_+\simeq 0.38$, and the peak amplitude denoted by $\langle\phi^2\rangle_{max}$ is estimated to be $8\pi^2r_+^2\langle\phi^2\rangle_{max}\simeq 0.0424$. We can clearly see a

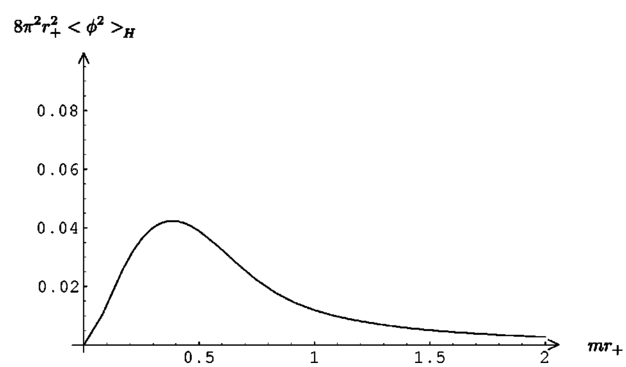

FIG. 1. The field-mass dependence of vacuum polarization $\langle\phi^2\rangle_H$ at the nearly extreme Reissner-Nordström horizon $r=r_+$. The amplitude has a resonance peak at $mr_+\simeq 0.38$ and a tail part decreasing in proportion to m^{-2} for very massive fields.

resonance behavior of the polarization amplitude for massive fields with the Compton wavelength $1/m$ of order of r_+ and also the tail part given by Eq. (99) in the mass range of $mr_+ \gg 1$.

VI. SUMMARY

We have studied vacuum polarization of quantized scalar fields in the Reissner-Nordström background by means of the Euclidean space Green's function. In particular, the renormalized expression $\langle\phi^2\rangle_H$ at the event horizon $r=r_+$ has been derived by revealing the contribution of the $n=0$ mode, which can cancel the logarithmic divergence.

We have found the dependence of $\langle\phi^2\rangle_H$ on field mass m: (1) The tail part observed in the large-mass limit $mr_+ \gg 1$ becomes equal to the DeWitt-Schwinger approximation. (2) For small-mass fields a suppression of temperature-induced excitation due to the coupling between m and κ occurs according to $\langle\phi^2\rangle_H = \langle\phi^2\rangle_T(1-3mr_+)$, where the massless part with the amplitude proportional to the black-hole temperature $T=\kappa/2\pi$ is given by $8\pi^2 r_+^2\langle\phi^2\rangle_T = \kappa r_+/3$. We can expect that mass-induced excitation becomes important for massive fields with $mr_+ \simeq 1$. Unfortunately, it is difficult to investigate in detail various aspects of the m-κ coupling in the case that both mr_+ and κr_+ are of order of unity. (3) Our main result therefore has been to show a resonance behavior of mass-induced excitation of vacuum polarization around nearly extreme Reissner-Nordström black holes with $\kappa r_+ \ll 1$: If the Compton wavelength $1/m$ of a massive field is of order of the black-hole radius r_+, the amplitude of vacuum polarization has a peak at the resonance mass given by $mr_+ \simeq 0.38$.

There should be a critical temperature $T_c = \kappa_c/2\pi$ of black holes in the range $0 < \kappa r_+ < 1/2$, below which a resonance peak of $\langle\phi^2\rangle_H$ is observed in the field-mass dependence. (If $\kappa > \kappa_c$, the polarization amplitude monotonously decreases with increase of m.) Though the value of κ_c remains uncertain within the analysis presented here, it is sure that dominant fields as quantum perturbations near the Schwarzschild horizon should be massless, while nearly extreme holes will have a quantum atmosphere dominated by fields with a resonance mass. The peak amplitude given by $8\pi^2 r_+^2\langle\phi^2\rangle_{max} \simeq 0.0424$ at the nearly extreme Reissner-Nordström horizon is not so much smaller than the massless part given by $8\pi^2 r_+^2\langle\phi^2\rangle_T = 1/6$ at the Schwarzschild horizon with the same area $4\pi r_+^2$. (If compared under the same black-hole mass M, the former becomes slightly larger than the latter evaluated by $8\pi^2 M^2\langle\phi^2\rangle_T = 1/24$.) Considering a black hole evolving toward the zero-temperature state with a fixed radius r_+, we conclude that the mass m of dominant fields generating vacuum polarization shifts from $mr_+ \ll 1$ to $mr_+ \simeq 0.38$ as the contribution of mass-induced excitation becomes important, without changing the polarization amplitude so much. Quantum back-reaction due to massive fields [9] will become very important for nearly extreme (low-temperature) black holes.

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