Gravitational scaling dimensions

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A model for quantized gravitation based on simplicial lattice discretization is studied in detail using a comprehensive finite size scaling analysis combined with renormalization group methods. The results are consistent with a value for the universal critical exponent for gravitation, $\nu = 1/3$, and suggest a simple relationship between Newton's constant, the gravitational correlation length and the observable average space-time curvature. Some perhaps testable phenomenological implications of these results are discussed. To achieve a high numerical accuracy in the evaluation of the lattice path integral a dedicated parallel machine was assembled.

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I. INTRODUCTION

One of the outstanding problems in theoretical physics is a determination of the quantum-mechanical properties of Einstein's relativistic theory of gravitation. Approaches based on linearized perturbation methods have had moderate success so far, as the underlying theory is known not to be perturbatively renormalizable [1,2]. Because of the complexity of even such approximate calculations, a fundamental coupling of the theory, the bare cosmological constant term, is usually set to zero, thus further restricting the potential physical relevance of the results. In addition gravitational fields are themselves the source for gravitation already at the classical level, which leads to the problem of an intrinsically non-linear theory where perturbative results are possibly of doubtful validity for sufficiently strong effective couplings. This is especially true in the quantum domain, where large fluctuations in the gravitational field appear at short distances. In general nonperturbative effects can give rise to novel behavior in a quantum field theory and, in particular, to the emergence of non-trivial fixed points of the renormalization group (a phase transition in statistical mechanics language). It has been realized for some time that in general the universal low and high energy behavior of field theories is almost completely determined by the fixed point structure of the renormalization group trajectories [3].

The situation described above bears some resemblance to the theory of strong interactions, quantum chromodynamics. Non-linear effects are known here to play an important role, and end up restricting the validity of perturbative calculations to the high energy, short distance regime, where the effective gauge coupling can be considered weak due to asymptotic freedom [4]. For low energy properties Wilson's discrete lattice formulation, combined with the renormalization group and computer simulations, has provided so far the only convincing evidence for quark confinement and chiral symmetry breaking, two phenomena which are invisible to any order in the weak coupling, perturbative expansion.

A discrete lattice formulation can be applied to the problem of quantizing gravitation. Instead of continuous metric fields, one deals with gravitational degrees of freedom which live only on discrete space-time points and interact locally with each other. In Regge's simplicial formulation of gravity [5] one approximates the functional integration over continuous metrics by a discretized sum over piecewise linear simplicial geometries [6–9]. In such a model, the role of the continuum metric is played by the edge lengths of the simplices, while curvature is described by a set of deficit angles, which can be computed via known formulas as functions of the given edge lengths. The simplicial lattice formulation of gravity is locally gauge invariant [10] and can be shown to contain perturbative gravitons in the lattice weak field expansion [6], making it an attractive and faithful lattice regularization of the continuum theory.

The discretized theory is restricted to a finite set of dynamical variables, once a set of suitable boundary conditions are imposed such as periodic or with some assigned boundary manifold. In the end the original continuum theory of gravity is to be recovered as the space-time volume is made large and the fundamental lattice spacing of the discrete theory is set to zero, possibly without having to rely, at least in principle, on any further approximation to the original continuum theory.

Quantum fluctuations in the underlying geometry are represented in the discrete theory by fluctuations in the edge lengths, which can be modeled by a well-defined, and numerically exact, stochastic process. In analogy with other field theory models studied by computer, calculations are usually performed in the Euclidean imaginary time framework, which is the only formulation amenable to a controlled numerical study, at least for the immediate foreseeable future. The Monte Carlo method, based on the concept of importance sampling, is well suited for evaluating the discrete path integral for gravity and for computing the required averages and correlation functions. By a careful and systematic analysis of the lattice results, the critical exponents can be extracted and the scaling properties of invariant correlation functions determined from first principles.

Studies on small lattices suggest a rich scenario for the ground state of quantum gravity [7,9,11,12]. The present evidence indicates that simplicial quantum gravity in four dimensions exhibits a phase transition (in the bare coupling G) between *two phases*: a strong coupling phase, in which the

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geometry is smooth at large scales and quantum fluctuations in the gravitational field eventually average out and are bounded, and a weak coupling phase, in which the geometry is degenerate and space-time collapses into a lower-dimensional manifold, bearing some physical resemblance to a branched polymer. Only the smooth, small negative curvature and thus anti–de Sitter-like phase appears to be physically acceptable. Phrased in different terms, the two phases of quantized gravity found in [12] can loosely be described as having, in one phase (with bare coupling $G<G_c$, the rough branched polymer-like phase),

$$\langle g_{\mu\nu}\rangle = 0, \tag{1}$$

while, in the other (with bare coupling $G>G_c$, the smooth phase),

$$\langle g_{\mu\nu}\rangle \approx c\,\eta_{\mu\nu}, \tag{2}$$

with a vanishingly small negative average curvature in the vicinity of the critical point at G_c. The existence of a phase transition at finite coupling G, usually associated in quantum field theory with the appearance of an ultraviolet fixed point of the renormalization group, implies in principle non-trivial, calculable non-perturbative scaling properties for correlations and effective coupling constants and, in particular, in the case at hand for Newton's gravitational constant. Since only the smooth phase with $G>G_c$ has acceptable physical properties, one would conclude on the basis of fairly general renormalization group arguments that at least in this lattice model the gravitational coupling can only *increase* with distance. Furthermore, the rise of the gravitational coupling in the infrared region rules out the applicability of perturbation theory to the low energy domain, to the same extent that such an approach is deemed to be inapplicable to study the low-energy properties of asymptotically free gauge theories.

It is a remarkable property of quantum field theories that a wide variety of physical properties can be determined from a relatively small set of universal quantities [13]: namely, the universal leading critical exponents, computed in the vicinity of some fixed point (or fixed line) of the renormalization group equations. In the lattice theory the presence of a fixed point or phase transition is often inferred from the appearance of non-analytic terms in invariant local averages, such as for example the average curvature

$$\langle l^2\rangle \frac{\left\langle \int d^4x\sqrt{g}R(x)\right\rangle}{\left\langle \int d^4x\sqrt{g}\right\rangle} \equiv \mathcal{R}(k) \underset{k\to k_c}{\sim} -A_{\mathcal{R}}(k_c-k)^{4\nu-1}, \tag{3}$$

where $k=1/8\pi G$. From such averages one can determine the value for ν, the correlation length exponent,

$$\xi(k) \underset{k\to k_c}{\sim} A_\xi (k_c-k)^{-\nu}. \tag{4}$$

An equivalent result, relating the quantum expectation value of the curvature to the physical correlation length ξ, is

$$\mathcal{R}(\xi) \underset{k\to k_c}{\sim} \xi^{1/\nu-4}. \tag{5}$$

Matching of dimensionalities in these equations is restored by supplying appropriate powers of the ultraviolet cutoff, the Planck length $l_P=\sqrt{G}$. The exponent ν is known to be related to the derivative of the beta function for G in the vicinity of the ultraviolet fixed point,

$$\beta'(G_c) = -1/\nu. \tag{6}$$

In addition, the correlation length ξ itself determines the long-distance decay of the connected, invariant two-point correlations at fixed geodesic distance d. For the curvature correlation one has, for distances much larger compared to the correlation length,

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c \underset{d\gg\xi}{\sim} d^{-\sigma}e^{-d/\xi}, \tag{7}$$

while for shorter distances one expects a slower power law decay

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c \underset{d\ll\xi}{\sim} \frac{1}{d^{2(4-1/\nu)}}. \tag{8}$$

The possibility of non-trivial scaling dimensions in the theory of gravitation is not new and was pointed out some time ago in a series of interesting papers [14]. Moreover, it is easy to see that the scale dependence of the effective Newton constant is given by

$$G(r) = G(0)[1+c(r/\xi)^{1/\nu}+O((r/\xi)^{2/\nu})], \tag{9}$$

with c a calculable numerical constant. In this last expression the momentum scale ξ^{-1} plays a role similar to the scaling violation parameter $\Lambda_{\overline{MS}}$ of QCD. It seems natural, although paradoxical at first, to associate ξ with some macroscopic cosmological length scale, such as the Hubble distance cH_0^{-1}, with the lack of screening of gravitational interactions ultimately accounting for such an unusual interpretation [12,15]. Of course an increase of the gravitational coupling at large distances signals a likely breakdown of perturbation theory for computing low energy properties of gravity.

It should be clear, even from this brief discussion, that the critical exponents by themselves already provide a significant amount of useful information about the continuum theory. In reality, the complexity of the lattice interactions and the practical need to sample many statistically independent field configurations contributing to the path integral, which is necessary for correctly incorporating into the model the effects of quantum-mechanical fluctuations, leads to the requirement of powerful computational resources. The results presented in this paper were obtained using a dedicated custom-built 20-GFlop 64-processor parallel computer, described in detail in [16].

Finally one should mention that recently there has been a significant resurgence of interest in the classical applications of the Regge formulation to gravity. A description of the methods as applied to several aspects of the initial value problem in general relativity can be found in the recent references in [17]. For a related approach to lattice gravity based on dynamical triangulations see also [18].

A brief outline of the paper is as follows. Section II contains a discussion of general and finite size scaling and related issues as they apply to the lattice theory of gravity. Section III touches on the issue of the unboundedness of the Euclidean gravitational action. Section IV defines local curvature averages and their fluctuations, while Sec. V introduces a set of exact sum rules for averages which follow from the scaling properties of the partition function. Section VI defines a set of invariant correlations and discusses how they relate to the local fluctuations defined previously. Section VII includes a general discussion of the expected properties of the theory in the presence of an ultraviolet fixed point, including expectations based on the analytical $2+\epsilon$ expansion. In Sec. VIII the numerical results are presented. Section IX contains a discussion of the possible future physical relevance of the results, while Sec. X contains the conclusions.

II. FINITE SIZE SCALING

One of the most important quantities used in establishing the continuum limit of a lattice field theory are the critical exponents. Reliable estimates for the exponents in a lattice field theory require a comprehensive finite-size analysis, a procedure by which accurate values for the critical exponents are obtained by taking into account the linear size dependence of the result computed in a finite volume V. One starts from the general Euclidean action (or statistical mechanics Hamiltonian)

$$H=\sum_i g_i\mathcal{O}_i \tag{10}$$

with g_i the coupling associated with the operator $\mathcal{O}_i$. In the gravitational case the couplings would correspond to the bare cosmological constant, the Newtonian gravitational constant and the higher derivative coupling. Close to a renormalization group fixed point denoted by $\{g_i^*\}$ one chooses the $\mathcal{O}_i$'s to be eigenvectors of the linearized renormalization group transformation, such that

$$g_i-g_i^*\rightarrow b^{y_i}(g_i-g_i^*), \tag{11}$$

where b is the scale factor of the transformation. In the simplest statistical mechanics systems, such as a ferromagnet in the absence of an external magnetic field, one has $\mathcal{O}\sim H$ as the only relevant operator (in the sense that $y>0$) and $g\sim t=T-T_c$. As will be discussed below, in the gravitational case the role of T is played by the bare gravitational coupling G. Additional operators appearing in the action are classified as marginal ($y=0$) or irrelevant. The relevance of the energy operator reflects the fact that close to the critical point t is the only parameter that needs to be tuned to achieve criticality, synonymous with long range correlations. Universality of critical behavior then accounts for the fact that many diverse physical systems exhibit the same scaling behavior in the vicinity of the critical point, as a consequence of a divergent correlation length [13].

In practice the renormalization group approach is brought in via a slightly different route, involving a change in the overall linear size of the system. The usual starting point for the derivation of the scaling properties of the theory is the renormalization group (RG) behavior of the free energy $F=-\log Z/V$:

$$F(t,\{u_j\})=F_{reg}(t,\{u_j\})+b^{-d}F_{sing}(b^{y_t}t,\{b^{y_j}u_j\}), \tag{12}$$

where F_{sing} is the singular, non-analytic part of the free energy, and F_{reg} is the regular part. b is the block size in the RG transformation, while y_t and y_j $(j\geqslant 2)$ are the relevant eigenvalues of the RG transformation (for more details see the review in [19]). One denotes here by $y_t>0$ the relevant eigenvalue, while the remaining eigenvalues $y_j\leqslant 0$ are associated with either marginal or irrelevant operators. Usually y_t^{-1} is called ν, while the next subleading exponent y_2 is denoted $-\omega$.

The correlation length ξ determines the asymptotic decay of correlations, in the sense that one expects for example, for the two-point function at large distances,

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \underset{|x-y|\gg\xi}{\sim} e^{-|x-y|/\xi}. \tag{13}$$

The scaling equation for the correlation length itself,

$$\xi(t)=b\,\xi(b^{y_t}t), \tag{14}$$

implies for $b=t^{-1/y_t}$ that $\xi\sim t^{-\nu}$ with a correlation length exponent

$$\nu=1/y_t. \tag{15}$$

Derivatives of the free energy F with respect to t then determine, after setting the scale factor $b=t^{-1/y_t}$, the scaling properties of physical observables, including corrections to scaling [20]. Thus for example, the second derivative of the free energy with respect to t yields the specific heat exponent $\alpha=2-d/y_t=2-d\nu$:

$$\frac{\partial^2}{\partial t^2}F(t,\{u_j\})\sim t^{-(2-d\nu)}. \tag{16}$$

In the gravitational case one identifies the scaling field t with k_c-k, where $k=1/16\pi G$ involves the bare Newton's constant. The appearance of singularities in physical averages, obtained from appropriate derivatives of F, is rooted in the fact that close to the critical point at $t=0$ the correlation length diverges.

The above results can be extended to the case of a finite lattice of volume V and linear dimension $L=V^{1/d}$. The volume-dependent free energy is then written as

$$F(t,\{u_j\},L^{-1})=F_{reg}(t,\{u_j\})+b^{-d}F_{sing}(b^{y_t}t,\{b^{y_j}u_j\},b/L). \tag{17}$$

For $b=L$ (a lattice consisting of only one point) one obtains the finite size scaling (FSS) form of the free energy (for a detailed presentation of this procedure see [21]; see also [22,23] for a field-theoretic justification). After taking derivatives with respect to the fields t and $\{u_j\}$, the FSS scaling form for physical observables follows. For a quantity O diverging like t^{-x_O} in the infinite volume limit one has

$$O(L,t)=L^{x_O/\nu}\left[\tilde{f}_O\left(\frac{L}{\xi(\infty,t)}\right)+\mathcal{O}(\xi^{-\omega},L^{-\omega})\right], \qquad (18)$$

with $\tilde{f}_O$ a smooth scaling function, and $\xi(\infty,t)$ the infinite volume correlation length. For sufficiently large volumes the correction to scaling term involving ω can be neglected, but in general one needs to be aware of their presence if either the volumes are not large enough or if the corrections are large due to a large amplitude or small exponent. Some properties of the scaling function $\tilde{f}_O(y)$ can be deduced on general grounds: it is expected to show a peak if the finite volume value for O is peaked; it is analytic at $x=0$ since no singularity can develop in a finite volume, and $\tilde{f}_O(y) \sim \tilde{y}^{-x_O}$ for large y for a quantity O which diverges as t^{-x_O} in the infinite volume limit.

The last expression is useful when the infinite-volume correlation length is known. But since close to the critical point $\xi \sim t^{-\nu}$, one can deduce the equivalent scaling from

$$O(L,t)=L^{x_O/\nu}[\tilde{f}_O(Lt^{\nu})+\mathcal{O}(L^{-\omega})], \qquad (19)$$

which relies on knowledge of t and, thus, of the critical point, instead. For a state of the art application of the above methods to the 3D Ising model see [24].

The previous discussion applies to continuous, second order phase transitions. First order phase transitions are driven by instabilities and are in general not governed by any renormalization group fixed point. The underlying reason is that the correlation length does not diverge at the transition point, and thus the system never becomes scale invariant. Exponents for continuous, second order phase transition in general obey the rigorous bound

$$y_t<d, \quad \text{or} \quad \nu>1/d. \qquad (20)$$

A first order phase transition in renormalization group theory, on the other hand, can be associated with the somewhat pathological case $\nu=1/d$, for which the first derivative of the free energy develops a step-function singularity. In a renormalization group framework the corresponding pseudocritical point is denoted as a discontinuity fixed point [25].

In the simplest case, a first order transition develops as the system tunnels between two neighboring minima of the free energy. In the metastable branch the free energy acquires a complex part with an essential singularity in the coupling located at the first order transition point [26,27]. As a consequence, such a singularity is not generally visible from the stable branch, in the sense that a power series expansion in the temperature is unaffected by such a singularity. Indeed in the infinite volume limit the singularity associated with a first order transition at T_c becomes infinitely sharp, like a δ- or θ-function type singularity. The singularity in the free energy at the endpoint of the metastable branch (at say T^*) then cannot be explored directly; it has to reached by an analytic continuation from the stable side of the free energy branch.

III. UNBOUNDEDNESS OF THE EUCLIDEAN THEORY

Perturbation theory on a lattice and in the continuum suggests the presence of an instability in the Euclidean formulation for sufficiently smooth manifold. It is also known that the above instability is associated with the appearance of a wrong sign for the conformal mode. On the lattice the instability seems to persist close to the critical point [12], which suggests that the continuum limit has to be reached by some sort of analytic continuation from the stable phase towards the critical point, naturally defined as the point in coupling constant space where the correlation length diverges.

In the weak-field expansion [28] the Einstein-Hilbert action contains both spin-2 (graviton) and spin-0 (conformal mode) contributions. In the continuum one can by a judicious choice of invariant correlation functions isolate physical properties of the graviton from the conformal mode. A similar result holds on the lattice, as can be seen by expanding the Regge action about a regular lattice and using the fact that the lattice and continuum actions are equivalent for sufficiently smooth manifolds [6,29]. In general, after expanding the metric around flat space (which requires $\lambda=0$),

$$g_{\mu\nu}=\eta_{\mu\nu}+\sqrt{16\pi G}h_{\mu\nu}, \qquad (21)$$

one can cast the lowest order quadratic contribution to the action in the form

$$I_E[h_{\mu\nu}]=\frac{1}{2}\int d^4x\, h_{\mu\nu}V_{\mu\nu\lambda\sigma}h_{\lambda\sigma}, \qquad (22)$$

where V is a matrix which can be expressed in terms of spin projection operators. In momentum space it can be written as

$$V=[P^{(2)}-2P^{(0)}]p^2, \qquad (23)$$

where $P^{(2)}$ and $P^{(0)}$ are spin-2 and spin-0 projection operators introduced in [30]. Physically, the two terms correspond to the propagation of the graviton and of the conformal mode, respectively, with the latter one appearing with the ''wrong'' sign. In the ''Landau'' gauge, with a gauge fixing term $\alpha^{-1}(\partial_\mu h^{\mu\nu})^2$ and $\alpha=0$, one obtains, for the graviton propagator in momentum space,

$$G_{\mu\nu\lambda\sigma}(p)=\frac{P^{(2)}_{\mu\nu\lambda\sigma}}{p^2}-\frac{\frac{1}{2}P^{(0)}_{\mu\nu\lambda\sigma}}{p^2}. \qquad (24)$$

The unboundedness of the Euclidean gravitational action shows up clearly in the weak field expansion, with the spin-0 mode acquiring a propagator term with the wrong sign.[1] It has been argued that in weak field perturbation theory and in order to avoid the unboundedness problem one should perform the functional integral over metrics by distorting the integration contour so as to include complex conformal factors [31]. One drawback of this prescription is that it only appears applicable within the framework of perturbation theory. For a recent review of the Euclidean instability problem see [32].

In the presence of a cosmological constant, things are further complicated by the fact that since flat space is no longer a solution of the classical equations of motion, and the above expansion for the metric loses part of its meaning due to the presence of the tadpole term. But after shifting to the correct 0th order solution, a similar result is obtained. One can further modify the action to include additional invariant terms, but things do not get any better. In the presence of higher derivative terms in the gravitational action, the above result is modified by terms $O(p^4)$, and becomes [33]

$$G_{\mu\nu\lambda\sigma}(p)=\frac{P^{(2)}_{\mu\nu\lambda\sigma}}{p^2+\frac{2a}{k}p^4}+\frac{\frac{1}{2}P^{(0)}_{\mu\nu\lambda\sigma}}{-p^2+\frac{a}{k}p^4}. \tag{25}$$

The p^4 terms improve the ultraviolet behavior of the theory, but do not remove the unboundedness problem, which reappears for sufficiently small p^2, in the low momentum or long-distance limit. Moreover, the resulting theory is most likely not unitary unless the coupling a is vanishingly small. The lack of positivity of physical correlations for $a>0$ can be seen explicitly even in a non-perturbative treatment [34], and makes such a modified theory of gravitation in the end somewhat unattractive.

IV. LOCAL AVERAGES AND FLUCTUATIONS

In the following the relevant definitions for gravitational averages and correlations on the lattice will be briefly recalled, in a form which will be used in later sections. The starting point for a non-perturbative study of quantum gravity is a suitable definition of the discrete Feynman path integral. In the simplicial lattice approach one starts from the discretized Euclidean path integral for pure gravity, with the squared edge lengths taken as fundamental variables:

$$Z_L=\int_0^\infty \prod_s (V_d(s))^\sigma \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2] \times \exp\left\{-\sum_h (\lambda V_h - k\,\delta_h A_h + a\,\delta_h^2 A_h^2/V_h+\cdots)\right\}. \tag{26}$$

[1]It should be noted that such an instability is not peculiar to gravitation. Indeed the Euclidean path integral for the one-dimensional Coulomb potential, an otherwise completely well-behaved quantum mechanical system, already exhibits such an instability. It would be premature to conclude from such a result that the problem is physically ill posed.

The above expression represents a lattice discretization of the continuum Euclidean path integral for pure quantum gravity,

$$Z_C=\int \prod_x (\sqrt{g(x)})^\sigma \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \times \exp\left\{-\int d^4x\sqrt{g}\left(\lambda-\frac{k}{2}R+\frac{a}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+\cdots\right)\right\}, \tag{27}$$

with $k^{-1}=8\pi G$, and G Newton's constant, and reduces to it for smooth enough field configurations. In the discrete case the integration over metrics is replaced by integrals over the elementary lattice degrees of freedom, the squared edge lengths. The discrete gravitational measure in Z_L can be considered as the lattice analogue of the DeWitt [35] continuum functional measure [15]. The δA term in the lattice action is the well-known Regge term [5], and reduces to the Einstein-Hilbert action in the lattice continuum limit [6,29]. A cosmological constant term is needed for convergence of the path integral, while the curvature squared term allows one to control the fluctuations in the curvature [7,9,11,12]. In practice, and for obvious phenomenological reasons, one is only interested in the limit when the higher derivative contributions are small compared to the rest, $a\to 0$. In this limit the theory depends, in the absence of matter and after a suitable rescaling of the metric, only on one bare parameter, the dimensionless coupling k^2/λ. Without loss of generality, one can therefore set the bare cosmological constant $\lambda=1$.

Some partial information about the behavior of physical correlations can be obtained indirectly from local invariant averages. In [7,12] gravitational observables such as the average curvature and its fluctuation were introduced. The appropriate lattice analogues of these quantities are readily written down by making use of the usual correspondences $\int d^4x\sqrt{g}\to\sum_{\mathrm{hinges}\ h} V_h$, etc. On the lattice the natural choices for invariant operators are

$$\sqrt{g}(x)\to \sum_{\mathrm{hinges}\ h\supset x} V_h$$

$$\sqrt{g}R(x)\to 2\sum_{\mathrm{hinges}\ h\supset x} \delta_h A_h$$

$$\sqrt{g}R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}(x)\to 4\sum_{\mathrm{hinges}\ h\supset x} (\delta_h A_h)^2/V_h \tag{28}$$

[we have omitted here on the right-hand side (RHS) an overall numeric coefficient, which will depend on how many hinges are actually included in the summation; if the sum extends over all hinges within a single hypercube, then there will be a total of 50 hinge contributions]. In this paper no higher derivative terms will be considered, and thus only the first and second operators will be used in the following discussion.

On the lattice one prefers to define quantities in such a way that variations in the average lattice spacing $\sqrt{\langle l^2\rangle}$ are compensated by the appropriate factor as determined from dimensional considerations. In the case of the average curvature one defines the lattice quantity $\mathcal{R}$ as

$$\mathcal{R}(k) \equiv \langle l^2 \rangle \frac{\left\langle 2 \sum_h \delta_h A_h \right\rangle}{\left\langle \sum_h V_h \right\rangle}, \tag{29}$$

which in the continuum corresponds to

$$\mathcal{R}(k) \sim \frac{\left\langle \int d^4x \sqrt{g} R(x) \right\rangle}{\left\langle \int d^4x \sqrt{g} \right\rangle}, \tag{30}$$

and similarly for the curvature fluctuation,

$$\chi_{\mathcal{R}}(k) \equiv \frac{\left\langle \left(\sum_h \delta_h A_h \right)^2 \right\rangle - \left\langle \sum_h \delta_h A_h \right\rangle^2}{\left\langle \sum_h V_h \right\rangle}, \tag{31}$$

which in the continuum corresponds to

$$\chi_{\mathcal{R}}(k) \sim \frac{\left\langle \left(\int \sqrt{g} R \right)^2 \right\rangle - \left\langle \int \sqrt{g} R \right\rangle^2 .}{\left\langle \int \sqrt{g} \right\rangle} \tag{32}$$

The latter is related to the connected curvature correlation at zero momentum:

$$\chi_{\mathcal{R}} \sim \frac{\int d^4x \int d^4y \langle \sqrt{g(x)} R(x) \sqrt{g(y)} R(y) \rangle_c}{\left\langle \int d^4x \sqrt{g(x)} \right\rangle}. \tag{33}$$

Both $\mathcal{R}$ and $\chi_{\mathcal{R}}$ are related to derivatives of Z_L with respect to k:

$$\mathcal{R}(k) \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z_L \tag{34}$$

and

$$\chi_{\mathcal{R}}(k) \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z_L \, . \tag{35}$$

One can contrast the behavior of the preceding quantities, associated strictly with the curvature, with the analogous quantities involving the local volumes (and which correspond to the square root of the determinant of the metric in the continuum). Consider the average volume per site,

$$\langle V \rangle \equiv \frac{1}{N_0} \left\langle \sum_h V_h \right\rangle, \tag{36}$$

and its fluctuation defined as

$$\chi_V(k) \equiv \frac{\left\langle \left(\sum_h V_h \right)^2 \right\rangle - \left\langle \sum_h V_h \right\rangle^2}{\left\langle \sum_h V_h \right\rangle}, \tag{37}$$

where one denotes by V_h the volume associated with the hinge h. In the continuum it corresponds to the expression

$$\chi_V(k) \sim \frac{\left\langle \left(\int \sqrt{g} \right)^2 \right\rangle - \left\langle \int \sqrt{g} \right\rangle^2}{\left\langle \int \sqrt{g} \right\rangle}. \tag{38}$$

The latter is related to the connected volume correlator at zero momentum:

$$\chi_V \sim \frac{\int d^4x \int d^4y \langle \sqrt{g(x)} \sqrt{g(y)} \rangle_c}{\left\langle \int d^4x \sqrt{g(x)} \right\rangle}. \tag{39}$$

The average volume per site, $\langle V \rangle$, and its fluctuation χ_V are simply related to derivatives of Z_L with respect to the bare cosmological constant λ:

$$\langle V \rangle \sim \frac{\partial}{\partial \lambda} \ln Z_L \tag{40}$$

and

$$\chi_V(k) \sim \frac{\partial^2}{\partial \lambda^2} \ln Z_L . \tag{41}$$

One would expect the fluctuations in the curvature to be sensitive to the presence of a spin-2 massless particle, while fluctuations in the volume would only probe the correlations in the conformal mode channel.

V. SUM RULES

In this section some useful sum rules will be derived, which follow from simple scaling properties of the discrete functional integral. These will be later used in the discussion of the numerical results. A simple scaling argument, based on neglecting the effects of curvature terms entirely (which vanish in the vicinity of the critical point), gives first of all an estimate of the average volume per edge:

$$\langle V_l \rangle \sim \frac{2(1+\sigma d)}{\lambda d} \underset{d=4,\sigma=0}{\sim} \frac{1}{2\lambda}. \tag{42}$$

In four dimensions the numerical simulations with $\sigma = 0$ agree quite well with the above formula.

Additional exact lattice identities can be obtained by examining the scaling properties of the action and measure. The bare couplings k and λ in the gravitational action are dimensionful in four dimensions, but one can define the dimensionless ratio k^2/λ and rescale the edge lengths so as to eliminate the overall length scale $\sqrt{k/\lambda}$. As a consequence the path integral for pure gravity,

$$Z_L(\lambda,k,a)=\int d\mu[l^2]e^{-I[l^2]}, \tag{43}$$

obeys the simple scaling property

$$\begin{aligned} Z_L(\lambda,k,a) &= \left(\frac{k}{\lambda}\right)^{N_1} Z_L\left(\frac{k^2}{\lambda},\frac{k^2}{\lambda},a\right) \\ &= (\lambda)^{-N_1/2} Z_L\left(1,\frac{k}{\sqrt{\lambda}},a\right) = (\lambda)^{-N_1} Z_L\left(\frac{\lambda}{k^2},1,a\right), \end{aligned} \tag{44}$$

where N_1 represents the number of edges in the lattice, and the dl^2 measure $(\sigma=0)$ has been selected [15], which is the lattice analog of the continuum DeWitt functional measure. This equation implies in turn a sum rule for local averages, which (again for the specific case of the dl^2 measure) reads

$$2\lambda\left\langle \sum_h V_h \right\rangle - k\left\langle \sum_h \delta_h A_h \right\rangle - N_1 = 0, \tag{45}$$

and is easily derived from Eq. (44) and the definitions in Eqs. (34) and (40). N_0 represents the number of sites in the lattice, and the averages are defined per site (for the hypercubic lattice used in this paper, $N_1=15N_0$, $N_2=50N_0$, $N_3=36N_0$ and $N_4=24N_0$). The coefficients on the LHS of the equation reflect the scaling dimensions of the various terms, with the last term on the LHS term arising from the scaling property of the functional measure. This last formula is very useful in checking the accuracy of numerical calculations and the convergence properties of the Monte Carlo sampling, and is usually satisfied to high accuracy $\tilde{\mathcal{O}}(10^{-4})$. It is easy to see that a similar sum rule holds for the fluctuations:

$$\begin{aligned} 4\lambda^2\left[\left\langle\left(\sum_h V_h\right)^2\right\rangle - \left\langle \sum_h V_h\right\rangle^2\right] - k^2\left[\left\langle\left(\sum_h \delta_h A_h\right)^2\right\rangle \right. \\ \left. - \left\langle \sum_h \delta_h A_h \right\rangle^2\right] - 2N_1 = 0. \end{aligned} \tag{46}$$

Further sum rules can be derived by considering even higher derivatives of $\ln Z_L$ with respect of λ and k. The last equation relates the fluctuation in the curvature to fluctuations in the volumes, and thus implies a relationship between their singular parts as well. In particular, a divergence in the curvature fluctuation implies a divergence of the same nature in the volume fluctuation. In light of the previous discussion, from now on we shall consider without loss of generality only the case of bare coupling $\lambda=1$. As a consequence, all lengths will be tacitly expressed in units of the fundamental microscopic length scale $\lambda^{-1/4}$.

VI. INVARIANT CORRELATIONS

In quantized gravity complications arise due to the fact that the physical distance between any two points x and y in a fixed background geometry,

$$d(x,y|g)=\min_{\xi}\int_{\tau(x)}^{\tau(y)} d\tau \sqrt{g_{\mu\nu}(\xi)\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}}, \tag{47}$$

is a fluctuating quantity dependent on the choice of background metric. In addition, the Lorentz group used to classify spin states is meaningful only as a local concept. Since the simplicial formulation is completely coordinate independent, the introduction of the local Lorentz group requires the definition of a tetrad within each simplex, and the notion of a spin connection to describe the parallel transport of tensors between flat simplices. Some of these aspects have recently been discussed from a continuum point of view in [36–39].

If the deficit angles are averaged over a number of contiguous hinges which share a common vertex, one is naturally lead to the connected correlator

$$G_R(d)\equiv\left\langle \sum_{h\supset x} \delta_h A_h \sum_{h'\supset y} \delta_{h'} A_{h'}\,\delta(|x-y|-d)\right\rangle_c, \tag{48}$$

which probes correlations in the scalar curvatures:

$$G_R(d)\sim\langle \sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c. \tag{49}$$

Similarly one can construct the connected correlator

$$G_V(d)\equiv\left\langle \sum_{h\supset x} V_h \sum_{h'\supset y} V_{h'}\,\delta(|x-y|-d)\right\rangle_c, \tag{50}$$

which probes correlations in the volume elements:

$$G_V\sim\langle \sqrt{g}(x)\sqrt{g}(y)\,\delta(|x-y|-d)\rangle_c. \tag{51}$$

The correlation length ξ is defined through the long-distance decay of the connected, invariant correlations at fixed geodesic distance d. For the curvature correlation one has, at large distances,

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c \underset{d\gg\xi}{\sim} e^{-d/\xi}. \tag{52}$$

At shorter distances one expects a slower, power law decay

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c \underset{d\ll\xi}{\sim} \left(\frac{1}{d}\right)^{2n}, \tag{53}$$

with a power characterized by the exponent n. In both cases, the distances considered are much larger than the lattice spacing, $d,\xi\gg l_0$. From scaling considerations one can show (see below) $n=4-1/\nu$.

Simple scaling arguments allow one to determine the scaling behavior of correlation functions from the critical exponents which characterize the singular behavior of local averages in the vicinity of the critical point. A divergence of the correlation length ξ,

$$\xi(k) \equiv m(k)^{-1} \underset{k\to k_c}{\sim} A_\xi (k_c - k)^{-\nu}, \tag{54}$$

signals the presence of a phase transition, and leads to the appearance of a singularity in the free energy $F(k)$. The presence of a phase transition usually inferred from non-analytic terms in invariant averages, such as the average curvature. The curvature critical exponent δ is introduced via

$$\mathcal{R}(k) \underset{k\to k_c}{\sim} -A_{\mathcal{R}}(k_c - k)^{\delta}. \tag{55}$$

An additive constant could be added, but the evidence up to now points to this constant being zero. Similarly one sets, for the curvature fluctuation,

$$\chi_{\mathcal{R}}(k) \underset{k\to k_c}{\sim} -A_{\mathcal{R}}(k_c - k)^{-(1-\delta)}. \tag{56}$$

Scaling [Eqs. (15)] relates the exponent δ to ν:

$$\nu = \frac{1+\delta}{d}. \tag{57}$$

From such averages one can determine the value for ν, the correlation length exponent. An equivalent result, relating the quantum expectation value of the curvature to the physical correlation length ξ , is

$$\mathcal{R}(\xi) \underset{k\to k_c}{\sim} \xi^{1/\nu - 4}, \tag{58}$$

which is obtained from Eqs. (54) and (55) using (57). Matching of dimensionalities in these equations is restored by supplying appropriate powers of the Planck length $l_P = \sqrt{G}$.

It is then easy to relate the critical exponent ν to the scaling behavior of correlations at large distances. The curvature fluctuation is related to the connected scalar curvature correlator at zero momentum:

$$\chi_{\mathcal{R}}(k) \sim \frac{\int d^4x \int d^4y \langle \sqrt{g}R(x)\sqrt{g}R(y)\rangle_c}{\left\langle \int d^4x \sqrt{g} \right\rangle} \underset{k\to k_c}{\sim} (k_c - k)^{\delta - 1}. \tag{59}$$

A divergence in the fluctuation is then indicative of long range correlations, corresponding to the presence of a massless particle. Very close to the critical point one would expect for large separations a power law decay in the geodesic distance:

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\rangle \underset{|x-y|\to\infty}{\sim} \frac{1}{|x-y|^{2n}}, \tag{60}$$

with the power n related to the exponent δ via $n = \delta d/(1+\delta) = d - 1/\nu$. *A priori* one cannot exclude to possibility that some states acquire a mass away from the critical point, in which case one would expect the following behavior for the correlation functions:

$$\langle \sqrt{g}R(x)\sqrt{g}R(y)\rangle \underset{|x-y|\gg\xi}{\sim} \exp(-|x-y|/\xi), \tag{61}$$

where ξ is the fundamental correlation length and $m = 1/\xi$ the associated mass. The above equation can in fact be considered as a definition for what is meant by the correlation length ξ.

VII. BETA FUNCTION AND CONTINUUM LIMIT

The long distance behavior of quantum field theories is determined by scaling behavior of the coupling constant under a change in the momentum scale. Asymptotically free theories such as QCD lead to vanishing gauge couplings at short distances, while the opposite is true for QED. In general the fixed point(s) of the renormalization group need not be at zero coupling, but can be located at some finite G_c, leading to a non-trivial fixed point or limit cycle [3,4,40].

In the $2+\epsilon$ perturbative expansion for gravity [41] one analytically continues in the spacetime dimension by using dimensional regularization, and applies perturbation theory about $d=2$, where Newton's constant is dimensionless. A similar method is quite successful in determining the critical properties of the $O(n)$-symmetric non-linear sigma model above two dimensions [42]. In this expansion the dimensionful bare coupling is written as $G_0 = \Lambda^{2-d}G$, where Λ is an ultraviolet cutoff (corresponding on the lattice to a momentum cutoff of the order of the inverse average lattice spacing, $\Lambda \sim 1/l_0$). There seem to be some technical difficulties with this expansion due to the presence of kinematic singularities for the graviton propagator in two dimension (the Einstein action is a topological invariant in $d=2$), but which seem to have been overcome recently. A double expansion in G and $\epsilon = d-2$ then leads in lowest order to a nontrivial fixed point in G above two dimensions:

$$\beta(G) \equiv \frac{\partial G}{\partial \log \Lambda} = (d-2)G - \beta_0 G^2 + \cdots, \tag{62}$$

with $\beta_0 > 0$ for pure gravity. To lowest order the ultraviolet fixed point is then at $G_c = 1/\beta_0(d-2)$. Integrating Eq. (62) close to the non-trivial fixed point one obtains, for $G > G_c$,

$$m = \Lambda \exp\left(-\int^G \frac{dG'}{\beta(G')}\right) \underset{G\to G_c}{\sim} \Lambda |G - G_c|^{-1/\beta'(G_c)}, \tag{63}$$

where m is an arbitrary integration constant, with the dimensions of a mass, and which should be associated with some physical scale. It would appear natural here to identify it with the inverse of the gravitational correlation length ($\xi = m^{-1}$) or some scale associated with the average curvature. The derivative of the beta function at the fixed point defines the critical exponent ν, which to this order is independent of β_0, $\beta'(G_c) = -(d-2) = -1/\nu$.

The previous results illustrate how the lattice continuum limit should be taken. It corresponds to $\Lambda \to \infty$, $G \to G_c$ with

m held constant; for fixed lattice cutoff the continuum limit is approached by tuning G to G_c. In four dimensions the exponent ν is defined by

$$m \underset{G\to G_c}{\sim} C\Lambda|G-G_c|^{\nu}, \tag{64}$$

where m is proportional to the graviton mass, and C is a calculable numerical coefficient. The value of ν determines the running of the effective coupling $G(\mu)$, where μ is an arbitrary momentum scale. The renormalization group tells us that in general the effective coupling will grow or decrease with length scale $r=1/\mu$, depending on whether $G>G_c$ or $G<G_c$, respectively. The physical mass parameter m is itself scale independent, and obeys the Callan-Symanzik renormalization group equation

$$\mu\frac{\partial}{\partial\mu}m=\mu\frac{\partial}{\partial\mu}\{C\mu|G(\mu)-G_c|^{\nu}\}=0. \tag{65}$$

As a consequence, for $G>G_c$, corresponding to the smooth phase, one expects, for the running, effective gravitational coupling [12,15],

$$G(r)=G(0)[1+c(r/\xi)^{1/\nu}+O((r/\xi)^{2/\nu})], \tag{66}$$

with c a calculable numerical constant. The physical mass $m=\xi^{-1}$ determines the magnitude of scaling corrections and plays a role similar to $\Lambda_{\overline{\rm MS}}$ in QCD. It cannot be determined perturbatively as it appears as an integration constant. Physically it separates the short distance, ultraviolet regime with characteristic momentum scale μ,

$$l_0^{-1}\gg\mu\gg m, \tag{67}$$

from the large distance, infrared region

$$m\gg\mu\gg L^{-1}, \tag{68}$$

where $L=\langle V\rangle^{1/4}$ is the linear size of the system.

The exponent ν is simply related to the derivative of the beta function for G in the vicinity of the ultraviolet fixed point:

$$\beta'(G_c)=-1/\nu. \tag{69}$$

Thus computing ν is equivalent to computing the derivative of the beta function in the vicinity of the ultraviolet fixed point. There are indications from the lattice theory that only the smooth phase with $G>G_c$ exists (in the sense that spacetime collapses onto itself for $G<G_c$), which would suggest that the gravitational coupling can only *increase* with distance.

One should also perhaps recall here the fact that a bare cosmological constant λ, which could appear in the original action [as indicated in Eq. (26)], has been scaled out when it was set equal to one by rescaling all the edge lengths. If one puts it back in, then the effective Newton's constant would have to be multiplied by that bare scale. As a result one obtains for the running of Newton's constant, valid for "short" distances $\mu\gg m$,

$$G(r)\underset{r\ll\xi}{\sim} l_0^2\lambda^{-1/2}\left[G_c+\left(\frac{r}{C\xi}\right)^{1/\nu}+\cdots\right], \tag{70}$$

where G_c is a pure number of order one, and below it will be argued that $1/\nu=3$. The quantity l_0 is the average lattice spacing, and the correct dimensions for $G(\mu)$ (length squared) have been restored. In addition a bare cosmological constant λ was re-introduced, which was previously set equal to one in Eq. (26) (it fixes the overall length scale in the functional integral over edge lengths).

VIII. NUMERICAL RESULTS

Next we come to a discussion of the numerical methods employed in this work and the analysis of the results. As in previous work, the edge lengths are updated by a straightforward Monte Carlo algorithm, generating eventually an ensemble of configurations distributed according to the action and measure of Eq. (26). Further details of the method as applied to pure gravity are discussed in [7,12], and will not be repeated here.

In this work lattices of size $4\times4\times4\times4$ (with 256 sites, 3840 edges, 6144 simplices), $8\times8\times8\times8$ (with 4096 sites, 6144 edges, 98304 simplices), $16\times16\times16\times16$ (with 65536 sites, 983040 edges, 1572864 simplices) were considered. Even though these lattices are not very large, one should keep in mind that as a result of the simplicial nature of the lattice there are many edges per hypercube with many interaction terms, and as a consequence the statistical fluctuations can be comparatively small, unless measurements are taken very close to a critical point and at rather large separation in the case of the potential. The results presented here are still preliminary, and in the future it should be possible to repeat such calculations with improved accuracy on much larger lattices.

The topology is restricted to a four-torus (periodic boundary conditions). We have argued before that one could perform similar calculations with lattices employing different boundary conditions or topology, but the universal infrared scaling properties of the theory should be determined only by short-distance renormalization effects.

It seems reasonable that based on physical considerations one needs to impose the constraint that the scale of the curvature be much smaller than the average lattice spacing, but still much larger than the overall size of the system, in other words,

$$\langle l^2\rangle\ll\langle l^2\rangle|\mathcal{R}|^{-1}\ll\langle V\rangle^{1/2}, \tag{71}$$

or that in momentum space the physical scales should be much smaller that the ultraviolet cutoff, but much larger than the infrared cutoff. An equivalent requirement is then

$$L^{-1}\lesssim m\lesssim l_0^{-1}, \tag{72}$$

where L is the linear size of the system, $m=1/\xi$, and l_0 the lattice spacing. It should be kept in mind that in this model, and contrary to ordinary gauge theories on a lattice, the lattice spacing is a dynamical quantity. Even close to the criti-

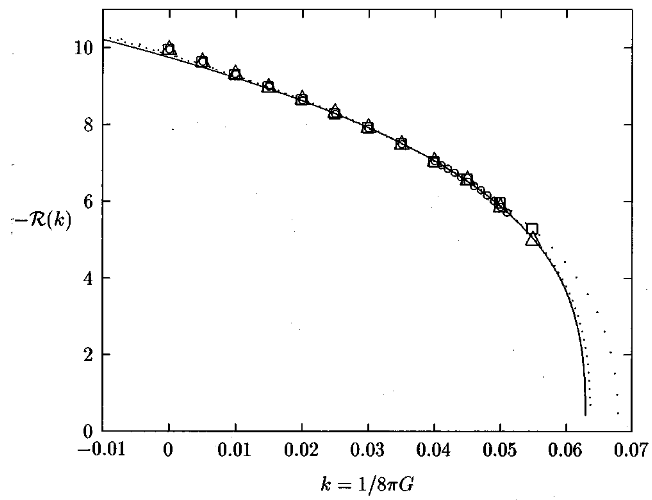

FIG. 1. Average curvature $\mathcal{R}$ as a function of k, on lattices with 4^4 ($\square$), 8^4 ($\triangle$) and 16^4 ($\bigcirc$) sites. Statistical errors $[\sim\mathcal{O}(10^{-3})]$ are much smaller than the size of the symbols. The thin-dotted, dotted and solid lines represent best fits of the form $\mathcal{R}(k) = A(k_c - k)^{\delta}$.

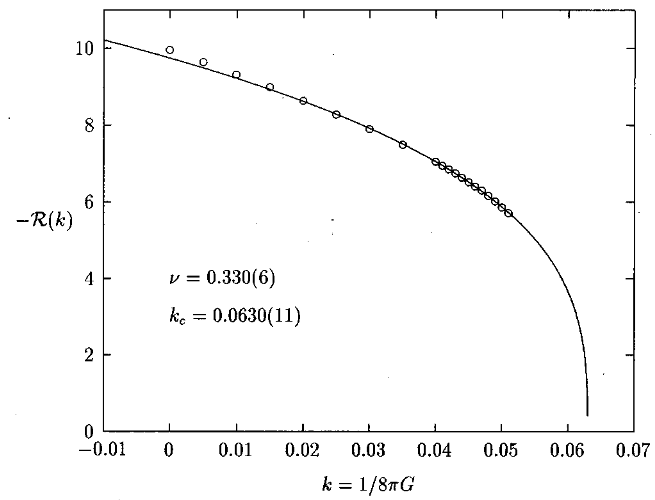

FIG. 2. Average curvature $\mathcal{R}$ as a function of k, on the 16^4 ($\bigcirc$) lattice only. Statistical errors $[\sim\mathcal{O}(10^{-3})]$ are much smaller than the size of the symbols. The solid line represents a best fit of the form $A(k_c - k)^{\delta}$ for $k \geq 0.02$, with $\delta = 4\nu - 1$.

cal point where the curvature vanishes the lattice is by no means regular, and the quantity $l_0 = \sqrt{\langle l^2 \rangle}$ only represents an ''average'' cutoff parameter.

The bare cosmological constant λ appearing in the gravitational action of Eq. (26) was fixed at 1 (since this coupling sets the overall length scale in the problem), and the higher derivative coupling a was set to 0 (pure Regge-Einstein action). For the measure in Eq. (26) this choice of parameters leads to a well-behaved ground state for $k < k_c \approx 0.053$ for $a = 0$ [12,11]. The system then resides in the ''smooth'' phase, with a fractal dimension close to 4; on the other hand, for $k > k_c$ the curvature becomes very large (''rough'' phase), and the lattice tends to collapse into degenerate configurations with very long, elongated simplices [7,9,11,12]. For $a = 0$ we investigated 22 values of k.

On the 16^4 lattice 36000 consecutive configurations were generated for each value of k and 22 different values for k were chosen. The results for different values of k can be considered as completely statistically uncorrelated, since they originated from unrelated configurations. On the smaller 8^4 lattice 100000 consecutive configurations were generated for each value of k. On the 4^4 lattice 500000 consecutive configurations were generated for each value of k. To accumulate the results, the machine ran continuously for about 14 months.

The results obtained for the average curvature $\mathcal{R}$ [defined in Eq. (29)] as a function of the bare coupling k are shown in Fig. 1, on lattices of increasing size with 4^4, 8^4 and 16^4 sites. Figure 2 shows the 16^4 data by itself. The errors in are quite small, of the order of a tenth of 1% or less, and are therefore not visible in the graph.

In [12] it was found that as k is varied, the curvature is negative for sufficiently small k (''smooth'' phase), and appears to go to zero continuously at some finite value k_c. For $k \geq k_c$ the curvature becomes very large, and the simplices tend to collapse into degenerate configurations with very small volumes ($\langle V \rangle / \langle l^2 \rangle^2 \sim 0$) (a similar two-phase structure has been found recently also in the dynamical triangulation approach [18], with the smooth phase replaced by a collapsed unphysical phase). This ''rough'' or ''collapsed'' phase is the region of the usual weak field expansion ($G \to 0$); in the continuum it is characterized by the unbounded fluctuations in the conformal mode. But there appears to be more structure to the data.

Accurate and reproducible curvature data can only be obtained for k below the instability point k_u since, as already pointed out in [12], for $k > k_u \approx 0.053$ an instability develops, presumably associated with the unbounded conformal mode. Its signature is typical of a sharp first order transition, beyond which the system tunnels into the rough, elongated phase which is two dimensional in nature and has no physically acceptable continuum limit. The instability is caused by the appearance of one or more localized singular configuration, with a spike-like curvature singularity. It is not associated with any sort of coherent effect or the appearance of long-range order, and remains localized around a few lattice points. In other words, the correlation length ξ remains finite at k_u. At strong coupling such singular configurations are suppressed by a lack of phase space due to the functional measure, which imposes non-trivial constraints due to the triangle inequalities and their higher dimensional analogues. In other language, the measure regulates the conformal instability at sufficiently strong coupling.

It is characteristic of first order transitions that the free energy develops only a delta-function singularity at k_u, with the metastable branch developing no non-analytic contribution at k_u. Indeed it is well known from the theory of first order transitions that tunneling effects will lead to a purely imaginary contribution to the free energy, with an essential singularity for $k > k_u$ [26]. In the following we shall clearly distinguish the instability point k_u from the true critical point k_c.

As a consequence, the non-analytic behavior of the free energy (and its derivatives which include for example the average curvature) has to be obtained by *analytic continuation* of the Euclidean theory into the metastable branch. This procedure, while unusual, is formally equivalent to the construction of the continuum theory exclusively from its strong coupling (small k, large G) expansion:

$$Z_L(k)=\sum_{n=0}^{\infty} a_n k^n, \tag{73}$$

$$\mathcal{R}(k)=\sum_{n=0}^{\infty} b_n k^n. \tag{74}$$

Given a large enough number of terms in this expansion, the nonanalytic behavior in the vicinity of the true critical point at k_c can then be determined using differential or Pade approximants [43], for appropriate combinations of thermodynamic functions which are expected to be meromorphic in the vicinity of the true critical point [44]. In the present case, instead of the analytic strong coupling expansion, one has at one's disposal a set of (in principle, arbitrarily) accurate data points to which the expected functional form can equally be fitted. And what is assumed is the kind of regularity which is always assumed in extrapolating finite series (whether convergent or asymptotic as in the case of QED or $\lambda\phi^4$ in $d<4$ [45]) to the boundary of their radius of convergence.

Ultimately it should be kept in mind that one is really only interested in the *pseudo-Riemannian* case, and not the Euclidean one for which an instability due to the conformal mode is to be expected. Indeed, had such an instability not occurred, one might wonder if the resulting theory still had any relationship to the original continuum theory. Arguments based on effective actions suggest that if the Euclidean (or, more appropriately, Riemannian) lattice theory eventually approaches the classical continuum theory at large distances and in the vicinity of the critical point, then an instability in the quantum lattice theory *must* develop, since the continuum classical theory is known to be unstable.

In the following only data for $k\leqslant k_u$ will be considered; in fact to add a margin of safety only $k\leqslant 0.051$ will be considered throughout the rest of the paper. This choice will avoid the inclusion in the fits of any data affected by the sharp turnover which appears, for large lattices, at $k=k_u\approx 0.053$.

To extract the critical exponent δ, one fits the computed values for the average curvature to the form [see Eq. (55)]

$$\mathcal{R}(k) \underset{k\to k_c}{\sim} -A_{\mathcal{R}}(k_c-k)^{\delta}. \tag{75}$$

It would seem unreasonable to expect that the computed values for $\mathcal{R}$ are accurately described by this function even for small k. Instead the data are fitted to the above functional form for either $k\geqslant 0.02$ or $k\geqslant 0.03$ and the difference in the fit parameters can be used as one more measure for the error. Additionally, one can include a subleading correction

$$\mathcal{R}(k) \underset{k\to k_c}{\sim} -A_{\mathcal{R}}[k_c-k+B(k_c-k)^2]^{\delta}, \tag{76}$$

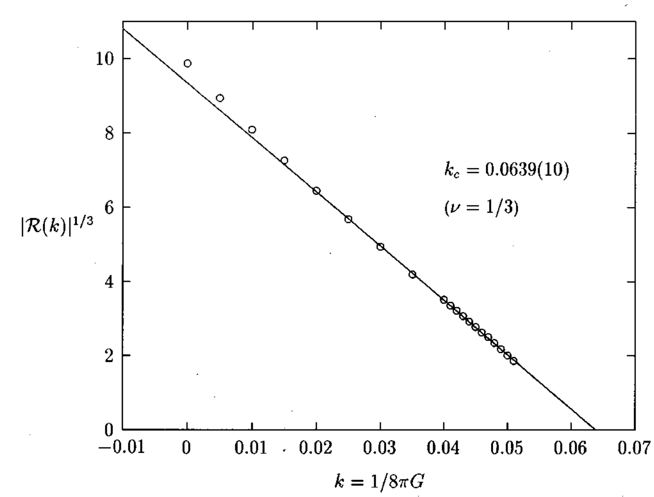

FIG. 3. Average curvature on the 16^4 lattice, raised to the third power. If $\delta=\nu=1/3$, the data should fall on a straight line. The solid line represents a linear fit of the form $A(k_c-k)$. The small deviation from linearity of the transformed data is quite striking.

and use the results to further constraint the errors on $A_{\mathcal{R}}$, k_c and $\delta=4\nu-1$.

Using this set of procedures one obtains on the lattice with 4^4 sites,

$$k_c=0.0676(20), \quad \nu=0.343(8), \tag{77}$$

and on the lattice with 8^4 sites one finds

$$k_c=0.0614(27), \quad \nu=0.322(16), \tag{78}$$

while on the lattice with 16^4 sites one finds

$$k_c=0.0630(11), \quad \nu=0.330(6). \tag{79}$$

These results suggest that ν is very close to 1/3, and can be compared to the older low-accuracy estimate on an 8^4 lattice obtained in [12] for $a=0$, $\nu=0.33(3)$.

Figure 3 shows a graph of the average curvature $\mathcal{R}(k)$ raised to the third power. One would expect to get a straight line close to the critical point if the exponent for $\mathcal{R}(k)$ is exactly 1/3. The numerical data indeed support this assumption, and in fact the linearity of the results close to k_c is quite striking. The computed data are quite close to a straight line over a wide range of k values, providing further support for the assumption of an algebraic singularity for $\mathcal{R}(k)$ itself, with exponent close to 1/3. Using this procedure one finds, on the 16^4-site lattice,

$$k_c=0.0639(10). \tag{80}$$

Since the critical exponents play such a central role in determining the existence and nature of the continuum limit, it appears desirable to have an independent way of estimating them, which either does not depend on any fitting procedure or at least analyzes a different and complementary set of data. By studying the dependence of averages on the physical size of the system, one can independently estimate the critical exponents.

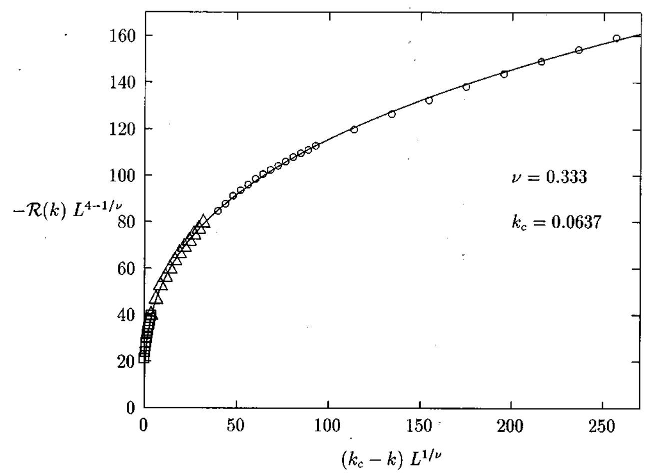

FIG. 4. Finite size scaling behavior of the scaled curvature versus the scaled coupling. Here $L=4$ for the lattice with 4^4 sites (□), $L=8$ for a lattice with 8^4 sites (△), and $L=16$ for the lattice with 16^4 sites (○). Statistical errors are comparable to the size of the dots. The solid line represents a best fit of the form $a+bx^c$. Finite size scaling predicts that all points should lie on the same universal curve. At $k_c=0.0637$ the scaling plot gives the value $\nu=0.333$.

Figure 4 shows a graph of the scaled curvature $\mathcal{R}(k)L^{4-1/\nu}$ for different values of $L=4,8,16$, versus the scaled coupling $(k_c-k)L^{1/\nu}$. If scaling involving k and L holds according to Eq. (19), with $x_{\mathcal{O}}=1-4\nu$ the scaling dimension for the curvature, then all points should lie on the same universal curve. From Eq. (19), with $t\sim k_c-k$ and $x_O=-\delta=1-4\nu$, one has

$$\mathcal{R}(k,L)=L^{-(4-1/\nu)}[\tilde{\mathcal{R}}((k_c-k)L^{1/\nu})+\mathcal{O}(L^{-\omega})] \quad (81)$$

where $\omega>0$ is a correction-to-scaling exponent. The data support well such scaling behavior, and provide a further stringent test on the value for ν, which appears to be consistent, within errors, with 1/3.

Figure 5 shows explicitly the size dependence of the average curvature. For small k the volume dependence is small, and gradually increases towards the critical point. Such a trend is in agreement with the expectation that the correlation length ξ is growing as one approaches the critical point, leading to a more marked volume dependence. For fixed $k\neq k_c$ one expects, on the four-torus,

$$\mathcal{R}_L(k) \underset{L\gg 1/m(k)}{\sim} \mathcal{R}_\infty(k)+Am(k)^{1/2}L^{-3/2}e^{-m(k)L}+\cdots, \quad (82)$$

where $L=V^{1/4}$ is the linear size of the system and $m=\xi^{-1}$ is the lightest mass in the theory. Combining and averaging the estimates from correlations [34], potential [12] and finite size corrections to the average curvature one can in fact estimate the magnitude of this mass directly. One obtains $m\sim 0.81(k_c-k)^{1/3}$, giving a correlation length of about two lattice spacings at $k=0.050$.

The value of k_c itself should depend on the size of the system. Indeed such a dependence is found when comparing

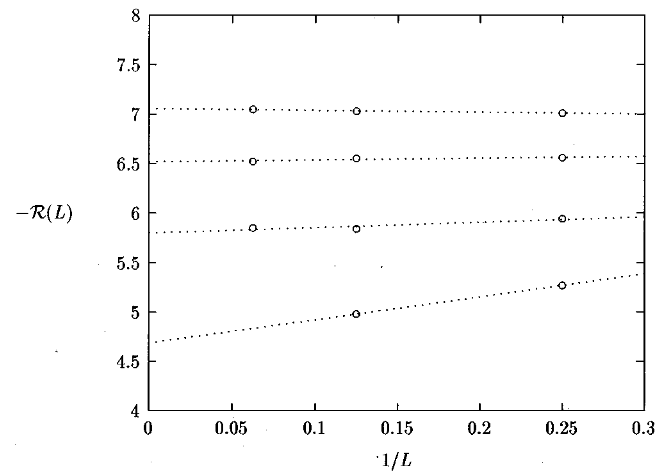

FIG. 5. Volume dependence of the average curvature, for $L=4,8,16$, and (from top to bottom) $k=0.040$, 0.045, 0.050 and 0.055.

k_c (as obtained from the algebraic singularity fits discussed previously) on different lattice sizes. One writes

$$k_c(L) \underset{L\to\infty}{\sim} k_c(\infty)+cL^{-1/\nu}+\cdots. \quad (83)$$

Figure 6 shows the size dependence of critical coupling k_c as obtained on different size lattices. In all three cases $k_c(L)$ is first obtained from a fit to the average curvature of the form $\mathcal{R}(k)=A(k_c-k)^\delta$ for $k\geq 0.02$. Furthermore, if one assumes $\nu=1/3$ and extracts k_c from a linear fit to $\mathcal{R}^3$, then the variations in k_c for different size lattices are substantially reduced (points labeled by circles in Fig. 6.). Because of the few values of L, it is not possible at this point to extract an

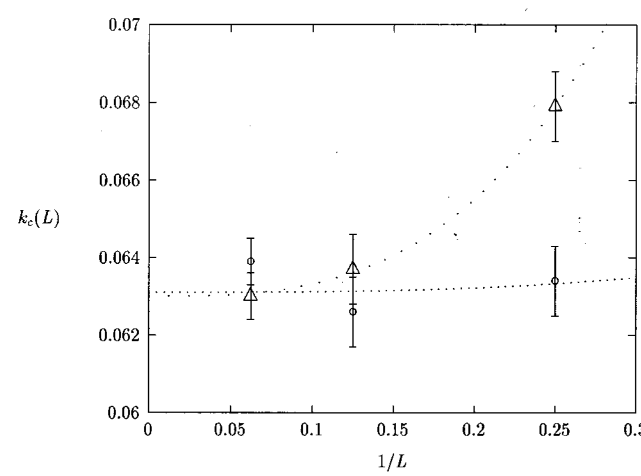

FIG. 6. Volume dependence of the critical coupling k_c, as determined from the singularity in the average curvature, for lattices with $L=4,8,16$. The points labelled by ○ are obtained assuming $\nu=1/3$; in both cases the lines represent simple fits of the type $k_c=a+b/L^3$.

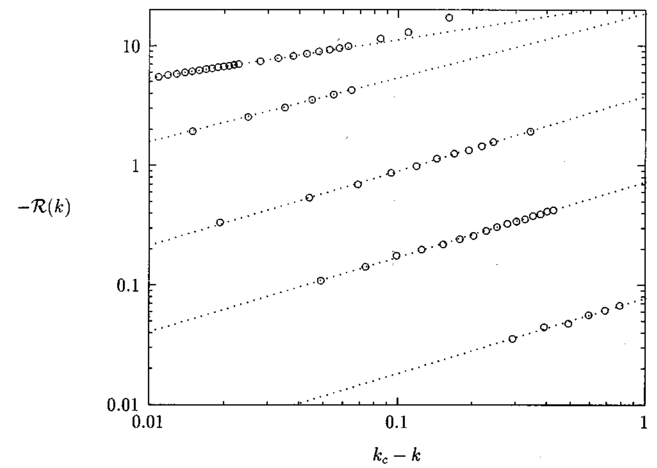

FIG. 7. Average curvature $\mathcal{R}$ versus reduced coupling k_c-k, on a log-log scale. From top to bottom, $a=0$, 0.0005, 0.005, 0.02, 0.1, with a the higher derivative coupling. Statistical error bars are comparable to the size of the dots. The slope of each straight lines determines the critical exponent $\delta = 4\nu - 1$. The slope is noticeably smaller for $a=0$, suggesting that the higher derivative terms mask the true critical behavior up to very small $k_c - k$.

estimate for ν from this particular set of data. But since ν is close to 1/3, it makes sense to use this value in Eq. (83) at least as a first approximation.

Figure 7 shows a plot of the average curvature $\mathcal{R}(k)$ versus reduced coupling $k_c - k$, for several values of a, the higher derivative coupling of Eq. (26). $a=0$ corresponds to the pure Regge action with no explicit higher derivative lattice contribution, for which the path integral is still well defined (at least for sufficiently small $|k|$), since the deficit angles are bounded, and the edge lengths fluctuate around some average value, which is determined by the interplay of the measure and the cosmological constant term. Alternatively, one can think of the fluctuations in the conformal mode as becoming bounded (again at least for sufficiently small $|k|$) when a momentum cutoff of order $\pi/\sqrt{\langle l^2 \rangle}$ is dynamically generated.

The slope of each straight lines determines the critical exponent $\delta = 4\nu - 1$, and it seems clear from the graph that the slope is noticeably smaller for $a=0$, suggesting that the higher derivative terms mask the true critical behavior up to very small $k_c - k$ (it was already noted in [12] that for $a=0$ the assumption of an algebraic singularity for the average curvature leads to a value for the curvature exponent which is much smaller than the estimate for $a>0$, namely $\delta \approx 0.30(4)$).

Indeed it seems that one of the effects of the higher derivative terms is to push the region of instability towards smaller and smaller values of $k_c - k$, until it becomes numerically undetectable. But we would argue that it is only close to this region that the correct continuum behavior is recovered. The situation is similar to what happens in the weak field expansion and perturbation theory: higher derivative terms do not cure the instability problems in the physically relevant region of small momenta and large correlation lengths.

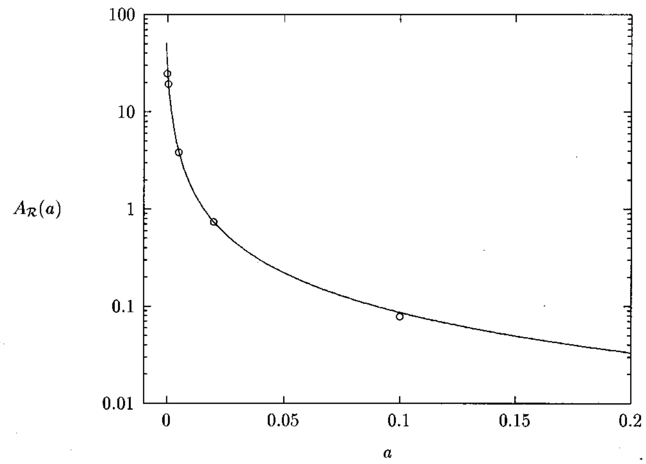

FIG. 8. Curvature amplitude $A_{\mathcal{R}}$ versus the higher derivative coupling a. The amplitude increases rapidly as a approaches zero, the pure Einstein-Regge limit.

Figure 8 shows a plot of the curvature amplitude $A_{\mathcal{R}}$ versus the higher derivative coupling a. The rapid growth close to $a=0$ is consistent with an expected catastrophic instability for $a<0$ (wrong sign for higher derivative terms).

A compilation of previous estimates for ν, together with the new value at $a=0$, is shown in Fig. 9. There seems to be a clear trend toward smaller values as a approaches zero, the Einstein-Regge limit. While the Einstein action contribution becomes the dominant one at large distances, this is no longer the case at intermediate distances in the presence of the higher derivative terms. One concludes that for $a>0$ the higher derivative terms tend to mask the true critical behavior, which requires $k_c - k \ll a^{-1}$.

Figure 10 shows a plot of the average volume per site, $\langle V \rangle$, in units of the average edge length $\sqrt{\langle l^2 \rangle}$. The curve is a fit of the form $a + b(k_c - k)^c$, and suggests a rather sudden

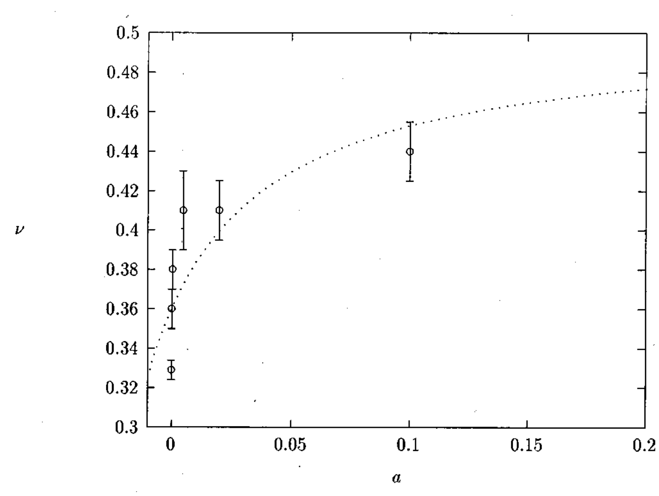

FIG. 9. Critical exponent ν computed from the average curvature versus the higher derivative coupling a. Note the small errorbar on the recent value for ν at $a=0$. For $a>0$ the higher derivative terms tend to mask the true critical behavior, which requires $k_c - k \ll a^{-1}$.

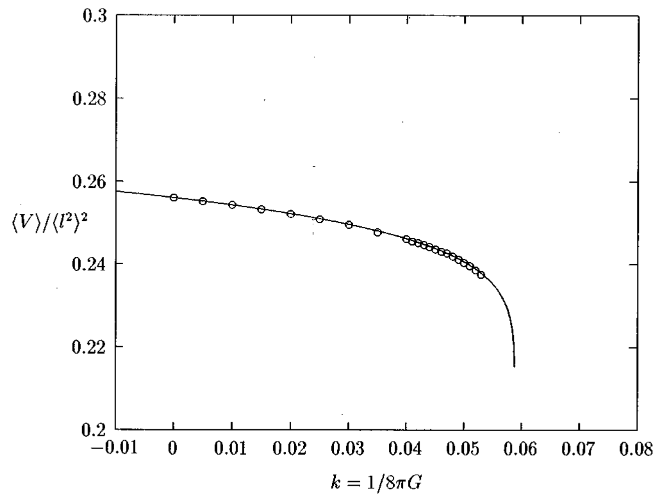

FIG. 10. Average volume per site, $\langle V\rangle$, in units of the average edge length. Statistical errors are much smaller than the size of the dot. The curve is a fit of the form $a+b(k_c-k)^c$. Note the resolution on the vertical scale.

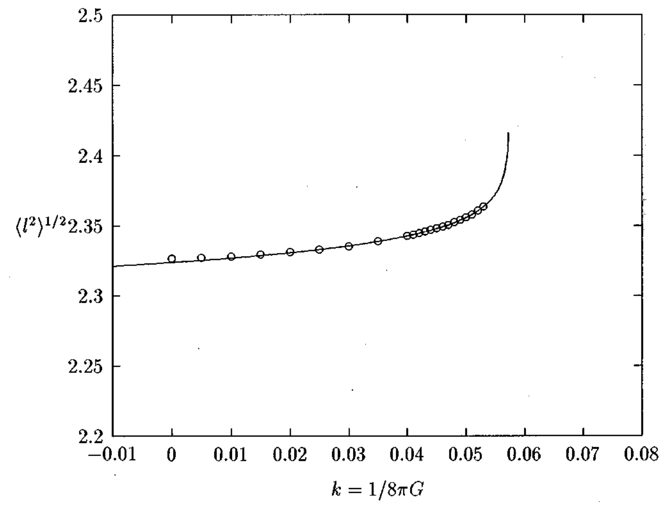

FIG. 11. Average edge length as a function of the bare coupling k. The curve is a fit of the form $a+b(k_c-k)^c$ for $k \geqslant 0.02$. Statistical errors are much smaller than the symbol size.

drop of the average volume in the vicinity of the critical point. A non-analiticity in $\langle V\rangle$ at k_c is in fact consistent with the sum rule of Eq. (45), which suggest that the singular behavior in the average curvature $\mathcal{R}(k)$ and the average volume $\langle V\rangle(k)$ are simply related. Typically, the sum rule of Eq. (45) is satisfied to one part in 10^3 or better.

As can be seen from Fig. 10, close to the transition at k_c the average volume per site expressed in units of the average lattice spacing, $\langle V\rangle/\langle l^2\rangle^2$, shows only a weak singularity when the critical point is approached from the smooth phase ($k<k_c$), and tends to a finite value. On the other hand, in the rough phase ($k>k_c$) the volume per site seems to approach smaller and smaller values as the lengths of the runs are extended. In fact it would seem that in the rough phase the volume per site can be made to approach zero, at least for some simplices. One refers therefore alternatively to this phase as the collapsed or polymer-like phase, since its effective dimension is 2.[2] Furthermore, the relaxation times in the rough phase become exceedingly long, with the system getting stuck in some degenerate, spike-like configurations without being able to get out of it again.

It seems difficult to see how the collapse of the simplices could be averted by choosing a different lattice structure (for example a random lattice), since its properties seem to be unaffected by changes in the measure or the action, at least to the extent they have been investigated. Indeed the collapsed, polymer-like phase appears even in the simplest models based on a regular tessellation of the four-sphere [7,8]. From a continuum point of view, the existence of such a pathological phase is not unexpected, and is interpreted as a reflection of the unbounded fluctuations in the conformal mode expected for sufficiently large k. Indeed unbounded fluctuations in the conformal mode in the continuum correspond to rapid fluctuations in the simplicial volumes, and this is precisely what is observed on the lattice for $k>k_c$, namely a rapid variation of simplicial volumes when going from one simplex to a neighboring one.

Figure 11 shows a plot of the average edge length $\sqrt{\langle l^2\rangle}$. The curve is a fit of the form $a+b(k_c-k)^c$, and suggests a rapid increase in this quantity towards the critical point at k_c. Indeed as the critical point is approached the number of fairly small and fairly large edge lengths proliferate, leading to an increasingly wide edge length distribution.

Figure 12 shows a plot of the average curvature fluctuation $\chi_{\mathcal{R}}(k)$ defined in Eq. (31). At the critical point the curvature fluctuation diverges, by definition. As in the case of the average curvature $\mathcal{R}(k)$ analyzed previously, one can extract the critical exponent δ and k_c by fitting the computed values for the curvature fluctuation to the form [see Eq. (56)]

$$\chi_{\mathcal{R}}(k) \underset{k\to k_c}{\sim} A_{\chi_{\mathcal{R}}}(k_c-k)^{-(1-\delta)}. \tag{84}$$

As for the curvature itself, it would seem unreasonable to expect that the computed values for $\mathcal{R}$ are accurately described by this function even for small k. Instead the data are fitted to the above functional form for either $k\geqslant 0.02$ or $k\geqslant 0.03$ and the difference in the fit parameters can be used as one more measure for the error. Additionally, one can include a subleading correction

$$\chi_{\mathcal{R}}(k) \underset{k\to k_c}{\sim} -A_{\chi_{\mathcal{R}}}[k_c-k+B(k_c-k)^2]^{-(1-\delta)}, \tag{85}$$

and use the results to further constraint the errors on $A_{\chi_{\mathcal{R}}}$, k_c and $\delta=4\nu-1$.

---

[2]An elementary argument can be given to explain the fact that the collapsed phase for $k>k_c$ has an effective dimension of 2, as was found in [12]. The instability is driven by the Euclidean Einstein term in the action and, in particular, its unbounded conformal mode contribution. As the manifold during collapse reaches an effective dimension of 2 this term turns into a topological invariant, unable to drive the instability further to a still lower dimension.

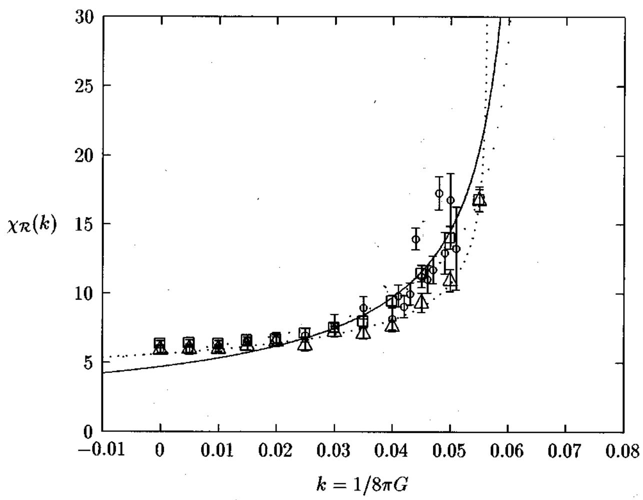

FIG. 12. Curvature fluctuation on lattices with 4^4 (□), 8^4 (△) and 16^4 (○) sites. The thin-dotted, dotted and solid lines represent best fits of the form $\chi_{\mathcal{R}}(k) = A(k_c - k)^{-(1-\delta)}$ for $k \geqslant 0.02$.

The values for δ and k_c obtained in this fashion are consistent with the ones obtained from the average curvature $\mathcal{R}(k)$, but with somewhat larger errors, since fluctuations are more difficult to compute accurately than local averages, and require much higher statistics. Using these procedures one obtains, on the lattice with 16^4 sites,

$$k_c = 0.0636(30), \quad \nu = 0.317(38). \tag{86}$$

Figure 13 shows a graph of the inverse curvature fluctuation $\chi_{\mathcal{R}}(k)$ on the 16^4-site lattice, raised to power 3/2. One would expect to get a straight line close to the critical point if the exponent for $\chi_{\mathcal{R}}(k)$ is exactly $-2/3$. The numerical data indeed support this assumption. The computed data are consistent with linear behavior for small $k \geqslant 0.02$, providing further support for the assumption of an algebraic singularity for $\chi_{\mathcal{R}}(k)$ itself, with exponent close to $-2/3$. Using this procedure one finds, on the 16^4-site lattice,

$$k_c = 0.0641(17), \tag{87}$$

which is completely consistent with the value obtained from $\mathcal{R}^3$ (see Fig. 3 and related discussion), and suggests again that the exponent ν must be close to 1/3.

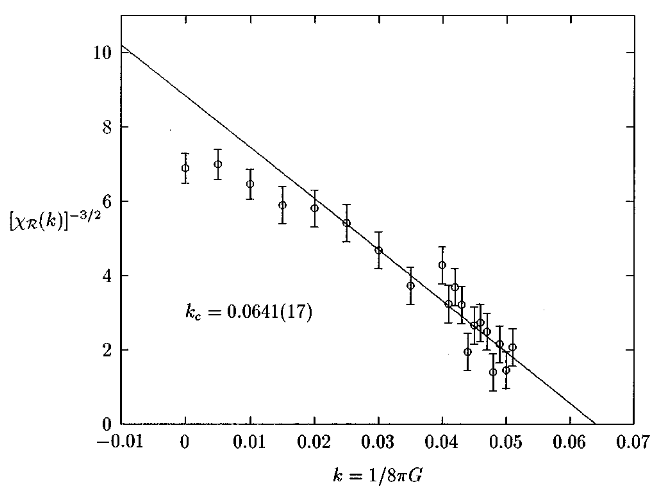

FIG. 13. Inverse curvature fluctuation raised to the power 3/2, on the 16^4 (○) lattice; data are scaled by a factor of $\times 100$. The straight line represents a linear fit of the form $A(k_c - k)$. The location of the critical point in k is consistent with the estimate obtained from the average curvature, but with a somewhat larger error.

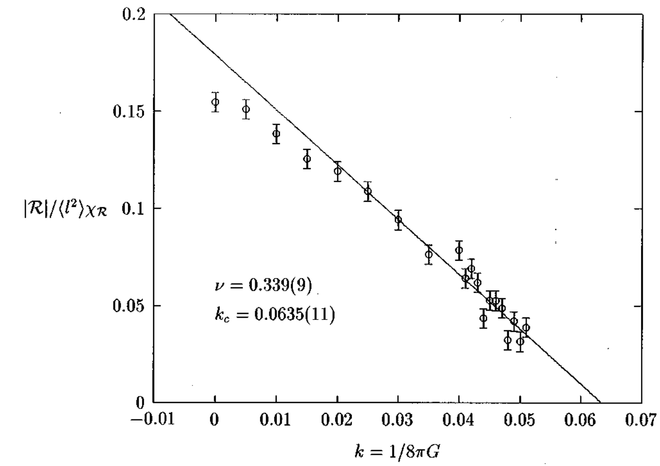

FIG. 14. Inverse of the logarithmic derivative of the average curvature $\mathcal{R}(k)$. The straight line represents a best fit of the form $A(k_c - k)$ for $k \geqslant 0.02$. The location of the critical point in k is consistent with the estimate coming from the average curvature $\mathcal{R}$. From the slope of the line one computes directly the exponent ν.

Figure 14 shows the results for the logarithmic derivative of the average curvature $\mathcal{R}(k)$, obtained from the data shown in Figs. 3 and 12. From the definition of the average curvature $\mathcal{R}$ and curvature fluctuation [Eqs. (29) and (31)], and the fact that both are proportional to derivatives of the free energy F with respect to k [Eqs. (34) and (35)], one notices, for the ratio,

$$\frac{2\langle l^2\rangle \chi_{\mathcal{R}}(k)}{\mathcal{R}(k)} \sim \left(\frac{\partial}{\partial k} \ln Z_L\right) \Bigg/ \left(\frac{\partial^2}{\partial k^2} \ln Z_L\right)$$
$$\sim \frac{\partial}{\partial k} \ln\left(\frac{\partial}{\partial k} \ln Z_L\right). \tag{88}$$

The assumption of an algebraic singularity in k for $\mathcal{R}$ and $\chi_{\mathcal{R}}$ [Eqs. (55) and (56)] then implies that the logarithmic derivative as defined above has a simple pole at k_c, with residue $\delta = 4\nu - 1$:

$$\frac{2\langle l^2\rangle \chi_{\mathcal{R}}(k)}{\mathcal{R}(k)} \underset{k \to k_c}{\sim} \frac{\delta}{k - k_c}, \tag{89}$$

with the critical amplitude dropping out of this particular expression. The above result is general and does not rely on k being real. This suggests that in principle the method of Padé rational approximants (which applies only to meromor-

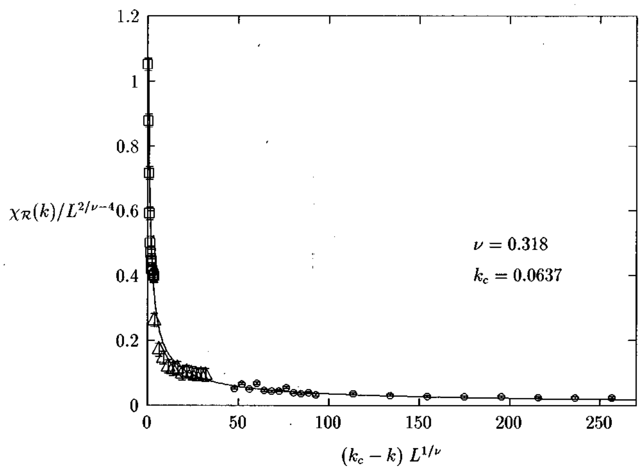

FIG. 15. Finite size scaling behavior of the scaled curvature fluctuation versus the scaled coupling. Here $L=4$ for the lattice with 4^4 sites ($\square$), $L=8$ for the lattice with 8^4 sites ($\triangle$), and $L=16$ for the lattice with 16^4 sites ($\bigcirc$). The solid line represents a best fit of the form $1/(a+bx^c)$. Finite size scaling predicts that all points should lie on the same universal curve. At $k_c=0.0637$ the scaling plot gives the value $\nu=0.318$.

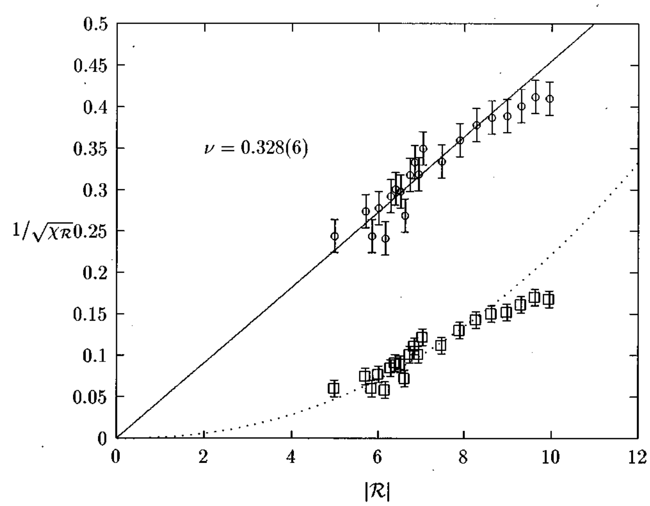

FIG. 16. Inverse curvature fluctuation, $1/\chi_{\mathcal{R}}$, versus the average curvature, $\mathcal{R}$ ($\square$), and $1/\sqrt{\chi_{\mathcal{R}}}$ versus $\mathcal{R}$ ($\bigcirc$). Points shown are for the largest, 16^4-site, lattice. For $\nu=1/3$, $1/\sqrt{\chi_{\mathcal{R}}}$ is expected to be linear in $\mathcal{R}$ for small $\mathcal{R}$.

phic functions) can be employed to locate singularities in $\chi_{\mathcal{R}}(k)$, even for complex k [43,44]. Using this method on the 16^4 lattice one finds

$$k_c=0.0635(11), \qquad \nu=0.339(9). \tag{90}$$

It is encouraging that the above estimates are in good agreement with the values obtained previously using the other methods.

Figure 15 shows a graph of the scaled curvature fluctuation $\chi_{\mathcal{R}}(k)/L^{2/\nu-4}$ for different values of $L=4,8,16$, versus the scaled coupling $(k_c-k)L^{1/\nu}$. If scaling involving k and L holds according to Eq. (19), with $t \sim k_c-k$ and $x_O=1-\delta=2-4\nu$, then all points should lie on the same universal curve. From the general Eq. (19) one expects, in this particular case,

$$\chi_{\mathcal{R}}(k,L)=L^{2/\nu-4}[\tilde{\chi}_{\mathcal{R}}((k_c-k)L^{1/\nu})+\mathcal{O}(L^{-\omega})], \tag{91}$$

where $\omega>0$ is again the correction-to-scaling exponent. Again the data support such scaling behavior, and provide a further estimate on the value for ν, close to 1/3.

Figure 16 shows a plot of the curvature fluctuation $\chi_{\mathcal{R}}$ versus the curvature $\mathcal{R}$. If the curvature approaches zero at the critical point where the curvature fluctuation diverges, one would expect the curvature fluctuation to diverge at $\mathcal{R}=0$. One has

$$\chi_{\mathcal{R}}(\mathcal{R}) \underset{k\to k_c}{\sim} A|\mathcal{R}|^{(1-\delta)/\delta} \sim A|\mathcal{R}|^{(4\nu-2)/(4\nu-1)}. \tag{92}$$

An advantage of this particular combination is that it does not require knowledge of k_c in order to estimate ν. Using all points corresponding to $k \geqslant 0.02$ one finds

$$\nu=0.328(6). \tag{93}$$

The error in ν can be estimated, for example, by using a more elaborate fit of the type

$$\chi_{\mathcal{R}} \underset{\mathcal{R}\to 0}{\sim} A|\mathcal{R}+B\mathcal{R}^2|^{(4\nu-2)/(4\nu-1)}. \tag{94}$$

For $\nu=1/3$ the exponent becomes equal to -2, and one has the simple result

$$\chi_{\mathcal{R}} \underset{\mathcal{R}\to 0}{\sim} A|\mathcal{R}|^{-2}. \tag{95}$$

One concludes that the evidence supports a vanishing curvature at the critical point, where the curvature fluctuation $\chi_{\mathcal{R}}$ and the correlation length ξ diverge. This result is further supported by the consistency of the values for k_c obtained independently from $\mathcal{R}(k)$ and $\chi_{\mathcal{R}}(k)$ (Figs. 2, 3, 4, 12, 13, 14, and 15).

As an independent measure of the fluctuation one can also investigate the behavior of the edge length fluctuation defined as

$$\chi_{l^2}(k)=\frac{1}{N_1}\left\{\left\langle\left(\sum_{i=1}^{N_1} l_i^2\right)^2\right\rangle-\left\langle\sum_{i=1}^{N_1} l_i^2\right\rangle^2\right\} \underset{k\to k_c}{\sim} (k_c-k)^{-\gamma}, \tag{96}$$

where γ is a critical exponent. Using an analysis similar to what is done for the curvature and curvature fluctuation, on the 16^4 lattice it is found to diverge at

$$k_c=0.0609(23) \tag{97}$$

in agreement within errors with the previous values quoted for k_c. One would expect such a fluctuation to be related to the fluctuations in the local volumes, and, by the sum rule of Eq. (45) which relates the fluctuations in the volume to fluctuations in the curvature, one would expect $\gamma=1-\delta=2-4\nu$. The numerical results for gamma have larger errors

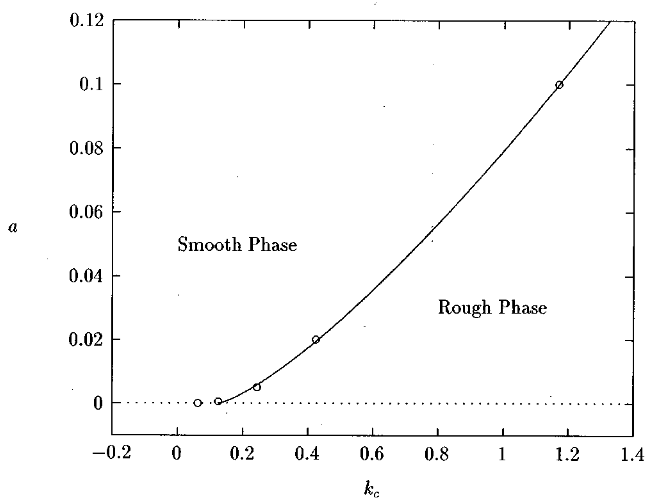

FIG. 17. Phase diagram for the model in the k-a plane. A critical line separates the smooth, strong coupling, phase from the rough, weak coupling, phase. The dotted line denotes the pure Einstein theory, without higher derivative terms.

but give values between 0.46 and 0.85, certainly consistent with a value of $\gamma = 2/3$ for $\nu = 1/3$.

Finally Fig. 17 summarizes the known information about the phase diagram in the k-a plane. The solid line separates the smooth phase with small negative curvature from the rough, polymer-like phase.

Table I summarizes the results obtained for the critical point $k_c = 1/8\pi G_c$ and the critical exponent ν. From the best data (with the smallest statistical uncertainties and the least systematic effects) one concludes

TABLE I. Summary table for the critical point k_c and the critical exponent ν, as obtained from the largest lattice with 16^4 sites. The last three entries assume a critical point at $k_c = 0.0636$.

Method	k_c	ν
$\mathcal{R}$ vs k	0.0630(11)	0.330(6)
$\mathcal{R}^3$ vs k	0.0639(10)	-
$\chi_{\mathcal{R}}$ vs k	0.0636(30)	0.317(38)
$\chi_{\mathcal{R}}^{3/2}$ vs k	0.0641(17)	-
$\chi_{\mathcal{R}}/(\langle l^2\rangle \mathcal{R})$ vs k	0.0635(11)	0.339(9)
$\chi_{\mathcal{R}}$ vs $\mathcal{R}$	-	0.328(6)
χ_{l^2} vs k	0.0609(23)	$\gamma = 0.46(8)$
χ_{l^2} vs $\mathcal{R}$	-	$\gamma = 0.54(7)$
$\mathcal{R}$ FS scaling	-	0.333(2)
$\chi_{\mathcal{R}}$ FS scaling	-	0.318(10)
χ_{l^2} FS scaling	-	$\gamma = 0.85(6)$

$$k_c = 0.0636(11), \quad \nu = 0.335(9), \tag{98}$$

which suggests $\nu = 1/3$ for pure gravity.[3]

IX. CRITICAL EXPONENTS AND PHENOMENOLOGY

In this section some consequences of the results presented above will be discussed, with ultimately an eye towards possible physical applications. Naively one would expect simply on the basis of dimensional arguments that the curvature scale gets determined by the correlation length

$$\mathcal{R} \underset{\mathcal{R}\to 0}{\sim} 1/\xi^2, \tag{99}$$

but one cannot in general exclude the appearance of some non-trivial exponent.

In the previous section arguments have been given in support of the value $\nu = 1/3$ for pure gravity. From Eq. (58) relating the average curvature to the correlation length one has

$$\mathcal{R}(\xi) \underset{k\to k_c}{\sim} \frac{1}{l_P^{2-d+1/\nu}\xi^{d-1/\nu}}, \tag{100}$$

and the correct dimension for the average curvature $\mathcal{R}$ has been restored by supplying appropriate powers of the ultraviolet cutoff, the Planck length $l_P = \sqrt{G}$. One notices that close to two dimensions the exponent of ξ indeed approaches 2, since $\nu \sim 1/(d-2)$, and the classical result is recovered.

For $\nu = 1/3$ in four dimensions[4] one then obtains the remarkably simple result

$$\mathcal{R}(\xi) \underset{k\to k_c}{\sim} \frac{1}{l_P \xi}. \tag{101}$$

An equivalent form can be given in terms of the curvature scale H_0, defined through $R = -12H_0^2$, and which has dimensions of a mass squared. One has close to the critical point

$$H_0^2 = C_H \mu_P m, \tag{102}$$

where $\mu_P = 1/\sqrt{G}$ is the Planck mass, $m = 1/\xi$ is the inverse gravitational correlation length, and $C_H \approx 4.9$ a numerical

[3]The value $\nu = 1/3$ does not correspond to any known field theory or statistical mechanics model in four dimensions. For dilute branched polymers it is known that $\nu = 1/2$ in three dimensions [46], and $\nu = 1/4$ at the upper critical dimension $d = 8$ [47], so one would expect a value close to 1/3 somewhere in between. I thank John Cardy for a discussion on this point.

[4]For all scalar field theories (spin $s = 0$) in four dimensions it is known that $\nu = 1/2$, while for the compact Abelian U(1) gauge theory ($s = 1$) one has $\nu = 2/5$ [49]. The value $\nu = 1/3$ for pure gravitation ($s = 2$) in four dimensions is then consistent with the simple formula $\nu = 1/(2 + s/2)$.

constant of order 1; the value for C is extracted from the known numerical values for $\mathcal{R}$ and m close to the critical point at k_c.

One can raise the legitimate concern of how these results are changed by quantum fluctuations of matter fields. In the presence of matter fields coupled to gravity (scalars, spin-$\frac{1}{2}$ fermions, vector bosons, spin-3/2 fields, etc.) one expects the value for ν to change due to vacuum polarization loops containing these fields. A number of arguments can be given though for why these effects should not be too dramatic, unless the number of light matter fields is very large. First, in the case of a single light scalar field the vacuum polarization effects are so small that they are barely detectable in the numerical evaluations of the path integral [48]. Furthermore one notices that to leading order in the $2+\epsilon$ expansion the exponent ν only depends on the dimensionality of space-time, irrespective of the number of matter fields and of their type [41]. Finally one can compute for example the effects of scalar matter fields on the one-loop beta function in the $2+\epsilon$ expansion for gravity, and finds $\beta_0=(2/3)(25-n_f)$ where n_f is the number of massless scalar fields [41]. Thus, unless n_f is large, the matter contribution is quite small even to next-to-leading order in the $2+\epsilon$ expansion. The present evidence would therefore suggest that the approximation in which vacuum polarization effects of light matter fields are neglected should not be too unreasonable.

It seems natural to identify H_0 with either some (negative) average spatial curvature or possibly with the Hubble constant determining the macroscopic expansion rate of the present universe [12,15]. In the Friedmann-Robertson-Walker (FRW) model of standard cosmology [50] one has, for the Ricci scalar,

$$R_{Ricci}=-6\left\{\left(\frac{\dot{R}}{R}\right)^2+\frac{k}{R^2}+\frac{\ddot{R}}{R}\right\}, \tag{103}$$

where $R(t)$ is the FRW scale factor, and $k=0,\pm 1$ for spatially flat, open or closed universes respectively. Today the Hubble constant is given by $H_0^2=(\dot{R}/R)^2_{t_0}$, but it is eventually expected to show some slow variation in time, and its characteristic length scale $cH_0^{-1}\approx 10^{28}$ cm today is comparable to the present extent of the visible universe. Under such circumstances from Eq. (102) one would expect the gravitational correlation ξ to be significantly larger than cH_0^{-1}. A potential problem arises though in trying to establish a relationship between quantities which are truly constants [such as the ones appearing in Eq. (102)], and H_0 which most likely depends on time.[5] In any case it is clear that some of these considerations are in fact quite general, to the extent that they rely on general principles of the renormalization group and are not tied to any particular value of ν, although $\nu=1/3$ clearly has some aesthetic appeal. Additional cosmological and astrophysical arguments and proposed tests can be found in a recent paper [51].

One further observation can be made regarding the running of G. Assuming the existence of an ultraviolet fixed point, the effective gravitational coupling is given by Eq. (66) for "short distances" $r\ll\xi$, but now with an exponent $\nu=1/3$:

$$G(r)=G(0)[1+c(r/\xi)^3+O((r/\xi)^6)], \tag{104}$$

with c a calculable numerical constant of order 1. The appearance of ξ in this equation, which is a very large quantity by Eq. (102), suggests that the leading scale-dependent correction, which gradually increases the strength of the effective gravitational interaction as one goes to larger and larger length scales, should be extremely small.[6]

It is only for distances comparable to or larger than ξ that the gravitational potential should start to weaken and fall off exponentially, with a range given by the gravitational correlation length ξ:

$$V(r)\underset{r\gg\xi}{\sim}-G(r)\frac{\mu_1\mu_2 e^{-r/\xi}}{r}. \tag{105}$$

In many ways these results appear qualitatively consistent with the expected behavior of the tree-level graviton propagator in anti–de Sitter space [52,53]. In the real world the range ξ must be of course very large. From the fact that super-clusters of galaxies apparently do form, one can easily set an observational lower limit $\xi>10^{25}$ cm.

It is unclear to what extent gravitational correlations can be measured directly. From the definition of the curvature correlation function in Eq. (53) one has for "short distances" $r\ll\xi$ and for the specific value $\nu=1/3$ the remarkably simple result

$$\langle\sqrt{g}R(x)\sqrt{g}R(y)\,\delta(|x-y|-d)\rangle_c\underset{d\ll\xi}{\sim}\frac{A}{d^2}, \tag{106}$$

with A a calculable numerical constant of order 1. One can contrast this behavior with the semiclassical result attained close to two dimensions (and which incidentally coincides with the lowest order weak field expansion result [38]), which gives instead for the power the value $2(d-1/\nu)\sim 2[d-(d-2)]\sim 4$, as expected on the basis of naive dimensional arguments $(R\sim\partial^2 h)$.

If one considers the curvature R averaged over a spherical volume $V_r=4\pi r^3/3$,

$$\overline{\sqrt{g}R}=\frac{1}{V_r}\int_{V_r}d^3\vec{x}\,\sqrt{g(\vec{x},t)}R(\vec{x},t), \tag{107}$$

one can compute the corresponding variance in the curvature:

[5]The only exception being the steady state cosmological models, where H_0 is truly a constant of nature. These models are not favored by present observations, including detailed features of the cosmic background radiation.

[6]And suggests that the deviations from classical general relativistic behavior for most physical quantities are in the end practically negligible.

$$[\delta(\sqrt{g}R)]^2 = \frac{1}{V_r^2}\int_{V_r} d^3\vec{x}\int_{V_r} d^3\vec{y}\langle\sqrt{g}R(\vec{x})\sqrt{g}R(\vec{y})\rangle_c = \frac{9A}{4r^2}. \tag{108}$$

As a result the rms fluctuation of $\sqrt{g}R$ averaged over a spherical region of size r is given by

$$\delta(\sqrt{g}R) = \frac{3\sqrt{A}}{2}\frac{1}{r}, \tag{109}$$

while the Fourier transform power spectrum at small $\vec{k}$ is

$$P_{\vec{k}} = |\sqrt{g}R_{\vec{k}}|^2 = \frac{4\pi^2 A}{2V}\frac{1}{k}. \tag{110}$$

One can use Einstein's equations to relate the local curvature to the (primordial) mass density. From Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{111}$$

for a perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu \tag{112}$$

one obtains for the Ricci scalar, in the limit of negligible pressure,

$$R(x) \approx 8\pi G\rho(x). \tag{113}$$

As a result one expects for the density fluctuations a power law decay of the form

$$\langle\rho(x)\rho(y)\rangle_c \underset{|x-y|\ll\xi}{\sim} \frac{1}{|x-y|^2}. \tag{114}$$

Similar density correlations have been estimated from observational data by analyzing known galaxy number density distributions, giving a value for the exponent of about 1.77 $\pm$0.04 for distances in the 10 kpc to 10 Mpc range [54].

X. CONCLUDING REMARKS

Numerical simulation methods combined with modern renormalization group arguments and finite size scaling can provide detailed information on non-perturbative aspects of a lattice model of quantum gravity. It has been known for some time that the lattice model has two phases, only one of which is physically acceptable. In this work we have described in some detail the properties of the latter smooth phase, and provided quantitative estimates for the critical point, the scaling dimensions and the behavior of correlations at distances large compared to the cutoff. In spite of the fact that the Euclidean theory becomes unstable as one approaches the critical point at k_c, it is still possible to determine by a straightforward analytic continuation the physical properties of the model in the vicinity of the true fixed point, defined as the point where a non-analyticity develops in the strong coupling branch of $Z_L(k)$, and where scaling implies that the physical correlation ξ diverges.

If this prescription is followed, an estimate for the non-perturbative Callan-Symanzik beta function in the vicinity of the fixed point can be obtained, to leading order in the deviation of the bare coupling from its critical value. The resulting scale evolution for the gravitational constant is then quantitatively quite small, if one assumes that the scaling violation parameter is related to an average curvature and its characteristic scale H_0. Its infrared growth, consistent with the general idea that gravitational vacuum polarization effects cannot exert any screening, suggests that low energy properties of quantum gravity are inaccessible by weak coupling perturbation theory: low energy quantum gravity is a strongly coupled theory. On a more quantitative side, as pointed out in the discussion there are a number of attractive features to the pure gravity result $\nu = 1/3$, including a simple form for the curvature correlations at short distances.

Compared to results found in dynamical triangulation models, where the average curvature is of the same order as the ultraviolet cutoff in the vicinity of the critical point, here the strong coupling, smooth phase is found not to be collapsed. A likely explanation for the unphysical behavior of the dynamical triangulation model in the strong coupling phase is the lack of perturbative gravitons in the weak field limit or equivalently, the absence of smooth deformations of the geometry (including gauge modes) and of classical gravitational waves.

It seems legitimate to ask the question whether the present lattice model for quantum gravity provides any insight into the problem of the cosmological constant. The answer is both yes and no. To the extent that a naive prediction of quantum gravity is that the curvature scale should be of the same order of the Planck length, $\mathcal{R} \sim 1/G$, the answer is definitely yes. Indeed it can be regarded as a non-trivial result of the lattice models for gravity that a region in coupling constant space can be found where space-time is stiff and the curvature can be made much smaller than $1/G$. In fact the evidence indicates that the average curvature $\mathcal{R}$ vanishes at the critical point k_c. And this is achieved with a bare cosmological constant λ which is of order 1 in units of the cutoff. Phrased differently, the dimensionless ratio between the renormalized and the bare cosmological constant becomes arbitrarily small towards the critical point.

At the same time the effective long distance cosmological constant is non-vanishing and of order $1/\xi$, and the value zero is only obtained when ξ is exactly zero, which happens only at the critical point k_c. Thus to make the effective cosmological constant small requires a fine-tuning, in the sense that the bare coupling k_c-k has to be small. But since the correlation length determines the corrections to the Newtonian potential (and in particular its eventual decrease for large enough distances), it would seem unnatural to have a short correlation length ξ: in such a world there would be no long-range gravitational forces, and separate space-time domains would have decoupled fluctuations. From this perspective, long range forces and a small cosmological constant go hand in hand. Quantum fluctuation effects show that hyperbolic space-times with small curvature radii cannot sustain long-range gravitational forces, at least in this model.

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