Cosmological evolution of general scalar fields and quintessence

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We study the cosmological evolution of scalar fields with arbitrary potentials in the presence of a baryotropic fluid (matter or radiation) without making any assumption on which term dominates. We determine what kind of potentials $V(\phi)$ permits a quintessence interpretation of the scalar field ϕ and to obtain interesting cosmological results. We show that all model dependence is given in terms of $\lambda \equiv -V'/V$ only and we study all possible asymptotic limits: λ approaching zero, a finite constant, or infinity. We determine the equation of state dynamically for each case. For the first class of potentials, the scalar field quickly dominates the universe behavior, with an inflationary equation of state allowing for a quintessence interpretation. The second case gives the extensively studied exponential potential, while in the last case, when λ approaches infinity, if it does not oscillate, then the energy density redshifts faster than the baryotropic fluid, but if λ oscillates, then the energy density redshift depends on the specific potential.

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Models with a cosmological constant term, intended as a constant vacuum contribution or as a slowly decaying scalar field, have recently received considerable attention for several reasons, both theoretical and observational.

From the theoretical point of view, we have to face the possible conflict between the age of the universe in the standard Einstein–de Sitter model and the age of the oldest stars, in globular clusters. Estimates of the Hubble expansion parameter from a variety of methods seem to point to $H_0 \approx 70 \pm 10$ km/s/Mpc (a recent review can be found in [1]; see, e.g., [2] for specific projects), leading to an expansion age of $t_U \approx 9 \pm 1$ Gyr for a spatially flat universe with null cosmological constant. On the other hand, the age of globular clusters has been estimated in the range $\simeq 13$–15 Gyr [3], although revised determinations based on the Hypparcos distance scale are lower by approximately 2 Gyr [4].

The requirements of structure formation models also suggest a cosmological constant term. Simulations of structure formation profit from the presence of matter that resists gravitational collapse and Λ cold dark matter (CDM) models provide a better fit to the observed power spectrum of galaxy clustering than does the standard CDM model [5,6].

On the observational side, we find direct evidence in recent works on spectral and photometric observations on type Ia supernova [7] that favor eternally expanding models with positive cosmological constant. Statistical fits to several independent astrophysical constraints support these results [8]. See, however, [9] for a different explanation of these observations. More indirect evidence comes from the observational support for a low matter density universe from x-ray mass estimations in clusters [10,11]. In these works, if the nucleosynthesis limits on the baryonic mass are to be respected, the total matter that clusters gravitationally is limited to $\lesssim 0.3$. In such a case, a cosmological term would reconcile the low dynamical estimates of the mean mass density with total critical density suggested by inflation and the flatness problem.

Many models with a scalar field playing the role of a decaying cosmological constant have been proposed up to now. Some of them are specific models motivated by physical considerations but most of them are phenomenological proposals for the desired energy density redshift [12–15]. As a first step in the study of these models, the age of the universe is calculated for several redshift laws of the energy density that resides in the dynamical scalar field [16,17]. Observational consequences of an evolving Λ component decaying to matter and/or radiation have been studied in [18,19], obtaining severe constraints on such models. Another possibility, the one that we consider here, is that the scalar field couples to matter only through gravitation. This kind of scalar field, with negative pressure and a time-varying, spatially fluctuating energy density, received the name of quintessence [20]. Its effects on the cosmic microwave background anisotropy are analyzed in [20] and [21] and the phenomenological difficulties of quintessence have been studied in [22]. Constraints on the equation of state of a quintessencelike component have been placed from observational data [23]. Recently, a potential for a cosmologically successful decaying Λ term has been constructed in [24].

As we have discussed above, the behavior of scalar fields is fundamental in understanding the evolution of the universe. In this paper we are interested in giving a general approach to the analysis of the cosmological evolution of scalar fields and to determine what kinds of potentials lead to a possible interpretation of the scalar field as quintessence and to a dominating energy density. However, we will not assume any kind of scale dependence for the potential or impose any condition on which energy density dominates [13–15,19,26,25,27]. We will show that all model dependence is given only in terms of the quantity $\lambda \equiv -V'/V$,

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where the prime denotes derivative with respect to the scalar field ϕ, and its limiting behavior at late times determines the evolution of the scalar field.

Our starting point is a universe filled with a baryotropic energy density, which can be either matter or radiation, and the energy density of a scalar field. The scalar field ϕ will have a self-interaction, given in terms of the scalar potential $V(\phi)$, but it will interact with all other fields only gravitationally. The baryotropic fluid is described by an energy density ρ_γ and a pression p_γ with a standard equation of state $p_\gamma=(\gamma_\gamma-1)\rho_\gamma$, where $\gamma_\gamma=1$ for matter and $\gamma_\gamma=4/3$ for radiation. We do not make any hypothesis on which energy density dominates, that of the barotropic fluid or that of the scalar field.

The equations to be solved, for a spatially flat Friedmann-Robertson-Walker (FRW) universe, are then given by

$$\dot{H}=-\frac{1}{2}(\rho_\gamma+p_\gamma+\dot{\phi}^2),$$
$$\dot{\rho}=-3H(\rho+p), \tag{1}$$
$$\ddot{\phi}=-3H\dot{\phi}-\frac{dV(\phi)}{d\phi},$$

where H is the Hubble parameter, $V(\phi)$ is the scalar field potential, $\dot{\phi}\equiv d\phi/dt$, $\rho(p)$ is the total energy density (pression), and we have taken $8\pi G=1$. It is useful to make a change of variables [27] $x\equiv\dot{\phi}/\sqrt{6}H$, $y\equiv\sqrt{V}/\sqrt{3}H$ and Eqs. (1) become

$$x_N=-3x+\sqrt{\frac{3}{2}}\lambda y^2+\frac{3}{2}x[2x^2+\gamma_\gamma(1-x^2-y^2)],$$
$$y_N=-\sqrt{\frac{3}{2}}\lambda xy+\frac{3}{2}y[2x^2+\gamma_\gamma(1-x^2-y^2)], \tag{2}$$
$$H_N=-\frac{3}{2}H[\gamma_\gamma(1-x^2-y^2)+2x^2],$$

where N is the logarithm of the scale factor a, $N\equiv\ln(a)$, $f_N\equiv df/dN$ for $f=x,y,H$, and $\lambda(N)\equiv-V'/V$. Notice that all model dependence in Eqs. (2) is through the quantities $\lambda(N)$ and the constant parameter γ_γ. Equations (2) must be supplemented by the Friedmann or constraint equation for a flat universe, $\rho_\gamma/3H^2+x^2+y^2=1$, and they are valid for any scalar potential as long as the interaction between the scalar field and matter or radiation is gravitational only. This means that it is possible to separate the energy and pression densities into contributions from each component, i.e., $\rho=\rho_\gamma+\rho_\phi$ and $p=p_\gamma+p_\phi$, where ρ_ϕ (p_ϕ) is the energy density (pressure) of the scalar field. We do not assume any equation of state for the scalar field. This is indeed necessary since one cannot fix the equation of state and the potential independently. For arbitrary potentials the equation of state for the scalar field, $p_\phi=(\gamma_\phi-1)\rho_\phi$, is determined once ρ_ϕ, p_ϕ have been obtained. Alternatively we can solve for x,y using Eqs. (2) and the quantity $\gamma_\phi=(\rho_\phi+p_\phi)/\rho_\phi=2x^2/(x^2+y^2)$ is, in general, time or scale dependent.

As a result of the dynamics, the scalar field will evolve to its minimum, and if we do not wish to introduce any kind of unnatural constant or fine-tuning problem, the minimum of the potential must have zero energy, i.e., $V|_{min}=V'|_{min}=0$ at ϕ_{min}. We will consider here only these kind of potentials. For finite ϕ_{min} the scalar field will naturally oscillate around its vacuum expectation value (VEV). If the scalar field has a nonzero mass or if the potential V admits a Taylor expansion around ϕ_{min}, then, using the Hôpital rule, one has $\lim_{t\to\infty}|\lambda|=\infty$ and it will oscillate. On the other hand, if $\phi_{min}=\infty$, then ϕ will not oscillate and λ will approach either zero, a finite constant, or infinity. The oscillating behavior of ϕ or λ is important in determining the cosmological evolution of x,y and $\Omega_\phi\equiv\rho_\phi/\rho=x^2+y^2$, and we will show that any scalar field with a nonvanishing mass redshifts as a matter field.

Before solving Eqs. (2) we define the useful cosmological acceleration parameter α and expansion rate parameter Γ. The acceleration parameter is defined as

$$\alpha\equiv\frac{\rho+3p}{(3\gamma_\gamma-2)\rho}=\frac{3\gamma-2}{3\gamma_\gamma-2}, \tag{3}$$

with $\gamma=(\rho+p)/\rho$. If $\alpha=1$, then the acceleration of the universe is the same as that of the baryotropic fluid and any deviation of α from 1 implies a different cosmological behavior of the universe due to the contribution of the scalar field. A positive accelerating universe requires a negative α while for $0<\alpha<1$ the acceleration of the universe is negative (deceleration) but smaller than that of the baryotropic fluid. For $\alpha>1$ the deceleration is larger than for the baryotropic fluid. In terms of the standard deceleration parameter $q\equiv-\ddot{a}a/\dot{a}^2$ one has $\alpha=2q/(3\gamma_\gamma-2)$ or in terms of x,y one finds

$$\alpha=1-\frac{3\gamma_\gamma}{3\gamma_\gamma-2}\left(y^2-x^2\frac{2-\gamma_\gamma}{\gamma_\gamma}\right)=1-3\Omega_\phi\frac{\gamma_\gamma-\gamma_\phi}{3\gamma_\gamma-2}.$$

It is also useful to define the normalized equation of the state parameter,

$$\Gamma=\frac{\gamma}{\gamma_\gamma}, \tag{4}$$

which gives the relative expansion rate of the universe with respect to the baryotropic fluid. A Γ smaller than 1 means that the universe expands slower than the baryotropic fluid and a Γ larger than 1 says that the universe expands faster due to the contribution of the scalar field. In our case α and Γ are not independent since $\Gamma=1-(1-\alpha)(3\gamma_\gamma-2)/3\gamma_\gamma=1-\Omega_\phi(\gamma_\gamma-\gamma_\phi)/\gamma_\gamma$.

A general analysis of Eqs. (2) can be done by noting that, given the constant parameter γ_γ, all model dependence is through the quantity $\lambda(N)$. For an arbitrary potential V, Eqs. (1) or (2) will be, in general, nonlinear and there will be no analytic solutions. We can, of course, solve them numerically but we need to do it for each particular case and initial conditions separately. In order to have an understanding of the evolution of the scalar field we will study the asymptotic limit. It is useful to distinguish the different limiting cases for the cosmologically relevant quantities x, y, and $\Omega_\phi = x^2+y^2$. Ω_ϕ will either approach 0, 1, or a finite constant value. For $\Omega_\phi \to 0$ the scalar field dilutes faster than ordinary matter or radiation, and if $\Omega_\phi \to 1$, then the scalar dominates the energy density of the universe. When $0<\Omega_\phi \to cte<1$, the scalar and barotropic energy density redshift at the same speed. Which behavior will x,y,Ω_ϕ have depends on λ and on γ_γ. We will separate the analysis of Eqs. (2) into three different behaviors of $|\lambda|$ at late times. In the first case we consider λ a finite constant (or approaching 1), $\lambda = c$. Second, we study the limit $\lambda \to 0$. In the third case, we take $\lambda \to \infty$, which is the natural case if the VEV of ϕ is finite but we can have the same limit for $\phi \to \infty$. We divide in this case the analysis into an oscillating and a not oscillating $|\lambda| \to \infty$.

Equations (2) admit five different critical solutions for x,y with λ constant [27]. A constant value of H requires $H=0$ or $y=1$. The latter case is a critical point of Eqs. (2) only if $\lambda=0$ while the former is the trivial $x=y=H=0$ case. For x,y the five different critical solutions $(x_c=1,\ y_c=0)$, $(x_c=-1,\ y_c=0)$, and $(x_c=0,\ y_c=0)$ are unstable (extreme) critical points. The other two depend on the value of $\lambda(N)$. For $\lambda^2>3\gamma_\gamma$ [27] one finds the critical and late time attractor values

$$x_c=\sqrt{\frac{3}{2}}\frac{\gamma_\gamma}{\lambda}, \quad y_c=\sqrt{\frac{3(2-\gamma_\gamma)\gamma_\gamma}{2\lambda^2}}, \quad \Omega_{\phi c}=\frac{3\gamma_\gamma}{\lambda^2},$$

$$\gamma_\phi=\gamma_\gamma \tag{5}$$

for the quantities x, y, and Ω_ϕ and an effective equation of state equivalent to that of the barotropic fluid [i.e., $\gamma_\phi = (\rho_\phi+p_\phi)/\rho_\phi=\gamma_\gamma$]. In this limiting case the redshift of the barotropic fluid and the scalar field is the same.

On the other hand, if $\lambda^2<6$, then one obtains

$$x_c=\frac{\lambda}{\sqrt{6}}, \quad y_c=\sqrt{1-\frac{\lambda^2}{6}}, \quad \Omega_{\phi c}=1, \quad \gamma_\phi=\frac{\lambda^2}{3}, \tag{6}$$

and the scalar energy density dominates the universe at late times. If the scalar field has $\gamma_\phi<\gamma_\gamma$, then the solutions in Eq. (6) are stable, and the redshift of the scalar field is slower than that of the barotropic fluid. However, if $\gamma_\phi=\lambda^2/3>\gamma_\gamma$, then the solution in Eq. (6) is unstable and the scalar field ends up in the regime of the solution given in Eq. (5).

If $\lambda(N)$ is constant, then Eqs. (5) or (6) are indeed solutions to $x_N=y_N=0$, but if $\lambda(N)$ is not constant, then the critical values in Eqs. (5) and (6) solve $x_N=y_N=0$ only on single points, not an interval. This means that the attractor solution to Eqs. (2) is only valid as an asymptotic limit and $x_c(N),y_c(N),\lambda(N)$ are functions of N. If x,y do not oscillate, since their value is constrained to $|x|\leqslant 1, |y|\leqslant 1$, this implies that at late times they will approach a constant value, given by the attractor solutions of Eqs. (2), and x_N, y_N will vanish. Therefore, we can generalize the attractor solutions of x,y given in Eqs. (5) and (6) for more complicated potentials that have a nonconstant $\lambda(N)$. It is the asymptotic behavior of λ which determines to which attractor solution x, y will evolve. If $\lambda\to\infty$ (without oscillating), then x, y have an asymptotic behavior given by Eqs. (5) and they end up at $x=y=0$ which is in this case a stable point. For $\lambda \to 0$, x,y are given by Eq. (6) and they go to $x=0$ and $y=1$.

Let us now start with our first case, i.e., λ constant. If $\lambda = -V'/V=c$, the scalar potential has an exponential form $V=he^{-c\phi}$. This case has been extensively studied [19,26,27] and one finds critical (i.e., constant) points for x and y at late times. The value of x,y depends on the value of $\lambda=c$, and their solutions is given by Eqs. (5) or (6). Since this case has been amply documented in the literature [19,26,27], we do not include its numerical analysis here. The cosmological parameters are $\alpha=\Gamma=1$ for $\lambda=c>\sqrt{3\gamma_\gamma}$ and $\alpha=(c^2-2)/(3\gamma_\gamma-2)$, $\Gamma=c^2/3\gamma_\gamma$ for $\lambda=c=\sqrt{6}$. Note that in the first case Ω_ϕ is finite, and even if it dominates the universe, the acceleration and expansion of the universe are the same as for the baryotropic fluid. On the other hand, for $\lambda=c=\sqrt{6}$ one has $\Omega_\phi=1$ and the acceleration parameter α is in general different than 1; it is negative if $c^2<2$ (assuming $\gamma_\gamma>2/3$, i.e., matter or radiation). In this case we could have interesting quintessence models.

For more complicated potentials, λ is not a constant and its evolution determines that of x and y. The evolution of the scalar field leads to nonlinear equations, and critical points may exist but analytic solutions are either more difficult or impossible to obtain. However, the solutions given in Eqs. (5) and (6) may give a good approximation of the limiting behavior of x and y.

Let us now consider the second case, i.e., $\lambda\to 0$. In this limit we can eliminate in Eq. (2) the term proportional to λ, and since $-3<H_N/H<0$ for all values of x, y, and γ_γ, we have

$$\frac{x_N}{x}=-\left(3+\frac{H_N}{H}\right)<0, \quad \frac{y_N}{y}=-\frac{H_N}{H}>0. \tag{7}$$

From Eqs. (7) we conclude that x will approach its minimum value (i.e., $x\to 0$) while y will increase to its maximum value (i.e., $y\to 1$). In the asymptotic region with $|x|\ll 1$, $|\lambda|\ll 1$ one can solve Eqs. (2) for x,y,H, giving

$$x(N)=\frac{e^{-3N}}{\sqrt{1-ce^{-3\gamma_\gamma N}}}, \quad y(N)=\frac{1}{\sqrt{1-ce^{-3\gamma_\gamma N}}},$$

$$H(N)=d\sqrt{1-ce^{-3\gamma_\gamma N/2}}, \tag{8}$$

with c,d integration constants. These solutions show that in the asymptotic region the scalar field dominates the energy density of the universe and the Hubble parameter goes to a constant value.

In the limit $\lambda\to 0$, the first derivative of the potential approaches zero faster than the potential itself and examples of this kind of behavior are given by potentials of the form $V=V_0\phi^{-n}, n>0$. The scalar field will dominate the energy of the universe, leading to a "true" nonvanishing cosmological

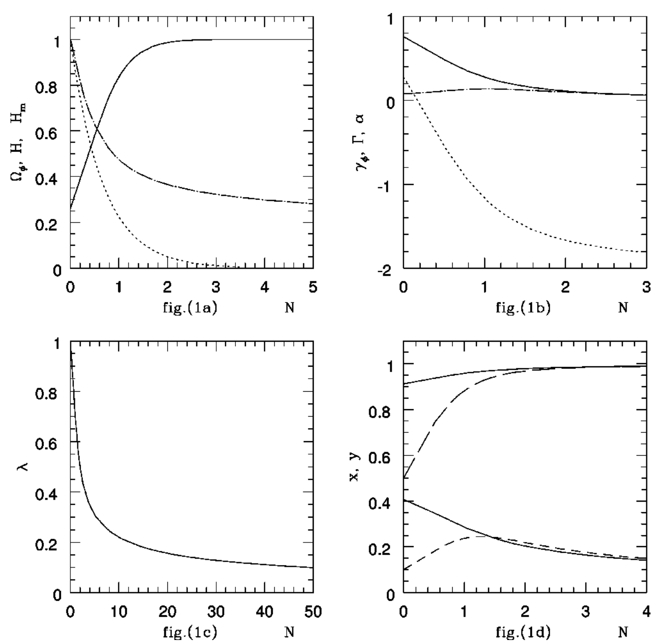

FIG. 1. Evolution of the universe filled with matter and a scalar field with $V=V_0\phi^{-1}$. The initial conditions are $x_0=0.1$, $y_0=0.5$, and $H_0=1$. In (a) we show Ω_ϕ (solid curve) quickly approaching 1; the Hubble parameter (dot-long-dashed curve) tends to a finite constant, and for comparison, we have drawn H_m for a standard matter-dominated universe (dotted curve); notice that with this type of potential the difference in the rate of expansion with the standard model is remarkable. In (b) we plot γ_ϕ (dot-long-dashed curve), Γ (solid curve), and acceleration parameter α (dotted curve). In (c) λ slowly evolves to zero with *N*. In (d), the numerical *x* and *y* solutions are plotted (short and long dashed curves, respectively) and compared to their attracting solutions, Eqs. (6), lower and upper solid curves, respectively.

constant at late times with $x\rightarrow 0$, $y\rightarrow 1$, $\Omega_\phi\rightarrow 1$, and $\gamma_\phi\rightarrow 0$. The analytic solution to Eq. (2) for x,y can be approximated by the expressions given in Eqs. (6). However, x,y are no longer constant since x_c, y_c depend on $\lambda(N)$, which is itself not constant, but these expressions are a good approximation at late times [see Fig. 1(d)] to the numerical results.

The cosmological parameters are in this case $\alpha=-2/(3\gamma_\gamma-2)$ and $\Gamma=0$. This means that the acceleration of the universe is positive (since $\alpha<0$) and the expansion of the universe is exponential. In Fig. 1 we show the behavior of x, y, γ_ϕ, and Ω_ϕ for a potential $V=V_0\phi^{-1}$. In Fig. 1(a) we can see that the scalar field quickly dominates the universe behavior and the Hubble parameter tends to a constant different than zero; i.e., the universe enters in an accelerated regime. This can also be seen in Fig. 1(b) where the acceleration parameter α is smaller than 0 and γ_ϕ; Γ are inflationary almost all the time. Figure 1(c) shows the behavior of $\lambda(N)$ for this case and in Fig. 1(d) we can see that the numerical solution has as asymptotic limit the solutions of Eqs. (6).

As our final case we take the limit $\lambda\rightarrow\infty$, and we will separate this case into two different possibilities. The first one is when λ approaches its limiting value without oscillating and the second case is when λ does oscillate.

In the nonoscillating case, in the region $|\lambda|\gg 1$ the leading term of Eqs. (2) is the one proportional to λ if $|y|,|x|$ are not much smaller than 1. In such a case the equations for x_N, y_N are

$$x_N=\sqrt{\frac{3}{2}}y^2\lambda, \quad y_N=-\sqrt{\frac{3}{2}}xy\lambda. \tag{9}$$

The sign of x_N is given by λ, and if it does not oscillate, then x would reach its maximum value $x=1$ while $y\rightarrow 0$ for $\lambda>0$ and $x\rightarrow -1$, $y\rightarrow 0$ for $\lambda<0$. However, in the region $y\rightarrow 0$ the other terms in Eqs. (2) become relevant and Eq. (9) is no longer a good approximation. In the region $|x|\gg|\lambda|y^2$ the evolution of x is given by $x_N/x=-(3+H_N/H)<0$ and x will approach its minimum absolute value $x\rightarrow 0$ like y. This region is the scaling region characterized by almost constant values of x, y, and λ. The end of the scaling region is when y_N/y changes sign and becomes positive. This happens at $\sqrt{3/2}\lambda x+H_N/H\simeq 0$ with $H_N/H\simeq -3\gamma_\gamma/2$. After a brief increase for y and x, they finally end up approaching the values given by the solution of Eqs. (5), $x\rightarrow x_c$, $y\rightarrow y_c$, and going to the extreme values of $x=y=0$. In this case x_c, y_c are not really critical (constant) points since λ is not constant. The kinetic and potential scalar energies will decrease faster than ρ, i.e., faster than ordinary matter or radiation. It is interesting to note that even though x,y approach zero, the equation of state of the scalar field becomes constant, i.e., $\gamma_\phi=2x^2/(x^2+y^2)=cte$, because x,y decrease at the same velocity. This leads to γ_ϕ approaching the value of γ_γ; i.e., the equation of state of the scalar field will be the same as that of the baryotropic fluid (matter or radiation). Even though the equation of state of the scalar field approaches that of the baryotropic fluid, its kinetic and potential energies decrease faster than that of the baryotropic fluid, the reason being that $\gamma_\phi\geq\gamma_\gamma$ at late times and the equality is only valid at $t=\infty$. The cosmological parameters are $\alpha=\Gamma=1$, giving the same asymptotic behavior for the universe with or without the scalar field.

Examples of this kind of behavior are given by potentials like $V=e^{-a\phi^2}$, $V=e^{-ae^\phi}$. In Fig. 2 we show the behavior of the dynamical variables and the cosmological parameters as a function of N for $V=Ae^{-ce^\phi}$ with $\lambda=-V'/V=ce^\phi$. This potential gives the asymptotic limit for string moduli fields [28]. The solution of Eqs. (2) shows that $\phi\rightarrow\infty$ minimizes the potential and $\lambda\rightarrow\infty$ at late times [Fig. 2(d)]. The limiting values are $x=y=\Omega_\phi=0$, as we can see in Figs. 2(a) and 2(h). In Fig. 2(a) we also show the evolution of the Hubble parameter in our model, as compared with the standard matter-dominated case; we can see that the scalar field can influence universe development only at early times. In Fig. 2(g) we show x,y for small N with $\gamma_\gamma=1$. For $N\gg 1$, x and y approach the value given in Eqs. (5) (λ^2 is larger than $3\gamma_\gamma$); see Fig. 2(h). Figure 2(b) shows the behavior of the γ_ϕ parameter as x and y evolve and the effective total Γ parameter [cf. Eq. (4)] for the ''fluid'' composed of matter and the scalar field. Figure 2(c) is the acceleration parameter defined in Eq. (3) Finally, Fig. 2(e) represents the phase space structure for (x,y) obtained with different initial

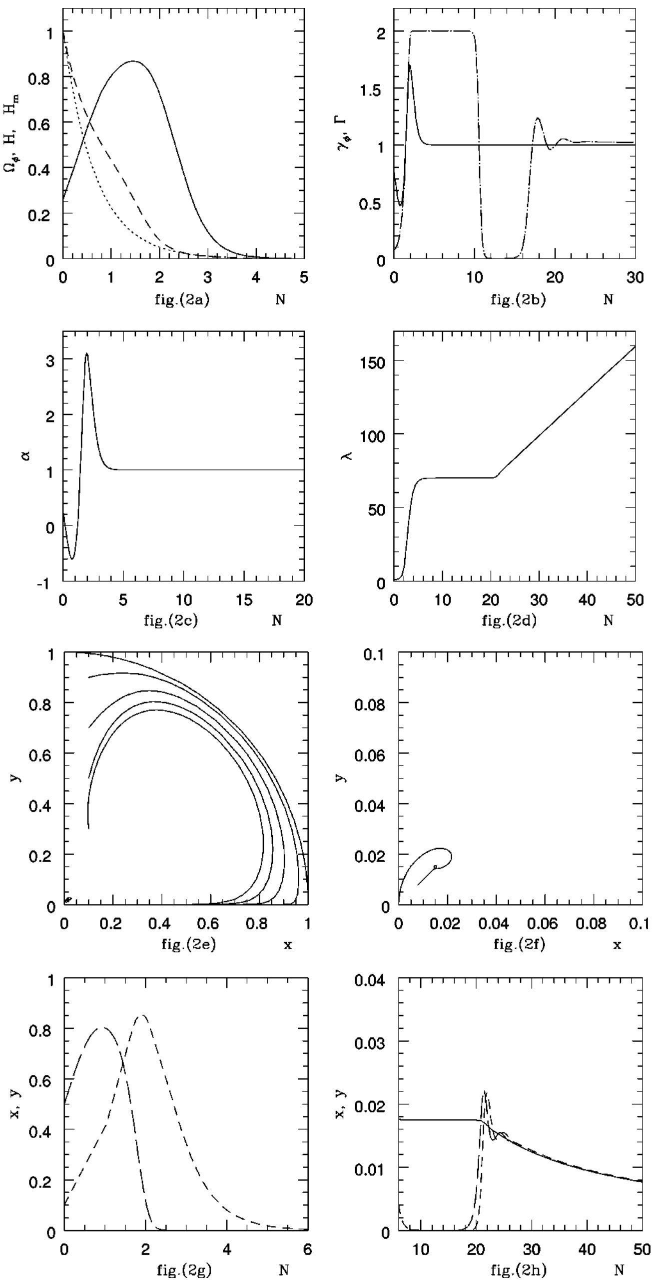

FIG. 2. Cosmological solution for $V=Ae^{-ce^{\phi}}$, $\gamma_\gamma=1$, with initial conditions $x_0=0.1$, $y_0=0.5$, $\lambda_0=1$, and $H_0=1$. In (a) the solid curve shows the evolution of Ω_ϕ, and the dashed line is the numerical solution for H, while the dotted line is the comparison value of H_m for a standard matter-dominated universe, as a function of N. In (b) we show the evolution of γ_ϕ (dot-long-dashed curve) and of the effective total Γ parameter (solid curve). In (c) the acceleration parameter α is displayed as a function of N. In (d) we have $\lambda(N)$. We show in (e) the phase plane (x,y) for different initial conditions (x_0,y_0): (0.0,1.0), (0.1,0.9), (0.1,0.7), (0.1,0.5), and (0.1,0.3), and the final evolution is amplified in (f). In (g) we show x and y (short and long dashed curves, respectively) for small N. Finally, (h) displays the asymptotic behavior of x and y [curves are marked in the same way as in (g)], showing the good agreement with the attracting solution, Eq. (5) (solid curve).

conditions, where the final behavior is amplified in Fig. 2(f). The plateau in the graph for λ in Fig. 2(d), for approximately 20 e-folds, corresponds to the scaling region, where x and y are constants (almost zero in this case), preceding the final evolution where the scalar field recovers a small quantity of kinetic and potential energy that finally go to zero.

We conclude that such fields are *not* good candidates for quintessence (parametrizing a slow varying cosmological constant) and they do not play a significant role at late times unless they are produced at a late stage. We would like to emphasize that these results are completely general and leave out a great number of candidate fields such as string moduli [28]. The only condition we have used to derive these results is that $\lambda\to\infty$ without oscillating.

We will now consider the case when $|\lambda|\to\infty$ but with an oscillating ϕ field. In this case the VEV of ϕ is finite and without loss of generality we can take it to be zero. Around the minimum the potential can be expressed as a power series in ϕ and keeping only the leading term we have $V=V_0\phi^n$ with $n>0$, even since the potential must be bounded. The condition that $V'|_{min}=0$ requires $n>1$ and a finite scalar mass requires $n=2$. For this potential $\lambda=-n/\phi$ and it oscillates approaching a value $|\lambda|=\infty$; see Fig. 3, below. As a first guess we could think that the limiting behavior of x,y is also given by Eq. (5) and therefore tends to zero as $\lambda\to\infty$. However, this asymptotic behavior is no longer a good approximation and we must solve the dynamical nonlinear equations (see Fig. 3 for the numerical solution of a quadratic potential).

We will now determine under which conditions Ω_ϕ will either dominate (approach 1), oscillate around a finite constant value, or vanish. Since asymptotically $H\propto 1/t$, a finite Ω_ϕ ($\neq 1$) requires that $\dot{\phi},(\phi)^{n/2}\propto 1/t$ or equivalently that x,y are either constant or oscillate. We can thus write $y=\phi^{n/2}/3H=BF_1^{n/2}[G(t)]$, $x=\dot{\phi}/6H=A\ F_2[G(t)]$ where F_1,F_2 are arbitrary oscillating functions depending on a single argument $G(t)$, and A,B are constants. The function $G(t)$ is for the time being an unspecified function of t. Of course, F_1 and F_2 are not independent since the functional dependence of ϕ determines the functional dependence of $\dot{\phi}$; however, this is not important at this stage. Taking the derivative with respect to N, we have $y_N=(nF_{1G}/2F_1)G_N y$ and $x_N=(F_{2G}/F_2)G_N x$, where $F_{iG}\equiv dF_i/dG$, $f_N\equiv df/dN$ with $f=x,y,G$ and $i=1,2$. Since F_1,F_2 are oscillating functions with a single argument $G(t)$, we have that the average of $\langle F_i^2\rangle=\langle F_{iG}^2\rangle$ and the average of y^2,x^2 is then

$$\langle y_N^2\rangle=\frac{n^2}{4}\langle G_N^2 y^2\rangle,\quad \langle x_N^2\rangle=\langle G_N^2 x^2\rangle. \tag{10}$$

In the asymptotic limit with x^2,y^2 oscillating and $\lambda\to\infty$ the evolution of x,y is given by Eqs. (9). Using Eqs. (10) and (9) we find that a potential $V=V_0\phi^n$ may have finite values of x,y at late times and

$$\frac{\langle y^2\rangle}{\langle x^2\rangle}=\frac{2}{n}, \tag{11}$$

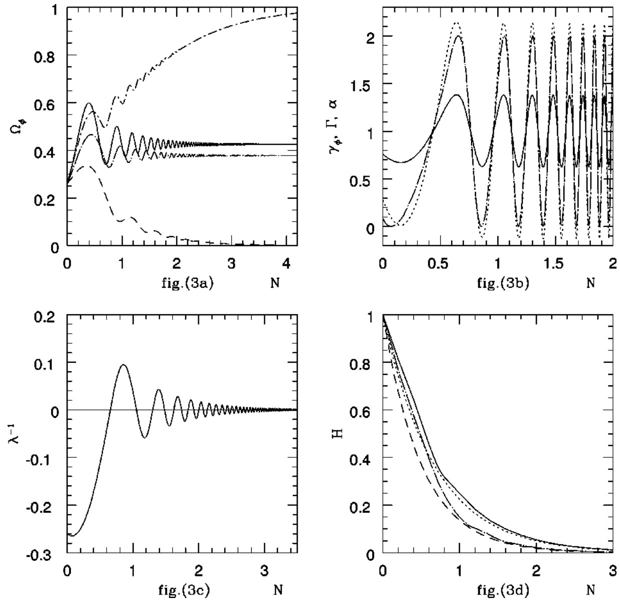

FIG. 3. Examples of evolution of the cosmological parameters for a universe filled with a perfect fluid ($\gamma_\gamma = 1, 3/4, 1/2$) and a scalar field with potential $V = V_0\phi^2$. The initial conditions are $x_0 = 0.1$, $y_0 = 0.5$, $H_0 = 1$, and $V_0 = 3\pi^2/32$. In (a) the numerical solution for Ω_ϕ (dot-long-dashed curve) with $\gamma_\gamma = 1$ is compared to the analytic expression $\Omega_\phi = x^2 + y^2$, calculated from Eq. (14) (solid curve). Also shown in the figure are Ω_ϕ for $\gamma_\gamma = 3/4$ (dot-short-dashed curve) and $\gamma_\gamma = 1/2$ (dashed curve). In (b) the equation of state and acceleration parameters are displayed as a function of N: γ_ϕ (dot-long-dashed line), Γ (solid line) and acceleration parameter (dotted line). In (c) we plot λ^{-1}, in order to display the oscillating behavior of $\lambda(N)$, as it approaches ∞. In (d) we plot the evolution of H for two different models. The solid curve represents the numerical solution for H, with $V = V_0\phi^2$ and $\gamma_\gamma = 1$ and is compared to H for a standard matter-dominated universe (dotted curve). The dot-long-dashed curve corresponds to our numerical solution for H, with $V = V_0\phi^4$ and $\gamma_\gamma = 4/3$, following a similar evolution as a standard radiation-dominated model (dashed curve).

giving $\langle\gamma_\phi\rangle = 2/(1 + \langle y^2\rangle/\langle x^2\rangle) = 2n/(2+n)$. Depending on whether γ_γ is larger, equal, or smaller than γ_ϕ, Ω_ϕ will go to 1, finite constant, or 0, respectively. Notice that this result is completely general and any massive scalar field redshifts at late times as matter fields with $\gamma_\phi = \gamma_\gamma = 1$ (i.e., $n = 2$).

In order to obtain the asymptotic solution analytically, we will solve Eqs. (2) in a region where only one component of the energy density dominates. This will be valid always in the asymptotic regime. However, we do not make any assumptions on which term dominates. If it is the baryotropic fluid which dominates, then the equation of state for ρ will have a parameter $\gamma = \gamma_\gamma$; however, if it is the scalar field that dominates, then we take $\gamma = \langle\gamma_\phi\rangle$ (average in time) since in this case γ_ϕ oscillates. In this regime we can solve for the Hubble parameter and we get the standard form $H(t) = 2/(3\gamma t)$. To determine the evolution of the scalar field we have to solve its equation of motion, $\ddot{\phi} + 3H\dot{\phi} + dV(\phi)/d\phi = 0$, with $V = V_0\phi^n$ and n positive and even, i.e.,

$$\ddot{\phi} + \frac{2}{\gamma t}\dot{\phi} + nV_0\phi^{n-1} = 0. \qquad (12)$$

For $n = 2$ the solution of Eq. (12) is given in terms of J_m and Y_m, the Bessel functions of the first and second kinds, respectively, $\phi(z) = z^{-m}(2V_0)^{m/2}[c_1 J_m(z) + c_2 Y_m(z)]$ and $\dot{\phi} = -z^{-m}(2V_0)^{(m-1)/2}[c_1 J_{m+1}(z) + c_2 Y_{m+1}(z)]$ with c_1, c_2 constants, $m \equiv (\gamma - 1/2)$, $z \equiv t\sqrt{2V_0}$, and we have used that $d[z^{-m}K_m(z)]/dz = -z^m K_{m+1}$ with $K_m = J_m, Y_m$ [14]. Using these solutions we have

$$y = rz^{1-m}[k\,J_m(z) + Y_m(z)],$$
$$x = -rz^{1-m}[k\,J_{m+1}(z) + Y_{m+1}(z)], \qquad (13)$$

with $r = (3\gamma/8)c_2(2V_0)^{m/2}$, $k \equiv c_2/c_1$. A simple analytic expression can be obtained using the asymptotic limit of the Bessel functions $J_m \simeq \sqrt{2/\pi z}\cos[z - \pi(2m+1)/4], Y_m \simeq \sqrt{2/\pi z}\sin[z - \pi(2m+1)/4]$ for $z \gg 1$ (i.e., $t \gg 1$). The amplitude of x and y in Eqs. (13) in the limit $z \gg 1$ goes as $x \simeq y \simeq z^{1/2-m}$. A finite value of x, y requires $m = \gamma - 1/2 = 1/2$ (i.e., $\gamma = 1$). For $\gamma > 1$ $(\gamma < 1)$, $x, y \to 1$ $(x, y \to 0)$ at large times. Furthermore, in the asymptotic limit $\gamma_\phi = 2\{\cos[z - \pi(2m+1)/4] - k\sin[z - \pi(2m+1)/4]\}^2/(1 + k^2)$ is an oscillating function with an average value $\langle\gamma_\phi\rangle = 1$. We can conclude, therefore, that if the barotropic fluid has $\gamma_\gamma < 1$, i.e., smaller than γ_ϕ, then $\Omega_\phi = x^2 + y^2 \to 0$, but if $\gamma_\gamma > 1 = \langle\gamma_\phi\rangle$, then the dominant energy density in the asymptotic regime will be that of the scalar field leading to $\Omega_\phi = 1$. Finally, if $\gamma_\gamma = 1 = \langle\gamma_\phi\rangle$, then the energy density of the scalar field dilutes as fast as the barotropic fluid and Ω_ϕ tends to a constant finite value. The solutions in Eq. (13) for $\gamma = 1$ can be given a completely analytic expression since in this case $m = 1/2$ and the Bessel functions take simple form $J_{1/2} = \sqrt{2/\pi z}\sin(z)$, $Y_{1/2} = -\sqrt{2/\pi z}\cos(z)$, $J_{3/2} = \sqrt{2/\pi z}[\sin(z)/z - \cos(z)]$ and $Y_{3/2} = -\sqrt{2/\pi z}[\cos(z)/z + \sin(z)]$. Putting these expressions into the definitions of x, y, Eq. (13), we get

$$y = y_0\sin(z) - \left(x_0 + \frac{y_0}{z_0}\right)\cos(z),$$
$$x = -y_0\left(\frac{\sin(z)}{z} - \cos(z)\right) + \left(x_0 + \frac{y_0}{z_0}\right) \times \left(\frac{\cos(z)}{z} + \sin(z)\right), \qquad (14)$$

where the initial conditions are given by y_0, x_0 at $z_0 = \pi/2$. In the limit $z \gg 1$ we have $\Omega_\phi \simeq y_0^2 + (x_0 + y_0/z_0)^2$.

The analytic solution in Eqs. (14) agrees reasonably well with the one obtained by solving Eqs. (2) numerically. This can be seen in Fig. 3(a), where we plot Ω_ϕ obtained numerically for $\gamma_\gamma = 1$ as a dot-long-dashed line and expressions (14) as a solid line for a potential $V = V_0\phi^2$. We also plot Ω_ϕ for $\gamma_\gamma = 4/3$ and $\gamma_\gamma = 1/2$ to illustrate the different asymptotic limits. For $\gamma_\gamma < 1$ we have $\Omega_\phi \to 0$, and $\gamma_\gamma = 1$ gives $\Omega_\phi \to cte$, and for $\gamma_\gamma = 4/3$ we have $\Omega_\phi \to 1$. These different

TABLE I. Asymptotic behavior of Ω_ϕ, γ_ϕ, the acceleration parameter $\alpha(\phi)=(3\gamma-2)/(3\gamma_\gamma-2)$, and the expansion rate parameter $\Gamma=\gamma/\gamma_\gamma$ for different limiting cases of $\lambda(\phi)$. In the last column we give an example of potential $V(\phi)$ which satisfies this limit.

$\lambda(\phi)=-V'/V$	$\Omega_\phi=\rho_\phi/\rho$	γ_ϕ	$\alpha(\phi)$	$\Gamma(\phi)$	$V(\phi)$
$c=$cte $(>\sqrt{3\gamma_\gamma})$	$\frac{3\gamma_\gamma}{c^2}$	γ_γ	1	1	$V_0e^{-c\phi}$
$c=$cte $(<\sqrt{6})$	1	$\frac{c^2}{3}$	$\frac{c^2-2}{3\gamma_\gamma-2}$	$\frac{c^2}{3\gamma_\gamma}$	$V_0e^{-c\phi}$
∞ (no oscil.)	0	γ_γ	1	1	$V_0e^{-ce^\phi}$
∞ (oscil.)	0	$\frac{2n}{2+n}(>\gamma_\gamma)$	1	1	
	cte	$\frac{2n}{2+n}(=\gamma_\gamma)$	1	1	$V_0\phi^n$, $n>0$ even
	1	$\frac{2n}{2+n}(<\gamma_\gamma)$	$\frac{3\gamma_\phi-2}{3\gamma_\gamma-2}$	$\frac{\gamma_\phi}{\gamma_\gamma}$	
0	1	0	$-\frac{2}{3\gamma_\gamma-2}$	0	$V_0\phi^{-n}$, $n>0$

limits can be understood by noting that the average of $\langle\gamma_\phi\rangle=1$, and therefore, if $\gamma_\gamma>\gamma_\phi(\gamma_\gamma<\gamma_\phi)$, the baryotropic fluid redshifts faster (slower) than the scalar field, while for $\gamma_\gamma=\langle\gamma_\phi\rangle=1$ both energy densities dilute at the same speed. The asymptotic value of Ω_ϕ in the limiting case $\lambda\to\infty$ with an oscillating ϕ field depends on the value of γ_γ and on the initial conditions $x_0=x(N_0), y_0=y(N_0)$. Figure 2(b) shows the oscillating effective equations of state for the scalar field and the mixture of matter and scalar fields, and the resulting acceleration features of the universe, α.

We have obtained an analytic solution of Eq. (12) for $n=2$. This is clearly the simplest case since Eq. (12) is linear in ϕ and its derivatives. For $n>2$, Eq. (12) becomes nonlinear in ϕ and no simple analytic solution exits. However, let us use in Eq. (12) the ansatz

$$\phi=t^{-2/n}[c_1\cos(\beta t^{2/n})+c_2\sin(\beta t^{2/n})]. \tag{15}$$

It can be easily seen that this ansatz has the correct asymptotic behavior $\phi^{n/2},\dot{\phi},H\propto 1/t$ and $\ddot{\phi}\propto\phi^{n-1}$ if we wish to have x, y finite. The t exponents in Eq. (15) are determined by solving Eqs. (12) and this equation also imposes the conditions $\gamma=2n/(2+n)$ and $\beta=\frac{1}{4}V_0n^3c_2^{n-2}[\sin(\beta t^{2/n})+k\cos(\beta t^{2/n})]^{n-2}$. Notice that the value of γ is precisely the value we obtained using general arguments only [cf. Eq. (11)]. Notice as well that the ansatz in Eq. (15) is not a "complete" answer to Eq. (12) since β is not a true constant. Only for $n=2$ is β indeed constant and the solution in Eq. (15) is the one we had previously obtained in terms of the Bessel functions [see Eqs. (13)]. For $n\neq 2$ we must take the average of β and work in the asymptotic region. Nevertheless, Eq. (15) gives a good analytic approximation to the numerical solution. In terms of Eq. (15), x,y take the following expressions:

$$x=\frac{x_0}{2/\pi+k}\left(k\sin(\alpha t^{2/n})-\cos(\alpha t^{2/n})+\frac{t^{-2/n}}{\alpha}[\sin(\alpha t^{2/n})+k\cos(\alpha t^{2/n})]\right),$$

$$y=y_0[\sin(\alpha t^{2/n})+k\cos(\alpha t^{2/n})]^{n/2}, \tag{16}$$

with $x_0=(1/n)\sqrt{3/2}\beta c_2\gamma$, $y_0=\frac{1}{2}\sqrt{3V_0}c_2^{n/2}\gamma$, and the initial conditions are taken at $t_0^{2/n}=\pi/2\beta$. Using Eq. (16) we obtain $\langle y^2\rangle/\langle x^2\rangle=2/n$, at late times, as in Eq. (11). We have, therefore, $\langle\gamma_\phi\rangle=2n/(2+n)$ and $\gamma=\langle\gamma_\phi\rangle$, i.e., the ansatz in Eq. (15) is a "solution" to Eq. (12) only when the dominant energy density redshifts as fast as the scalar field. This is, of course, no surprise since we imposed on the ansatz, Eq. (15), the limit $x,y\to cte$.

The cosmological parameters are in this case $\alpha=1-3\Omega_\phi(\gamma_\gamma-\gamma_\phi)/(3\gamma_\gamma-2)$ and $\Gamma=1-\Omega_\phi(\gamma_\gamma-\gamma_\phi)/\gamma_\gamma$ with $\gamma_\phi=2n/(2+n)$. From these expression we conclude that when Ω_ϕ remains finite $\alpha=\Gamma=1$ since in this case $\gamma_\gamma=\gamma_\phi$, leading to the same behavior of the universe with or without the contribution of the scalar field. However, when $\Omega_\phi\to 1$, $\alpha=(3\gamma_\phi-2)/(3\gamma_\gamma-2)$, $\Gamma=\gamma_\phi/\gamma_\gamma$, and one could have an accelerating universe if $\gamma_\phi<2/3$ which requires that $n<1$ and this is not acceptable since the first derivative of the scalar potential must vanish at the minimum. If $\Omega_\phi\to 0$, then clearly $\alpha=\Gamma=1$ and the scalar field plays asymptotically no important role.

To conclude this part of the analysis, we have established that if the initially dominant energy density component has a γ parameter larger (smaller) than $\langle\gamma_\phi\rangle=2n/(2+n)$, then Ω_ϕ will approach 1 (0). For $n=2$ we have $\langle\gamma_\phi\rangle=1$; for $n=4$ we have $\langle\gamma_\phi\rangle=4/3$. Since the condition $V'|_{min}=0$ requires $1<n$, we have that $\gamma_\phi>2/3$ for all n and the scalar field will not give an accelerating universe. For $n>4$ the energy density will decrease faster than radiation, and since at late times the universe is matter dominated, only a scalar field with a nonvanishing mass could lead to a significant contribution to the energy density of the universe. However, since its redshift goes as matter, it is not a candidate for a cosmological constant but it could serve as dark matter. Figure 3(d) illustrates these characteristics of a power-law potential for the Hubble parameter H for $V=V_0\phi^n$, $n=2,4$, a radiation- and a matter-dominated universe.

To summarize and conclude, we have studied the cosmological evolution of the universe filled with a baryotropic fluid and a scalar field with an arbitrary potential but only with a gravitational interaction with all other fields. The analysis done is completely general, and we do not assume any kind of scale or time dependence of the scalar potential or any assumption on which the energy density (baryotropic or scalar) dominates. Our results are summarized in Table I. We showed that all model dependence is given by $\lambda\equiv -V'/V$ and γ_γ. Any scalar potential leads to one of the three different limiting cases of λ: finite constant, zero, or infinity. In the first case, Ω_ϕ approaches a finite constant (different than zero) depending on the value of $\lambda=c$. For $\lambda\rightarrow 0$ we obtained $x\rightarrow 0$, $y\rightarrow 1$ with a constant Hubble parameter H and an accelerating universe. Finally, for $\lambda\rightarrow\infty$ we concluded that if λ does not oscillate, $x,y,\Omega_\phi\rightarrow 0$, and if λ oscillates, then all cases are possible (i.e., $\Omega_\phi\rightarrow 0$, 1, or a finite constant) depending on the value of γ_γ and the power of the leading term in the scalar potential.

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[1] W. L. Freedman, Phys. Rep. (to be published), astro-ph/9909076.

[2] L. Ferrarese *et al.*, in ''Proceedings of the Cosmic Flows Workshop, 1999,'' astro-ph/9909134; M. A. Hendry and S. Rauzy, in ''Proceedings of the Cosmic Flows Workshop, 1999,'' astro-ph/9908343.

[3] B. Chaboyer, P. Demarque, P. J. Kernan, and L. M. Krauss, Science **271**, 957 (1996).

[4] B. Chaboyer, P. Demarque, P. J. Kernan, and L. M. Krauss, Astrophys. J. **494**, 96 (1998).

[5] G. Efstathiou, S. Maddox, and W. Sutherland, Nature (London) **348**, 705 (1990).

[6] J. Primack and A. Klypin, Nucl. Phys. B (Proc. Suppl.) **51**, 30 (1996); J. F. Navarro and M. Steinmetz, astro-ph/9908114.

[7] A. G. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.*, Astrophys. J. **517**, 565 (1999); P. M. Garnavich *et al.*, *ibid.* **509**, 74 (1998).

[8] G. Efstathiou, S. L. Bridle, A. N. Lasenby, M. P. Hobson, and R. S. Ellis, Mon. Not. R. Astron. Soc. **303L**, 47 (1999); M. Roos and S. M. Harun-or Rashid, astro-ph/99010234.

[9] E. I. Sorokina, S. I. Blinnikov, and O. S. Bartunov, astro-ph/9906464; A. G. Riess, A. V. Filippenko, W. Li, and B. P. Schmidt, astro-ph/9907038.

[10] S. D. M. White, J. F. Navarro, A. E. Evrard, and C. S. Frenk, Nature (London) **366**, 429 (1993).

[11] G. Steigman and J. E. Felten, in *Proceedings of the St. Petersburg Gamow Seminar*, edited by A. M. Bykov and R. A. Chevalier [Space Sci. Rev. **74**, 245 (1995)].

[12] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).

[13] J. M. Overduin and F. I. Cooperstock, Phys. Rev. D **58**, 043506 (1998); V. Mendez, Class. Quantum Grav. **13**, 3229 (1996); J. Uzan, Phys. Rev. D **59**, 123510 (1999); A. P. Billyard, A. A. Coley, R. J. van den Hoogen, J. Ibanez, and I. Olasagaste, Class. Quantum Grav. **16**, 4035 (1999).

[14] A. R. Liddle and R. J. Scherrer, Phys. Rev. D **59**, 023509 (1999).

[15] W. Chen and Y. Wu, Phys. Rev. D **41**, 695 (1990); Y. Fujii and T. Nishioka, *ibid.* **42**, 361 (1990); D. Pavon, *ibid.* **43**, 375 (1991); J. Matyjasek, *ibid.* **51**, 4154 (1995); M. S. Berman, *ibid.* **43**, 1075 (1991); J. C. Carvalho, J. A. S. Lima, and I. Waga, *ibid.* **46**, 2404 (1992); J. A. S. Lima and J. M. F. Maia, *ibid.* **49**, 5597 (1994); V. Silveira and I. Waga, *ibid.* **50**, 4890 (1994).

[16] T. S. Olson and T. F. Jordan, Phys. Rev. D **35**, 3258 (1987); J. W. Moffat, Phys. Lett. B **357**, 526 (1995).

[17] P. J. E. Peebles and B. Ratra, Astrophys. J., Lett. Ed. **325**, L17 (1988); J. W. Moffat, Phys. Lett. B **357**, 526 (1995).

[18] K. Freese, F. C. Adams, J. A. Frieman, and E. Mottola, Nucl. Phys. **B287**, 797 (1987); M. Birkel and S. Sarkar, Astropart. Phys. **6**, 197 (1997).

[19] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).

[20] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).

[21] V. Silveira and I. Waga, Phys. Rev. D **56**, 4625 (1997); L. Amendola, astro-ph/9908023.

[22] D. Lyth and C. Koldo, Phys. Lett. B **458**, 197 (1999); I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999); D. Huterer and M. S. Turner, astro-ph/9808133; P. Brax and J. Martin, Phys. Lett. B **465**, 40 (1999); T. Chiba, Phys. Rev. D **60**, 083508 (1999).

[23] M. S. Turner and M. White, Phys. Rev. D **56**, 4439 (1997); G. Efstathiou, astro-ph/9904356; L. Wang *et al.*, astro-ph/9901388.

[24] A. Albrecht and C. Skordis, Phys. Rev. Lett. **84**, 2076 (2000).

[25] C. Wetterich, Nucl. Phys. **B302**, 668 (1998).

[26] P. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998); E. J. Copeland, A. Liddle, and D. Wands, Ann. (N.Y.) Acad. Sci. **688**, 647 (1993).

[27] E. J. Copeland, A. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[28] A. de la Macorra, hep-ph/9910330.