Nonequilibrium dynamics of quantum tunneling

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We consider dynamics during phase transitions by quantum tunneling using nonequilibrium quantum field theory. We define an order parameter to represent a state of the system. The evolution equation of this order parameter and fluctuations around it are derived. In some cases long wave fluctuations are increased by instabilities during tunneling processes and these fluctuations enhance the tunneling.

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I. INTRODUCTION

Phase transitions in the early Universe are very important processes since they affected the time evolution of the Universe [1,2]. Therefore, to understand these phase transitions it is necessary to set up a formalism for describing the dynamics during phase transitions. Since in this process the system deviates from the equilibrium state, static quantities such as effective potential in quantum field theory at finite and zero temperatures are not suitable for understanding these dynamical aspects of phase transitions.

Recently there has been great progress in nonequilibrium field theory and it is applied to cosmology [3–15]. These methods enable us to describe the evolution of the out-of-equilibrium situation including phase transitions. For instance, these descriptions are utilized for parametric particle decay in preheating of chaotic inflation [16–21] and spinodal instability in new inflation [22,23]. In these phenomena the static description is not applicable and the method including time evolution in quantum field theory can be used to understand these processes.

In addition to new and chaotic inflation, there is another mechanism of inflation, old inflation, in which the inflation proceeds by quantum tunneling [24]. Although the original old inflation was unsuccessful [25], there are still inflation models including tunneling process. Since in chaotic and new inflation the dynamical description revealed properties unknown in ordinary methods, we would expect that the dynamical description is also important to understand tunneling. Therefore it will be needed to describe tunneling in a formulation including the nonequilibrium dynamics during this process. However, until now such an approach has not existed. (There are many papers discussing dynamics of bubbles, i.e., growth and percolation, but we do not consider it in this paper.)

Our ordinary picture of tunneling is as follows. At first a true vacuum bubble appears in a false vacuum quantum mechanically. If this bubble is larger than the critical bubble, in which the loss of the surface energy is compensated by the gain of the volume energy, then this bubble expands. Finally these bubbles percolate each other and the entire system becomes true vacuum. This picture is an analogy of a first order phase transition in statistical mechanics [26] except that in statistical mechanics bubbles are created thermally not quantum mechanically.

In this scenario, the basic quantity is a nucleation probability per unit time per unit volume Γ/V and quantum mechanics is only related to it. For the calculation of the decay constant, we usually use bounce method [27,28]. In this method, time is rotated to imaginary time and quantum problem becomes solving a classical equation of motion in imaginary time.

This method has been used in various areas but there are some difficulties. First, we calculate the decay constant in homogeneous false vacuum. Certainly in tunneling we usually compute a tunneling rate in homogeneous circumstances. This is because environment effects are model dependent and, if we do not concern these specific effects, we assume the system is initially inhomogeneous. However, even before true vacuum bubbles larger than the critical bubble are created, quantum mechanically true vacuum bubbles that are smaller than the critical bubble are produced and disappearing continuously. This production and collapse of subcritical bubbles will produce some fluctuations inherently and it is unclear whether these quantum fluctuations in a metastable false vacuum remain small or not. It is probable that these fluctuations have a strong influence on the tunneling. The formulation including the dynamics during tunneling enables us to consider this effect.

This is similar to the subcritical bubble effect in a weakly first-order phase transition in electroweak baryogenesis [29,30]. In this scenario, if the electroweak phase transition is weakly first order, this phase transition proceeds by production of subcritical bubbles not the formation of critical bubbles. Recently this problem has been discussed in a different way in Ref. [31].

Second, in imaginary time the evolution of wave function is not unitary. Moreover, when a Lagarangian has direct dependence on time, for example, expansion of the universe, or there are another fields not tunneling, it is difficult to interpret the meaning of time expansion or time evolution. For example, simple harmonic oscillator becomes inverted harmonic oscillator.

Moreover the amplification of fluctuations is thought as particle creation during tunneling. This problem is treated in various methods [32–34] but all of these methods use imaginary time. Owing to this, nonunitary Bogoliubov transformation is used in Ref. [32]. Therefore we think that describing the dynamics of tunneling in real time is essential for more understanding and application of tunneling.

Motivated by these reasons we treat the dynamics during tunneling process in flat space time with real time. To investigate the dynamics of tunneling we use the functional Schrödinger equation. This formulation makes it possible to describe the evolution of the system including the dynamics during tunneling. As in statistical phase transitions, we define the order parameter to represent the state of the system. We also consider the evolution of fluctuations around the order parameter.

In tunneling not all of these fluctuations are important. In a decay process with a survival probability $P=\exp(-\Gamma t)$ where Γ is a decay constant, the typical time scale of this process is Γ^{-1}. In tunneling, Γ is extremely small and the typical time scale is greatly large. Therefore we are usually concerned only an event rate par time Γ. However, to understand the dynamics we need to know the time evolution of the system over the long period of time and then rapid fluctuations compared to this large time scale are neglected or thought of as noise. The effect of noise surely influences the transition rate [35] but this does not drastically change the tunneling process and the order of Γ. Therefore in this paper we do not consider these rapid fluctuations.

This article is organized as follows. In the next section we formulate the equations to describe the dynamics of tunneling, one for order parameter and other for fluctuations around an order parameter. In Secs. III and IV the evolution of an order parameter and fluctuations around it are considered. In Sec. V the back reaction of fluctuations to an order parameter is discussed. Finally in Sec. VI we conclude this paper with some comments.

II. EVOLUTION EQUATIONS OF ORDER PARAMETER AND FLUCTUATIONS

In this section we derive the equations to describe the evolution of the system during tunneling. The Lagrangian we consider is

$$L=\int dx^3\left\{\frac{1}{2}(\partial_\mu\phi)^2-V(\phi)\right\}, \tag{2.1}$$

$$V(\phi)=\frac{m^2}{2}\phi^2+\frac{g}{3!}\phi^3+\frac{\lambda}{4!}\phi^4, \tag{2.2}$$

where m^2 and λ are positive and g is negative. This potential $V(\phi)$ has two minima, one is a symmetric false vacuum at $\phi=0$, and other is a symmetry breaking true vacuum at $\phi=\sigma$. We assume the system is in a finite volume and introduce the discrete Fourier transform of the field as

$$\phi(\mathbf{x})=\sum_{\mathbf{k}}\phi_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{2.3}$$

Then the above Lagrangian becomes

$$L=\Omega\left\{\frac{1}{2}\sum_{\mathbf{k}}\dot{\phi}_{-\mathbf{k}}\dot{\phi}_{\mathbf{k}}-\frac{1}{2}\sum_{\mathbf{k}}(\mathbf{k}^2+m^2)\phi_{-\mathbf{k}}\phi_{\mathbf{k}}-\frac{g}{3!}\sum_{\mathbf{k},\mathbf{p}}\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{-\mathbf{k}-\mathbf{p}}-\frac{\lambda}{4!}\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{\mathbf{q}}\phi_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right\}, \tag{2.4}$$

where the overdot means time derivative and Ω is the volume of the system we are considering. The canonical momentum conjugate to $\phi_{\mathbf{k}}$ is $\pi_{\mathbf{k}}=\Omega\dot{\phi}_{-\mathbf{k}}$ and the Hamiltonian is

$$H=\frac{1}{2\Omega}\sum_{\mathbf{k}}\pi_{-\mathbf{k}}\pi_{\mathbf{k}}+\Omega\left\{\frac{1}{2}\sum_{\mathbf{k}}(m^2+\mathbf{k}^2)\phi_{-\mathbf{k}}\phi_{\mathbf{k}}+\frac{g}{3!}\sum_{\mathbf{k},\mathbf{p}}\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{-\mathbf{k}-\mathbf{p}}+\frac{\lambda}{4!}\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{\mathbf{q}}\phi_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\right\}. \tag{2.5}$$

With this Hamiltonian the Schrödinger equation becomes

$$i\frac{\partial}{\partial t}\Psi=H\Psi. \tag{2.6}$$

where Ψ is a wave functional and fields in the Hamiltonian are in the Schrödinger representation.

To specify the state of the system we define the order parameter ϕ_c as

$$\phi_c=\frac{1}{\Omega}\int dx^3\langle\phi(t,\mathbf{x})\rangle=\frac{1}{\Omega}\int dx^3\int[d\phi]\Psi^*\phi(\mathbf{x})\Psi, \tag{2.7}$$

where $[d\phi]=\Pi_{\mathbf{y}}d\phi(\mathbf{y})$. The vacuum in which the state resides is represented by the value of this parameter. If the system is in the symmetric false vacuum, the value of the order parameter is zero. If the system is in symmetry breaking true vacuum, the value of the order parameter is σ.

Ordinarily we derive the equation of motion for ϕ_c using above definition of ϕ_c and the Schrödinger equation and then determine the evolution of the order parameter. This is a classical equation with quantum corrections. For phase transitions in statistical physics and new and chaotic inflations, this equation will be suitable for describing the evolution of the system. However, tunneling is basically quantum phenomenon and cannot be described by the classical equation of motion for the order parameter ϕ_c. For this reason we will take the following way. Since the order parameter is independent to spatial coordinates, only the zero momentum mode of ϕ contributes to ϕ_c. Therefore instead of the classical equation of motion for the order parameter, we derive the Schrödinger equation for the zero mode and define the evolution of the order parameter ϕ_c.

We assume the wave functional of the system can be factorized to the wave functions of zero mode and nonzero modes $\Psi=\Psi_{\mathrm{zero}}\Psi_{\mathrm{nonzero}}$. This is possible for the homogeneous configuration, which is an initial condition we take.

Product $\Psi^*_{\rm nonzero}$ to the Schrödinger equation (2.6) and integrate out non zero modes. Then the Schrödinger equation for the zero mode is

$$i\frac{\partial}{\partial t}\Psi_{\rm zero}=H_{\rm zero}\Psi_{\rm zero}, \tag{2.8}$$

with the Hamiltonian

$$H_{\rm zero}=-\frac{1}{2\Omega}\frac{\partial^2}{\partial\phi_{\mathbf{0}}^2}+\Omega\Bigg\{V(\phi_{\mathbf{0}})+\frac{1}{2}V''(\phi_{\mathbf{0}})\sum_{\mathbf{k}\neq\mathbf{0}}\langle\phi_{-\mathbf{k}}\phi_{\mathbf{k}}\rangle+\frac{1}{3!}V'''(\phi_{\mathbf{0}})\sum_{\substack{\mathbf{k},\mathbf{p}\neq\mathbf{0}\\ \mathbf{k}+\mathbf{p}\neq\mathbf{0}}}\langle\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{-\mathbf{k}-\mathbf{p}}\rangle$$
$$+\frac{1}{4!}V''''(\phi_{\mathbf{0}})\sum_{\substack{\mathbf{k},\mathbf{p},\mathbf{q}\neq\mathbf{0}\\ \mathbf{k}+\mathbf{p}+\mathbf{q}\neq\mathbf{0}}}\langle\phi_{\mathbf{k}}\phi_{\mathbf{p}}\phi_{\mathbf{q}}\phi_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\rangle+\left\langle\left(H_{\text{non zero}}-i\frac{\partial}{\partial t}\right)\right\rangle\Bigg\} \tag{2.9}$$

where primes stand for derivatives with respect to $\phi_{\mathbf{0}}$ and $\langle\cdots\rangle$ are expectation values with nonzero mode wave function $\Psi_{\rm nonzero}$ and $H_{\rm nonzero}$ includes only nonzero mode of ϕ. The terms independent to $\phi_{\mathbf{0}}$ only contribute to an over all phase factor of the wave function $\Psi_{\rm zero}$ and we will neglect them in the following.

This is the evolution equation of the order parameter during tunneling. The terms include expectation value of $\phi_{\mathbf{k}}$ is back reaction of fluctuations to the evolution of the order parameter. The value of the order parameter ϕ_c is evaluated by the definition (2.7) with the wave function of this Schrödinger equation.

Next we consider fluctuations around the order parameter. At first we write the field as

$$\phi=\phi_c+\eta=\phi_c+\sum_{\mathbf{k}}\eta_{\mathbf{k}}e^{-i\mathbf{k}\cdot x}, \tag{2.10}$$

where $\eta_{\mathbf{k}}$ are equivalent to $\phi_{\mathbf{k}}$ for $\mathbf{k}\neq\mathbf{0}$. Substitute this into the Lagrangian (2.2) and obtain

$$L=\Omega\Bigg\{\frac{1}{2}\dot{\phi}_c^2-V(\phi_c)-[\ddot{\phi}_c+V'(\phi_c)]\eta_{\mathbf{0}}+\frac{1}{2}\sum_{\mathbf{k}}\dot{\eta}_{-\mathbf{k}}\dot{\eta}_{\mathbf{k}}$$
$$-\frac{1}{2}\sum_{\mathbf{k}}[\mathbf{k}^2+V''(\phi_c)]\eta_{-\mathbf{k}}\eta_{\mathbf{k}}$$
$$-\frac{1}{3}V'''(\phi_c)\sum_{\mathbf{k},\mathbf{p}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}}$$
$$-\frac{1}{4!}V''''(\phi_c)\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\Bigg\}, \tag{2.11}$$

where we have partially integrated and discarded the surface term. The momentum conjugate to $\eta_{\mathbf{k}}$ is

$$\pi_{\eta_{\mathbf{k}}}=\Omega\,\dot{\eta}_{-\mathbf{k}} \tag{2.12}$$

and the Hamiltonian for fluctuations is

$$H_\eta=\frac{1}{2\Omega}\sum_{\mathbf{k}}\pi_{-\mathbf{k}}\pi_{\mathbf{k}}+\Omega\Bigg\{V(\phi_c)+[\ddot{\phi}_c+V'(\phi_c)]\eta_{\mathbf{0}}+\frac{1}{2}\sum_{\mathbf{k}}\Big([\mathbf{k}^2+V''(\phi_c)]\eta_{-\mathbf{k}}\eta_{\mathbf{k}}+\frac{1}{3!}V'''(\phi_c)\sum_{\mathbf{k},\mathbf{p}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathrm{p}}$$
$$+\frac{1}{4!}V''''(\phi_c)\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\Bigg\}. \tag{2.13}$$

The quantum fluctuations around the order parameter is determined by the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi_\eta=H_\eta\Psi_\eta. \tag{2.14}$$

Equations (2.8), (2.9), (2.13), and (2.14) are equations to describe the dynamics during tunneling process and we consider each equations in the following sections.

III. ORDER PARAMETER

In this section we consider time evolution of the order parameter ϕ_c in the Schrödinger equation without back reaction

$$i\frac{\partial}{\partial t}\Psi=H\Psi, \tag{3.1}$$

$$H=-\frac{1}{2\Omega}\frac{\partial^2}{\partial\phi^2}+\Omega V(\phi), \tag{3.2}$$

where H and Ψ are $H_{\rm zero}$ and $\Psi_{\rm zero}$ in the previous section, respectively. In this equation the potential has two minima, one is symmetric vacuum, whose positions we call σ_0 and adjust to the origin, and another is symmetry breaking vacuum, we call σ_2.

For sufficiently deep wells we can approximately define eigenstates near the bottoms of each minimum and expand the state with these eigenstates. As we are concerned to the transition from one minimum to another minimum, we expand a state Ψ with these eigenfunctions in two minima:

$$\Psi=\sum_n c_n\psi_n+\sum_{\tilde n}\tilde c_{\tilde n}\tilde\psi_{\tilde n}, \tag{3.3}$$

where ψ_n and $\tilde\psi_{\tilde n}$ are eigenfunctions in the minima σ_0 and σ_2, respectively. This is an extention of Ref. [12]. After substitution of this expansion into Eq. (3.1) and production of ψ_n and $\tilde\psi_{\tilde n}$, we obtain an evolution equation for coefficients (see the Appendix in detail):

$$i\frac{d}{dt}\begin{pmatrix}c_n\\ \tilde c_{\tilde n}\end{pmatrix}=\begin{pmatrix}H_{nm} & H_{n\tilde m}\\ H_{\tilde n m} & H_{\tilde n\tilde m}\end{pmatrix}\begin{pmatrix}c_m\\ \tilde c_{\tilde m}\end{pmatrix}, \tag{3.4}$$

where $H_{n\tilde m}=\langle\psi_n H\tilde\psi_{\tilde m}\rangle$ and c_m and $c_{\tilde m}$ are column vectors. In this expression we omitted the terms whose order is $O(\langle\psi_n\tilde\psi_{\tilde m}\rangle)$.

We calculate the matrix elements. As the states ψ_n and $\tilde\psi_{\tilde n}$, we use approximate solutions of equations:

$$H\psi=E\psi \quad \text{at} \quad \phi<\sigma_1, \tag{3.5}$$

$$H\tilde\psi=\tilde E\tilde\psi \quad \text{at} \quad \phi>\sigma_1, \tag{3.6}$$

where σ_1 is a value where the potential is local maximum and ψ is small in $\phi>\sigma_1$ and $\tilde\psi$ is small in $\phi<\sigma_1$. Therefore $H_{nm}=E_n\delta_{nm}$ and $H_{\tilde n\tilde m}=\tilde E_{\tilde n}\delta_{\tilde n\tilde m}$. As the matrix elements $\langle\psi\tilde\psi\rangle$ is extremely small, we use WKB approximation for the off diagonal elements $\langle\psi H\tilde\psi\rangle$ [36]. In the classically forbidden region WKB solutions are

$$\psi=\frac{C}{\sqrt{W}}\exp\left(-\int_a^\phi d\phi W\right), \tag{3.7}$$

$$\tilde\psi=\frac{\tilde C}{\sqrt{\tilde W}}\exp\left(-\int_\phi^b d\phi\tilde W\right), \tag{3.8}$$

where $W=\sqrt{2\Omega(\Omega V-E)}$, $\tilde W=\sqrt{2\Omega(\Omega V-\tilde E)}$, and $W(a)=0$ and $\tilde W(b)=0$. C, $\tilde C$ are constants and we set $C=(\Omega\omega_0/2\pi)^{1/2}$ and $\tilde C=(\Omega\tilde\omega_0/2\pi)^{1/2}$ with $\omega_0=V''(\sigma_0)$ and $\tilde\omega_0=V''(\sigma_2)$. This normalizes ψ and $\tilde\psi$ in $\phi<\sigma_1$ and $\phi>\sigma_1$, respectively.

Then consider

$$\langle\psi H\tilde\psi\rangle=\int_{-\infty}^{\infty}d\phi\psi\left(-\frac{1}{2\Omega}\frac{\partial^2}{\partial\phi^2}+\Omega V\right)\tilde\psi. \tag{3.9}$$

For $\phi>\sigma_1$, $\tilde\psi$ is an eigenfunction [see Eq. (3.6)] and the integrand becomes $\tilde E\psi\tilde\psi$. However, in this region this integration is negligible as ψ is extremely small and the integral is restricted in the region $\phi<\sigma_1$. By partial integrations, this integration becomes

$$\langle\psi H\tilde\psi\rangle=-\frac{1}{2}\left[\psi\frac{\partial\tilde\psi}{\partial\phi}-\frac{\partial\psi}{\partial\phi}\tilde\psi\right]_{-\infty}^{\sigma_1}+\int_{-\infty}^{\sigma_1}d\phi\tilde\psi\left(-\frac{1}{2\Omega}\frac{\partial^2}{\partial\phi^2}+\Omega V\right)\psi. \tag{3.10}$$

The last term is negligible by the same argument applied to the integration in the region $\phi>\sigma_1$. Since $\psi=\tilde\psi=0$ at $\phi=-\infty$,

$$\langle\psi H\tilde\psi\rangle=-\frac{1}{2}C^*\tilde C\left(\sqrt{\frac{\tilde W}{W}}+\sqrt{\frac{W}{\tilde W}}\right)\Bigg|_{\sigma_1}\times\exp\left(-\int_a^{\sigma_1}d\phi W-\int_{\sigma_1}^b d\phi\tilde W\right). \tag{3.11}$$

In the case $\tilde E=E$,

$$\langle\psi H\tilde\psi\rangle=-C^*\tilde C\exp\left(-\int_a^b d\phi W\right). \tag{3.12}$$

To gain a qualitative picture we restrict to two states, one localized in $\phi\sim\sigma_0$ and other in $\phi\sim\sigma_2$. In this case Eq. (3.4) can be easily solved and the solution is

$$\begin{pmatrix}c\\ \tilde c\end{pmatrix}=\frac{1}{\lambda_--\lambda_+}\begin{pmatrix}e^{-i\lambda_+t}(\lambda_+-E)-e^{-i\lambda_-t}(\lambda_--E)\\ -\Delta(e^{-i\lambda_+t}-e^{-i\lambda_-t})\end{pmatrix}, \tag{3.13}$$

where $\lambda_\pm=\frac{1}{2}[E+\tilde E\pm\sqrt{(E-\tilde E)^2+4\Delta^2}]$ and $\Delta=\langle\psi H\tilde\psi\rangle$. The order parameter is $\phi_c=|\tilde c|^2\sigma$. This shows oscillatory behavior between two states with a period $T=2\pi/(\lambda_+-\lambda_-)$. This period is

$$T=\frac{2\pi}{|E-\tilde E|}, \tag{3.14}$$

for two states with large energy difference and

$$T=\frac{\pi}{|C^*\tilde C|}\exp\left(\int_a^b d\phi W\right), \tag{3.15}$$

for two states with $E=\tilde E$. The oscillation is rapid when the energy gap is large and slow when the energy gap of two states is small. In general, the solution is superposition of these oscillations. Since in these oscillations the rapid fluctuations can be considered as noise, the state of the system is mainly ruled by the slowest motion, i.e., the oscillation between the false vacuum and the nearest energy state in true minimum. Therefore the order parameter ϕ_c is regarded as a slowly oscillating variable between 0 and σ. It looks effectively as if the classical field ϕ_c moves in an extremely flat potential with effective mass $(\lambda_+ - \lambda_-)$.

From the Ω dependence of T for $E=\tilde{E}$, the period T grows with Ω except small Ω. Therefore for large Ω the order parameter ϕ varies slowly.

In the thin-wall approximation, $a=0$ and $b=\sigma$ and for a false vacuum $E=0$. In this case the inverse of the half period is

$$(T/2)^{-1} \propto \Omega \exp\left(-\Omega \int_0^\sigma d\phi \sqrt{2V} \right). \tag{3.16}$$

For the critical bubble in the thin wall approximation the bubble radius is [27]

$$R = 3\frac{\int_0^{\sigma_2} d\phi \sqrt{2V(\phi)}}{V(0)-V(\sigma_2)}, \tag{3.17}$$

and the exponent of Eq. (3.16) agrees with that of a decay constant of the false vacuum that is calculated with the bounce method, except numerical factor.

The oscillational property of this solution will be terminated by the dissipative effect ignored in this paper. But until the dissipation becomes effective, the above time evolution is preserved and the conclusion in this paper is consistent at the early stage. The decay rate of this oscillation is approximately proportional to $|\langle \tilde{\psi}_n H_{\rm int} \psi \rangle|^2$ where $H_{\rm int}$ is an interaction term with another degree of freedom. ψ is a false vacuum state and $\tilde{\psi}_n$ are lower energy states. By definition ψ is the lowest energy state localized in $\phi=0$ and then $\tilde{\psi}_n$ are all localized in $\phi=\sigma$. Therefore the matrix elements $\langle \tilde{\psi}_n H_{\rm int} \psi \rangle$ are exponentially suppressed and the oscillatory behavior remains sufficiently long period.

IV. FLUCTUATIONS AROUND THE ORDER PARAMETER

In this section we consider the evolution of fluctuations around the order parameter $\phi_c(t)$. Once the motion of the order parameter is specified the treatment of fluctuations around the order parameter follows the previous works [5]. We deal with the equation

$$i\frac{\partial}{\partial t}\Psi_\eta = H_\eta \Psi_\eta \tag{4.1}$$

with the Hamiltonian

$$H_\eta = -\frac{1}{2\Omega}\sum_{\mathbf{k}} \frac{\partial^2}{\partial \eta_{-\mathbf{k}} \partial \eta_{\mathbf{k}}} + \Omega \left\{ [\ddot{\phi}_c + V'(\phi_c)]\eta_{\mathbf{0}} + \frac{1}{2}\sum_{\mathbf{k}} [\mathbf{k}^2 + V''(\phi_c)]\eta_{-\mathbf{k}}\eta_{\mathbf{k}} + \frac{1}{3!}V'''(\phi_c)\sum_{\mathbf{k},\mathbf{p}} \eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}} + \frac{1}{4!}V''''(\phi_c)\sum_{\mathbf{k},\mathbf{p},\mathbf{q}} \eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}} \right\}, \tag{4.2}$$

where we have discarded the terms independent to the fields $\eta_{\mathbf{k}}$ because it becomes over all phase factor of wave function.

At first we concentrate on the terms quadratic in $\eta_{\mathbf{k}}$ because as an initial condition we assume there are no fluctuations. As derived in the previous section, the order parameter ϕ_c moves slowly from 0 to σ. During this, the square of frequency

$$\omega_{\mathbf{k}}^2 = \mathbf{k}^2 + V''(\phi_c) \tag{4.3}$$

can be negative for small wavenumber $\mathbf{k}$ in some period because the potential is nonconvex, i.e., $V''(\phi_c)<0$ when ϕ_c is in spinodal region. When the square of frequency is negative this mode increases exponentially. This amplification in the long wave modes of fluctuations is common to the phase transitions in statistical physics and quantum field theory.

The maximum value of $|\mathbf{k}|^2$ with negative $\omega_{\mathbf{k}}^2$ is the square root of the absolute value of the minimum value of V'', we call $V''_{\rm min}$, then $|\mathbf{k}|<|V''_{\rm min}|^{1/2}$. On the other hand, in the region with the volume Ω, the minimum value of $|\mathbf{k}|$ is $2\pi\Omega^{-1/3}$, then $|\mathbf{k}|>2\pi\Omega^{-1/3}$.

Therefore if the system we consider has the volume smaller than $|V''_{\rm min}|^{-3/2}$, the square of the frequency $\omega_{\mathbf{k}}^2$ remains positive for all ϕ_c and the long wave fluctuations does not increase. This indicates that the fluctuations remain initial small quantum fluctuations and the system stays in homogeneous and the entire volume transits to the true vacuum simultaneously.

On the contrary, if the region has the volume larger than $|V''_{\rm min}|^{-3/2}$, the square of the frequency $\omega_{\mathbf{k}}^2$ can be negative for some value of ϕ_c and the long wave modes are amplified. This indicates that the fluctuations are amplified and the system becomes inhomogeneous and phase separation occurs and domains are formed. Therefore the entire volume will not transit to the true vacuum simultaneously but each region transits to the true vacuum separately. This is the spinodal decomposition in statistical mechanics.

In usual picture of tunneling, the phase transition proceeds by the production of the critical bubbles. The radius of this critical bubble is larger than $|V''_{\rm min}|^{-3/2}$ in a nearly degenerated potential. This suggests that for the models we usually use thin wall approximation, that is the nearly degenerate potential, the homogeneous bubble formation is improbable. Before formation of critical bubbles, subcritical bubbles are produced and these make the system inhomogeneous. This will be quantum mechanical counterpart of subcritical bubbles in a weakly first-order phase transition in electroweak baryogenesis [29].

In tunneling phase transition, quantum mechanically true vacuum subcritical bubbles are formed and destroyed continuously. These attempts produce fluctuations. Owing to the instability of the system, some of these fluctuations do not remain small but amplified. On the other hand, in the tunneling process we assume that the entire space is in the false vacuum and there is no inhomogeneity. Therefore, we are concerned with the bubble nucleation rate in the homogeneous background in the ordinary method. However, the above amplified fluctuations are originated to the quantum fluctuations and the instability of this system. These effects are, for that reason, intrinsic and we cannot ignore these effects in the tunneling process. When we treat the tunneling, we must include these effects in some cases.

In each region, except near the boundary, the order parameter defined in this region takes the definite value, that is zero for false vacuum and σ for true vacuum. The original order parameter is average of these order parameters in each region, $\phi_c=(0\Omega_{\rm false}+\sigma\Omega_{\rm true})/(\Omega_{\rm false}+\Omega_{\rm true})$, where $\Omega_{\rm false}$ and $\Omega_{\rm true}$ mean the total volume in false and true vacuum, respectively. Therefore the approximate fraction of the volume in true vacuum is $\Omega_{\rm true}/\Omega_{\rm total}=\phi_c/\sigma$.

A. Linear approximation

We consider the equation quadratic in η,

$$i\frac{\partial}{\partial t}\Psi=\sum_{\mathbf{k}}\left\{-\frac{1}{2\Omega}\frac{\partial^2}{\partial\eta_{\mathbf{k}}\partial\eta_{-\mathbf{k}}}+\frac{1}{2}\Omega\omega_{\mathbf{k}}^2\eta_{\mathbf{k}}\eta_{-\mathbf{k}}+\Omega[\ddot{\phi}_c+V'(\phi_c)]\eta_{\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}}\right\}\Psi. \tag{4.4}$$

In this case wave function decompose to each mode. In each mode this equation can be solved by the Gaussian ansatz

$$\Psi=\prod_{\mathbf{k}}\Psi_{\mathbf{k}}, \tag{4.5}$$

where

$$\Psi_{\mathbf{k}}=A_{\mathbf{k}}\exp\left(-\frac{1}{2}\Omega B_{\mathbf{k}}\eta_{\mathbf{k}}\eta_{-\mathbf{k}}+\Omega C\eta_{\mathbf{0}}\delta_{\mathbf{k},\mathbf{0}}\right). \tag{4.6}$$

The Schrödinger equation becomes equation of $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ and C.

$$\dot{A}_{\mathbf{k}}=-\frac{i}{2}(B_{\mathbf{k}}-\Omega C^2\delta_{\mathbf{k},\mathbf{0}})A_{\mathbf{k}}, \tag{4.7}$$

$$\dot{B}_{\mathbf{k}}=-iB_{\mathbf{k}}^2+i\omega_{\mathbf{k}}^2, \tag{4.8}$$

$$\dot{C}=-iB_{\mathbf{0}}C+\ddot{\phi}_c+V'(\phi_c). \tag{4.9}$$

By the first equation, $A_{\mathbf{k}}$ can be expressed with $B_{\mathbf{k}}$ and C. The second equation becomes a linearized equation by the replacement

$$B_{\mathbf{k}}=-i\frac{\dot{\varphi}_{\mathbf{k}}}{\varphi_{\mathbf{k}}}, \tag{4.10}$$

then

$$\ddot{\varphi}_{\mathbf{k}}+\omega_{\mathbf{k}}^2\varphi_{\mathbf{k}}=0. \tag{4.11}$$

The solution of this equation increases exponentially when the square of frequency is negative.

Initial conditions are $B_{\mathbf{k}}(t_0)=\omega_{\mathbf{k}}(t_0)$ and for convenience we take for $\varphi_{\mathbf{k}}$

$$\varphi_{\mathbf{k}}(t_0)=\frac{1}{\sqrt{\omega_{\mathbf{k}}(t_0)}}, \tag{4.12}$$

$$\dot{\varphi}_{\mathbf{k}}(t_0)=i\sqrt{\omega_{\mathbf{k}}(t_0)}. \tag{4.13}$$

From Eq. (4.9) and an initial condition $C(t_0)=0$,

$$C=\frac{1}{\varphi_{\mathbf{0}}(t)}\int_{t_0}^{t}dt'(-i)\{\ddot{\phi}_c(t')+V'[\phi_c(t')]\}\varphi_{\mathbf{0}}(t'). \tag{4.14}$$

With theses quantities two point functions are expressed simply

$$\langle\eta_{\mathbf{k}}\eta_{-\mathbf{k}}\rangle=\frac{|\varphi_{\mathbf{k}}|^2}{2\Omega}\quad\text{for}\quad\mathbf{k}\neq\mathbf{0}, \tag{4.15}$$

$$\langle\eta_{\mathbf{0}}^2\rangle=\frac{|\varphi_{\mathbf{0}}|^2}{\Omega}\left(\frac{1}{2}+\Omega|\varphi_{\mathbf{0}}|^2(\mathrm{Re}C)^2\right), \tag{4.16}$$

where ReC is a real part of C. These increase exponentially for long wave modes during $\omega_{\mathbf{k}}^2$ is negative.

In this wave function the expectation value of a linear term

$$\langle\eta_{\mathbf{0}}\rangle=|\varphi_{\mathbf{0}}|^2\mathrm{Re}C, \tag{4.17}$$

is nonzero by the existence of C unless C is pure imaginary. This contradicts with the fact $\langle\eta\rangle=0$ that follows from the definition of η in Eq. (2.10). This is due to the linear approximation in this subsection and then we redefine the two point function of zero mode as

$$\langle\eta_{\mathbf{0}}^2\rangle|_{\rm new}=\langle\eta_{\mathbf{0}}^2\rangle-\langle\eta_{\mathbf{0}}\rangle^2=\frac{|\varphi_{\mathbf{0}}|^2}{2\Omega}. \tag{4.18}$$

In the following $\langle\eta_{\mathbf{0}}^2\rangle$ means this redefined expectation value.

The solution of Eq. (4.11) is solved by WKB approximation. In the region $t<t_{\mathbf{k}}$ where $t_{\mathbf{k}}$ is a time $\omega_{\mathbf{k}}(t_{\mathbf{k}})=0$,

$$\varphi_{\mathbf{k}}(t)=e^{i\delta}\frac{1}{\sqrt{p_{\mathbf{k}}(t)}}\exp\left(-i\int_t^{t_{\mathbf{k}}}dt p_{\mathbf{k}}(t)\right) \tag{4.19}$$

then

$$|\varphi_{\mathbf{k}}(t)|^2=\frac{1}{p_{\mathbf{k}}(t)}, \tag{4.20}$$

where $p_{\mathbf{k}}(t)=\sqrt{\omega_{\mathbf{k}}^2}$ and δ is a constant. In the region $t>t_{\mathbf{k}}$ the solution is

$$\varphi_{\mathbf{k}}(t)=e^{i\delta+i\pi/4}\frac{1}{\sqrt{\pi_{\mathbf{k}}(t)}}\exp\left(\int_{t_{\mathbf{k}}}^{t}dt\,\pi_{\mathbf{k}}(t)\right) \tag{4.21}$$

then

$$|\varphi_{\mathbf{k}}(t)|^2=\frac{1}{\pi_{\mathbf{k}}(t)}\exp\left(2\int_{t_{\mathbf{k}}}^{t}dt\,\pi_{\mathbf{k}}(t)\right) \tag{4.22}$$

where $\pi_{\mathbf{k}}(t)=\sqrt{-\omega_{\mathbf{k}}^2}$.

With these quantities we consider two point correlation function. For large Ω, which is a case in thin wall approximation,

$$\langle\eta(\mathbf{x},t)\,\eta(\mathbf{0},t)\rangle=\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{|\varphi_{\mathbf{k}}|^2}{2\Omega}=\frac{1}{2}\int dk^3e^{-i\mathbf{k}\cdot\mathbf{x}}|\varphi_{\mathbf{k}}|^2. \tag{4.23}$$

For late time $t\gg t_0$, $|\varphi_{\mathbf{k}}|^2$ is given by Eq. (4.22). In this integration small k region contribute to this integration. The result is

$$\langle\eta(\mathbf{x},t)\,\eta(\mathbf{0},t)\rangle=\frac{1}{16\pi^{3/2}}\frac{1}{\sqrt{-V''}}\frac{e^ce^{-r^2/4d}}{d^{3/2}}\left\{1+\frac{1}{4}\frac{1}{-V''}\frac{1}{d}\times\left(3-\frac{r^2}{2d}\right)\right\}, \tag{4.24}$$

where $V''=\partial^2V/\partial t^2$ and

$$r=|\mathbf{x}| \tag{4.25}$$

$$c=2\int_{t_0}^{t}dt\sqrt{-V''}, \tag{4.26}$$

$$d=\int_{t_0}^{t}dt\frac{1}{\sqrt{-V''}}. \tag{4.27}$$

From this expression a typical length of the correlated domain is

$$l(t)\sim2\sqrt{d}=2\left(\int_{t_0}^{t}dt\frac{1}{\sqrt{-V''}}\right)^{1/2}. \tag{4.28}$$

This length grows in time as $l(t)\sim(t-t_0)^{1/4}$ in the beginning and $l(t)\sim(t-t_0)^{1/2}$ in the late times.

B. Hartree approximation

Until now we have neglected the terms η^3 and η^4 but since the fluctuations increase exponentially we cannot neglect these terms. Furthermore we cannot use the perturbation for the terms with η^3 and η^4 as these are large even if small couplings. We need nonperturbative method.

Nonperturbative method we can use in this paper is Hartree approximation. This is achieved by the replacement

$$\sum_{\mathbf{k},\mathbf{p}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}}\rightarrow\sum_{\mathbf{k},\mathbf{p}}3\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\rangle\eta_{-\mathbf{k}-\mathbf{p}}, \tag{4.29}$$

$$\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\rightarrow\sum_{\mathbf{k},\mathbf{p},\mathbf{q}}(6\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\rangle\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}-3\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\rangle\langle\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\rangle) \tag{4.30}$$

in the Hamiltonian (4.2).

By Hartree approximation, the square of the frequency becomes

$$\omega_{\mathbf{k}}^{(H)2}=\mathbf{k}^2+V''(\phi_c)+\frac{1}{2}\sum_{\mathbf{k}}\langle\eta_{-\mathbf{k}}\eta_{\mathbf{k}}\rangle, \tag{4.31}$$

where superscript H means this quantity is Hartree approximation. Last term prevents the square of frequency becomes negative for large $\Sigma_{\mathbf{k}}\langle\eta_{-\mathbf{k}}\eta_{\mathbf{k}}\rangle$.

By the last term in Eq. (4.31), $\omega_{\mathbf{k}}^{(H)2}$ does not take negative value for large $\langle\eta^2\rangle$. In a finite volume, there exists minimum value of $|\mathbf{k}|$, we call $k_{\min}$. Therefore the maximum value of $\Sigma_{\mathbf{k}}\langle\eta_{-\mathbf{k}}\eta_{\mathbf{k}}\rangle$ is

$$\sum_{\mathbf{k}}\langle\eta_{-\mathbf{k}}\eta_{\mathbf{k}}\rangle|_{\max}=-V''(\phi_c)|_{\min}-k_{\min}^2. \tag{4.32}$$

V. BACK REACTION OF FLUCTUATIONS TO TUNNELING

In this section we will consider a back reaction of nonzero mode fluctuations to zero mode. Before including this effect in the zero mode Schrödinger equation we will need to subtract the ultraviolet divergences in the expectation value $\langle\eta^2\rangle$. This is implemented by ordinary renormalization. Since this subtracts only ultraviolet divergences, the infrared part, i.e., long wave mode amplification, remains almost unchanged.

Formally we must renormalize in dynamical situations but in this paper we simply subtract expectation value at initial time $\langle\eta^2(0)\rangle$ from $\langle\eta^2(t)\rangle$. This is because ultraviolet part is not strongly influenced by the evolution during tunneling process and hence, initially regularized, ultraviolet divergence will not appear. In the following $\langle\eta^2\rangle$ means this subtracted value.

After the subtraction of ultraviolet divergence the Schrödinger equation for zero mode becomes

$$i\frac{\partial}{\partial t}\Psi=\left(-\frac{1}{2\Omega}\frac{\partial^2}{\partial\phi_0^2}+\Omega V_{\mathrm{corr}}\right)\Psi, \tag{5.1}$$

where V_{corr} is a corrected potential

$$\begin{aligned}V_{\mathrm{corr}}=&V(\phi_0)+\frac{1}{2}V''(\phi_0)\sum_{\mathbf{k}\neq\mathbf{0}}\langle\eta_{-\mathbf{k}}\eta_{\mathbf{k}}\rangle\\&+\frac{1}{3!}V'''(\phi_0)\sum_{\substack{\mathbf{k},\mathbf{p}\neq\mathbf{0}\\ \mathbf{p}+\mathbf{k}\neq\mathbf{0}}}\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}}\rangle\\&+\frac{1}{4!}V''''(\phi_0)\sum_{\substack{\mathbf{k},\mathbf{p},\mathbf{q}\neq\mathbf{0}\\ \mathbf{k}+\mathbf{p}+\mathbf{q}\neq\mathbf{0}}}\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{\mathbf{q}}\eta_{-\mathbf{k}-\mathbf{p}-\mathbf{q}}\rangle.\end{aligned} \tag{5.2}$$

In this potential the last term is ϕ_0 independent and in our approximation, $\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}}\rangle=0$. Therefore this potential depends only on $\langle\eta_{\mathbf{k}}\eta_{-\mathbf{k}}\rangle$. As $\langle\eta^2\rangle$ becomes larger, this back reaction lower the potential barrier and tunneling transition enhanced. Extreme large $\langle\eta^2\rangle$ potential has only one minimum.

This disagrees with Caldeira-Legget result, i.e., the environment prevents tunneling [35]. This is due to the fact that we have neglected the dissipative effect. To include the nonperturbative effect we have employed the Hartree approximation. As is well known, this approximation cannot include collision effect. Since these effects transfer energy to another mode, they become important especially in late times when the system approaches asymptotic state. This effect causes nonzero $\langle\eta_{\mathbf{k}}\eta_{\mathbf{p}}\eta_{-\mathbf{k}-\mathbf{p}}\rangle$ and the third term in Eq. (5.2) prevents tunneling process. However, if these dissipative effects are not so strong, then before these effect becomes significant the above conclusion that tunneling is enhanced by the fluctuations is correct. To include collision effect nonperturbatively, we can use the classical approximation [37].

The above alteration of the potential is also subject to the volume Ω of the system we are considering as the amplification of fluctuations depend on it. For small Ω, fluctuations remain small and the potential is almost unchanged. For large Ω, fluctuations are enhanced and the potential is strongly changed.

In the statistical phase transition the supercooled state is unstable and even a small fluctuation from the external environment enhances the transition to the true vacuum. But the increase of fluctuations and the enhance of the transition in quantum tunneling is by itself.

VI. CONCLUSION

We have investigated in this paper the dynamics during tunneling phase transition. We set up the problem in a manner similar to the phase transition in statistical physics, except the Langevin equation replaced with a Schrödinger equation.

We defined the order parameter to specify the state of the system and derived the evolution equation of it. We also derived the evolution equation of fluctuations around the order parameter. In this formalism we obtain the quantum time evolution of the order parameter and fluctuations around it.

In some cases the long wave fluctuations are amplified by instability during tunneling process. This enhances the tunneling of this region. This will be interpreted that creation and destruction of small bubbles, subcritical bubbles, enhance the transition of this region. This effect is purely intrinsic property and cannot be ignored when we consider the tunneling phase transition even initially inhomogeneous.

This amplification of fluctuations cannot be included in the bounce method and other static treatments in previous works. Only the dynamical treatment by nonequilibrium quantum field theory can achieve this result.

Finally, a few comments. We have not included the rescattering effect in this paper. This becomes important for rate time when the system approaches equilibrium state. Transition of volume energy to surface energy of the true vacuum bubbles will be carried by this effect.

Our motivation to consider the dynamics of tunneling is to investigate tunneling in the early Universe. In this case the expansion of regions suppresses the transition rate. On the other hand, this expansion causes the redshift of physical wave vectors and leads to the amplification of these red shifted modes. This results in the enhancement of tunneling of this region.

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APPENDIX: DERIVATION OF THE EQUATION OF THE COEFFICIENTS

In this appendix we derive Eq. (3.4) in more detail. We treat the time evolution of the system described by the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi=H\Psi. \tag{A1}$$

In this equation a potential has two minimum, one for false vacuum and other for true vacuum. In each minimum we define the eigenfunctions, ψ_n for false vacuum and $\tilde{\psi}_{\tilde{n}}$ for true vacuum. Since we are interested in the evolution of the initial false vacuum state we expand the wave function such as

$$\Psi=\sum_{n}^{N}c_n\psi_n+\sum_{\tilde{n}}^{\tilde{N}}\tilde{c}_{\tilde{n}}\tilde{\psi}_{\tilde{n}}, \tag{A2}$$

where N and $\tilde{N}$ are finite integers as we are concerned with the slow movement of the system. Insert to the Schrödinger equation, we obtain

$$i\dot{c}_n\psi_n+c_n i\frac{\partial}{\partial t}\psi_n+i\dot{\tilde{c}}_{\tilde{n}}\tilde{\psi}_{\tilde{n}}+\tilde{c}_{\tilde{n}}i\frac{\partial}{\partial t}\tilde{\psi}_{\tilde{n}}=c_nH\psi_n+\tilde{c}_{\tilde{n}}H\tilde{\psi}_{\tilde{n}}, \tag{A3}$$

where an overdot means time derivative. Product $\psi_{n'}$ and $\tilde{\psi}_{m'}$, we obtain the differential equation for the coefficients

$$i\begin{pmatrix} 1 & \langle \psi_n \tilde{\psi}_{m'} \rangle \\ \langle \tilde{\psi}_m \psi_{n'} \rangle & 1 \end{pmatrix}\begin{pmatrix} \dot{c}_{n'} \\ \dot{\tilde{c}}_{m'} \end{pmatrix} = \begin{pmatrix} M_{nn'} & M_{nm'} \\ M_{mn'} & M_{mm'} \end{pmatrix}\begin{pmatrix} c_{n'} \\ \tilde{c}_{m'} \end{pmatrix}, \tag{A4}$$

where

$$M_{nn'} = \langle \psi_n H \psi_{n'} \rangle - \left\langle \psi_n i \frac{\partial}{\partial t} \psi_{n'} \right\rangle, \tag{A5}$$

$$M_{nm'} = \langle \psi_n H \tilde{\psi}_{m'} \rangle - \left\langle \psi_n i \frac{\partial}{\partial t} \tilde{\psi}_{m'} \right\rangle, \tag{A6}$$

$$M_{mn'} = \langle \tilde{\psi}_m H \psi_{n'} \rangle - \left\langle \tilde{\psi}_m i \frac{\partial}{\partial t} \psi_{n'} \right\rangle, \tag{A7}$$

$$M_{mm'} = \langle \tilde{\psi}_m H \tilde{\psi}_{m'} \rangle - \left\langle \tilde{\psi}_m i \frac{\partial}{\partial t} \tilde{\psi}_{m'} \right\rangle. \tag{A8}$$

In this equation $\langle \psi_n O \psi_m \rangle$ (where O is some operator) means the matrix with elements

$$\langle \psi_n O \tilde{\psi}_m \rangle = \int dx\, \psi_n^* O \tilde{\psi}_m . \tag{A9}$$

The condition that this equation does not degenerate is

$$\det\begin{pmatrix} 1 & \langle \psi_n \tilde{\psi}_{m'} \rangle \\ \langle \tilde{\psi}_m \psi_{n'} \rangle & 1 \end{pmatrix} \neq 0. \tag{A10}$$

For small $\langle \psi_n \tilde{\psi}_m \rangle$ this determinant becomes approximately

$$\det\begin{pmatrix} 1 & \langle \psi_n \tilde{\psi}_{m'} \rangle \\ \langle \tilde{\psi}_m \psi_{n'} \rangle & 1 \end{pmatrix} = 1 + O(\langle \psi_n \tilde{\psi}_{\tilde{m}} \rangle^2). \tag{A11}$$

In this case the inverse of this matrix is

$$\begin{pmatrix} 1 & \langle \psi_n \tilde{\psi}_m \rangle \\ \langle \tilde{\psi}_m \psi_n \rangle & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\langle \psi_n \tilde{\psi}_m \rangle \\ -\langle \tilde{\psi}_m \psi_n \rangle & 1 \end{pmatrix} + O(\langle \psi_n \tilde{\psi}_m \rangle^2). \tag{A12}$$

Therefore the differential equation of the coefficients becomes

$$i\frac{d}{dt}\begin{pmatrix} c_n \\ \tilde{c}_m \end{pmatrix} = \begin{pmatrix} 1_{nn'} & -\langle \psi_n \tilde{\psi}_{m'} \rangle \\ -\langle \tilde{\psi}_m \psi_{n'} \rangle & 1_{mm'} \end{pmatrix} \times \begin{pmatrix} M_{n'n''} & M_{n'm''} \\ M_{m'n''} & M_{m'm''} \end{pmatrix}\begin{pmatrix} c_{n''} \\ \tilde{c}_{m''} \end{pmatrix}. \tag{A13}$$

This equation becomes with the neglect of the terms $O(\langle \psi_n \tilde{\psi}_{\tilde{m}} \rangle^2)$,

$$i\frac{d}{dt}\begin{pmatrix} c_n \\ \tilde{c}_m \end{pmatrix} = \begin{pmatrix} H_{nn''} & H_{nm''} \\ H_{mn''} & H_{mm''} \end{pmatrix}\begin{pmatrix} c_{n''} \\ \tilde{c}_{m''} \end{pmatrix}, \tag{A14}$$

where

$$H_{nn''} = \langle \psi_n H \psi_{n''} \rangle - \left\langle \psi_n i \frac{\partial}{\partial t} \psi_{n''} \right\rangle, \tag{A15}$$

$$H_{nm''} = \langle \psi_n H \tilde{\psi}_{m''} \rangle - \left\langle \psi_n i \frac{\partial}{\partial t} \tilde{\psi}_{m''} \right\rangle - \langle \psi_n \tilde{\psi}_{m'} \rangle \langle \tilde{\psi}_{m'} H \tilde{\psi}_{m''} \rangle + \langle \psi_n \tilde{\psi}_{m'} \rangle \left\langle \tilde{\psi}_{m'} i \frac{\partial}{\partial t} \tilde{\psi}_{m''} \right\rangle, \tag{A16}$$

$$H_{mn''} = \langle \tilde{\psi}_m H \psi_{n''} \rangle - \left\langle \tilde{\psi}_m i \frac{\partial}{\partial t} \psi_{n''} \right\rangle - \langle \tilde{\psi}_m \psi_{n'} \rangle \langle \psi_{n'} H \psi_{n''} \rangle + \langle \tilde{\psi}_m \psi_{n'} \rangle \left\langle \psi_{n'} i \frac{\partial}{\partial t} \psi_{n''} \right\rangle, \tag{A17}$$

$$H_{mm''} = \langle \tilde{\psi}_m H \tilde{\psi}_{m''} \rangle - \left\langle \tilde{\psi}_m i \frac{\partial}{\partial t} \tilde{\psi}_{m''} \right\rangle. \tag{A18}$$

For time-independent ψ_n and the neglect of $O(\langle \psi \tilde{\psi} \rangle)$ this equation becomes

$$H_{nn''} = \langle \psi_n H \psi_{n''} \rangle, \tag{A19}$$

$$H_{nm''} = \langle \psi_n H \tilde{\psi}_{m''} \rangle - \langle \psi_n \tilde{\psi}_{m'} \rangle \langle \tilde{\psi}_{m'} H \tilde{\psi}_{m''} \rangle, \tag{A20}$$

$$H_{mn''} = \langle \tilde{\psi}_m H \psi_{n''} \rangle - \langle \tilde{\psi}_m \psi_{n'} \rangle \langle \psi_{n'} H \psi_{n''} \rangle, \tag{A21}$$

$$H_{mm''} = \langle \tilde{\psi}_m H \tilde{\psi}_{m''} \rangle. \tag{A22}$$

For $N=1$ and $\tilde{N}=1$ this is Eq. (3.4).

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