# Light quark masses from lattice quark propagators at large momenta

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We compute nonperturbatively the average up-down and strange quark masses from the large momentum (short-distance) behavior of the quark propagator in the Landau gauge. This method, which has never been applied so far, does not require the explicit calculation of the quark mass renormalization constant. Calculations were performed in the quenched approximation, by using $\mathcal{O}(a)$ improved Wilson fermions. The main results of this study are $m_l^{\rm RI}(2\ {\rm GeV})=5.8(6)$ MeV and $m_s^{\rm RI}(2\ {\rm GeV})=136(11)$ MeV. Using the relations between different schemes, obtained from the available four-loop anomalous dimensions, we also find $m_l^{\rm RGI}=7.6(8)$ MeV and $m_s^{\rm RGI}=177(14)$ MeV, and the modified minimal subtraction scheme ($\overline{\rm MS}$) masses $m_l^{\overline{\rm MS}}(2\ \ {\rm GeV})=4.8(5)$ MeV and $m_s^{\overline{\rm MS}}(2\ \ {\rm GeV})=111(9)$ MeV.

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## I. INTRODUCTION

Determination of quark masses is becoming one of the most intensive fields of investigation in lattice QCD [1–14]. The accuracy of the predictions is significantly improving mainly because of two recent theoretical developments:

Nonperturbative renormalization procedures have been introduced [15,16] in order to remove the systematic uncertainties coming from the truncation of perturbative series in the calculation of the relevant renormalization constants. These procedures also provide an appropriate nonperturbative, short-distance definition of the quark masses either in the so-called regularization invariant (RI) momentum subtraction (MOM) or Schrödinger functional schemes. The relation between the mass in the RI-MOM scheme (which will be used in this study) and in the modified minimal subtraction scheme ($\overline{\rm MS}$) scheme,[1] or the renormalization group invariant mass, is known at next-to-next-to-leading order (NNLO) [17], and very recently even at next-to-NNLO (NNNLO) [18], in continuum perturbation theory.

A second important theoretical progress is the reduction of finite cutoff ($\mathcal{O}(a)$) effects obtained by improving the lattice fermion action and operators. The perturbative procedure, proposed in Refs. [19],[20], has been recently extended to a fully nonperturbative $\mathcal{O}(a)$ improvement by the ALPHA Collaboration [21], so that the remaining discretization errors are only of $\mathcal{O}(a^2)$.

In the past year, several independent lattice determinations of the light quark masses [9–11] have been presented, adopting both nonperturbative renormalization procedures and nonperturbative improvement.

[1]The conversion of the results to the $\overline{\rm MS}$ scheme is only necessary for comparison with other calculations for which the quark masses are renormalized perturbatively. Otherwise, the method of Ref. [15] allows us, in principle, to obtain the renormalized quark masses for the RI-MOM scheme in a completely nonperturbative way.

The two standard definitions of the lattice quark masses are based on the vector (VWI) and the axial-vector (AWI) chiral Ward identities [22]. The VWI relates the bare quark mass to the value of the Wilson hopping parameter, $2am=(1/\kappa-1/\kappa_{crit})$. With this definition, one can easily show that the mass renormalization constant is $Z_m(\mu)=Z_S^{-1}(\mu)$. The definition based on the AWI is $2a\overline{m}=\langle\alpha|\partial_\mu A_\mu|\beta\rangle/\langle\alpha|P|\beta\rangle$, where $\partial_\mu A_\mu$ and $P$ are the divergence of the (improved) axial-vector current and the pseudoscalar density, respectively. In this case, $Z_{\overline{m}}(\mu)=Z_A/Z_P(\mu)$.

In this paper, in order to calculate the renormalized quark mass, we adopt a new method based on the study of the large $p^2$ behavior of the renormalized quark propagator. The method is based on the idea that at large Euclidean momenta it is possible to match lattice and continuum correlators by requiring the vanishing of chirality violating form factors [23,25]. This procedure is justified by the following two observations. The first is that at large momenta the renormalized perturbation theory becomes chirally invariant (explicit chiral symmetry breaking effects induced by the regularization are reabsorbed by imposing the validity of the chiral Ward identities, while violations from the nonvanishing quark masses disappear at large momenta). The second observation is that the contributions due to the spontaneous breaking of chiral symmetry, which are absent in perturbation theory, die off at large momenta. Thus, both effects decrease as we go deeper into the Euclidean region.

The simplest application of this idea is the possibility of relating the quark mass to the renormalized quark propagator

$$\hat{S}(p)=\frac{i\not{p}}{p^2}\hat{\sigma}_1(p^2)+\frac{\hat{\sigma}_2(p^2)}{p^2}, \tag{1}$$

since, at large $p^2$, we expect [26]

$$\hat{\sigma}_2(p^2)\simeq m+\langle\overline{q}q\rangle\frac{4\pi\alpha_s}{3p^2}+\mathcal{O}(1/p^4). \tag{2}$$

 

In particular, the quark mass renormalized in the RI-MOM scheme can be directly extracted from the quark propagator renormalized in the same scheme by using

$$m_q^{\mathrm{RI}}(\mu)=\frac{1}{12}\mathrm{Tr}[\hat{S}^{-1}(p;\mu)]_{p^2=\mu^2}, \tag{3}$$

where the trace is over both color and spin indices.[2] $\hat{S}(p;\mu)$ is the (improved) quark propagator renormalized at some scale $\mu$. Since the quark propagator is a gauge dependent quantity, the definition of the RI-MOM mass also depends on the gauge.

At large momenta and up to discretization errors, Eq. (3) is equivalent to the definition of the quark mass based on the AWI. Chiral symmetry provides a relation between the inverse quark propagator, $\hat{S}(p;\mu)^{-1}$, and the amputated Green function of the pseudoscalar density, $\hat{\Lambda}_5(p;\mu)$, computed between external (off-shell) quark states of equal momenta $p$. The AWI then reads

$$2\hat{m}_q(\mu)\hat{\Lambda}_5(p;\mu)=\gamma_5\hat{S}^{-1}(p;\mu)+\hat{S}^{-1}(p;\mu)\gamma_5. \tag{4}$$

All quantities in Eq. (4) are assumed to be renormalized (and improved) in the same scheme and at the same scale $\mu$. In the RI-MOM scheme (and in a fixed gauge), the Green function $\hat{\Lambda}_5(p;\mu)$ satisfies the following renormalization condition [15]:

$$\frac{1}{12}\mathrm{Tr}[\gamma_5\hat{\Lambda}_5(p;\mu)]_{p^2=\mu^2}=1. \tag{5}$$

By tracing both sides of Eq. (4) with $\gamma_5$ and by using Eq. (5), the relation (3) is readily derived. Note that if $\mu$ in Eq. (3) is not chosen in the perturbative region, i.e., $\mu \gg \Lambda_{\mathrm{QCD}}$, the definition of the quark mass will be affected by nonperturbative, chirally breaking contributions proportional to the quark condensate and higher-dimensional operators appearing in higher power corrections ($\propto 1/p^{2n}$).

The advantage in determining the masses from the quark propagators is that it is not necessary to calculate explicitly the mass renormalization constants [i.e., $Z_S(\mu)$ or $Z_P(\mu)$]. This is merely a consequence of the fact that the renormalized quark propagator is directly expressed in terms of the renormalized quark mass. Unlike in the case of the VWI, the critical value of the hopping parameter $\kappa_{crit}$ is also not needed (which is the advantage inherent to the use of the AWI). There is, however, one renormalization constant for any quark-mass definition; in our case, using Eq. (3), this is the quark-field renormalization constant $Z_q$. In the RI-MOM scheme $Z_q$ is fixed by the following renormalization condition:

$$\frac{i}{48}\mathrm{Tr}\left[\gamma_\mu\frac{\partial\hat{S}^{-1}(p;\mu)}{\partial p_\mu}\right]_{p^2=\mu^2} \equiv \frac{i}{48}Z_q(\mu)\mathrm{Tr}\left[\gamma_\mu\frac{\partial S^{-1}(p)}{\partial p_\mu}\right]_{p^2=\mu^2}=1. \tag{6}$$

In summary, Eqs. (3) and (6) are all we need to extract quark masses from propagators. The procedure becomes rather complicated, however, if we want to extend it to the nonperturbatively improved case. The drawback with the method of Ref. [15] is that the improvement program (which was initially carried out for on-shell quantities) must be extended to off-shell Green functions on nongauge invariant states and involves additional counterterms for a full $\mathcal{O}(a)$ improvement [23]. The strategy followed in this case will be illustrated in detail in Sec. II.

In this paper, Eqs. (3) and (6) have been used to compute the average up-down and the strange quark masses by performing a lattice QCD calculation in the quenched approximation. We use the nonperturbatively improved action [21], and improve the quark propagator in the chiral limit. The $\mathcal{O}(a)$ improvement procedure for off-shell (gauge noninvariant) quantities has been discussed in Ref. [23]. Since that paper has not been published yet, we will describe here in some detail the specific case of the quark propagator. For technical reasons, which are related to the mixing with non–gauge-invariant higher-dimensional operators (see below), we are not able to improve the propagator out of the chiral limit.[3] Therefore, our determination of the quark masses is affected by $\mathcal{O}(g_0^2 am)$ systematic errors. Since the value of the inverse lattice spacing in this simulation is $a^{-1}\simeq 2.72$ GeV, these errors are expected to be negligible for the strange and the light quark masses.

We conclude this section by summarizing the main results of this paper. From the study of the quark propagator, we extract the (quenched) light and strange quark masses in the RI-MOM scheme:

$$m_l^{RI}(2\ \mathrm{GeV})=5.8(6)\ \mathrm{MeV},\quad m_s^{RI}(2\ \mathrm{GeV})=136(11)\ \mathrm{MeV}. \tag{7}$$

These results are in very good agreement with those of Ref. [9], namely $m_s^{\mathrm{RI}}(2\ \mathrm{GeV})=138(15)$ MeV and $m_l^{\mathrm{RI}}(2\ \mathrm{GeV})=5.6(5)$ MeV.

Using the NNNLO perturbative formulas of Ref. [18], we obtain the renormalization group invariant quark masses

$$m_l^{\mathrm{RGI}}=7.6(8)\ \mathrm{MeV},\quad m_s^{\mathrm{RGI}}=177(14)\ \mathrm{MeV}, \tag{8}$$

where the renormalization group invariant quark mass is defined according to the convention usually adopted in perturbative calculations [17,18,29], i.e.,

[2]We note, in passing, that by using this method we were not able to extract the value of the quark condensate (2).

[3]The general problem of the improvement out of the chiral limit has not been solved yet, although several interesting proposals exist [23,24,27,28].

$$m_q^{\rm RGI}=\lim_{\mu\to\infty} m_q(\mu)(\alpha_s(\mu))^{-\gamma_m^{(0)}/\beta_0}, \tag{9}$$

with $\gamma_m^{(0)}$ and $\beta_0$ being scheme independent.[4]

Finally, the quark masses in the $\overline{\rm MS}$ read

$$\begin{array}{lccc} & \text{NLO} & \text{NNLO} & \text{NNNLO} \\ m_l^{\overline{\rm MS}}(2\ {\rm GeV})=\{5.2(5); & 4.9(5); & 4.8(5)\}\ {\rm MeV}, \\ m_s^{\overline{\rm MS}}(2\ {\rm GeV})=\{120(9); & 114(9); & 111(9)\}\ {\rm MeV}, \end{array} \tag{11}$$

where the numbers within the curly brackets are obtained after converting the RI-MOM results to the $\overline{\rm MS}$ one to NLO, NNLO, and NNNLO accuracy, respectively. The details on anomalous dimensions and beta function are listed in the Appendix.

We note that in most of the phenomenological applications, for example with QCD sum rules, the theoretical expressions are only known to the NLO and, for consistency, the quark masses at the same accuracy should be used.

Similarly, we stress that lattice calculations of quark masses, in which the mass renormalization constants have been determined by using (one-loop) perturbation theory, should be compared with our NLO results of Eq. (11), since they have been derived at the same order of accuracy.

Preliminary results obtained with the method discussed in this paper were already presented: see Ref. [30].

## II. IMPROVED QUARK PROPAGATOR

The general problem of improving gauge noninvariant, off-shell correlation functions has been studied in Ref. [23]. Since this paper is still unpublished, in this section we will discuss in some detail the nonperturbative improvement of the lattice quark propagator to $\mathcal{O}(a)$.

### A. The subtracted quark propagator

The original idea of improvement [31] (later developed in Ref. [32] for gauge theories) consists of adding, to both the action and operators, a complete set of higher-dimensional (''irrelevant'') operators, the coefficients of which are tuned so as to cancel finite cutoff effects (to a desired order of lattice spacing). Specifically, the improvement of the Wilson action to $\mathcal{O}(a)$ is achieved by adding a set of dimension-five operators [21],

$$S=S_W+a\sum_{i=1}^{n} c_i\int d^4x\,\mathcal{O}_i^{(d=5)}(x), \tag{12}$$

allowed by gauge invariance and discrete lattice symmetries, namely

$$\mathcal{O}_1=\frac{i}{4}\bar{q}\sigma_{\mu\nu}F_{\mu\nu}q,\quad \mathcal{O}_2=\frac{1}{2g_0^2}m\,{\rm Tr}(F_{\mu\nu}F_{\mu\nu}),$$

$$\mathcal{O}_3=m^2\bar{q}q,\quad \mathcal{O}_4=m\bar{q}(\overleftrightarrow{\not{D}}+m_0)q,\quad \mathcal{O}_5=\bar{q}(\overleftrightarrow{\not{D}}+m_0)^2q, \tag{13}$$

where $m$ is the bare subtracted mass and $\not{D}+m_0$ is the bare Dirac operator appearing in $S_w$. The operators $\mathcal{O}_2$ and $\mathcal{O}_3$ can be reabsorbed in the definition of the bare strong coupling and the quark mass. A major simplification comes from the restriction of improvement to physical amplitudes, for which the equations of motion can be used. In this way, one is left with one (Clover) counterterm only, $\mathcal{O}_1$, the coefficient of which was computed nonperturbatively in Ref. [21].

The equations of motion cannot be used to improve off-shell quantities (such as the quark propagator); in this case, one must also consider the operators $\mathcal{O}_{4,5}$. As for operators which are Becchi-Rouet-Stora-Tyutin (BRST) allowed but not gauge invariant, only the BRST variation of $\bar{c}^a\partial_\mu A_\mu^a$ (where $\bar{c}^a$ and $A_\mu^a$ are the antighost and gauge fields, respectively) may contribute. This term, however, can be absorbed into a redefinition of the gauge-fixing parameters [23]. Besides the terms in the action considered above, in Ref. [23] it has been also shown that, for the quark propagator, there is another operator, which is not BRST invariant but may contribute to off-shell correlation functions (because its presence is not excluded by Slavnov-Taylor identities). The effects of $\mathcal{O}_{4,5}$ and of this extra operator can be eliminated with a simple redefinition of the quark field:

$$\hat{q}(x)=Z_q^{(0)-1/2}(1+b_q ma)\{1+ac_q'(\not{D}+m_0)+ac_{\rm NGI}\not{\partial}\}q(x). \tag{14}$$

We now discuss how the unknown coefficients $Z_q$ $[Z_q^{-1/2}=Z_q^{(0)-1/2}(1+b_q ma)]$, $c_q'$ (corresponding to the coefficient of the operator $\mathcal{O}_5$), and $c_{\rm NGI}$, present in Eq. (14), can be determined from the analysis of the lattice bare quark propagator, $S_L(p)$.

From Eq. (14), it follows that the relation between $S_L(p)$ and the improved, renormalized quark propagator, $\hat{S}(p)$, constructed in terms of the quark fields, $q$ and $\hat{q}$, respectively, has the form

$$S_L(p)=(1-2ac_{\rm NGI}i\not{p})Z_q\hat{S}(p)-2ac_q'. \tag{15}$$

In this equation, it is convenient to express the renormalized quark propagator $\hat{S}(p)$ in terms of the two invariant scalar form factors $\hat{\sigma}_1(p^2)$ and $\hat{\sigma}_2(p^2)$ defined in Eq. (1). For further use, we remark that at large $p^2$, up to power-suppressed $(\sim 1/p^2)$ and logarithmic corrections, $\hat{\sigma}_1(p^2)\simeq 1$ and $\hat{\sigma}_2(p^2)\simeq m$, where $m$ is the renormalized quark mass. After substituting Eq. (1) into Eq. (15), one finds

$$\sigma_{1L}(p^2)=\frac{1}{12}{\rm Tr}[-i\not{p}S_L(p)]=Z_q(\hat{\sigma}_1(p^2)-2ac_{\rm NGI}\hat{\sigma}_2(p^2)), \tag{16}$$

[4]Note that in Ref. [16], another convention has been used:

$$m_q^{\rm RGI}=\lim_{\mu\to\infty} m_q(\mu)[(2\beta_0/\pi)\alpha_s(\mu)]^{-\gamma_m^{(0)}/\beta_0}. \tag{10}$$

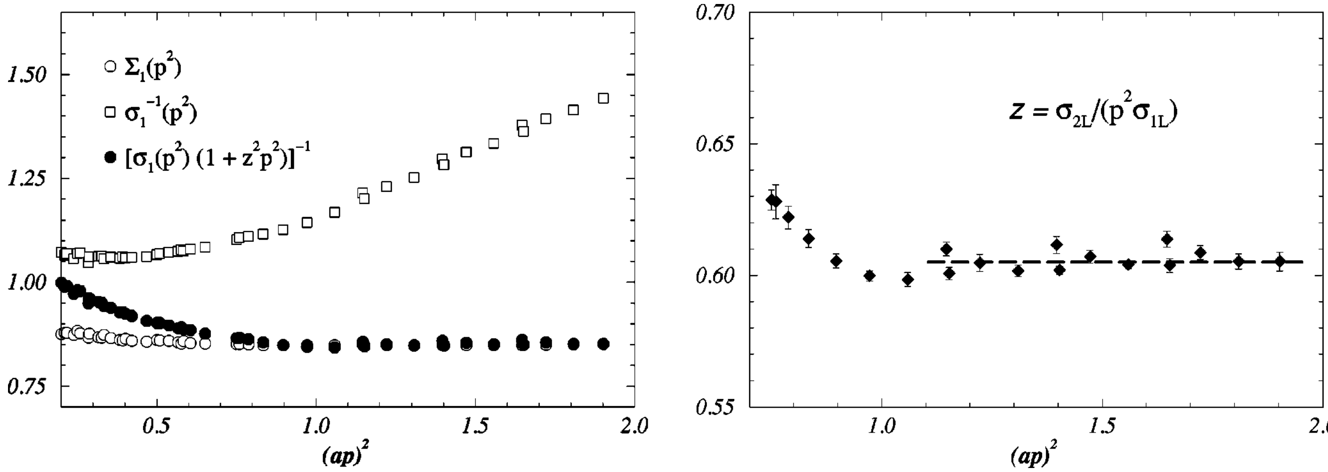


FIG. 1. The left figure shows the effect of the correction due to the factor $(1+z^2p^2)$. Empty symbols denote the quantities extracted directly from the lattice quark propagator. The filled circles denote the effect of the correction. Note that the error bars are smaller the symbols used. The value of the constant $z$ as obtained from a fit to our data in the large $p^2$-region is shown in the right figure. Both figures correspond to $\kappa=0.1344$.

$$\frac{\sigma_{2L}(p^2)}{p^2}=\frac{1}{12}\mathrm{Tr}[S_L(p)]=-2ac'_q+2ac_{\mathrm{NGI}}Z_q\hat{\sigma}_1(p^2)+Z_q\frac{\hat{\sigma}_2(p^2)}{p^2}, \tag{17}$$

where $\sigma_{1L,2L}(p^2)$ are the analog of $\hat{\sigma}_{1,2}(p^2)$ for the lattice bare propagator.

Using Eqs. (16) and (17), the coefficients $c'_q$ and $Z_qc_{\mathrm{NGI}}$ can be determined as follows: at large $p^2$ and in the chiral limit, since $\hat{\sigma}_2\sim 1/p^2\rightarrow 0$, it is in principle possible to separate $c'_q$ and $Z_qc_{\mathrm{NGI}}$ using the $p^2$ dependence of $\hat{\sigma}_1$; the overall renormalization constant $Z_q$, including its $\mathcal{O}(a)$ mass dependence, can then be determined by combining Eq. (16) with the renormalization condition (6). In practice, however, this procedure is very difficult to implement. The reason is that the logarithmic $p^2$ dependence of $\hat{\sigma}_1(p^2)$, entering the right-hand side of Eq. (17), is very mild. This is especially true in the Landau gauge, where it starts at order $\alpha_s^2$ is perturbation theory. Thus, it is very hard (if not impossible) to disentangle the contributions coming from the two coefficients,[5] $c'_q$ and $c_{\mathrm{NGI}}$.

Let us return to the large $p^2$ behavior of the lattice quark propagator, in the limit where power and logarithmic corrections can be neglected. In this limit, Eqs. (16) and (17) become

$$\sigma_{1L}(p^2)\simeq Z_q(1-2ac_{\mathrm{NGI}}m)\equiv\tilde{Z}_q \tag{18}$$

and

$$\frac{\sigma_{2L}(p^2)}{p^2}\simeq-2ac'_q+2ac_{\mathrm{NGI}}Z_q\equiv-2a\tilde{c}'_q. \tag{19}$$

[5]A promising way to separately compute $c'_q$ and $c_{\mathrm{NGI}}$ nonperturbatively is from the study of the quark-gluon vertex.

In the large $p^2$ region, by using Eq. (18), it is then possible to determine an ''effective'' renormalization constant, $\tilde{Z}_q$, which reduces to $Z_q$ in the chiral limit. Moreover, in terms of the coefficient $\tilde{c}'_q$, computed through Eq. (19), we can define a ''subtracted'' quark propagator, $\tilde{S}(p)$, as

$$\tilde{S}(p)=\tilde{Z}_q^{-1}[S_L(p)+2a\tilde{c}'_q]. \tag{20}$$

If the coefficient $c_{\mathrm{NGI}}$ were equal to zero, the subtracted propagator $\tilde{S}(p)$ would correspond to the improved, renormalized quark propagator, $\hat{S}(p)$, as can be seen from Eq. (15). In the presence of $c_{\mathrm{NGI}}$, however, $\tilde{S}(p)$ and $\hat{S}(p)$ differ by terms of $\mathcal{O}(c_{\mathrm{NGI}}am)$, up to (small) logarithmic corrections. We conclude that, by following the procedure outlined above, we are able to exactly improve the quark propagator in the chiral limit. Out of the chiral limit, since $c_{\mathrm{NGI}}$ is of $\mathcal{O}(g_0^2)$ in perturbation theory, the propagator is affected by $\mathcal{O}(g_0^2am)$ discretization errors. In the range of quark masses considered in this paper, these terms are expected to be smaller than other statistical and systematic uncertainties. They may be important, however, in the calculation of heavy quark masses.

### B. Practical implementation

Let us now discuss how the improvement procedure for the lattice quark propagator works in practice. As a preliminary (and instructive) step, we consider the inverse unsubtracted lattice propagator expressed in terms of the usual form factors, $\Sigma_{1L}$ and $\Sigma_{2L}$,

$$S_L^{-1}(p)=-i\not{p}\Sigma_{1L}(p^2)+\Sigma_{2L}(p^2). \tag{21}$$

$\Sigma_{1L}$ is special in that its $p^2$ behavior is protected by the VWI, $\Sigma_{1L}(p^2)\sim\mathrm{const}=Z_V$, as can be seen in Fig. 1 (empty circles).[6] By using Eqs. (1) and (17) in the large $p^2$ limit one finds

$$\sigma_{1L}(p^2)=\Sigma_{1L}^{-1}(p^2)\left[1+\left(\frac{\sigma_{2L}(p^2)}{p^2\sigma_{1L}(p^2)}\right)^2 p^2\right]^{-1} \to\Sigma_{1L}^{-1}(p^2)[1+\mathcal{O}(a^2p^2)]. \quad (22)$$

The effect of the $\mathcal{O}(a^2)$ term in Eq. (22) is important at large $p^2$, as shown in Fig. 1. $\Sigma_{1L}(p^2)$ (empty circles) is indeed flat, whereas $\sigma_{1L}^{-1}(p^2)$ (empty squares) is not. The reason is the presence of the contact term $\tilde{c}_q'$ in the factor

$$1+z^2p^2\equiv1+\left(\frac{\sigma_{2L}(p^2)}{p^2\sigma_{1L}(p^2)}\right)^2 p^2\to1+4\tilde{c}_q'^2a^2p^2, \quad (23)$$

at large $p^2$. For $\tilde{c}_q'=0$, instead, the factor $1+z^2p^2\sim1+m^2/p^2\to1$. Thus the dangerous $\mathcal{O}(a^2)$ terms of Eq. (22) jeopardize the $p^2$ behavior that $\sigma_1$ should have in the continuum.

The following procedure has been adopted to get rid of the $\mathcal{O}(a^2)$ corrections induced by the contact terms.[7] Motivated by Eq. (22), we first multiply the original lattice propagator by the overall factor, $(1+z^2p^2)$. Since the action is only $\mathcal{O}(a)$ improved, this subtraction is formally irrelevant. The multiplication, however, removes the dependence of $\sigma_{1L}(p^2)$ on $p^2$, coming from the $\mathcal{O}(a^2)$ effect discussed above. This can be seen in Fig. 1 by comparing the squares and filled circles. It is important to add that $z$ (for each $\kappa$) has been fixed by fitting to a constant the ratio

$$z=\frac{1}{p^2}\frac{\sigma_{2L}}{\sigma_{1L}} \quad (24)$$

in the large $p^2$ region. The data and the fit are displayed in Fig. 1 for $\kappa=0.1344$. The numerical values for $z$ will be given in the next section. The multiplication by $(1+z^2p^2)$ flattens the $p^2$ dependence of both $\sigma_{1L}$ and $\sigma_{2L}$. This effect is particularly pronounced for $\sigma_{1L}$, as can be seen in Fig. 1.

We now describe the procedure followed to remove the constant contact term $c_q'$, in Eq. (20). After the multiplication of the propagator by $(1+z^2p^2)$, we fit $\sigma_{2L}(p^2)$ to the form expected from the operator product expansion (OPE), namely (up to logarithmic corrections)

$$\frac{\sigma_{2L}(p^2)}{p^2}=\mathcal{A}+\frac{\mathcal{B}}{a^2p^2}+\mathcal{C}a^2p^2, \quad (25)$$

in the region of large Euclidean momenta where the OPE applies. We have used the interval $0.5\leqslant a^2p^2\leqslant2.0$, which at

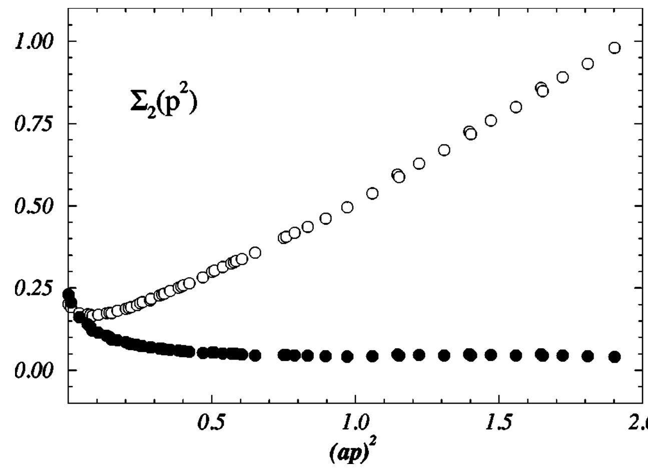


FIG. 2. Empty symbols denote $\Sigma_{2L}(p^2)$, extracted directly from the lattice quark propagator. The filled symbols correspond to the subtracted form factor $\tilde{\Sigma}_{2L}(p^2)$. Illustrated is the case with $\kappa=0.1344$.

$\beta=6.2$ corresponds to $2\ \mathrm{GeV}^2\leqslant p^2\leqslant15\ \mathrm{GeV}^2$ in physical units. By comparison with Eq. (19), the coefficient $\mathcal{A}$ is evidently $\mathcal{A}=-2a\tilde{c}_q'$. From the fit of our data to Eq. (25), we obtain $\mathcal{A}=0.617(11)$, to be compared to $\mathcal{A}^{(\mathrm{BPT})}=0.573$, as computed in one-loop (boosted) perturbation theory [33].[8] According to Eq. (3), the parameter $\mathcal{B}$ is proportional to the quark mass, so that it is expected to vanish in the chiral limit. The value that we obtain, $\mathcal{B}=0.003(6)$, is consistent with expectations. It clearly demonstrates that the contribution of the term proportional to the quark condensate is negligible in the range of momenta chosen for the fit. This point was recently questioned in Ref. [35]. From our fit this contribution appears to be completely negligible for $p^2\gtrsim3\ \mathrm{GeV}^2$, as expected from OPE, if $\langle\bar{q}q\rangle\sim\Lambda_{\mathrm{QCD}}^3$. Finally, we also obtain $\mathcal{C}=0.022(4)$, for the parameter which contains the information on residual $\mathcal{O}(a^2)$ effects. Note that without the correcting factor $(1+z^2p^2)$ we would have obtained $\mathcal{C}=-0.070(3)$, which is larger than the above results. This shows that the residual $\mathcal{O}(a^2)$ effects are smaller than those induced by the contact terms.

To summarize, we subtract the unwanted discretization effects using

$$\tilde{S}(p)=\frac{i\not{p}}{p^2}\tilde{\sigma}_1(p^2)+\frac{\tilde{\sigma}_2(p^2)}{p^2}=S_L(p)(1+z^2p^2)-\mathcal{A}-\mathcal{C}a^2p^2. \quad (26)$$

The resulting improved propagator, $\tilde{S}(p)$, exhibits a good chiral behavior at large $p^2$ and its inverse has small $\mathcal{O}(a^2)$ corrections. $\Sigma_2(p^2)$ is expected to be a slowly varying function of the momentum at large $p^2$, with a logarithmic $p^2$ dependence governed by the quark mass anomalous dimension. As shown in Fig. 2, the bare form factor $(\Sigma_{2L}(p^2))$

[6]Throughout this paper, we adopt a continuum notation in which $ap_\mu$ stands for $\sin(ap_\mu)$. Thus, for instance, $a^2p^2$ corresponds to $\Sigma_\mu\sin^2(ap_\mu)$ and $a\not{p}$ is equal to $\Sigma_\mu\gamma_\mu\sin(ap_\mu)$.

[7]Although not mentioned before, it is clear that there are other $\mathcal{O}(a^2)$ effects, besides those induced by the $\mathcal{O}(a)$ contact term, resulting in the factor $(1+z^2p^2)$. These, however, are found to be much smaller, see below.

[8]The perturbative calculation of $\tilde{c}_q'$ indicates explicitly that this coefficient is gauge dependent.

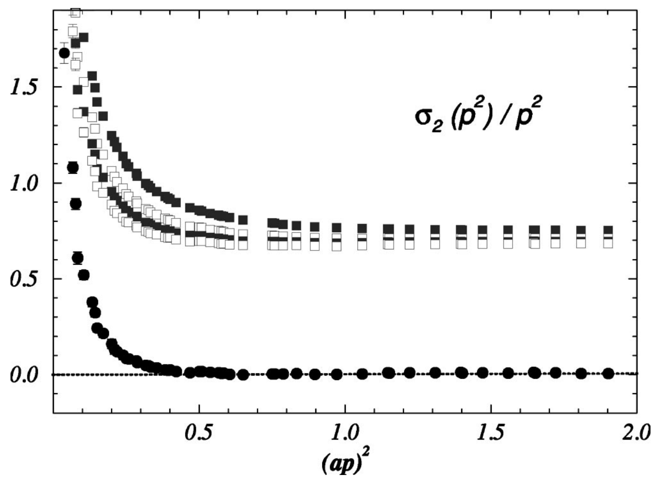


FIG. 3. The effect of subtraction of the contact term, $\tilde{c}'_q$, from the scalar part of the quark propagator. Squares denote $\sigma_2/p^2$, as obtained from unsubtracted [albeit corrected by the $(1+z^2p^2)$ factor] propagators corresponding to the four values of the hopping parameters used in this study. The filled symbols denote $\tilde{\sigma}_2/p^2$, obtained after the subtraction of the term due to $\tilde{c}'_q$ [see Eq. (26)], and extrapolated to the chiral limit.

exhibits, instead, a strong linear dependence in $(pa)^2$, induced by the $\mathcal{O}(a)$ contact term proportional to $\tilde{c}'_q$. This effect disappears in the subtracted form factor $\tilde{\Sigma}_{2L}(p^2)$, defined from $\tilde{S}^{-1}(p)$, as shown in the same plot.

As a further confirmation of the effectiveness of the subtraction, we show $(1+z^2p^2)\sigma_{2L}(p^2)/p^2$ in Fig. 3 for different values of $\kappa$. At large $p^2$ and for all the values of the quark masses, this quantity has a good plateau corresponding, up to further $\mathcal{O}(a^2)$ corrections, to the contact term $\sim\tilde{c}'_q$ that we want to subtract. In the same figure, we also show $\tilde{\sigma}_2(p^2)/p^2$, after the subtraction and extrapolated to the chiral limit (filled circles). This curve demonstrates the accuracy of the subtraction procedure, since in this limit we expect $\tilde{\sigma}_2(p^2)/p^2\sim\langle\bar{q}q\rangle/p^4$.

## III. NUMERICAL DETAILS AND PHYSICAL RESULTS

In this section we briefly recall some elements of the lattice calculation, which are explained in great detail in Ref. [34], and present our physical results. These results are obtained on a sample of 200 quenched gauge fields configurations, on a $24^3\times 64$ lattice, and at $\beta=6.2$. The value of the inverse lattice spacing, $a^{-1}=2.72(11)$ GeV, is obtained from the $m_{K*}$ mass. The quark propagators have been computed by using the nonperturbatively $\mathcal{O}(a)$ improved Wilson action for four different values of the light quark masses which correspond to the following set of the hopping parameters:

$$\kappa\in\{0.1352,0.1349,0.1344,0.1333\}. \tag{27}$$

All other details concerning the analysis of the light hadron spectrum can be found in Ref. [34] where a subset of the 100 configuration was analyzed.

As discussed in the previous section, Eq. (26) involves the determination of the constant $z$. From the fit of our data to Eq. (24) in the interval $1.1\leqslant(ap)^2\leqslant 2$, we obtain

$$z=\{0.581(3),0.591(2),0.605(3),0.639(2)\}, \tag{28}$$

in decreasing order with respect to the $\kappa$ parameter. With the value of $z$ at hand, we follow the subtraction procedure described in the previous section and obtain the renormalized propagator, from which the quark masses can be derived. We now present the numerical results for $Z_q$ and for the quark masses in different schemes.

### A. The quark field renormalization constant $Z_q$

We fit the form factor $\tilde{\Sigma}_{1L}(p^2)$ to a constant, in the large momentum region. According to Eq. (6), the value of that constant corresponds, up to tiny $\mathcal{O}(\alpha_s^2)$ corrections [17,18], to the value of the quark field renormalization constant, $Z_q$, in the RI-MOM scheme. From a fit in the region of large $(ap)^2\in[1,1.8]$ (corresponding to $7.4\lesssim p^2\lesssim 13.2$ GeV$^2$ in physical units), we obtain

$$\tilde{Z}_q=\{0.849(2),0.847(2),0.846(3),0.839(2)\}. \tag{29}$$

These values extrapolated (linearly) to the chiral limit give[9]

$$Z_q^{(0)}(2\ \mathrm{GeV})=0.853(3), \tag{30}$$

which is to be compared with $Z_q^{(0)}=0.875$, from one-loop (boosted) perturbation theory [33]. Note that the quark-mass dependence of $\Sigma_{1L}(p^2)$, which according to Eq. (18) comes from both the mass dependent term in $Z_q$ and the term proportional to $c_{\mathrm{NGI}}$, is rather weak.

### B. Extracting the physical quark mass

From the subtracted inverse propagator, $\tilde{S}^{-1}(p)$, we get the renormalized quark mass in the RI-MOM scheme which, according to Eq. (3), corresponds at large $p^2$ to

$$m^{\mathrm{RI}}(\mu)=\left.\frac{\tilde{\Sigma}_2(p^2)}{\tilde{\Sigma}_1(p^2)}\right|_{p^2=\mu^2}. \tag{31}$$

This quantity is unaffected by the correcting factor $(1+z^2p^2)$. In Fig. 4, we display $m^{\mathrm{RI}}(\mu)$ for different values of $\kappa$. The numerical value of $m^{\mathrm{RI}}(\mu)$ can be read off directly from Fig. 4. Note that $m^{\mathrm{RI}}(\mu)$ is derived in a completely nonperturbative way.

In practice, to reduce the statistical fluctuations we extract $m^{\mathrm{RI}}(\mu_0)$, from a fit in the interval $1.1\leqslant(\mu a)^2\leqslant 1.8$ and corresponding to $7.5\ \mathrm{GeV}^2\lesssim\mu^2\lesssim 13\ \mathrm{GeV}^2$, by using

$$m^{\mathrm{RI}}(\mu)=\frac{c^{\mathrm{RI}}(\mu)}{c^{\mathrm{RI}}(\mu_0)}m^{\mathrm{RI}}(\mu_0), \tag{32}$$

[9]The value of $\kappa_{crit}$ is 0.135855(19).

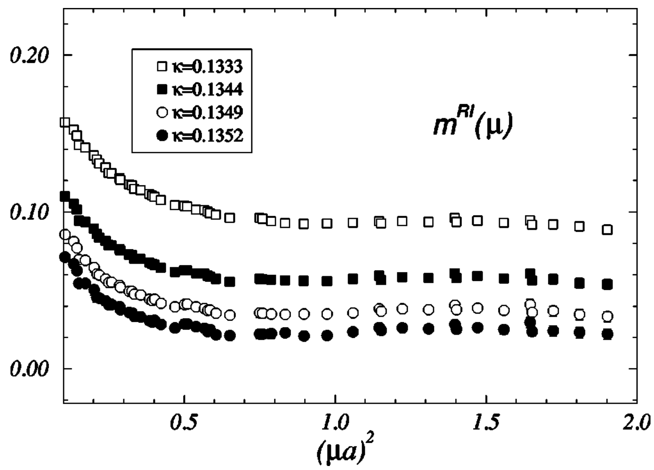


FIG. 4. The lattice quark masses corresponding to the indicated $\kappa$ values, in the RI-MOM scheme. They have been obtained using Eq. (31). The error bars are not visible, because they are smaller than the symbols used in this figure.

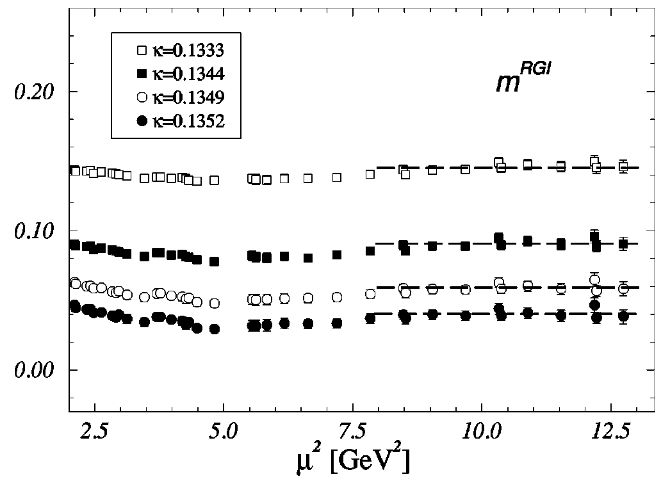


FIG. 5. Renormalization group invariant quark masses, obtained after dividing out the scale depending part in the RI-MOM scheme to N$^3$LO accuracy, $c^{\mathrm{RI}}(\mu)$, from the quark masses depicted in Fig. 4. Dashed lines correspond to the fitting interval used in (32).

with $c^{\mathrm{RI}}(\mu)$ computed in perturbation theory. We have chosen $(\mu_0 a)^2=1.45$, in the middle of the fitting interval, corresponding to $\mu_0=3.28$ GeV. With this procedure the error induced by the use of perturbation theory is negligible; namely, we find that the differences between NLO and NNNLO are less than 1%. The reason is that the expression in Eq. (32) depends on the ratio of $c^{\mathrm{RI}}$'s evaluated at different but close scales.

The numerical values of $c^{\mathrm{RI}}(\mu)$ have been obtained by using the quenched expression for $\alpha_s(\mu)$ with $\Lambda_{\mathrm{QCD}}=318$ MeV [36]. We checked that by using $\Lambda_{\mathrm{QCD}}=238$ MeV, as found in Ref. [16], the central value of the masses is increased by less than 1%.

To compute the physical values of the light and the strange quark masses, we invoke the procedure described in Ref. [9],[34]. The masses are fitted as quadratic functions of the squared pseudoscalar meson masses. By using the method of ''physical lattice planes'' [4], we fix the average up-down and strange quark masses, from the $\pi$ and $K$ meson, respectively.[10] The results are the following:

$$m_l^{\mathrm{RI}}(3.28\,\mathrm{GeV})=5.1(5)\ \mathrm{MeV},$$

$$m_s^{\mathrm{RI}}(3.28\,\mathrm{GeV})=118(9)\ \mathrm{MeV}. \tag{33}$$

Since it is customary to give the quark masses at the renormalization scale $\mu=2$ GeV, we again use Eq. (32) to rescale $m^{\mathrm{RI}}(3.28\ \mathrm{GeV})$ to $m^{\mathrm{RI}}(2\ \mathrm{GeV})$. This time, we have to run the mass to a lower scale than the one used in the fitting procedure ($7.5\ \mathrm{GeV}^2 \lesssim \mu^2 \lesssim 13\ \mathrm{GeV}^2$). For this reason the uncertainty due to higher orders is larger than before, namely of the order of 4%. The results are those given in Eq. (7). We note, in passing, that there is no reason, if not for comparison with other calculations, to evolve the masses down to $\mu=2$ GeV. Indeed, it would be much more convenient to work at a scale $\mu \gtrsim 3$ GeV where the uncertainty induced by higher order perturbative corrections is negligible.

We now illustrate the procedure adopted to obtain the quark masses in different schemes. The renormalization group invariant quark mass, which is a scheme and scale independent quantity, is related to $m^{\mathrm{RI}}(\mu)$ by the expression

$$m_q^{\mathrm{RGI}}=\frac{m_q^{\mathrm{RI}}(\mu)}{c^{\mathrm{RI}}(\mu)}, \tag{34}$$

whereas for the $\overline{\mathrm{MS}}$ mass we have

$$m^{\overline{\mathrm{MS}}}(\mu)=c^{\overline{\mathrm{MS}}}(\mu)m_q^{\mathrm{RGI}}=\frac{c^{\overline{\mathrm{MS}}}(\mu)}{c^{\mathrm{RI}}(\mu)}\,m_q^{\mathrm{RI}}(\mu). \tag{35}$$

The resulting values of $m_q^{\mathrm{RGI}}$ should be flat in a large range of $\mu^2$, as confirmed by the data shown in Fig. 5. By using Eq. (34), we obtain the following results

$$\begin{array}{llll} & \mathrm{NLO} & \mathrm{NNLO} & \mathrm{NNNLO} \\ m_l^{\mathrm{RGI}}=\{ & 8.5(9); & 7.8(8); & 7.6(8)\}\ \mathrm{MeV}, \\ m_s^{\mathrm{RGI}}=\{ & 196(15); & 182(14); & 177(14)\}\ \mathrm{MeV}. \end{array} \tag{36}$$

In the $\overline{\mathrm{MS}}$ case, from Eq. (35) we get the results quoted in Eq. (11).

In the calculation of $m_q^{\mathrm{RGI}}$ and $m^{\overline{\mathrm{MS}}}(\mu)$, we have used the physical value of $\alpha_s(\mu)$, corresponding to $\alpha_s(M_Z)=0.118$ [38], computed with the appropriate number of active flavors, e.g., $n_f=4$ at 2 GeV. This choice can be justified by assuming that the masses in Eq. (7) are the physical ones, up to some unknown quenching errors. We checked, however,

[10]Note that consistent results are obtained when the $\phi$ meson is used to extract the strange quark mass.

that the results by using the quenched $\alpha_s$ with $\Lambda_{\rm QCD} = 318$ MeV would be different by less than 2% in all of the cases considered in this paper.

## CONCLUSION

We have applied a new method to compute the renormalized quark masses from the lattice quark propagator using the OPE. We have discussed the subtleties related to the improvement of the propagator and especially the troubles arising from the presence of contact terms. Some of these problems could be avoided by working with $S_L(x)$ instead of $S_L(p)$. Feasibility studies are underway. The main results, given in the introduction, are very compatible with the values of the masses obtained by standard lattice methods. They are also in very good agreement with the recent result of Ref. [39], $m_s^{\overline{\rm MS}}(2\ {\rm GeV}) = 114 \pm 24$ MeV, obtained by using the model independent QCD sum rule analysis at a NNNLO (see Ref. [40] also).

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## APPENDIX

In this appendix we list the formulas which have been used to compute the perturbative scale dependence of the quark masses.

The effective QCD coupling is governed by the $\beta$ function which is known to four loops,

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_s(\mu)}{\pi}\right) = -\sum_{n=0}^{3} \beta_n \left(\frac{\alpha_s(\mu)}{\pi}\right)^{n+2} + \mathcal{O}(\alpha_s^6(\mu)), \tag{A1}$$

where the coefficients are [37]:

$$\beta_0 = \frac{1}{4}\left(11 - \frac{2}{3} n_f\right), \quad \beta_1 = \frac{1}{16}\left(102 - \frac{38}{3} n_f\right),$$

$$\beta_2^{\overline{MS}} = \frac{1}{64}\left(\frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2\right),$$

$$\beta_3^{\overline{MS}} = \frac{1}{256}\left[\frac{149753}{6} - 3564\zeta(3) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right) n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right) n_f^2 + \frac{1093}{729} n_f^3\right]. \tag{A2}$$

The coefficients of the mass anomalous dimension, which describes the running of the quark mass,

$$\mu^2 \frac{d}{d\mu^2} m_q(\mu) = -m(\mu) \sum_{n=0}^{3} \gamma_m^{(n)} \left(\frac{\alpha_s(\mu)}{\pi}\right)^{n+1} + \mathcal{O}(\alpha_s^5(\mu)), \tag{A3}$$

are also known up to four loops in both RI [18] and $\overline{\rm MS}$ [29] schemes. We list them all;

$$\gamma_m^{(0)} = 1,$$

$$(\gamma_m^{(1)})^{\overline{\rm MS}} = \frac{1}{16}\left(\frac{202}{3} - \frac{20}{9} n_f\right), \quad (\gamma_m^{(1)})^{\rm RI} = \frac{1}{16}\left(126 - \frac{52}{9} n_f\right),$$

$$(\gamma_m^{(2)})^{\overline{\rm MS}} = \frac{1}{64}\left[1249 - \left(\frac{2216}{27} + \frac{160}{3}\zeta(3)\right) n_f - \frac{140}{81} n_f^2\right],$$

$$(\gamma_m^{(2)})^{\rm RI} = \frac{1}{64}\left[\frac{20911}{3} - \frac{3344}{3}\zeta(3) - \left(\frac{18386}{27} - \frac{128}{9}\zeta(3)\right) n_f + \frac{928}{81} n_f^2\right],$$

$$(\gamma_m^{(3)})^{\overline{\rm MS}}=\frac{1}{256}\Bigg[\frac{4603055}{162}+\frac{135680}{27}\zeta(3)-8800\zeta(5)-\left(\frac{91723}{27}+\frac{34192}{9}\zeta(3)-880\zeta(4)-\frac{18400}{9}\zeta(5)\right)n_f$$
$$+\left(\frac{5242}{243}+\frac{800}{9}\zeta(3)-\frac{160}{3}\zeta(4)\right)n_f^2-\left(\frac{332}{243}-\frac{64}{27}\zeta(3)\right)n_f^3\Bigg],$$
$$(\gamma_m^{(3)})^{\rm RI}=\frac{1}{256}\Bigg[\frac{300665987}{648}-\frac{15000871}{108}\zeta(3)+\frac{6160}{3}\zeta(5)-\left(\frac{7535473}{108}-\frac{627127}{54}\zeta(3)-\frac{4160}{3}\zeta(5)\right)n_f$$
$$+\left(\frac{670948}{243}-\frac{6416}{27}\zeta(3)\right)n_f^2-\frac{18832}{719}n_f^3\Bigg]. \tag{A4}$$

The corresponding evolution parts in the running quark masses are

$$c^{\rm RI}(\mu)=\alpha_s(\mu)^{4/11}\Bigg\{1+\frac{489}{242}\left(\frac{\alpha_s(\mu)}{\pi}\right)+\left[\frac{25335863}{1405536}-\frac{19}{6}\zeta(3)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$
$$+\left[\frac{48247704573745}{220410533376}-\frac{170324909}{2509056}\zeta(3)+\frac{35}{36}\zeta(5)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\Bigg\}, \tag{A5}$$

$$c^{\overline{\rm MS}}(\mu)=\alpha_s(\mu)^{4/11}\Bigg\{1+\frac{499}{726}\left(\frac{\alpha_s(\mu)}{\pi}\right)+\frac{6375961}{4216608}\left(\frac{\alpha_s(\mu)}{\pi}\right)^2+\Bigg[\frac{344717507317}{55102633344}+\frac{6293}{3564}\zeta(3)$$
$$-\frac{25}{6}(5)\Bigg]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\Bigg\}, \tag{A6}$$

for $n_f=0$, and

$$c^{\rm RI}(\mu)=\alpha_s(\mu)^{12/25}\Bigg\{1+\frac{8803}{3750}\left(\frac{\alpha_s(\mu)}{\pi}\right)+\left[\frac{5679460183}{337500000}-\frac{119}{30}\zeta(3)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$
$$+\left[\frac{14533180260067051}{91125000000000}-\frac{1437607219}{21600000}\zeta(3)+\frac{19}{4}\zeta(5)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\Bigg\}, \tag{A7}$$

$$c^{\overline{\rm MS}}(\mu)=\alpha_s(\mu)^{12/25}\Bigg\{1+\frac{3803}{3750}\left(\frac{\alpha_s(\mu)}{\pi}\right)+\left[\frac{793412683}{337500000}-\frac{4}{5}\zeta(3)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^2$$
$$+\left[\frac{57222640693973}{7593750000000}-\frac{2202791}{337500}\zeta(3)+\frac{5}{3}\zeta(4)-\frac{7}{18}\zeta(5)\right]\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\Bigg\}, \tag{A8}$$

for $n_f=4$.


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