Zero mode quantization of multi-Skyrmions

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A zero mode quantization of the minimal energy SU(2) Skyrmions for nucleon numbers four to nine and seventeen is described. This involves quantizing the rotational and isorotational modes of the configurations. For nucleon numbers four, six and eight the ground states obtained are in agreement with the observed nuclear states of helium, lithium and beryllium. However, for nucleon numbers five, seven, nine and seventeen the spins obtained conflict with the observed isodoublet nuclear states.

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I. INTRODUCTION

In this article a simple quantization of higher charge Skyrmions is described and the results are compared to experimental nuclear data. The methods described may be used for Skyrmions of any nucleon number B, once the minimal energy solution is known. The minimal energy solutions are now known for $B \leqslant 9$ [1] and a conjectured solution exists for $B=17$ [2]. We use the moduli space approximation [3], which truncates the infinite dimensional configuration field space to a finite dimensional space consisting of classical configurations which are relevant to the low energy dynamics. The moduli space will necessarily include all minimal energy configurations and to obtain accurate results one should include all configurations corresponding to B Skyrmions with arbitrary separations and relative isospin orientations. Obviously, the more configurations that are included in the moduli space, the more difficult their analysis becomes. As a first approximation one may restrict the moduli space to be generated by the zero modes of the minimal energy solution. Any Skyrmion configuration can be translated, rotated or isorotated without changing its energy and these are the only zero modes. We shall ignore the translational modes since their quantization only gives a total momentum to the quantum state. The interesting physics arises when the rotational and isorotational degrees of freedom are quantized.

The minimal energy Skyrmions for $B=1$ and $B=2$ have spherical and axial symmetry respectively. For higher nucleon numbers the minimal energy solutions only have a discrete symmetry [1,4]. This means that the classical configuration $U_B(\mathbf{x})$ is invariant under a discrete group, H, of combined rotations and isorotations. Thus the moduli space of zero modes is given by a quotient space $\mathcal{C}=(\mathrm{SO}(3)\times\mathrm{SO}(3))/H$. This may be equivalently written as a quotient of the covering group $\mathcal{C}=(\mathrm{SU}(2)\times\mathrm{SU}(2))/K$, where K is a discrete subgroup of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ related to the discrete subgroup H of SO(3). Elements of K correspond to rotations and isorotations in H combined with 2π rotations and isorotations. In the cases where we need to be specifically concerned with K, as opposed to H, it has the form $K=\bar{H}\times\mathbb{Z}_2$ where $\bar{H}$ is the double group of H [$\bar{H}$ is a subgroup of SU(2) with two elements, h and $-h$ in $\bar{H}$ for every element of H in SO(3)]. Elements in K distinguish a clockwise rotation by θ about some axis from an anticlockwise rotation by $2\pi-\theta$ about the same axis. In the even nucleon sector 2π rotations are trivial, so it is sufficient to consider the group H. But for odd B it is necessary to consider K as opposed to H.

Semiclassical quantization of the configuration is achieved by quantizing on this quotient space. There are a number of inequivalent ways to quantize on a quotient space G/K; when G, which here is $\mathrm{SU}(2)\times\mathrm{SU}(2)$, is simply connected these are labeled by the irreducible representations of the group K. In general, the wave functions are defined on $\mathrm{SU}(2)\times\mathrm{SU}(2)$, but they transform under some irreducible representation of K. The reason for working with the double cover, $\mathrm{SU}(2)\times\mathrm{SU}(2)$, is that as is well known, 2π rotations have nontrivial consequences in the quantum theory, this enables single Skyrmions to be quantized as fermions. To determine which quantization is appropriate here, one must consider the Finkelstein-Rubenstein (FR) constraints [5]. They state that, in order for a single Skyrmion to be quantized as a fermion, wave functionals are sections of a line bundle over the classical configuration space whose holonomy around any noncontractible loop in the configuration space is (-1). In our case, quantizing on $\mathcal{C}$, wave functions are sections of a line bundle over $\mathcal{C}$ whose holonomy is (-1) for loops which remain noncontractible when $\mathcal{C}$ is extended to the full Skyrmion field configuration space. This is equivalent to defining wave functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ which are eigenstates of the operators which correspond to a rotation and isorotation by an element of K, with eigenvalues (-1) $+1$ depending on whether this operation is (non)contractible in the full Skyrmion configuration space. The effect of 2π rotations or isorotations is well known. A 2π rotation or isorotation of a configuration with nucleon number B is contractible if B is even and noncontractible if B is odd. Thus states with odd B are fermionic and states with even B are bosonic. The states define a one dimensional representation of the symmetry group K. If K has no nontrivial one dimensional representations then all the FR constraints must be $+1$. If there are nontrivial one dimensional representations of K then one needs to carefully examine the closed loop corresponding to elements of K which have character (-1) of

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this nontrivial one dimensional representation. It must be determined whether these loops are contractible or not.

For this, it is often necessary to split the configuration into individual or pairs of well separated Skyrmions, and then analyze the closed loop. The (non)contractibility of the $B=1$ and $B=2$ Skyrmions are known under such closed loops and from this the (non)contractibility of the loop may be determined. However, it is necessary that the configuration retains the symmetry of the specific element of K being considered, as it is being split into a well separated configuration of Skyrmions, i.e. the loop is closed throughout the deformation. This is not obvious from the Skyrme picture since there is no analytical data.

To proceed we can use the recently discovered rational map ansatz for Skyrmions [2]. These authors describe how, given an SU(2) monopole which can be uniquely described by a rational map, one may associate to it a Skyrmion. Using this method they were able to accurately approximate the known minimal energy Skyrmion configurations for nucleon numbers one to nine and the predicted solution for nucleon number seventeen. The minimal energy Skyrme configuration obtained in this manner has the same symmetries as the monopole from which it is derived. This ansatz has the advantage of clearly illustrating what combination of rotations and isorotations leave the solution invariant. It is also useful in that the reflection symmetries of the solution can easily be worked out which enables one to determine how the parity operator can be represented on $\mathcal{C}$.

As verified in [6] this ansatz also extends to describe some of the Skyrmions vibrational modes. There, the vibrational spectra of the minimal energy $B=2$ and $B=4$ Skyrmions was calculated. The vibrations form representations of the symmetry group of the minimal energy Skyrmion. The vibrational modes of the Skyrmions come in two different types. The modes of lower frequency correspond to the Skyrmion configuration breaking up into separated Skyrmions. The modes of higher frequencies correspond to the well known ''breather'' and generalizations of it whereby the local nucleon charge expands or contracts in places (in [7] a mechanism was given for describing these modes). It is also possible to look at vibrations of the rational maps. This corresponds to monopole motion on the monopole moduli space. Again, small variations from the symmetric configuration form representations of its symmetry group. In [6], they found that vibrations with frequency below that of the ''breather'' type modes form the same representations of the symmetry group as do the monopole vibrations. The implication of this is that, if a monopole configuration can be separated a small distance while respecting some discrete symmetry, then the same process can occur for Skyrmions. We wish to extend this correspondence to arbitrary monopole-Skyrmion separations. However, the rational map ansatz breaks down as the monopole separates into individual Skyrmions. Nonetheless we conjecture that the correspondence can be extended beyond this region, such that any monopole motion can be mapped to an equivalent path in the Skyrmion configuration space. In effect, this amounts to an embedding of the monopole moduli space into the Skyrmion configuration space. Evidence for this is seen by considering the possible scattering processes for the known cases of monopoles and Skyrmions [8]. In the above paper it was seen that for well known cases of monopole scattering, an equivalent Skyrmion scattering process could occur with the same symmetry. In fact, all we really need to assume is that, if the Skyrmion can be vibrated remaining invariant under some symmetry group element, then the continuation of this path in the Skyrmion configuration space, which will remain invariant under the symmetry, eventually becomes a configuration of well separated Skyrmions. For the monopoles this is always the case.

Assuming the results in [6] are true for general nucleon numbers, and that there is a 1-1 correspondence between monopole motion and Skyrmion motion for low vibrational energies then, if the monopole configuration can be deformed keeping a symmetry, so can the Skyrmions. But monopoles are in an exact 1-1 correspondence with rational maps [9,10]. The set of monopoles which have a discrete rotational symmetry is easily determined from the rational maps (because they have a simple action of the rotation group). Also, it is easy to see how the rational map of a symmetric multi-monopole changes when the multi-monopole splits up into well separated monopoles. So, rational maps can be used to determine whether a multi-monopole can be split into a specific configuration of well separated monopoles while respecting a certain symmetry group element. Thus by our above assumption it can be determined how a Skyrmion configuration can be split up while keeping a certain symmetry. In the cases considered here we can always separate into a configuration of $B=1$ and $B=2$ solutions whose behavior under rotations and isorotations is known. Using this method we shall determine the FR constraints. Once these are found it is a simple exercise to find the allowed quantum states.

In [11] and [12], such an analysis was carried out for the axially symmetric charge two solution and for the tetrahedrally symmetric charge three solution. For the $B=2$ case a ground state with the correct quantum numbers of the deuteron was obtained. And for $B=3$ it was found that the ground state had spin $\frac{1}{2}$, isospin $\frac{1}{2}$ in agreement with the observed isodoublet nucleus $({}^{3}_{1}\mathrm{H}, {}^{3}_{2}\mathrm{He})$. Here we will use the terminology for the sake of compactness that spin and isospin are the eigenvalues of the operators corresponding to rotations and isorotations of the Skyrmions respectively. Then spin corresponds to the total angular momentum of the nuclear state which is the experimentally observed quantity. In nuclear physics the total angular momentum is the sum of orbital and intrinsic spin angular momentum. Such a decomposition of total angular momentum into orbital and intrinsic spin parts does not exist in the Skyrme model. We comment on this further in Sec. VI.

In this paper we extend the zero mode analysis to the minimal energy Skyrmions with nucleon numbers four to nine and seventeen. We find that for $B=4$, 6, 8 the ground state has the correct spin, parity and isospin assignments as for ${}^{4}_{2}\mathrm{He}^{+}$, ${}^{6}_{3}\mathrm{Li}^{+}$ and ${}^{8}_{4}\mathrm{Be}^{+}$. However for odd nucleon numbers $B=5$, 7, 9 or 17 the ground states found by this method do not agree with the observed isodoublet states. For nucleon numbers 5, 7 and 9 the experimentally observed ground

states are isodoublets with spin $\frac{3}{2}$ and for $B=17$ the observed ground state is an isodoublet with spin $\frac{5}{2}$ [13]. However the zero mode quantization of Skyrmions results in the ground state for $B=5$ and $B=9$ both to be isodoublets with spin $\frac{1}{2}$, for $B=7$ and $B=17$ the ground state are both found to be isodoublets with spin $\frac{7}{2}$. As discussed below we do not try to predict the parity of the states with odd B. The ground states we find here exist experimentally as excited states. The experimentally observed ground state for $B=5$ appears here as an excited state. The experimentally observed ground states for $B=7$, $B=9$ and $B=17$ can be obtained here by including the vibrational modes but they will also appear here as excited states.

The vibrational modes form representations of the symmetry group of the minimal energy solution. Knowing this it is possible to combine the rotational and vibrational modes resulting in an enlarged configuration space. The vibrational spectra has been worked out for the Skyrme model for nucleon numbers two [6,14] and four [6], it is also possible to understand some aspects of the vibrational spectra for other values of B using the rational map ansatz. The vibrational modes of the Skyrmions with frequencies below the breather modes can be described by monopole motion and thus the representations they form of the symmetry group can be determined. The configuration space is now a fiber bundle over $(\mathrm{SU}(2)\times\mathrm{SU}(2))/K$, the fiber being the vector space corresponding to the vibrations. This space was described in [15]. States are now given by the direct product of Wigner functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and harmonic oscillator wave functions on the vibrational space. The states must satisfy a K invariance condition described below which restricts the allowed set of states. Using this formalism further excited states of the multi-Skyrmions may be described. It is possible that this approach may resolve the above problem of the ground state for $B=7$. A spin $\frac{3}{2}$ rotational state can be combined with a vibrational state to give an allowed state. If the vibrational energy of this state is not too large it may have lower energy than the state with spin $\frac{7}{2}$ and thus predict the correct ground state. To check this, the energies of the vibrational states need to worked out directly from the Skyrme model as the rational map approach has no information about the frequencies of the specific vibrations. The inclusion of vibrational modes may also fix the problem for $B=17$ but it will not work for $B=5$ and $B=9$.

Naturally, one would not expect that the quantization of zero modes and vibrations would give accurate results on binding energies of the states, etc., and the inclusion of more degrees of freedom are needed to accurately describe such properties. Nonetheless it is not obvious that the inclusion of other modes (allowing the Skyrmions to separate, calculating the zero point energies of the radiative pion modes) will resolve this difficulty. A possible resolution of this is that the solutions found in [1] are not well defined minima, i.e. there may be a number of local minima with approximately equal energies and so an expansion about just one of these minima is not valid. This seems to occur for the $B=10$ case, to answer the question here requires further numerical investigation of the proposed minimal energy solutions.

In the following section we review the zero mode quantization discussed in [11] paying special attention to the FR constraints. Section III describes the rational map ansatz for Skyrmions and how it may be used to determine the FR constraints. Then in Sec. IV the quantization procedure is treated for each of the Skyrmions $B=4$ to $B=9$ and $B=17$. Section V describes how to include vibrational modes and gives the predicted excited states for the $B=4$ sector by considering the vibrations together with the zero modes. Finally in Sec. VI we calculate the expectation value of the nucleon density of the quantum ground states and compare to the classical nucleon densities. A criticism raised about the classical solutions of the Skyrme model is that they bear no resemblance to real nuclei. The classical nucleon densities have the symmetry of some discrete group. To find the nucleon density in the quantum state, following [16] we integrate the classical nucleon density times the norm squared of the wave function over the moduli space. We find that in all cases considered, the nucleon density in the quantum state is almost spherically symmetric, being exactly so in a number of cases. For example we find the ground state for $B=4$ to be spherically symmetric and for $B=6$ to be mainly S-wave with a small P-wave admixture. This agrees with the nucleon densities of helium 4 and lithium 6 respectively and shows how the nucleon density of the classical solutions is smeared by quantum effects to a more uniform angular dependence.

II. SEMI-CLASSICAL QUANTIZATION

The Skyrme model has the Lagrangian

$$L=\int d^3x\left\{-\frac{f_\pi^2}{16}\mathrm{Tr}(R^\mu R_\mu)+\frac{1}{32e^2}\mathrm{Tr}([R_\mu,R_\nu][R^\mu,R^\nu])\right\} \tag{2.1}$$

where $R_\mu=\partial_\mu U U^\dagger$, U is the SU(2) valued Skyrme field, and e, f_π are free parameters of the model whose values are chosen to best fit experimental data. The above Lagrangian has soliton solutions of finite energy. Finite energy implies that U tends to a constant at spatial infinity. Space is then compactified to S^3 and thus each soliton solution has an associated integer, the degree, corresponding to the element of $\pi_3(S^3)$ to which U belongs. Solitons of degree B are interpreted as B nucleons [17].

The symmetry group of the Skyrme Lagrangian is $\mathrm{SO}(3)\times$Poincaré Group$\times$P. P is the parity operator which acts as $\mathrm{P}: U[\mathbf{x}]\rightarrow U^\dagger[-\mathbf{x}]$. For time-independent fields such as static solitons the symmetry group is reduced to

$$\mathrm{SO}(3)\times\text{Euclidean Group of }\mathbb{R}^3\times\mathrm{P}. \tag{2.2}$$

The minimal energy solutions to the Skyrme model, $U_B[\mathbf{x}]$, have for $B\geqslant 3$ a discrete symmetry group. This means that the classical configuration $U_B[\mathbf{x}]$ is invariant under a discrete group H, of combined rotations and isorotations. To every element $S\in H$ there will exist an element $\Gamma(S)\in\mathrm{SO}(3)$ such that the rotation S, has the same effect on the configuration as the isorotation $\Gamma(S)$. Or alternatively, the combined rotation S and isorotation $\Gamma^{-1}(S)$ leaves the con-

figuration unchanged. The elements $\Gamma(S)$ form a representation of the group H. This is true because to each rotation S, $\Gamma(S)$ is unique. If $\Gamma(S)$ was not unique then the Skyrmion would be invariant under an isorotation, without any compensating rotation. Assuming this isorotation is about the x_3 axis, a simple argument shows that the right currents R_i are proportional to τ_3 (with τ_i the Pauli matrices). But this implies that the nucleon density $\mathcal{B}$ must vanish because it is given by

$$\mathcal{B}=\frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}\,R_iR_jR_k. \tag{2.3}$$

For $B=3$ and $B=9$ U_B has tetrahedral symmetry, $B=4$ and $B=7$ have octahedral and icosahedral symmetry respectively and the $B=5$, $B=6$ and $B=8$ solutions have D_{2d}, D_{4d}, and D_{6d} symmetries respectively.

One can act on the classical solutions with $\mathbb{R}^3\times\mathrm{SO}(3)\times\mathrm{SO}(3)$ in the following fashion to give a family of solutions with the same energy. This generates the zero mode moduli space. The transformations correspond to translations, rotations and isospin rotations. Explicitly

$$U_B[\mathbf{x}]\rightarrow A'U_B[D(A)(\mathbf{x}-\mathbf{a})]A'^{\dagger} \tag{2.4}$$

where A, A' are in the fundamental representation of SU(2), $\mathbf{a}\in\mathbb{R}^3$, and $D(A)$ is the SO(3) element associated to A, given by $D(A)_{ij}=\frac{1}{2}\mathrm{Tr}\,\tau_iA\tau_jA^{-1}$. So generally the minimal energy solution will have nine zero modes. We will henceforth ignore the translational $\mathbb{R}^3$ symmetry. The above is an $\mathrm{SO}(3)\times\mathrm{SO}(3)$ action since A has the same effect on $U_B[\mathbf{x}]$ as $-A$ does, and similarly for A'. If we label an element in the moduli space by $\{A,A'\}$ we have the identifications

$$\{A,A'\}\cong\{A,-A'\}\cong\{-A,A'\}\cong\{-A,-A'\}. \tag{2.5}$$

The moduli space approximation to multi-Skyrmion dynamics involves letting A, A' become time-dependent and substituting Eq. (2.4) into the Skyrme Lagrangian (2.1). The reduced Lagrangian is quadratic in the time derivatives $a_k=-i\,\mathrm{Tr}\,\tau_kA'^{\dagger}\dot{A}'$, $b_k=-i\,\mathrm{Tr}\,\tau_kA\dot{A}^{\dagger}$, and is given by

$$L_B=\frac{1}{2}a_iU_{ij}a_j+\frac{1}{2}b_iV_{ij}b_j-a_iW_{ij}b_j-M_B \tag{2.6}$$

with M_B is the mass of the solution and the tensors U_{ij},V_{ij},W_{ij} are dependent on the classical solution $U[\mathbf{x}]$, given by [11]

$$\begin{aligned}
U_{ij}&=\frac{1}{8}\int d^3x\,\mathrm{Tr}\left\{U^{\dagger}\left[\frac{1}{2}\tau_i,U\right]U^{\dagger}\left[\frac{1}{2}\tau_j,U\right]+\left[U^{\dagger}\partial_kU,U^{\dagger}\left[\frac{1}{2}\tau_i,U\right]\right]\left[U^{\dagger}\partial_kU,U^{\dagger}\left[\frac{1}{2}\tau_j,U\right]\right]\right\}\\
W_{ij}&=\frac{i}{8}\int d^3x\,\mathrm{Tr}\left\{U^{\dagger}\left[\frac{1}{2}\tau_i,U\right]U^{\dagger}(\mathbf{x}\times\nabla)_jU+\left[U^{\dagger}\partial_kU,U^{\dagger}\left[\frac{1}{2}\tau_i,U\right]\right][U^{\dagger}\partial_kU,U^{\dagger}(\mathbf{x}\times\nabla)_jU]\right\}\\
V_{ij}&=-\frac{1}{8}\int d^3x\,\mathrm{Tr}\{U^{\dagger}(\mathbf{x}\times\nabla)_iUU^{\dagger}(\mathbf{x}\times\nabla)_jU\}-\frac{1}{8}\int d^3x\,\mathrm{Tr}\{[U^{\dagger}\partial_kU,U^{\dagger}(\mathbf{x}\times\nabla)_iU]\\
&\quad\times[U^{\dagger}\partial_kU,U^{\dagger}(\mathbf{x}\times\nabla)_jU]\}
\end{aligned} \tag{2.7}$$

where we have set $f_\pi^2=8$ and $e^2=1/2$.

This Lagrangian may now be quantized in the manner described in [11]. The momenta conjugate to a_i and b_i become the body-fixed spin and isospin angular momentum operators called K_i and L_i which satisfy the SU(2) commutation relations, $[K_i,K_j]=i\epsilon_{ijk}K_k$, and similarly for L_i. There also exist space-fixed spin and isospin angular momentum operators denoted by J_i and I_i related to the body-fixed operators by

$$J_i=-D_{ij}(A)^TL_j,\quad I_i=-D_{ij}(A')K_j. \tag{2.8}$$

The commutation relations are

$$[L_i,A]=-\frac{1}{2}\tau_iA,\quad [J_i,A]=\frac{1}{2}A\tau_i,$$

$$[I_i,A']=-\frac{1}{2}\tau_iA',\quad [K_i,A']=\frac{1}{2}A'\tau_i \tag{2.9}$$

and all other commutators vanish. This means that $\mathbf{L}^2=\mathbf{J}^2$ and $\mathbf{I}^2=\mathbf{K}^2$. The Hamiltonian becomes that of a rigid body in space and isospace. However, the above derivation of the rigid body Hamiltonian is not complete since we have not considered the discrete symmetry group $H\subset\mathrm{SO}(3)$, of the solution. This means that rotating the configuration by an element $S\in\mathrm{SO}(3)$ has the same effect on the configuration as the isorotation $\Gamma(S)$. The isorotations need not be the same as the rotations, but they do form a three dimensional representation of H. Labeling $\{R,R'\}$ as the set of zero modes corresponding to rotations and isorotations, $\mathrm{SO}(3)\times\mathrm{SO}(3)$, we have the following identification:

$$\{R,R'\}\cong\{SR,\ R'\Gamma^{-1}(S)\},\quad S\in H. \tag{2.10}$$

Thus the moduli space is $(SO(3)\times SO(3))/H$ with the above quotient. But we really need to consider the covering space $SU(2)\times SU(2)$ because 2π rotations or isorotations can be noncontractible. If we view the moduli space as a quotient of $SU(2)\times SU(2)$ then each closed loop corresponding to $\{S,\ \Gamma(S)\}$ will correspond to four closed loops since both S and $\Gamma(S)$ can be lifted in two ways to SU(2). We now have the identifications

$$\{A,A'\}\cong\{hA,\ A'h'^{-1}\},\quad h\in\overline{H}, \tag{2.11}$$

where h and h' are in the fundamental representation of SU(2) and $D(h)=S$ and $D(h')=\Gamma(S)$. So $\pm h$ and $\pm h'$ are lifts to SU(2) of S and $\Gamma(S)$ respectively. The elements h form the double group $\overline{H}$ of H. Equation (2.11) includes Eq. (2.5) and determines the moduli space as $SU(2)\times SU(2)/K$ where K is the subgroup of $SU(2)\times SU(2)$ consisting of the elements $\{\pm h,\pm h'\}$. If the representation Γ of H lifts to a representation $\tilde{\Gamma}$ of $\overline{H}$ i.e. such that $D(\tilde{\Gamma}(h))=\Gamma(D(h))$ for $h\in\overline{H}$, with $\tilde{\Gamma}(h_1h_2)=\tilde{\Gamma}(h_1)\tilde{\Gamma}(h_2)$ and $\tilde{\Gamma}(-1)=-1$, then K has the form $\overline{H}\times\mathbb{Z}_2$, but it is not always possible that Γ can be lifted in this way.

Each element of the group K corresponds to one of the four ways of lifting a symmetry group element in H to the covering space. For each element of K it is necessary to determine whether this transformation is a contractible loop in the Skyrmion configuration space or not. When B is even this task is simpler since 2π rotations and isorotations are contractible so there is no need to distinguish a rotation from the same rotation plus a 2π rotation. But in the odd B case this distinction matters and so it is necessary to be more careful. We deal with this on a case by case basis in Sec. IV.

As discussed in the Introduction, to determine which quantization is appropriate here we need to consider the Finkelstein-Rubenstein (FR) constraints [5]. These authors showed that it is possible to quantize the solitons as fermions if one lifts the classical configuration space to its simply connected covering space. A quantization scheme which treats single Skyrmions as fermions is to multiply states by a phase $+1$ (-1) when acted on by operators which implement contractible (noncontractible) loops in the classical configuration space. They also showed that the exchange of two identical Skyrmions and the 2π rotation of one of the Skyrmions are homotopic loops thus proving that the usual notion of spin-statistics holds in the Skyrme model. Also, as a result of the fact that $\pi_4(SU(2))=\mathbb{Z}_2$ there are only two topologically distinct loops in the space. Williams [18] verified that the $B=1$ Skyrmion can be quantized as a fermion by showing that a 2π rotation of it is a noncontractible loop in the Skyrmion configuration space. This was extended in [19] whereby it was shown that the 2π rotation of a charge B Skyrmion is contractible if B is even and noncontractible is B is odd.

Thus the operator which corresponds to implementing a closed loop on the configuration space acts on states with eigenvalue ±1 according to the contractibility of the loop. In our case the closed loops always correspond to rotations or isorotations and the operators which generate such transformations are $\mathbf{L}_i$ and $\mathbf{K}_i$. If the symmetry group element is of the form

$$\{h,\ h'\}=\left\{e^{-\frac{i\theta_1}{2}\hat{\mathbf{n}}_1\cdot\boldsymbol{\tau}},\quad e^{-\frac{i\theta_2}{2}\hat{\mathbf{n}}_2\cdot\boldsymbol{\tau}}\right\}, \tag{2.12}$$

then using Eqs. (2.9) and (2.11) the constraints on the quantum states arising from the symmetries of the classical solution may be expressed as

$$e^{i\theta_1\hat{\mathbf{n}}_1\cdot\mathbf{L}}e^{i\theta_2\hat{\mathbf{n}}_2\cdot\mathbf{K}}|\Psi\rangle=\pm|\Psi\rangle, \tag{2.13}$$

the $\pm$ depending on whether the loop corresponding to $\{h,\ h'\}$ is contractible or not in the full configuration space. A 2π rotation or isorotation of a Skyrmion of nucleon number B is contractible if B is even and noncontractible if B is odd. So, physical states $|\Psi\rangle$ also satisfy

$$e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{K}}|\Psi\rangle=e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{L}}|\Psi\rangle=(-1)^B|\Psi\rangle. \tag{2.14}$$

This means that for even B, I and J are integral and for odd B, I and J are half integral.

Returning to the Lagrangian in Eq. (2.6), in general U_{ij}, V_{ij}, W_{ij} are diagonal. The number of different eigenvalues of U_{ij}, V_{ij}, W_{ij} depends on the symmetry of the solution. Tetrahedral, octahedral or icosahedral symmetry implies the matrices have a single eigenvalue if the fields transform according to a three dimensional irreducible representation of the group. For instance, the $B=4$ solution has octahedral symmetry whereby a rotation by an element of the octahedral group combined with an isorotation leaves the solution invariant. The rotations form the defining representation of the octahedral group and so V_{ij} is proportional to the identity matrix with one common moment of inertia. But the corresponding isorotations are in a reducible representation, comprising irreducible representations of dimensions one and two and this means that U_{ij} has two distinct eigenvalues.[1] It also turns out that the cross term W_{ij} vanishes because the symmetry is realized differently between the rotations and isorotations.

If the matrices have only one eigenvalue the Hamiltonian is that of a spherical top [20] (in space and isospace). If the Skyrmion has an axis of symmetry above the second order (for $B=6$ and $B=8$) then U_{ij}, V_{ij}, W_{ij} have two distinct eigenvalues and the Hamiltonian is that of a symmetrical top (in such cases we take the axis of symmetry to be the x_3 axis), otherwise (for $B=5$) U_{ij}, V_{ij}, W_{ij} has three different eigenvalues and the Hamiltonian is that of an asymmetrical top. A basis for the Hilbert space of states is given by $|J,J_3,L_3\rangle\otimes|I,I_3,K_3\rangle$, with $-J\leqslant J_3$, $L_3\leqslant J$ and $-I\leqslant I_3$, $K_3\leqslant I$. In all that follows, the third component of the space and isospace angular momentum J_3 and I_3 are omitted. The value of J_3 corresponds to the angular momentum eigenvalue of the state about a fixed axis in space and is not physically relevant. States with differing values of I_3 correspond to the different states in an isospin multiplet, e.g. I_3

[1] I thank K. Baskerville for pointing this out to me.

$=2$ means the state has two more protons than neutrons etc. These states will be energy eigenstates in all cases except possibly for $B=5$, where the energy eigenstates will not generally have definite values of K_3, L_3.

It is an easy numerical task to calculate the moments of inertia from the rational map generated Skyrmions. As described in the next section the Skyrme field is approximated by $U[r,\theta,\phi]=\exp(if(r)\hat{\mathbf{n}}_R\cdot\boldsymbol{\tau})$ where $\hat{\mathbf{n}}_R$ is derived from a rational function of $z=\tan(\theta/2)e^{i\phi}$, where r, θ and ϕ are polar coordinates and $f(r)$ is determined numerically. Inserting this into Eq. (2.7) the moments of inertia are obtained by radial and angular integrations. It is found that the rotational moments of inertia (V_{ij}) become much larger than the isorotational moments of inertia (U_{ij}) as B increases. For example, for $B=1$ the moment of inertia is $U_{ij}=V_{ij}=106.4\delta_{ij}$ in units of $1/e^3 f_\pi$, the rotational and isorotational moments of inertia being equal due to spherical symmetry. But already at $B=4$ we have $U_{11}=U_{22}=254.0$, $U_{33}=306.4$ and $V_{11}=V_{22}=V_{33}=1162.9$. U_{ij} and W_{ij} increase approximately like B while V_{ij} increases like B^2 (of course in certain cases some symmetry can imply that some moments of inertia are zero). The energies of rotational states are like $\frac{1}{2}J(J+1)/V$, and isorotational states are like $\frac{1}{2}I(I+1)/U$ where V and U indicate the rotational and isorotational moments of inertia and J and I indicate the spin and isospin eigenvalues. We see that states with the lowest energy will always have I as small as possible (there is a contribution from the W moments of inertia but since these are of order U it does not change the outcome). So states with high isospin are energetically unfavorable and will not exist, this is true of real nuclei whose nucleon number is small. As the nucleon number increases, electromagnetic effects will favor neutrons over protons but for all small nuclei ($B\leqslant 30$) the ground state has the smallest possible value of isospin. So to find the lowest energy states we will always set the isospin to its lowest possible value.

To obtain the correct quantum states we need to determine the (non)contractibility of the closed loops, corresponding to elements of K, in the configuration space. To do this we use the rational map description of Skyrmions which we now review.

III. RATIONAL MAP GENERATED SKYRMIONS

To describe the symmetries of the Skyrmions, and thus evaluate the FR constraints, we shall use the rational map ansatz for Skyrmions which was introduced in [2]. Jarvis has shown that there is a 1-1 correspondence between SU(2) monopoles of charge k and holomorphic rational maps from S^2 to S^2 of degree k [9]. The rational map may be written as $F(z)=p(z)/q(z)$, $p(z)$ and $q(z)$ are degree k polynomials in z where k is the monopole charge and z is a complex coordinate on the two sphere which can be written in terms of usual polar coordinates as $z=\tan(\theta/2)e^{i\phi}$. The point z corresponds to the unit vector

$$\hat{\mathbf{n}}_z=\frac{1}{1+|z|^2}[2\mathrm{Re}(z),2\mathrm{Im}(z),1-|z|^2]. \tag{3.1}$$

The value of the rational map corresponds to the unit vector

$$\hat{\mathbf{n}}_R=\frac{1}{1+|R|^2}[2\mathrm{Re}(R),2\mathrm{Im}(R),1-|R|^2]. \tag{3.2}$$

Skyrmions are given by maps from $\mathbb{R}^3$ to S^3. The idea in [2] is to identify the domain S^2 of the rational map with concentric spheres in $\mathbb{R}^3$, and the target of the rational map S^2 with spheres of latitude in S^3. A point in $\mathbb{R}^3$ can parametrized by (r,z); r denotes radial distance and z specifies the direction. The ansatz for the Skyrme field may then be written as

$$U[r,z]=\exp(if(r)\hat{\mathbf{n}}_R\cdot\boldsymbol{\tau}) \tag{3.3}$$

where $f(r)$ is a radial function satisfying $f(0)=\pi$, $f(\infty)=0$. $f(r)$ is determined numerically to give the closest approximation to the actual Skyrme configuration. In [2] this ansatz was used to accurately approximate the known minimal energy Skyrmion solutions for $B=1$ to $B=9$ and the conjectured buckyball solution of charge 17 was shown to exist.

The Jarvis rational maps have a natural action of SO(3) given by SU(2) Möbius transforms on the complex coordinate z,

$$z\rightarrow\frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}},\quad \begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\in \mathrm{SU}(2). \tag{3.4}$$

This corresponds to a rotation of $\hat{\mathbf{n}}_z$ and generates space rotations of the Skyrme field. For example, $z\rightarrow e^{i\theta}z$ is an anticlockwise rotation by θ about the x_3 axis, and $z\rightarrow 1/z$ is a π rotation about the x_1-axis. An SU(2) Möbius transform on the target S^2 of the rational map corresponds to a rotation of $\hat{\mathbf{n}}_R$, and thus to an global isospin rotation of the Skyrme field. A rational map $F(z)$ is symmetric under a subgroup H of SO(3) if there is a set of pairs $\{h,h'\}$ such that $F(z)$ satisfies

$$F(hz)=h'F(z), \tag{3.5}$$

h and h' are SU(2) matrices with $D(h)\in H$ and h' acts on F in the same manner as Eq. (3.4). Note that $-h$ has the same action as h in Eq. (3.5) and similarly for h' so these are SO(3) actions. $D(h')$ forms a representation, Γ, of H. Here we are concerned only with what the symmetry group, H, and the representation Γ are, we are not at this point discussing the contractibility of any loops. So the double cover of H does not enter here. Given a rational map it is easy to determine its symmetries H and the representation Γ. The rational map ansatz accurately models the known minimal energy Skyrmion configurations and clearly shows how the symmetry of the Skyrmion is realized, i.e. what combination of rotations and isorotations leave the solution invariant.

As explained in the Introduction the rational map approach is also useful in determining the FR constraints in cases where the Skyrme configuration needs to be split up into well separated configurations. We will assume that whenever a monopole configuration can be split up, respecting some symmetry then the same can be done for Skyrmi-

ons. Generally we begin with the minimal energy polyhedral shaped solution and end with some configuration of well separated Skyrmions both having some discrete symmetry. From the correspondence between monopole and Skyrmion vibrations [6] we can see that the Skyrmion can be vibrated keeping the relevant symmetry group element. The configurations are now separated maintaining invariance of the relevant symmetry group element until they are far apart. What is important is how the relative isospin orientation of the final configurations are aligned. This is determined by the initial vibration. In the cases we consider the vibration corresponds to a monopole motion and thus the vibration is of low frequency, so the Skyrmions separate in an attractive channel. Thus the asymptotic isospin orientations are aligned to give an attractive configuration. This will be unambiguous in the cases we consider.

To determine whether a monopole configuration can be separated while keeping a certain symmetry, again, it is easiest to use the rational map description of monopoles. The previously described Jarvis rational maps are suited to the description of monopoles which are symmetric under some subgroup of SO(3). But there is no natural action on these rational maps which corresponds to translations of the monopoles in space. There is an equivalent rational map description of monopoles due to Donaldson which allows one to see how the monopole configuration can be separated.

In [9], the Jarvis rational map is defined by considering solutions to the scattering equation for monopoles

$$(D_i - i\Phi)v = 0 \tag{3.6}$$

along all radial lines through some point in $\mathbb{R}^3$. D_i is the covariant derivative, Φ is the Higgs field and v is a complex doublet in the fundamental representation of SU(2). The rational maps have a natural action of SO(3) given by Eq. (3.4) but not a simple action of translations, since the choice of a point in $\mathbb{R}^3$ used to define the map breaks translational symmetry. There also exists the Donaldson rational map which is defined by solutions to the scattering equation (3.6) along all lines in $\mathbb{R}^3$ that point in a particular direction [10] (which we take here to be the x_3 axis). Donaldson rational maps for charge k monopoles are defined as based maps from $\mathbb{C} \to \mathbb{C} \cup \infty$, i.e. $F(w) = p(w)/q(w)$, $w \in \mathbb{C}$ with $q(w)$ a monic polynomial of degree k and $p(w)$ a polynomial of degree less than k with no factors in common with $q(w)$. Here w represents a point in the (x_1, x_2) plane. The choice of such a direction breaks rotational symmetry and in general there is no simple action which generates rotations on the rational maps. But rotations about the preferred axis used to define the map have a simple action since they preserve this axis. This will be enough for our purposes. Also, it is easy to see how translations act on the monopole. A rotation of angle θ about the preferred axis (x_3) and a translation in space, (v_1, v_2, x), acts on the map as follows:

$$F(w) \to e^{2x} e^{-2ik\theta} F(e^{-i\theta}(w - v)) \tag{3.7}$$

where $v = v_1 + iv_2$ and k is the monopole charge. The Jarvis rational maps are obviously suited to the construction of monopoles and Skyrmions which are symmetric under some subgroup of SO(3). But to see how monopoles can be separated in space the Donaldson maps are better suited. The two approaches are completely equivalent as descriptions of the monopole moduli space [we will use the notation $F(z)$ for Jarvis maps and $F(w)$ for Donaldson maps].

In the next section the FR constraints will be worked out using these methods. Once this is done it is easy to find the allowed states using Eqs. (2.13) and (2.14). This determines the spin and isospin of the states. To determine the parities of these states it is necessary to know how the classical solution behaves under P. For Jarvis rational maps, inversion corresponds to $z \to -1/\bar{z}$ ($\bar{z}$ denotes complex conjugate of z). If the rational map has a reflection symmetry, then on the restricted configuration space of rotations and isorotations P may be represented by some combination of body fixed rotations and isorotations. So P may be represented by some body fixed operator which can easily be evaluated on the angular momentum eigenstates to give the parity eigenvalue.

IV. THE $B=4$ TO $B=9$ AND $B=17$ GROUND STATES

The minimal energy $B=4$ solution has octahedral symmetry [4]. The octahedral group, O_h, is generated by three elements, a $2\pi/3$ rotation cyclically permutating the Cartesian axes and a $\pi/2$ rotation about the x_3 axis, and also the inversion element. In [21], using the instanton ansatz, it was determined how the octahedral symmetry is realized so we do not need to use the rational map in this case (the rational map approach gives the same result). The SU(2) Skyrme field may be written as $U_4[\mathbf{x}] = \sigma + i\pi^i \tau^i$ and the cubic symmetry is realized as follows:

$$\begin{aligned} C_4: &\quad (\pi^1, \pi^2, \pi^3)(-y, x, z) = (-\pi^2, -\pi^1, -\pi^3)(x, y, z) \\ C_3: &\quad (\pi^1, \pi^2, \pi^3)(y, z, x) = (\pi^2, \pi^3, \pi^1)(x, y, z) \\ \text{Inv}: &\quad (\pi^1, \pi^2, \pi^3)(-x, -y, -z) = (\tilde{\pi}^1, \tilde{\pi}^2, \tilde{\pi}^3)(x, y, z) \end{aligned} \tag{4.1}$$

where $\tilde{\pi}^1 = \frac{1}{3}(\pi^1 - 2\pi^2 - 2\pi^3)$ and cyclically permutating. Next, we need to work out the FR constraints associated with the C_3 and C_4 elements (the inversion element cannot be represented as a closed loop in the configuration space and there is no FR constraint associated with it). Since we are in the even nucleon number sector a 2π rotation in space or in isospace is a closed contractible loop and is associated with a phase of $(+1)$.

The FR constraint for the C_3 element is easy to determine. The C_3 element implies that a $2\pi/3$ rotation combined with a $-2\pi/3$ isorotation leaves $U_4[\mathbf{x}]$ invariant. Simply repeat the action three times to get a 2π rotation and a -2π isorotation. 2π rotations or isorotations are contractible. Since the C_3 action repeated three times is contractible this implies the C_3 element itself must be contractible so all permissible states must be eigenstates of the C_3 operator with eigenvalue $(+1)$. In the orientation of the Skyrme field given above the contractibility of the C_4 element is not obvious. It is helpful to do a global isorotation which makes this more transparent.

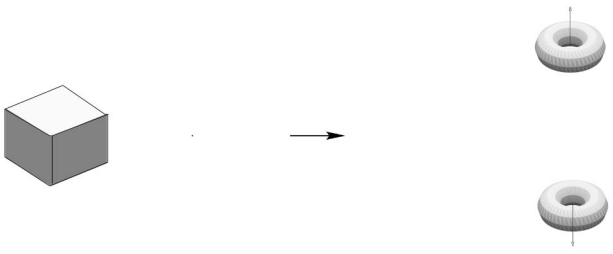

FIG. 1. $B=4$ Skyrmion separating to two $B=2$ Skyrmions.

If $U[\mathbf{x}]=h'U[D(h)\mathbf{x}]h'^{\dagger}$ then a global isospin transformed field $\tilde{U}[\mathbf{x}]=AU[\mathbf{x}]A^{\dagger}$ satisfies $\tilde{U}[\mathbf{x}]=\tilde{h}'\tilde{U}[D(h)\mathbf{x}]\tilde{h}'^{\dagger}$with $\tilde{h}'=Ah'A^{\dagger}$. In the orientation of Eq. (4.1) the $\pi/2$ rotation in space about the x_3 axis is accompanied by a π rotation in isospace about the $(x_1\text{-}x_2)$ axis. We choose A so that for the C_4 element above, the $\pi/2$ rotation in space about the x_3 axis is accompanied by a π rotation in isospace about the x_3 axis, [as an SO(3) rotation $D(A)$ maps the $(x_1\text{-}x_2)$ axis to the x_3 axis]. By a simple homotopy argument it is clear that a constant isorotation at every point on the closed loop will not affect its (non)contractibility. To show the contractibility of the C_4 loop we can continuously deform the loop into one in which is obviously contractible. Since the contractibility of a loop is invariant under homotopy, this will show that the original loop is contractible. The charge four cube can be deformed into two well separated charge two doughnuts along the x_3 axis. It is known from the vibrational spectra of the $B=4$ Skyrmion [6] that it is possible to do this while keeping the C_4 symmetry. The dipole moments of the two $B=2$ doughnuts will point in opposite directions so they attract. This may be seen schematically in Fig. 1 (for accurate pictures of the $B=4$ to $B=9$ and $B=17$ solutions see [1] or [2]).

A similar type of scattering process also occurs for monopoles and the C_4 symmetry is respected at all separations of the two 2-monopole clusters [22]. The two doughnuts are positioned at $(0,0,s)$ and $(0,0,-s)$ with $s\to\infty$, and are denoted $M1$ and $M2$ respectively. The field may be expressed as

$$U_4[\mathbf{x}]=U_2[\mathbf{x}-s\mathbf{e}_3]AU_2[\mathbf{x}+s\mathbf{e}_3]A^{\dagger} \tag{4.2}$$

with $A=i(\cos\phi\tau_1+\sin\phi\tau_2)$ for some $\phi\leqslant 2\pi$. $U_2[\mathbf{x}]$ is the axially symmetric (about x_3) charge two solution and $\mathbf{e}_i$ is a unit vector along the i axis in space. The form of A implies that the dipole moments of $M1$ and $M2$ are in opposite directions. The C_4 symmetry implies that a simultaneous $\pi/2$ rotation about the x_3 axis with a π isorotation about the x_3 axis leaves the configuration unchanged, i.e.

$$U_4[\mathbf{x}]\to e^{2i\lambda(\tau_3/2)}U_4[D(e^{-i\lambda(\tau_3/2)})\mathbf{x}]e^{-2i\lambda(\tau_3/2)} \tag{4.3}$$

with $0\leqslant\lambda\leqslant\pi/2$. Because it is axially symmetric $U_2[\mathbf{x}]$ satisfies [11]

$$U_2[D(e^{-i\lambda(\tau_3/2)})\mathbf{x}]=e^{-2i\lambda(\tau_3/2)}U_2[\mathbf{x}]e^{2i\lambda(\tau_3/2)} \tag{4.4}$$

for all values of λ. Thus the effect of the C_4 transformation is a 2π isorotation about the x_3 axis on $M2$ while leaving $M1$ unchanged

$$U_4[\mathbf{x}]\to U_2[\mathbf{x}-s\mathbf{e}_3]e^{4i\lambda(\tau_3/2)}AU_2[\mathbf{x}+s\mathbf{e}_3]A^{\dagger}e^{-4i\lambda(\tau_3/2)} \tag{4.5}$$

using $Ae^{i\lambda(\tau_3/2)}=e^{-i\lambda(\tau_3/2)}A$. Since the $B=2$ doughnut is a boson a 2π isorotation is contractible and thus the C_4 action on the cube is a contractible loop and so a $(+1)$ phase is associated to the operator representing the C_4 element. For the above argument to work it is crucial that the C_4 symmetry is respected at all times as the configuration is separated.

So the allowed states, $|\Psi\rangle$ are of the form $|J,L_3\rangle\otimes|I,K_3\rangle$, with the constraints

$$e^{(2\pi i/3\sqrt{3})(L_1+L_2+L_3)}e^{(2\pi i/3\sqrt{3})(K_1+K_2+K_3)}|\Psi\rangle=|\Psi\rangle$$
$$e^{i(\pi/2)L_3}e^{i(\pi/\sqrt{2})(K_1-K_2)}|\Psi\rangle=|\Psi\rangle \tag{4.6}$$

reverting to the generators used in Eq. (4.1). To find the allowed states is just a matter of finding simultaneous eigenvalues of the operators in Eq. (4.6). The ground state is given by $|\Psi\rangle=|0,0\rangle\otimes|0,0\rangle$; the first excited state with $I=0$ has $J=4$ and is

$$|\Psi\rangle=\left(|4,4\rangle+\sqrt{\frac{14}{5}}|4,0\rangle+|4,-4\rangle\right)\otimes|0,0\rangle. \tag{4.7}$$

If $I=1$, the lowest state has $J=2$ and is given by

$$|\Psi\rangle=\sqrt{6}|2,0\rangle\otimes\{(i-1)|1,-1\rangle+(i+1)|1,1\rangle\}+\{|2,2\rangle$$
$$+|2,-2\rangle\}\otimes\{2\sqrt{2}|1,0\rangle+(1-i)|1,1\rangle-(1+i)|1,-1\rangle\}. \tag{4.8}$$

To compute the parities of these states we know that from the Inv transform $U^{\dagger}(-\mathbf{x})=WU(\mathbf{x})W^{\dagger}$, where $W=e^{(i\pi/\sqrt{3})(K_1+K_2+K_3)}$. The parity operator P is defined as P: $U(\mathbf{x})\to U^{\dagger}(-\mathbf{x})$. So, on the configuration space of zero modes the parity operator P can be represented by $e^{(i\pi/\sqrt{3})(K_1+K_2+K_3)}$. We may act with P on the physical states to determine their parity. The $I=J=0$ and $I=0$, $J=4$ states both have $(+1)$ parity, and the $I=1$, $J=2$ state has (-1) parity. Thus we find that the ground state for $B=4$ has spin and isospin zero and positive parity in agreement with the ground state ${}^4_2\mathrm{He}^+$. The negative parity state with $I=1$, $J=2$ is observed as the lowest isospin triplet state $({}^4_1\mathrm{H}^-$, ${}^4_2\mathrm{He}^-$, ${}^4_3\mathrm{Li}^-)$ [13]. From nuclear tables there are a large number of states with $I=0$ that have energies less than the $J=4$ state. Our scheme for quantization is obviously very restrictive, the configuration is not allowed to vibrate in any fashion. Including the vibrational modes and allowing the Skyrmions to separate accounts for some of the missing states. This we will do in Sec. V.

The $B=4$ case has been previously considered by Walhout [23]. There, a perturbative analysis was considered about the full one parameter family of tetrahedally symmetric Skyrmions. The FR constraints corresponding to the ele-

ments of the tetrahedal group and the 2π rotations and isorotations of the individual Skyrmions were implemented and the ground state was found to have $I=0$, $J=0$ agreeing with the present, more restrictive, analysis. We included this case as it appears that the question of the (non)contractibility of the C_4 symmetry element of the cubic configuration has not been previously considered (in Walhout's case this transformation is not a symmetry).

$B=6$

The minimal energy $B=6$ Skyrmion has D_{4d} symmetry. It can be described in terms of a Jarvis rational map given by [2]

$$F(z)=\frac{z^4+a}{z^2(az^4+1)}, \quad a=0.16i. \tag{4.9}$$

The D_4 subgroup is generated by two elements, a π rotation about the x_1 axis and a π rotation about the (x_1+x_2) axis (combining these two elements gives a C_4 rotation about the x_3 axis). The elements act on the rational map by $F(1/z)=1/F(z)$ and $F(-i/z)=-1/F(z)$, i.e. a π rotation in space about the x_1 axis combined with a π isorotation about the x_1 axis leaves the solution invariant; and a π rotation about the (x_1+x_2) axis combined with a π isorotation about the x_2 axis leaves the solution invariant. A closed loop corresponding to the first symmetry group element is

$$U_6[\mathbf{x}]\to e^{i\lambda(\tau_1/2)}U_6[D(e^{-i\lambda(\tau_1/2)})\mathbf{x}]e^{-i\lambda(\tau_1/2)} \tag{4.10}$$

with $0\leqslant\lambda\leqslant\pi$.

To determine how the FR constraints act we need to know whether the closed loops generated by the C_2 elements are contractible or not. To see that the loop in Eq. (4.10) is noncontractible is not obvious by looking at the polyhedral solution. It is helpful to continuously deform the minimal energy solution into three well separated charge two doughnuts, one at the origin and the other two equidistant along the x_3 axis with their separation $2s$ very large. We now show that it is possible to do this for monopoles keeping the C_2 symmetry about the x_1 axis at all times, therefore by our earlier assumption the same can be done for Skyrmions. It is easiest to see this using Donaldson rational maps with x_3 as the preferred direction. Rotations about the x_3 axis have a simple action on the rational map, given by Eq. (3.7). Also, reflections can be defined on the maps [22], so a π rotation about the x_1 axis can be defined by combining a reflection in the (x_1,x_3) plane and a reflection in the (x_1,x_2) plane. A rational map of degree k, $F(w)=p(w)/q(w)$ has π rotational symmetry about the x_1 axis if

$$\frac{p(w)}{q(w)}=\frac{\overline{I(p(\overline{w}))}}{\overline{q(\overline{w})}}. \tag{4.11}$$

Here $I(p)$ is the unique polynomial of degree less than k that satisfies $I(p)p=1 \mod q$. Since we are determining the contractibility of the C_2 rotation about the x_1 axis we only need the configuration to remain invariant under this C_2 element. But in fact we can separate the configuration keeping all of the D_4 symmetry and it is convenient if we do this. The most general charge six monopole with D_{4d} symmetry is given by the Donaldson map

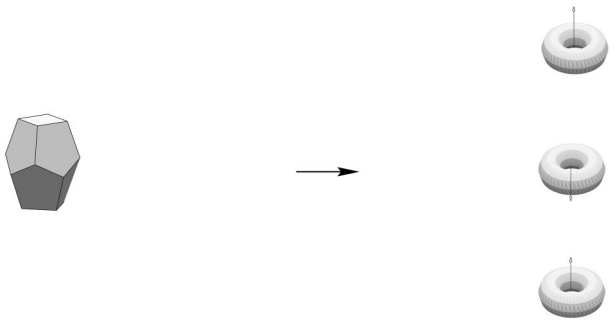

FIG. 2. $B=6$ Skyrmion separating to three $B=2$ Skyrmions.

$$F(w)=\frac{itw^4+1}{w^6}, \quad t\in\mathbb{R}. \tag{4.12}$$

Some value of t corresponds to the minimal energy Skyrmion. Now let $t=e^{2s}\to\infty$; $F(w)$ is given by $ie^{2s}/w^2+1/w^6$. Using the formula given in [24], this corresponds to three charge two monopoles lying on the x_3 axis, one at the origin and the other two at $(0,0\pm s)$. The charge two monopoles must approach axially symmetric monopoles as $s\to\infty$ since the overall configuration has C_4 symmetry about the x_3 axis. By our previous arguments we assume that the $B=6$ Skyrmion can be split up in the manner keeping D_4 symmetry. This is shown schematically in Fig. 2.

The dipole moments of the $B=2$ Skyrmion at $(0,0,s)$ and $(0,0,-s)$ point in the same direction and opposite to that of the $B=2$ Skyrmion at the origin, so the configuration is attracting. If $U_6[\mathbf{x}]$ is of the following form it will be D_4 symmetric as $s\to\infty$:

$$U_6[\mathbf{x}]=U_2[\mathbf{x}-s\mathbf{e}_3]\tau_1U_2[\mathbf{x}]\tau_1U_2[\mathbf{x}+s\mathbf{e}_3]. \tag{4.13}$$

Again $U_2[\mathbf{x}]$ is axially symmetric about the x_3 axis. Acting with the C_2 element has the effect of rotating and isorotating each of the charge two doughnuts about an axis in the plane of the doughnuts and also exchanging the Skyrmion at $(0,0,s)$ with the one at $(0,0,-s)$, this is

$$U_6[\mathbf{x}]\to U_2^a[\mathbf{x}]U_2^b[\mathbf{x}]U_2^c[\mathbf{x}] \tag{4.14}$$

where

$$\begin{aligned}
U_2^a[\mathbf{x}]&=e^{i\lambda(\tau_1/2)}U_2[D(e^{-i\lambda(\tau_1/2)})(\mathbf{x}-\mathbf{s}(\lambda))]e^{-i\lambda(\tau_1/2)}\\
U_2^b[\mathbf{x}]&=e^{i\lambda(\tau_1/2)}\tau_1U_2[D(e^{-i\lambda(\tau_1/2)})\mathbf{x}]\tau_1e^{-i\lambda(\tau_1/2)}\\
U_2^c[\mathbf{x}]&=e^{i\lambda(\tau_1/2)}U_2[D(e^{-i\lambda(\tau_1/2)})(\mathbf{x}+\mathbf{s}(\lambda))]e^{-i\lambda(\tau_1/2)}
\end{aligned} \tag{4.15}$$

$\mathbf{s}(\lambda)=sD(e^{i\lambda(\tau_1/2)})\mathbf{e}_3$, and $0\leqslant\lambda\leqslant\pi$. The interchange of two identical doughnuts is contractible since they are bosons but rotating and isorotating each of the doughnuts about an axis

in their plane is a noncontractible loop [11] and thus doing it for three doughnuts the total loop must be noncontractible. As mentioned earlier, it is the noncontractibility of the above C_2 element for the charge two torus which ensures the ground state obtained by zero mode quantizing the $B=2$ solution gives the correct quantum numbers of the deuteron i.e. $I=0$, $J=1$ [11]. If the loop was contractible then the ground state obtained by zero mode quantization would have $I=J=0$.

The other C_2 loop may be treated in a similar manner to see that it is also noncontractible. It is easiest to transform the field by a global isorotation so that the rotation and isorotation act about the same axis, then the analysis is identical to that above. We thus find

$$e^{i\pi(L_1+K_1)}|\Psi\rangle=-|\Psi\rangle$$

$$e^{i\pi/\sqrt{2}(L_1+L_2)}e^{i\pi K_2}|\Psi\rangle=-|\Psi\rangle \tag{4.16}$$

This gives the ground state as $|1,0\rangle\otimes|0,0\rangle$. The first excited state with $I=0$ is $|3,0\rangle\otimes|0,0\rangle$. The lowest state with $I=1$ is given by $|0,0\rangle\otimes|1,0\rangle$. To determine the parity of the states we use the reflection symmetry of the rational map $-iF(\sqrt{i}z)=\overline{F(\bar{z})}$. This implies that on the zero modes the parity operator can be represented as $P=e^{-i(\pi/2)K_3}e^{i(\pi/4)L_3}$. However there is an ambiguity here since the parity operator can also be represented by the above operator times any element of D_4, since this has the same effect on the classical solution, i.e. we could also write P as $e^{-i(\pi/2)K_3}e^{i(\pi/4)L_3}e^{i\pi(L_1+K_1)}$. But the C_2 elements of D_4 in the (x_1,x_2) plane are noncontractible so the operators corresponding to them act on the states with eigenvalue (-1). So different choices of P can give different results. The above two choices of P give opposite parity eigenvalues for all states. We see no theoretical reason to choose one above the other. The three states found above have the correct spins of the corresponding ground and first excited states of $^6_3$Li and the ground state of the isospin triplet ($^6_2$He, $^6_3$Li, $^6_4$Be). If we choose P as $e^{-i(\pi/2)K_3}e^{i(\pi/4)L_3}$ then this gives the three states each having positive parity in agreement with experiment. So we can choose P so as to give the correct parities of the states but theoretically there is an ambiguity in its definition.

A similar problem happens in the odd B case. The $B=1$ Skyrmion is spherically symmetric so P can be represented as the identity operator, or alternatively, as a 2π rotation. Since B is odd the two choices differ on the quantum states. Using the convention that the nucleon have positive parity, for $B=1$ we can take P to be the identity operator. For all odd B, 2π rotations are noncontractible so again there are two choices of P acting on the states, P_0 and $e^{2\pi i\hat{\mathbf{n}}\cdot\mathbf{L}}P_0$ where P_0 is the operator which corresponds classically to inversion. As in the $B=6$ situation we see no way of deciding which choice is correct. Thus, in these cases we will make no prediction for the parities of the states. This ambiguity may be cured by lifting to the full configuration space. This space is doubly connected for all B and in the quantum theory states are defined on this double cover. We need to lift the operator $P:U[\mathbf{x}]\rightarrow U^\dagger[-\mathbf{x}]$ to the double cover. For $B=1$ one chooses a lift of P to the double cover of the $B=1$ configuration space and this should determine how P should be lifted for all other B. But it is not obvious to us how to do this is practice. The role of parity in the Skyrme model has also been discussed recently in [25].

To summarize, for $B=6$ the states found are in agreement with the lowest energy states for nucleon number six, modulo our assumption about the parity. The ground state has spin 1 and positive parity, $^6_3$Li$^+$. The first excited state has spin 3 and positive parity. The lowest state $I=1$ triplet ($^6_2$He$^+$, $^6_3$Li$^+$, $^6_4$Be$^+$), is observed to have spin 0 and positive parity in agreement with that found above.

$B=8$

The $B=8$ case is similar to the $B=6$ case treated above. The minimal energy $B=8$ Skyrmion has D_{6d} symmetry. It can be described in terms of a Jarvis rational map given by [2]

$$F(z)=\frac{(z^6-ia)}{z^2(iaz^6-1)},\quad a=0.14. \tag{4.17}$$

The D_6 subgroup is generated by two elements, a C_2 rotation about the x_1 axis and a C_6 rotation about the x_3 axes. These act as $F(1/z)=1/F(z)$ and $F(e^{i\pi/3}z)=e^{-2\pi i/3}F(z)$. This means that a π rotation in space about the x_1 axis combined with a π isorotation about the x_1 axis leaves the classical solution invariant; and a $\pi/3$ rotation about the x_3 axis combined with a $2\pi/3$ isorotation about the x_3 axis leaves the solution invariant. Again, for the C_2 loop it is necessary to continuously deform the minimal energy solution into three well separated charge two doughnuts, one of charge four at the origin and one each of charge two equidistant along the x_3 axis with their separation $2s$ very large. Then, the charge four doughnut at the origin can be separated into two charge two doughnuts along the x_1 axis. This process can be seen to occur for monopoles in the following way.

The most general charge eight monopole with D_{6d} symmetry is given by the Donaldson map

$$F(w)=\frac{itw^6+1}{w^8},\quad t\in\mathbb{R}. \tag{4.18}$$

Again, some value of t corresponds to the minimal energy Skyrmion. Let $t=e^{2s}\rightarrow\infty$; the formula given in [24] implies that this corresponds to two charge two monopoles lying on the x_3 axis at $(0,0\pm s)$, and a charge four monopole at the origin. The monopoles must approach axially symmetric monopoles as $t\rightarrow\infty$ since the overall configuration has C_6 symmetry about the x_3 axis. Next, the charge four torus can be separated into two charge two doughnuts well separated along the x_1 axis keeping the C_2 symmetry about the x_1 axis. The charge four doughnut has a Donaldson rational map $F(w)=1/w^4$. This can be deformed to $F(w)=1/(w^2-v^2)^2$ for $v\in\mathbb{R}$ with C_2 symmetry about the x_1 axis preserved, cf. Eq. (4.11). As $v\rightarrow\infty$ this becomes two charge two doughnuts

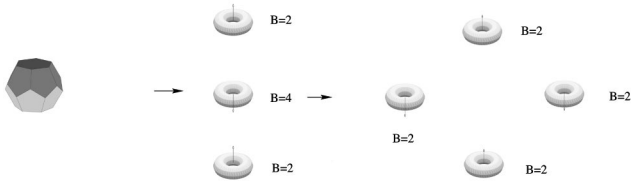

FIG. 3. $B=8$ Skyrmion separating to four $B=2$ Skyrmions.

separated along the x_1-axis. Again, a similar process is possible for the charge eight Skyrmions. This is indicated in Fig. 3.

The dipole moments of the Skyrmions at $(0,0,s)$ and $(0,0,-s)$ point in the same direction and opposite to that of the charge four Skyrmion at the origin, so that the configuration is attracting. Acting with the symmetry group element, which is a π rotation and isorotation about the x_1 axis, has the effect of rotating and isorotating each of the four charge 2 doughnuts about an axis in the plane of the doughnuts and also exchanging the Skyrmion at $(0,\ 0,\ s)$ with the one at $(0,\ 0,\ -s)$. The interchange of two doughnuts is contractible since they are bosons; rotating and isorotating each of the doughnuts about an axis in their plane is a noncontractible loop, so doing it for four doughnuts the total loop is contractible. The C_6 element can be written as a product of the above C_2 element with a C_2 element in the $(x_1,\ x_2)$ plane at an angle $\pi/6$ to the x_1 axis. This C_2 loop may be seen to be contractible in a similar manner to that above. So, physical states must satisfy

$$e^{i\pi(L_1+K_1)}|\Psi\rangle=|\Psi\rangle$$

$$e^{i\pi/3(L_3-2K_3)}|\Psi\rangle=|\Psi\rangle. \tag{4.19}$$

This gives the ground state as $|0,0\rangle\otimes|0,0\rangle$. The first excited state is given by $|2,0\rangle\otimes|0,0\rangle$. Their parity may be determined from the reflection symmetry of the rational map $e^{(4\pi i/3)}F(e^{(i\pi/6)}z)=\overline{F(\bar{z})}$. This implies that on the zero modes the parity operator can be represented as $P=e^{(i\pi/6)(L_3-2K_3)}$. There is no parity ambiguity here since B is even and all the FR constraints are $+1$. Thus, both states have positive parity. Again, this is in agreement with the spin 0 positive parity ground state of Beryllium 8, ${}^{8}_{4}\mathrm{Be}^{+}$, and the first excited state has spin 2 and positive parity [13].

$B=5$

The minimal energy $B=5$ Skyrmion has D_{2d} symmetry. It can be described in terms of a Jarvis rational map given by

$$F(z)=\frac{z(z^4-ibz^2-a)}{az^4+ibz^2-1},\quad a=3.07,\ b=3.94. \tag{4.20}$$

The rational map has the symmetries $F(-z)=-F(z)$ and $F(1/z)=1/F(z)$ (for all a,b). This is a simultaneous rotation and isorotation by π about the x_3 axis and a simultaneous rotation and isorotation by π about the x_1 axis.

The spin and isospin of the states must be half-integral since the nucleon number is odd. In the odd nucleon sector it is necessary to be careful when considering the (non)contractibility of the closed loops since 2π rotations are noncontractible. For instance if a configuration is invariant under a π rotation about some axis then the clockwise rotation has a different FR constraint to the anticlockwise rotation since they differ by a 2π rotation.

To determine the (non)contractibility of the closed loops first deform $F(z)$ until $b=0$. $F(z)$ now has D_4 symmetry including a C_4 rotation and isorotation about the x_3 axis. Explicitly, $F(z)=-iF(iz)$. Or the path

$$z\to e^{i\lambda}z,\quad F\to e^{-i\lambda}F,\quad 0\leqslant\lambda\leqslant\pi/2 \tag{4.21}$$

is a closed loop on the configuration corresponding to $b=0$. This path corresponds to an anti-clockwise rotation by $\pi/2$ about the x_3 axis combined with a clockwise isorotation by $\pi/2$ about the x_3 axis. The path traversed twice is a contractible loop since it is the product of two closed loops. This loop must also be contractible for the minimal energy solution. The path is now given by Eq. (4.21) with $0\leqslant\lambda\leqslant\pi$. So an anti-clockwise rotation combined with a clockwise isorotation by π about the x_3 axis is a contractible loop. From Eqs. (2.12) and (2.13), this implies that the operator $e^{i\pi(L_3+K_3)}$ acting on the allowed states gives $(+1)$.

Just to be clear about this suppose instead that for the solution with $b=0$, we rotated it by $\pi/2$ anticlockwise and isorotated it by $3\pi/2$ anti-clockwise, again this is a closed loop, i.e.

$$z\to e^{i\lambda}z,\quad F\to e^{3i\lambda}F,\quad 0\leqslant\lambda\leqslant\pi/2. \tag{4.22}$$

Repeated this loop twice gives a contractible loop which can be written as

$$z\to e^{i\lambda}z,\quad F\to e^{-i\lambda}e^{4i\lambda}F,\quad 0\leqslant\lambda\leqslant\pi. \tag{4.23}$$

This is the product of the loop in Eq. (4.21) (with $0\leqslant\lambda\leqslant\pi$) with a 4π isorotation which is contractible so we reach the same conclusion. Note that the operator $e^{i\pi(L_3-K_3)}$, which acts on states with eigenvalue (-1), does not correspond to a closed loop traversed twice when acting on the configuration with $b=0$ i.e. when $b=0$, $F(z)\neq iF(iz)$.

Next consider the C_2 symmetry group element. It is possible to deform the minimal energy charge five Skyrmion into a configuration of a $B=3$ tetrahedron and two $B=1$ Skyrmions on opposite sides of the tetrahedron. This is indicated in Fig. 4.

The $B=3$ looks like an anti-Skyrmion at large distances from its center so the total configuration is attracting. The $B=5$ solution was originally found by relaxing such a configuration [1]. The $B=1$ Skyrmions will be on the x_3 axis equidistant from the origin with the same isospin orientation. The $B=3$ tetrahedron is oriented so that its axes of second order are the x_1, x_2 and x_3 axes. It is easily seen that such a configuration of monopoles can be separated keeping the C_2 symmetry about the x_1 axis at all times since the set of $k=5$ monopoles with C_2 symmetry about the x_1 axis is connected. We take the C_2 element to act by an anticlockwise rotation combined with a clockwise isorotation. The effect of this is to rotate anti-clockwise and isorotate clockwise the

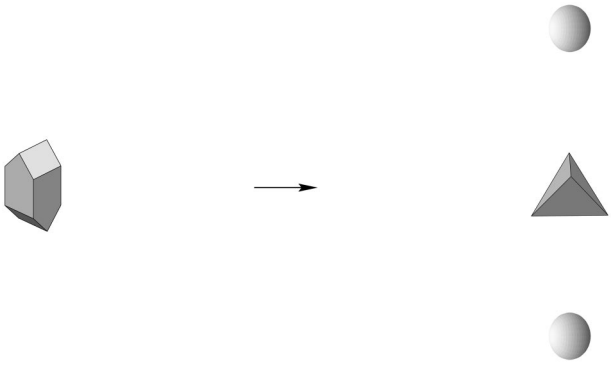

FIG. 4. $B=5$ Skyrmion separating to two $B=1$ Skyrmions and a $B=3$ Skyrmion.

tetrahedron and the $B=1$ Skyrmions by π about the x_1 axis and interchange the two $B=1$ Skyrmions. The zero mode analysis for the $B=3$ tetrahedron was considered in [12]. The loop corresponding to the π anti-clockwise rotation and π clockwise isorotation turns out to be contractible. We will review this for the $B=9$ case which has tetrahedral symmetry, the analysis is the same as for the $B=3$ case. For the $B=1$ Skyrmions a rotation combined with the opposite isorotation about the same axis leaves the configuration unchanged due to their hedgehog nature. The interchange of two identical $B=1$ Skyrmions is a noncontractible loop. Thus the overall loop is noncontractible. So, physical states satisfy

$$e^{i\pi(L_3+K_3)}|\Psi\rangle=|\Psi\rangle$$

$$e^{i\pi(L_1+K_1)}|\Psi\rangle=-|\Psi\rangle. \tag{4.24}$$

Since the rotations and isorotations act in the same way we can rewrite Eq. (4.24) as

$$e^{i\pi M_3}|\Psi\rangle=|\Psi\rangle$$

$$e^{i\pi M_1}|\Psi\rangle=-|\Psi\rangle \tag{4.25}$$

where $M_i=L_i+K_i$. The ground state is $|M,M_3\rangle=|1,0\rangle$. In terms of I, J this is

$$|\Psi\rangle=\left|\frac{1}{2},\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle-\left|\frac{1}{2},-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle. \tag{4.26}$$

As discussed earlier we will ignore the question of parity in the odd B sector. We recall that this is the only case where the states $|I,K_3\rangle\otimes|J,L_3\rangle$ are not necessarily eigenstates of the Hamiltonian since the symmetry group does not have an axis of order higher than the second. But it is easy to see that states with $I=\frac{1}{2}$, $J=\frac{1}{2}$ are eigenstates of the Hamiltonian, since the Hamiltonian only causes transitions from L_3 to L_3+2, L_3, and L_3-2, and similarly for K_3. So the $I=\frac{1}{2}$, $J=\frac{1}{2}$ state is an energy eigenstate. This is inconsistent with the observed isodoublet ground state ($^5_2$He, $^5_3$Li) which has spin $\frac{3}{2}$ [13]. This state can be obtained from $|M,M_3\rangle=|2,2\rangle-|2,-2\rangle$, which satisfies Eq. (4.25) but this has higher energy than $|M,M_3\rangle=|1,0\rangle$. For the helium-lithium isodoublet the first excited state is a spin $\frac{1}{2}$ state at excitation energy approximately 5 MeV. So the ground state we obtain is the first experimentally observed excited state of ($^5_2$He, $^5_3$Li).

The inclusion of the vibrational modes will give new states but the lowest energy state will still be the $I=\frac{1}{2}$, $J=\frac{1}{2}$ state. It is possible that a more careful quantization which allows the Skyrmions to separate will raise the energy of the spin $\frac{1}{2}$ state above that of the spin $\frac{3}{2}$ state but this is not at all obvious and would be a very challenging project.

$B=7$

The minimal energy $B=7$ Skyrmion has icosahedral symmetry Y. It can be described in terms of a Jarvis rational map given by

$$F(z)=\frac{bz^6-7z^4-bz^2-1}{z(z^6+bz^4+7z^2-b)},\quad b=\pm\sqrt{7}/5. \tag{4.27}$$

The icosahedral group is generated by two elements, a C_5 rotation and a C_3 rotation. The rotations form the defining F_1 representation of Y (using the notation of [26]) and one can check that the accompanying isospin transformations are in the other three dimensional irreducible representation F_2 (which only differs from F_1 in that, elements which in F_1 are represented by a $2\pi/5$ rotation are represented in F_2 by a $4\pi/5$ rotation). Again we are in the odd nucleon number sector and so the spin and isospin of the states must be half-integral.

To determine the FR constraints is more complicated in this case. We want to use the representation theory of the icosahedral group to determine the allowed states. But as discussed in Sec. II we need to lift the SO(3) elements to SU(2). Generally it is not possible to embed a group into its double group while maintaining the group structure, i.e. to choose a subgroup isomorphic to H in the group $\overline{H}$. This means we cannot immediately use the representation theory of the icosahedral group Y. We need to consider the group K consisting of the elements $\{\pm h,\pm h'\}$ where the elements $D(h)$ form the F_1 representation of Y and the elements $D(h')$ form the F_2 representation of Y. The elements h form the fundamental or defining representation, denoted Γ_6, of the double group $\overline{Y}$. The elements h' form the other irreducible two dimensional representation of $\overline{Y}$, denoted Γ_7, as can be seen from examining the character table of $\overline{Y}$ (see Table I). The character table of the double group $\overline{Y}$ is given above, with $\tau=(1+\sqrt{5})/2$. Both the representations Γ_6 and Γ_7 are representations of the double group $\overline{Y}$ and are not representations of Y, i.e. $\Gamma_i(-y)=-\Gamma_i(y)$ for $i=6,\ 7$, where y is an abstract group element of $\overline{Y}$. This means that the elements in K are of the following form:

$$K=\{(\Gamma_6(y),\Gamma_7(y)),(\Gamma_6(y),-\Gamma_7(y)),y\in\overline{Y}\}. \tag{4.28}$$

Or as a group, $K=\overline{H}\times\mathbb{Z}_2$. We now restrict to the group elements $(\Gamma_6(y),\Gamma_7(y))$, these form a subgroup of K which is isomorphic to $\overline{Y}$ [note that elements $(\Gamma_6(y),-\Gamma_7(y))$ do

TABLE I. Character table for $\overline{Y}$.

	E	$\overline{E}$	$12C_5$	$12\overline{C}_5$	$12C_5^2$	$12\overline{C}_5^2$	$20C_3$	$20\overline{C}_3$	$30C_2$
$\Gamma_1(A)$	1	1	1	1	1	1	1	1	1
$\Gamma_2(F_1)$	3	3	τ	τ	$1-\tau$	$1-\tau$	0	0	-1
$\Gamma_3(F_2)$	3	3	$1-\tau$	$1-\tau$	τ	τ	0	0	-1
$\Gamma_4(G)$	4	4	-1	-1	-1	-1	1	1	0
$\Gamma_5(H)$	5	5	0	0	0	0	-1	-1	1
Γ_6	2	-2	τ	$-\tau$	$-1+\tau$	$1-\tau$	1	-1	0
Γ_7	2	-2	$1-\tau$	$-1+\tau$	$-\tau$	τ	1	-1	0
Γ_8	4	-4	1	-1	-1	1	-1	1	0
Γ_9	6	-6	-1	1	1	-1	0	0	0

not form a subgroup of K]. Each element in this group corresponds to a symmetry of the $B=7$ Skyrmion and there exists a corresponding operator which acts on the allowed states with eigenvalue ± 1. Because $\overline{Y}$ forms a subgroup of K, this implies that the states transform by a representation of the group $\overline{Y}$. Since the states acquire only a ± 1 phase under each operation the representation must be one dimensional. The only such representation is the trivial one. This means that the allowed states have eigenvalue $+1$ corresponding to each of the above transforms. So here we can determine the FR constraints without any need of separating the configuration into individual Skyrmions. This is because there are no nontrivial one dimensional representations of the group $\overline{Y}$. In the previous cases of $B=4,6,8$ the symmetry group of the minimal energy configuration had O, D_4 and D_6 symmetries respectively. Each of these groups have nontrivial one dimensional representations. Thus in these cases, group theory alone cannot give the answer and it was necessary to examine a configuration of well separated Skyrmions in order to determine the contractibility of the loops.

Returning to the $B=7$ case, to find physical states of spin J, isospin I we need to decompose the spin J representation of SU(2) into representations of $\overline{Y}$ and the spin I representation of SU(2) into representations of $\overline{Y}$. We then take tensor products of these representations and look for values of I, J that give singlets of $\overline{Y}$. We need to take into account here that the isorotations are in the F_2 representation of Y, recall Eqs. (2.12) and (2.13). We keep $I=\frac{1}{2}$, because states of high isospin are energetically unfavorable, this means that the isospin states transform by the Γ_7 representation of $\overline{Y}$. The lowest allowed J is that which its decomposition into representations of $\overline{Y}$ contains the Γ_7 representation, since $\Gamma_7 \otimes \Gamma_7$ contains the trivial representation. We find that the lowest J is $\frac{7}{2}$ [26], in contradiction with the observed isodoublet of spin $\frac{3}{2}$. The spin $\frac{7}{2}$ state we found appears as the second excited state of the lithium-beryllium doublet at 4.6 MeV. The first excited state has spin $\frac{1}{2}$ at 0.5 MeV.

As noted earlier, it is possible to combine the vibrational modes with the rotational modes. This will give an enlarged set of states. The experimentally observed ground state with $I=\frac{1}{2}$ and $J=\frac{3}{2}$ can be obtained in this manner. The first observed excited state with spin $\frac{1}{2}$ can also be obtained. Since the vibrational frequencies are as yet unknown it is not clear whether in our analysis these states will have lower energy than the $I=\frac{1}{2}$, $J=\frac{7}{2}$ state. The ground state may be written as

$$|\Psi> = \left\{ \sqrt{\frac{7}{10}}\left|\frac{7}{2},-\frac{3}{2}\right\rangle - \sqrt{\frac{3}{10}}\left|\frac{7}{2},\frac{7}{2}\right\rangle \right\} \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle + \left\{ \sqrt{\frac{7}{10}}\left|\frac{7}{2},\frac{3}{2}\right\rangle + \sqrt{\frac{3}{10}}\left|\frac{7}{2},-\frac{7}{2}\right\rangle \right\} \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle. \tag{4.29}$$

$B=9$

The minimal energy $B=9$ Skyrmion has tetrahedral symmetry. It can be described in terms of a rational map given in [2]. The rotational subgroup is generated by two elements, a C_2 rotation about the x_3 axis and a C_3 rotation about the $(x_1+x_2+x_3)$ axis. The rotations form the defining F representation of the tetrahedral group T and one can check that the accompanying isorotations are also in the representation F. Here we are in the odd nucleon number sector and again the spin and isospin of the states must be half-integral. To determine the FR constraints here is similar to that for the $B=7$ case. The fundamental representation of $\overline{T}$, the double group of the tetrahedral group T, is denoted ϕ. By analogy with Eq. (4.28) the group K is of the form

$$K=\{(\phi(y),\phi(y)),(\phi(y),-\phi(y)),y\in\overline{T}\}. \tag{4.30}$$

So again K is of the form $K=\overline{T}\times\mathbb{Z}_2$, and since $\overline{T}$ is a subgroup of $K=\overline{T}\times\mathbb{Z}_2$, states transform by a representation of $\overline{T}$ which must be one dimensional. There are no nontrivial representations of $\overline{T}$ [20]. Since the rotations act in the same way as the isorotations the constraints can be expressed in terms of the operators $M_i=L_i+K_i$ just as for the $B=5$ case, again using Eqs. (2.12) and (2.13). Since all constraints are trivial we get

$$e^{i\pi M_3}|\Psi\rangle = |\Psi\rangle$$
$$e^{i(2\pi/3\sqrt{3})(M_1+M_2+M_3)}|\Psi\rangle = |\Psi\rangle. \tag{4.31}$$

This analysis is the same as that presented by Carson in [12] for the tetrahedrally symmetric $B=3$ solution, where he found the ground state to be $I=J=\frac{1}{2}$. The state is $|M,M_3\rangle = |0,0\rangle$. In terms of I, J this is

$$|\Psi\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle - \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle. \tag{4.32}$$

Again this is not in agreement with the isodoublet of beryllium and boron of spin $\frac{3}{2}$ ($^9_4$Be, $^9_5$B). The state obtained is the first excited state with excitation energy 1.6 MeV [13]. The observed ground state can be obtained here by including the vibrational modes but it will have higher energy than the spin $\frac{1}{2}$ state. This is a similar situation to above for $B=5$ with no obvious way around this difficulty even if the vibrational modes are included.

$B=17$

For $B \geqslant 9$ the minimal energy Skyrmion configurations are not yet known. From [1] it is expected that the minimal energy solution will look like a polygon with 12 pentagons and $2(B-7)$ hexagons. But as B increases there are many such polygons and it turns out that the energy difference between these solutions is very small, so it is hard to identify the minimal energy solution. But for $B=17$ a particularly symmetric configuration arises, the buckyball solution with icosahedral symmetry. Due to its enhanced symmetry, it is believed that this is the minimal energy solution for $B=17$.

This solution is described by the rational map [2]

$$F(z)=\frac{17z^{15}-187z^{10}+119z^5-1}{z^2(z^{15}+119z^{10}+187z^5+17)}. \tag{4.33}$$

This case is similar to that for $B=7$ which also has icosahedral symmetry. It can be checked from Eq. (4.33) that the rotations form the defining representation F_1 and the isorotations form the representation F_2. This is exactly the same as for $B=7$. So we find the ground state has $I=\frac{1}{2}$, $J=\frac{7}{2}$. However, from [13] this state is the eighth excited state of the isodoublet ($^{17}_{8}$O, $^{17}_{9}$F) whose ground state has spin $\frac{5}{2}$.

V. VIBRATIONAL MODES

To go beyond the first approximation of just considering the zero modes it is appropriate to include the vibrations of the Skyrmions. These have been calculated for the minimal energy $B=2$ and $B=4$ solutions [6]. The approximation of treating the interaction potential of the Skyrme configurations as a harmonic oscillator potential is not very accurate, since, as the minimal energy configuration separates into individual Skyrmions the potential flattens out. A more accurate treatment will involve estimating the inter-Skyrmion potential at intermediate and large separations. Thus it should not be expected that the inclusion of vibrational modes will yield accurate results for masses, binding energies of states etc.

Including the vibrational modes involves the coupling of harmonic oscillator wave functions to the rotational and isorotational wave functions. However, they do not combine in an arbitrary way; the interaction of the rotations and vibrations is described in [15] for general soliton models. The space of rotations and isorotations is $(\mathrm{SO}(3)\times\mathrm{SO}(3))/H$; again H is the symmetry group of the minimal energy solution. The vibrations fall into representations of H and the space of vibrations is a vector space denoted by V. V is a direct sum of vector spaces V_i with H acting irreducibly on each V_i.

The total configuration space $\mathcal{F}$ say, is now a vector bundle over $(SO(3)\times SO(3))/H$. For ease of notation we will restrict here to the case of even B so we do not need to worry about the double covering. It can be included without much difficulty. $\mathcal{F}$ can be defined by taking the product space $\mathrm{SO}(3)\times\mathrm{SO}(3)\times V$ with the following equivalence:

$$(R,R',v)\cong(SR,R'\Gamma^{-1}(S),\rho^{-1}(S)v),$$

$$S\in H,\quad (R,R')\in \mathrm{SO}(3)\times\mathrm{SO}(3),\quad v\in V. \tag{5.1}$$

$\Gamma(S)$ is as before and $\rho(S)$ is the action of H on the space of vibrations. As an example to see that this gives the correct configuration space consider the $B=4$ Skyrmion which has a cubic shape. One of the vibrational modes is the so called tetrahedral mode which can be imagined as follows. The vertices of the cube form two interlocking tetrahedra. The vibrating cube alternately separates into four Skyrmions on the vertices of one of the tetrahedra (positive mode), then contracts to the cube and then separates into four Skyrmions on the vertices of the dual tetrahedron (negative mode). Acting with the $\pi/2$ rotation and π isorotation about the x_3 axis (which is a symmetry of the cube) is equivalent to interchanging the positive and negative modes. So as not to overcount the configuration space we must identify rotating and isorotating the configuration about the x_3 axis with interchanging positive and negative vibrating modes.

Quantum states are given by the direct product of Wigner functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ with harmonic oscillator wave functions on V with the proviso that the states are H invariant. Again, in a manner similar to that treated for the zero modes the FR constraints determine how H invariance is to be implemented. The FR constraints for the closed loops corresponding to the above action of H are identical to those when just considering zero modes. This is because the loops are closed for all vibrational amplitudes, so the loop can be deformed to the case of amplitude zero, i.e. the zero mode case. When the classical solution has a reflection symmetry the vibrations corresponding to the vector space V_i have a definite parity $\rho_i=\pm 1$. It is possible to check that the parity operator for the rotational and vibrational states is given by $P\Pi_i\rho_i^n$ where P is the parity operator acting on the zero modes and the ith vibrational state is in the nth excited mode.

Here we will concentrate on the $B=4$ case since the vibrational spectra has been calculated [6]. The spectra was calculated at finite pion mass, whereas we are working with zero pion mass. But the vibrational frequencies found in [6] do not appear to vary greatly with the value of the pion mass used, so we will use their values. Anyway we are not interested in obtaining accurate numbers here, we just want to indicate how to couple the rotational and vibrational modes.

To find the allowed states is quite easy. If one is only interested in what states are allowed and not their dependence in terms of L_3, K_3 etc., then this can be determined by the representation theory of the cubic symmetry group O, alone. The configuration space is $\mathrm{SO}(3)\times\mathrm{SO}(3)\times V$ quotiented by O as described above. Since all the FR constraints all $+1$, the allowed states are O singlets of $\mathrm{SO}(3)\times\mathrm{SO}(3)\times V$. From Eq. (4.1) we know that the rotational SO(3) transforms as the defining F_1 representation of O and that the isorotational SO(3) transforms as the $E\oplus A_2$ representation of O, using the notation of [20]. From this we can work out how a spin J, isospin I state decomposes under O. The representations of O that the vibrations form were computed in [6] and so we can determine how the product $\mathrm{SO}(3)\times\mathrm{SO}(3)\times V$ transforms under O and so we can easily read off which combinations of I, J, and vibrations are allowed

as states. For $B=4$ the rotational moments of inertia are all equal, $V_{ij}=\delta_{ij}$ $(18\ \mathrm{MeV})^{-1}$, the isorotational moments of inertia are $U_{11}=U_{22}=(82.2\ \mathrm{MeV})^{-1}$, $U_{33}=(68.2\ \mathrm{MeV})^{-1}$, and the cross term between rotations and isorotations vanishes, $W_{ij}=0$. These values are obtained using the values of f_π, e from [27]. Thus the Hamiltonian is

$$H=41.1\mathbf{K}^2-7.0K_3^2+9.0\mathbf{L}^2, \tag{5.2}$$

in units of MeV. The energies of the vibrational states, $\hbar\omega$, and their representations of the cubic group are $(E^+,94$ MeV), $(A_2^-,104$MeV), $(F_2^+,107$ MeV), $(F_2^-,132$ MeV),$(A_1^+,$ 155 MeV), $(F_2^-,$ 168 MeV), and $(F_2^+,$ 189 MeV), the $\pm$ denotes parity. Restricting to $K=0$, i.e. ${}^4_2$He, the first few excited states are $J=2^+$ at 147 MeV, $J=0^+$ at 155 MeV, $J=2^+$ at 160 MeV, and then the first excited zero mode state, $J=4^+$ at 178 MeV. The observed excited states of ${}^4_2\mathrm{He}^+$ differ considerably from this [13]. The first few excited states are 0^+ at 20.1 MeV, 0^- at 21.1 MeV, and 2^- at 22.1 MeV. The most obvious discrepancy is the overestimation of the excitation energies, this is partly due to treatment of the potential as of harmonic oscillator type. Nonetheless this shows that the vibrational states are important and are of the same order of energy as the pure rotational states.

The experimentally observed ground state of $({}^7_3\mathrm{Li},\ {}^7_4\mathrm{Be})$ has $J=\frac{3}{2}$. For $B=7$ the lowest state with isospin $I=\frac{1}{2}$ was found to have $J=\frac{7}{2}$. By the same methods as above, using the monopole vibrations as a prediction for the low lying Skyrmion vibration frequencies, a state of $J=\frac{3}{2}$ can be obtained. If the vibrational frequency of this state is not too high it may have lower energy than the $J=\frac{7}{2}$ and thus give the correct ground state.

VI. NUCLEON DENSITIES OF THE STATES

Given the expressions for the states in terms of Wigner functions, other physical properties may be calculated such as the nucleon density of the quantum state. The nucleon density of the classical configurations are quite symmetrical and it is of interest to know how quantum effects change this. Given a state Ψ, we want an expression for the probability distribution $p_\Psi(\mathbf{x})$ on physical space which is interpreted as the nucleon density. This is done by averaging the classical nucleon density over the space of zero modes weighted with $|\Psi|^2$ [16] (we restrict here to zero mode states). Denoting the classical nucleon density by $B(\mathbf{x})$, the spatial probability distribution for the quantum state is defined as

$$p_\Psi(\mathbf{x})=\frac{1}{2}\int B(D(A)\mathbf{x})|\Psi(A,\ A')|^2\sin\tilde{\theta}d\tilde{\theta}d\tilde{\phi}d\tilde{\psi} \tag{6.1}$$

where $D(A)$ is parametrized by the Euler angles $(\tilde{\theta},\tilde{\phi},\tilde{\psi})$. $p_\Psi(\mathbf{x})$ is evaluated by expanding $B(D(A)\mathbf{x})$ in terms of spherical harmonics $Y_{mn}(\hat{\tilde{\mathbf{x}}})$, with $\tilde{\mathbf{x}}=D(A)\mathbf{x}$, then using the transformation properties of spherical harmonics under rotations

$$Y_{lm}(\hat{\tilde{\mathbf{x}}})=\sum_k D^l_{mk}(A)^*Y_{lk}(\hat{\mathbf{x}}) \quad (\text{no sum on } l) \tag{6.2}$$

and the fact that $|\Psi|^2$ can be written as a sum of terms $D^J_{ab}(A)D^J_{cb}(A)^*$, where $D^J_{ab}(A)$ are Wigner functions. The direct product and orthogonality properties of the Wigner functions are then used to compute $p_\Psi(\mathbf{x})$. We choose the space fixed angular momentum in the x_3 direction, b, equal to J, i.e. ''spin up.'' Considering only rotational and isorotational wave functions, $p_\Psi(\mathbf{x})$ will have the same radial dependence as the classical solution. But the angular dependence will be changed by quantum effects. In the Skyrme model there is no decomposition of angular momentum into orbital and intrinsic spin angular momentum. However, calculating the spatial probability distribution can give some insight into what the intrinsic spin and orbital contributions of the nuclear state are. If the nuclear state is mostly in a orbital S-state its nucleon density will also be almost spherically symmetric. For all the examples treated below the quantum nucleon density is more spherically symmetric than the classical nucleon density, it being exactly S-wave in a number of cases.

For $B=4$ we found the ground state to have $I=J=0$, the first excited state with $I=0$ has $J=4$ and the lowest state with $I=1$ has $J=2$. Inserting the above states into Eq. (6.1) we trivially find the probability distribution of the $I=0$, $J=0$ state to be spherically symmetric. This is also true of the ground state for $B=8$. For the $I=0$, $J=4$ state of $B=4$ we find the angular dependence to be mostly S-wave with $l=4$ contributions and some very small $l=6$ and $l=8$ contributions,

$$p_\Psi(\theta,\phi)\propto\{Y_{00}-0.045Y_{40}-0.027(Y_{44}+Y_{4-4})+0.0002Y_{60}+0.00003Y_{80}\}. \tag{6.3}$$

Here (θ,ϕ) are the angular coordinates on physical space, as opposed to the coordinates on $D(A)$. And for the $I=1$, $J=2$ state we again find the nucleon density to be mostly spherically symmetric with a small $l=4$ contribution

$$p_\Psi(\theta,\ \psi)\propto\{Y_{00}-0.01Y_{40}-0.01(Y_{44}+Y_{4-4})\}. \tag{6.4}$$

Thus when quantum effects are included the nucleon density becomes spherical or near spherical. It is known that the ground state of ${}^4_2$He is completely S-wave. In real nuclei the nucleon density is large up to a certain radius and then falls off quickly. Our quantum states have the same radial dependence as the classical solutions which is somewhat hollow, this becomes very noticeable for larger nucleon numbers.

For the $I=0$, $J=1$ ground state of $B=6$ it is found that

$$p_\Psi(\theta,\phi)\propto\{Y_{00}-0.03Y_{20}\}. \tag{6.5}$$

This result is slightly different than for the $B=2$ deuteron. In both cases the ground state is given by $I=0$ and $J=1$ with the same L_3 dependence but for the deuteron the quantum probability distribution is of a dumbbell shape [16]. Here, for the $B=6$ solution the quantum probability distribution is of a toroidal shape. The difference arises because the classical

nucleon densities of the two solutions are different. Nonetheless, the wave function is predominately S-wave and this is also in agreement with experiment.

The ground state for $B=7$ may be written as

$$|\Psi\rangle=\left\{\sqrt{\frac{7}{10}}\left|\frac{7}{2},-\frac{3}{2}\right\rangle-\sqrt{\frac{3}{10}}\left|\frac{7}{2},\frac{7}{2}\right\rangle\right\}\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle+\left\{\sqrt{\frac{7}{10}}\left|\frac{7}{2},\frac{3}{2}\right\rangle+\sqrt{\frac{3}{10}}\left|\frac{7}{2},-\frac{7}{2}\right\rangle\right\}\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle. \tag{6.6}$$

From this we can see that the probability distribution of this state must be spherically symmetric. This is so because in Eq. (6.1) we take $|\Psi|^2$ and integrate it with the classical nucleon density. The classical nucleon density has icosahedral symmetry and for $l\leqslant 7$ the only spherical harmonics which are icosahedrally symmetric are $l=0$ and an $l=6$ harmonic [26]. But $|\Psi|^2$ expanded in terms of Wigner functions has no $l=6$ term and so $p_\Psi(\mathbf{x})$ is spherically symmetric. The same analysis applies to the ground state of $B=17$.

For the $B=9$ ground state it is easy to show that the nucleon density is spherically symmetric. Since the spin is $\frac{1}{2}$ the nucleon density could only have $l=0$ and $l=1$ components. But the $l=1$ component is associated with a vector in space and this is incompatible with tetrahedral symmetry so the wave function is completely S-wave. It can also be checked that the $B=5$ ground state is completely S-wave.

So we see that when one includes quantum effects the classical picture of the nucleon density having a discrete point symmetry group is changed so that in the quantum state it is smeared forming a spherical or near spherically symmetric configuration.

VII. OUTLOOK

We have described the ground states of the $B=4$ to $B=9$ and $B=17$ Skyrmions obtained by quantizing the zero modes of the classical solutions. We did not attempt to calculate the masses, binding energies and other observables since a zero mode quantization is too restrictive to get accurate results. Nonetheless we expected to obtain the correct quantum numbers of the ground states. However our results are not promising; for $B=4$, $B=6$ and $B=8$ the correct ground states are obtained. But in the odd nucleon sector we have obtained the incorrect ground states. For nucleon numbers 5, 7 and 9 the experimentally observed ground states are isodoublets with spin $\frac{3}{2}$ and for nucleon number 17 the observed ground state is an isodoublet with spin $\frac{5}{2}$. However we obtained isodoublets with spin $\frac{1}{2}$ for $B=5$ and $B=9$, and an isodoublet with spin $\frac{7}{2}$ for $B=7$ and $B=17$. The symmetry of the classical solutions which can give spin $\frac{3}{2}$ states is C_4 symmetry, and C_6 symmetry can give a spin $\frac{5}{2}$ state. But the classical solutions in these cases do not have C_4 or C_6 symmetry.

The main assumption we made was that certain closed loops in the configuration space remain closed as the minimal energy configuration is separated into $B=1$ or $B=2$ Skyrmions. This was necessary in order to determine the FR constraints. The vibrational spectra of the minimal energy $B=2$ and $B=4$ Skyrmions for low frequencies is in correspondence with monopole vibrations about the corresponding monopoles. We assumed that this correspondence holds true for higher B. We consider this very likely, but the vibrational spectra for the Skyrmions needs to be found to confirm this. It was also assumed that if the solution could be vibrated, remaining invariant under a certain symmetry, then the continuation of the symmetric path in the configuration space results in a configuration of well separated Skyrmions. We have seen that this is true for monopoles in the cases considered and presumed it also holds for Skyrmions. Again, this does not seem to be a particularly strong assumption. In any case, the outcome for $B=7$, $B=9$ and $B=17$ is independent of these assumptions, since the FR constraints can be determined from the group theory alone, and the ground states obtained are not in agreement with experiment.

To obtain the experimentally observed ground states it will be necessary to include modes whereby the Skyrmions separate. It is not difficult to see that a quadratic approximation (by just considering the vibrational modes) will not cure this problem for the $B=5$ and $B=9$ cases. If the Skyrme model is to correctly predict the ground states of these nuclei it will be necessary to include configurations of Skyrmions with intermediate or long range separations which is a highly nontrivial problem.

Another possible resolution is that the solutions found in [1] are not well defined minima, i.e. there may be a number of solutions with approximately equal energies and so an expansion about just one of these minima is not valid. However we view that the more likely answer is that the zero mode configuration space is too restrictive. The zero mode approximation allows only for a collective motion of the Skyrmions with nine parameters, while the space that approximates the low energy behavior of B Skyrmions should be $6B$ dimensional. As B increases the validity of the zero mode approximation should break down.

Our final comment concerns the question of renormalization. Casimir energies arise from the renormalization of the vibrational modes around any classical Skyrmion solution. The energy corrections that arises from this will be dependent on the particular classical solution that one expands about. They will however be independent of the collective coordinates and thus be identical for each quantum state that arises from quantizing the zero modes of a given classical solution, see e.g. [28]. Thus the energy shift will be the same for each Skyrmion state. Further corrections to the soliton mass arise from the renormalization of the interaction terms which couple the vibrational modes to the collective coordinates of the Skyrmion. These give corrections which depend on the specific quantum state of the Skyrmion, i.e. the energy corrections are different for states of differing I, J. This leaves the possibility that the ground states which were found here may in fact be exotic, excited states after renormalization. Whether this approach can reconcile the Skyrmion ground states with observed nucleii is unclear, it is not obvious to us how the magnitude of these corrections will depend on the quantum numbers of the Skyrmion state. Of course, this is a difficult matter to test explicitly and we will not expand on it here.

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