

Mass hierarchies and the seesaw neutrino mixing

T. K. Kuo*

Department of Physics, Purdue University, West Lafayette, Indiana 47907

Guo-Hong Wu[†]

Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403

Sadek W. Mansour[‡]

Department of Physics, Purdue University, West Lafayette, Indiana 47907

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We give a general analysis of neutrino mixing in the seesaw mechanism with three flavors. Assuming that the Dirac and u -quark mass matrices are similar, we establish simple relations between the neutrino parameters and individual Majorana masses. They are shown to depend rather strongly on the physical neutrino mixing angles. We calculate explicitly the implied Majorana mass hierarchies for parameter sets corresponding to different solutions to the solar neutrino problem.

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One of the most pressing questions in particle physics has been the determination of the intrinsic properties of the neutrinos, namely, their masses and mixing angles. The recent atmospheric neutrino data [1] suggest strongly that neutrinos do have masses, and that, unlike the quark sector, at least some of the mixing angles are large. The most appealing model for small neutrino masses derives from the seesaw mechanism [2]. In doing so, however, one also introduces additional unknowns in the form of the Dirac and Majorana mass matrices. We do have a handle on the Dirac matrix, since, from the ideas of grand unified theories (GUTs), it should be similar to that of the quark sector. Not much is known about the Majorana mass matrix. The challenge is then to find out what the Majorana matrix is like in order for the effective neutrino mass matrix to come out correctly.

In this paper, we will analyze the general structure of the seesaw mass matrix. Without loss of generality, we will work in the basis in which the charged lepton and Majorana mass matrices are diagonal. As a starting point, we assume that the Dirac mass matrix, in analogy with the quark mass matrix, has hierarchical eigenvalues and small left-handed mixing angles. Even in this case, large mixing can occur through the interplay of the Dirac and Majorana matrices [3–7].

In the seesaw mechanism, the effective neutrino mass matrix is given by $m^{\text{eff}} = m_D M^{-1} m_D^T$. The Dirac matrix can be written as $m_D = U_0 m_D^{\text{diag}} V_0$ in the basis where M^{-1} is diagonal,

$$M^{-1} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}. \quad (1)$$

Here U_0 and V_0 are left-handed (LH) and right-handed (RH) rotations, respectively. m_D^{diag} is a diagonal matrix with eigenvalues m_i ($i=1,2,3$). The Majorana mass eigenvalues are given by $M_i = 1/R_i^2$ ($i=1,2,3$). For simplicity, we assume that U_0 and V_0 are real (i.e., we ignore CP violating effects). Note that V_0 also contains contribution from the diagonalization of M^{-1} , and thus it may contain large angles. However, we will restrict our discussions to reasonably small angles ($< \pi/4$) in V_0 .

It is convenient to write $U_0^{-1} m^{\text{eff}} U_0 = NN^T$, where

$$N = m_D^{\text{diag}} V_0 M^{-1/2} = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} V_0 \begin{pmatrix} R_1 & & \\ & R_2 & \\ & & R_3 \end{pmatrix}. \quad (2)$$

For hierarchical Dirac masses, to a good approximation

$$\begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix} V_0 \simeq \begin{pmatrix} m_1 V_{11} & 0 & 0 \\ m_2 V_{21} & m_2 V_{22} & 0 \\ m_3 V_{31} & m_3 V_{32} & m_3 V_{33} \end{pmatrix}. \quad (3)$$

Here $V_{ij} \equiv (V_0)_{ij}$ are the matrix elements of V_0 . More precisely [8], the leading correction to this approximation is a LH rotation with rotation angles

$$(\phi_{12}, \phi_{13}, \phi_{23}) \simeq \left(\frac{m_1}{m_2} \frac{V_{12}}{V_{22}}, \frac{m_1}{m_3} \frac{V_{13}}{V_{33}}, \frac{m_2}{m_3} \frac{V_{23}}{V_{33}} \right).$$

This can be absorbed into U_0 , and will be ignored henceforth.

If $M^{-1} \propto I$, then NN^T will be diagonal and m^{eff} will be diagonalized by U_0 . When M^{-1} deviates from the identity matrix, the product $V_0 M^{-1/2}$ is no longer orthogonal; then $N = U N^{\text{diag}} W$, where U and W are LH and RH rotations, respectively. The LH rotation U is induced by $M^{-1/2}$ so that $(U^{-1} U_0^{-1}) m^{\text{eff}} (U_0 U) = (N^{\text{diag}})^2$. Thus, the mixing angles for the effective neutrino mass matrix come from $U_0 U$, while the eigenvalues of N are just $\sqrt{m_i^{\text{eff}}}$, with m_i^{eff} denoting

*Email address: tkuo@physics.purdue.edu

[†]Email address: wu@dirac.uoregon.edu

[‡]Email address: mansour@physics.purdue.edu

the effective neutrino masses. To the extent that we may assume U_0 to be nearly the identity matrix, the case of large mixing angles receives its main contribution from U , which is induced by M .

From Eq. (2), we have

$$N \simeq \begin{pmatrix} R_1 m_1 V_{11} & 0 & 0 \\ R_1 m_2 V_{21} & R_2 m_2 V_{22} & 0 \\ R_1 m_3 V_{31} & R_2 m_3 V_{32} & R_3 m_3 V_{33} \end{pmatrix}. \quad (4)$$

While we have assumed that the Dirac mass eigenvalues m_i have the hierarchical structure of the quark masses, little is known about the values of R_i .

The problem within the seesaw model is then twofold. One, given the parameters m_i , R_i , and V_{ij} , what is the LH rotation matrix U as defined by $N = UN^{\text{diag}}W$? Conversely, if we know U , i.e., the physical mixing angles for the neutrinos, what can we deduce about these parameters? We will discuss these issues in this paper.

Let us start from a general 3×3 matrix

$$N = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}. \quad (5)$$

Anticipating the applications to the neutrino sector, we will assume $a^2 \ll b^2 \sim c^2$, where we have used the notation $\vec{a} = (a_1, a_2, a_3)$, etc. We have shown elsewhere [8] that when a matrix is brought into the upper triangular form, we can obtain the LH mixing angles easily. Now, it is always possible to find a RH rotation so that N becomes upper triangular. In fact, geometrically, this amounts to a new coordinate system where \vec{c} is aligned with the third axis, while the second and first axes are in the directions $\vec{c} \times (\vec{b} \times \vec{c})$ and $\vec{b} \times \vec{c}$, respectively. It is then clear that, for an appropriate RH rotation R ,

$$N_t = NR = \begin{pmatrix} \vec{a} \cdot \hat{i} & \vec{a} \cdot \hat{j} & \vec{a} \cdot \hat{c} \\ 0 & |\vec{b} \times \hat{c}| & \vec{b} \cdot \hat{c} \\ 0 & 0 & c \end{pmatrix}, \quad (6)$$

where $\hat{i} = \vec{b} \times \vec{c} / |\vec{b} \times \vec{c}|$, $\hat{j} = \vec{c} \times (\vec{b} \times \vec{c}) / (|\vec{c}| |\vec{b} \times \vec{c}|)$, and $\hat{c} = \vec{c} / |\vec{c}|$. Note that, since the physical mass matrix is given by NN^T , the matrix N in Eq. (5) is arbitrary up to any RH rotation. The matrix in Eq. (6), however, is constructed entirely from rotational invariants, and is given in terms of physical quantities. There is still some ambiguity in Eq. (6), in that the diagonal elements can be of either sign, corresponding to the choice of orientation of the axes.

To diagonalize N , we first diagonalize its (23) submatrix explicitly by $R_L(23)$ and $R_R(23)$, with

$$\tan 2\theta_{23}^L = \frac{2|\vec{b} \cdot \vec{c}|}{c^2 - b^2}, \quad \tan 2\theta_{23}^R = \frac{2|\vec{b} \cdot \vec{c}| |\vec{b} \times \vec{c}|}{c^4 + (\vec{b} \cdot \vec{c})^2 - (\vec{b} \times \vec{c})^2}. \quad (7)$$

Further, the eigenvalues of N are μ_2 and μ_3 , with

$$\mu_{2,3}^2 = \frac{1}{2}(c^2 + b^2) \mp \Delta, \quad \Delta^2 = \frac{1}{4}(c^2 + b^2)^2 - |\vec{b} \times \vec{c}|^2. \quad (8)$$

Thus we find

$$N'_t = R_L(23)N_tR_R(23) = \begin{pmatrix} \vec{a} \cdot \hat{i} & \alpha_2 & \alpha_3 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad (9)$$

where $(\alpha_2, \alpha_3) = (\vec{a} \cdot \hat{j}, \vec{a} \cdot \hat{c})R_R(23)$. Note that the mixing angle θ_{23}^L is maximal when $b^2 = c^2$. Also, a mass hierarchy, $\mu_3 \gg \mu_2$, implies that $\vec{b} \cdot \vec{c} \approx c^2$, i.e., \vec{b} and \vec{c} are nearly parallel. Since $|\alpha_3| \leq a \ll \mu_3$, a LH (13) rotation $R_L(13)$, with $\theta_{13}^L \sim |\alpha_3|/\mu_3 \ll 1$, removes the (13) element of N'_t without changing the other elements appreciably. Summarizing, we see that, if $a^2 \ll c^2$,

$$\bar{N} = R_L(13)R_L(23)NRR_R(23) \simeq \begin{pmatrix} \vec{a} \cdot \hat{i} & \alpha_2 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad (10)$$

where $R_L(13) \simeq I$, R is defined in Eq. (6), and the rotation angles in $R_{L,R}(23)$ are given in Eq. (7). The final diagonalization of \bar{N} can be achieved by a combined LH and RH (12) rotation. In particular, maximal mixing is possible if $\alpha_2 \sim \mu_2$. We emphasize that as long as the (13) rotation $R_L(13)$ is small, the diagonalization of N can be decomposed into that of two 2×2 matrices. This is a very useful simplification.

Of particular interest in neutrino physics is the possibility of large mixing angles. Assuming $|\vec{a}| \ll |\vec{c}|$, which ensures that $\theta_{13}^L \ll 1$, maximal (23) rotation, $\theta_{23}^L = \pi/4$, is obtained if $|\vec{b}| = |\vec{c}|$. There are now two possibilities:

(A) $|\vec{b} \cdot \vec{c}| \gg |\vec{b} \times \vec{c}|^2$.

In this case, from Eq. (8) we find that $\mu_2^2 \approx |\vec{b} \times \vec{c}|^2 / (c^2 + b^2) \ll \mu_3^2$. Under this condition, we can have bimaximal mixing, $\theta_{12}^L \approx \pi/4$. This is achieved if $\alpha_2 \approx \mu_2$. Note that, for $|\vec{b} \times \vec{c}| \ll |\vec{b} \cdot \vec{c}|$, we have the bound $|\tan 2\theta_{23}^R| \leq |\vec{b} \times \vec{c}| / |\vec{b} \cdot \vec{c}|$. Thus $|\sin \theta_{23}^R \vec{a} \cdot \hat{c}| \ll \mu_2$, so the condition $\alpha_2 \approx \mu_2$ can also be written as $\vec{a} \cdot \hat{j} \approx \mu_2$.

(B) $|\vec{b} \cdot \vec{c}| \sim |\vec{b} \times \vec{c}|$.

In this case $\mu_2 \sim \mu_3$. All of the components of \vec{a} are smaller than μ_2 , in particular, $|\mu_2| \gg \alpha_2$. Thus, $\theta_{12}^L \sim |\alpha_2|/|\mu_2| \ll 1$, and we can only have single maximal mixing.

Knowing N , the above analysis yields the neutrino mixing angles θ_{ij}^L . In Eq. (4), N is given in terms of the parameters m_i , R_i , and V_{ij} . There are generally two classes of possibilities: (A) All R_i are of the same order, or (B) there is a strong hierarchy in the Majorana sector.

For the case of no Majorana hierarchy, $R_1 \sim R_2 \sim R_3$, we can find approximately the rotation angles, which give

$$\bar{R}_L = R_{12} \begin{pmatrix} m_1 R_1^2 & \\ & m_2 R_2^2 \\ & & m_3 R_3^2 \end{pmatrix} V_{21} R_{13} \begin{pmatrix} m_1 R_1^2 & \\ & m_2 R_2^2 \\ & & m_3 R_3^2 \end{pmatrix} V_{31} R_{23} \begin{pmatrix} m_2 R_2^2 & \\ & m_3 R_3^2 \\ & & m_3 R_3^2 \end{pmatrix} V_{32}, \quad (11)$$

so that $\bar{R}_L N N^T \bar{R}_L^T = N_{\text{diag}}^2$. In other words, the induced neutrino mixing angles are given approximately by $\theta_{IJ}^L \approx (m_I M_J / m_J M_I) V_{JI}$, $(I, J) = (1, 2), (2, 3), (1, 3)$. Unless there

is a stronger hierarchy in the Majorana sector in comparison to the Dirac sector, all neutrino mixing angles are negligible when V_{ij} is reasonably small.

To analyze the situation when there is a strong hierarchy in R_i , we will limit our discussion to situations which are physically interesting. For this purpose, it is actually more convenient to reverse our procedure above, i.e., given the physical mixing angles, we deduce the form of the matrix N . We note that, experimentally, it is quite suggestive that the neutrino masses are hierarchical, and that the neutrino mixing matrix $U_{\alpha i}$ ($\alpha = e, \mu, \tau; i = 1, 2, 3$) is given approximately by

$$U = R_{23}(\phi)R_{13}(\epsilon)R_{12}(\theta) \simeq \begin{pmatrix} c_\theta & s_\theta & \epsilon \\ -(s_\theta c_\phi + \epsilon c_\theta s_\phi) & (c_\theta c_\phi - \epsilon s_\theta s_\phi) & s_\phi \\ (s_\theta s_\phi - \epsilon c_\theta c_\phi) & -(c_\theta s_\phi + \epsilon s_\theta c_\phi) & c_\phi \end{pmatrix}, \quad (12)$$

where $\tan\phi \sim \mathcal{O}(1)$ and $\epsilon \ll 1$ [9]. We have kept only the leading terms of ϵ in U . Here, $\tan\theta$ can be either of order 1 (corresponding to the ‘‘bi-maximal’’ mixing scenario), or could be smaller (‘‘single-maximal’’ scenario). In the latter situation, we will discuss two possibilities: (A) $\epsilon \ll s_\theta$, (B) $\epsilon \gg s_\theta$. Appropriate approximations can be obtained for each of them.

For the case $\epsilon \ll s_\theta$, the matrix N is

$$N = U \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{pmatrix} = \begin{pmatrix} c_\theta n_1 & s_\theta n_2 & \epsilon n_3 \\ -s_\theta c_\phi n_1 & c_\theta c_\phi n_2 & s_\phi n_3 \\ s_\theta s_\phi n_1 & -c_\theta s_\phi n_2 & c_\phi n_3 \end{pmatrix}. \quad (13)$$

up to an arbitrary RH rotation. From Eq. (13), with the assumption $n_1 \ll n_2 \ll n_3$, it is readily seen that $b^2/s_\phi^2 \approx c^2/c_\phi^2 = n_2^2 = m_3^{\text{eff}}$, and

$$\frac{|\vec{b} \times \vec{c}|}{|\vec{b} \cdot \vec{c}|} \approx \frac{c_\theta n_2 n_3}{s_\phi c_\phi n_3^2} = \frac{c_\theta n_2}{s_\phi c_\phi n_3} \ll 1. \quad (14)$$

These two equations imply that \vec{b} and \vec{c} are nearly parallel to leading order in n_2/n_3 . Also $a \ll b, c$, so that our general analysis is applicable to N .

We can put Eq. (13) in the lower triangular form through a RH rotation. We have

$$N = \begin{pmatrix} a & 0 & 0 \\ \vec{b} \cdot \vec{a}/a & |\vec{a} \times \vec{b}|/a & 0 \\ \vec{c} \cdot \vec{a}/a & \frac{(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})}{a|\vec{a} \times \vec{b}|} & \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{|\vec{a} \times \vec{b}|} \end{pmatrix} \simeq \begin{pmatrix} \sqrt{c_\theta^2 n_1^2 + s_\theta^2 n_2^2 + \epsilon^2 n_3^2} & 0 & 0 \\ (c_\phi s_\theta c_\theta n_2^2 + s_\phi \epsilon n_3^2)/a & s_\theta s_\phi n_2 n_3/a & 0 \\ (-s_\phi s_\theta c_\theta n_2^2 + c_\phi \epsilon n_3^2)/a & s_\theta c_\phi n_2 n_3/a & n_1/s_\theta s_\phi \end{pmatrix}. \quad (15)$$

Note that, as in Eq. (6), the entries of N in Eq. (15) are uniquely determined by the physical neutrino parameters, while Eq. (13) is not.

A similar analysis can be done for $\epsilon \gg s_\theta$. Here, Eqs. (13) and (15) are no longer valid. To lowest order in s_θ and ϵ and with $\phi = \pi/4$, we obtain

$$N \simeq \begin{pmatrix} \epsilon n_3 & 0 & 0 \\ \frac{n_3}{\sqrt{2}} \left(1 + \frac{s_2 \theta n_2^2}{2\epsilon n_3^2} \right) & \frac{n_2}{\sqrt{2}} \left(c_\theta - \frac{s_\theta}{\epsilon} \right) & 0 \\ \frac{n_3}{\sqrt{2}} \left(1 - \frac{s_2 \theta n_2^2}{2\epsilon n_3^2} \right) & -\frac{n_2}{\sqrt{2}} \left(c_\theta + \frac{s_\theta}{\epsilon} \right) & \frac{\sqrt{2}n_1}{\epsilon} \end{pmatrix}. \quad (16)$$

We may now compare Eq. (4) to Eqs. (15) and (16). Given the neutrino parameters, n_i , s_θ , and ϵ , this method gives immediately the Majorana masses M_i , assuming that we may identify the Dirac masses with the quark masses. We emphasize that this comparison is viable because the lower triangular form for the N matrix is unique, there being no more ambiguities due to RH rotations. Also, the structure in these equations shows immediately that they are consistent only if there is a very strong hierarchy, $R_1 \gg R_2 \gg R_3$. Next, we will discuss in detail the physical consequences of this result.

Assuming hierarchical Dirac neutrino masses and small RH rotation angles, it is seen from Eqs. (4) that the N matrix is naturally of lower-triangular form. The discussion of physical constraints can be done most conveniently in this lower triangular basis. In this section, we will start from Eqs. (4), (15), and (16) and present the results for the neutrino parameters derived from different solutions to the solar and atmospheric neutrino observations. To be concrete, we will use $\phi = 45^\circ$, and $m_i = (m_u, m_c, m_t)$ (at the seesaw scale), although different choices can be accommodated.

(A) $\epsilon \ll s_\theta$:

Depending on ϵ , Eq. (15) gives rise to three different patterns, with the largest elements located in the (a) (2,2), (3,2), (b) (i, j) , $i = 1, 2$, (c) (2,1), (3,1) positions, respectively. It can be seen that only type (a) with $\epsilon \ll m_2^{\text{eff}}/m_3^{\text{eff}}$ is natural, which we will concentrate on. In this limit, with $\tan\theta > n_1/n_2$,

$$N \simeq \begin{pmatrix} s_\theta n_2 & 0 & 0 \\ \frac{1}{\sqrt{2}} c_\theta n_2 & \frac{1}{\sqrt{2}} n_3 & 0 \\ -\frac{1}{\sqrt{2}} c_\theta n_2 & \frac{1}{\sqrt{2}} n_3 & \sqrt{2} \frac{n_1}{s_\theta} \end{pmatrix}, \quad (17)$$

where we identify $n_i^2 = m_i^{\text{eff}}$. Comparison to Eq. (4) immediately yields (with $V_{11} \approx V_{22} \approx V_{33} \approx 1$)

$$\begin{aligned} m_2^{\text{eff}} = n_2^2 &= (R_1 m_1)^2 / s_\theta^2 = m_u^2 / s_\theta^2 M_1, \\ m_3^{\text{eff}} = n_3^2 &= 2(R_2 m_2)^2 = 2m_c^2 / M_2, \\ m_1^{\text{eff}} = n_1^2 &= s_\theta^2 (R_3 m_3)^2 / 2 = s_\theta^2 m_t^2 / 2M_3. \end{aligned} \quad (18)$$

$$-V_{31}/V_{21} = V_{32}/V_{22} = m_2/m_3 = m_c/m_t. \quad (19)$$

Quite surprisingly, m_3^{eff} scales as m_c^2 rather than m_t^2 . This is because in Eq. (17) the (22) element is one of the largest. This gives a scale for M_2 much lower than one would expect,

$$M_2 \approx \frac{2m_c^2}{\sqrt{\Delta m_{\text{atm}}^2}} = 6 \times 10^9 \text{ GeV}, \quad (20)$$

where we have used $m_c(M_2) \approx 0.4 \text{ GeV}$, and $m_3^{\text{eff}} = \sqrt{\Delta m_{\text{atm}}^2} = \sqrt{3} \times 10^{-3}$, which will also be used in the following.

Note also that V_{32} is independent of the Majorana mass M_i and that it scales linearly with m_c/m_t . This means that the Majorana sector decouples owing to its very large hierarchy. Similarly, we have

$$\frac{V_{21}}{V_{11}} = \frac{m_1}{\sqrt{2}\tan\theta m_2} = \frac{m_u}{\sqrt{2}\tan\theta m_c}, \quad (21)$$

$$\frac{V_{31}}{V_{11}} = -\frac{m_1}{\sqrt{2}\tan\theta m_3} = -\frac{m_u}{\sqrt{2}\tan\theta m_t}. \quad (22)$$

From Eqs. (19), (20), (21), we see that all of the RH angles θ_{ij}^R are small.

The other two heavy Majorana mass values depend on the different solutions to the solar neutrino problem [10]. We will use $m_3^{\text{eff}} = \sqrt{\Delta m_{\text{atm}}^2}$, $m_2^{\text{eff}} = \sqrt{\Delta m_{\text{solar}}^2}$, while m_1^{eff} is not known. However, we can derive a bound for M_3 using the parameter $r = m_2^{\text{eff}}/m_1^{\text{eff}} \gg 1$. From Eq. (18), we note that M_1 and M_3 , in addition to the usual Dirac mass squared, depend sensitively on s_θ as well as on the effective neutrino masses. This can be seen directly in the following equations:

$$\frac{M_1}{M_2} = \frac{m_u^2 m_3^{\text{eff}}}{2m_c^2 s_\theta^2 m_2^{\text{eff}}}, \quad \frac{M_2}{M_3} = \frac{4m_c^2 m_1^{\text{eff}}}{m_t^2 s_\theta^2 m_3^{\text{eff}}}. \quad (23)$$

We will now turn to numerical estimates with inputs coming from the three solutions to the solar neutrino problem, vacuum oscillations (VO), large angle Mikheyev-Smirnov-Wolfenstein (MSW) (LAM), and small angle MSW (SAM).

(A1) VO

The neutrino masses and mixings are given by [10] $\theta = 45^\circ$, $m_2^{\text{eff}} \approx \sqrt{\Delta m_{\text{solar}}^2} = \sqrt{7} \times 10^{-11} \text{ eV}$. From Eq. (23), we find

$$M_1 = 2m_u^2/m_2^{\text{eff}} \approx 5 \times 10^8 \text{ GeV}, \quad (24)$$

$$M_3/r \approx 4 \times 10^{17} \text{ GeV} \quad (r \equiv m_2^{\text{eff}}/m_1^{\text{eff}} \gg 1). \quad (25)$$

(A2) LAM

Here we take [10],

$$\sin^2 2\theta_{\text{solar}} \approx 0.8, \quad m_2^{\text{eff}} \approx \sqrt{\Delta m_{\text{solar}}^2} = \sqrt{3} \times 10^{-5} \text{ eV}. \quad (26)$$

Going through the same analysis as in the VO case, we have $M_1 \approx 1 \times 10^6 \text{ GeV}$, $M_2 \approx 6 \times 10^9 \text{ GeV}$, $M_3/r \approx 4 \times 10^{14} \text{ GeV}$ ($r \gg 1$).

(A3) SAM-I

We now have [10]

$$\sin^2 2\theta_{\text{solar}} \approx 5 \times 10^{-3}, \quad m_2^{\text{eff}} \approx \sqrt{\Delta m_{\text{solar}}^2} = \sqrt{5} \times 10^{-6} \text{ eV}. \quad (27)$$

They imply $M_1 \approx 7 \times 10^8 \text{ GeV}$, $M_2 \approx 6 \times 10^9 \text{ GeV}$, $M_3/r \approx 4 \times 10^{12} \text{ GeV}$ ($r \gg 1$). These results were obtained under the assumption that $\epsilon \rightarrow 0$. Thus, if $\epsilon \gg n_2/n_3$, Eq. (17) is no longer valid. Also, if $\epsilon \gg s_\theta$, Eq. (15) should be replaced by Eq. (16). This condition is actually very likely to be valid for the case of the SAM solution. We treat this case in detail next.

(B) $\epsilon \gg s_\theta$ (SAM-II):

The parameters for θ and m_2^{eff} are taken as in SAM-I. However, we now use Eq. (16) instead of Eq. (15). Also, the numerical result depends on the value of ϵ , which, for definiteness, we will take to be $\epsilon = 0.1$. We obtain $M_1 \approx 2 \times 10^6 \text{ GeV}$, $M_2 \approx 1 \times 10^{11} \text{ GeV}$, $M_3/r \approx 2 \times 10^{13} \text{ GeV}$ ($r \gg 1$). In summary, we find that if the physical neutrino parameters are known, we can obtain the Majorana masses directly when we identify the Dirac masses with the quark masses. The Majorana masses have rather strong dependence on the physical mixing angles, so that their values span a wide range. Numerically, it is noteworthy that $M_2 \approx 6 \times 10^9 \text{ GeV}$ for a wide range of parameters. Also, the VO solution for M_3 ($\gg 4 \times 10^{17} \text{ GeV}$) seems too large for it to be viable. Finally, in some cases M_1 can be rather low ($\sim 10^6 \text{ GeV}$).

The observation that neutrino masses are tiny has a natural explanation in the seesaw model. However, its complicated structure also means that the neutrino mixing angles do not derive simply from the seesaw components, viz., the Dirac matrix m_D and the Majorana matrix M^{-1} . In fact, if we write $m_D = U_0 m_D^{\text{diag}} V_0$, $M^{-1} = U_M (M_{\text{diag}}^{-1/2})^2 U_M^T$, m_{eff} depends, roughly speaking, quadratically on all of the components that we displayed.

In this paper, we analyze the problem in several steps. First, we write $m_{\text{eff}} = NN^T$, so that N depends linearly on the aforementioned components. But N has the further ambiguity of an arbitrary RH rotation. We will eliminate this ambiguity by reducing N to the triangular form. The lower triangular form arises naturally if m_D has a hierarchy. However, the upper triangular form is the easiest to use in order to extract the LH, physical, neutrino mixing angles. By expressing the triangular matrix elements in terms of RH rotational invariants, it is easy to transform back and forth between the different forms of N . Thus, given the parameters in m_D and M , we can deduce the neutrino mixing angles. Conversely, given the physically plausible values of the neutrino masses and mixing angles, we can obtain the constraints that must be satisfied by the parameters in m_D and M . Experimentally, we have a fairly good idea about the intrinsic neutrino parameters. Their masses are most probably hierarchical, the (23) mixing angle is almost maximal, and the (13) mixing angle is small. From these parameters, we can infer the properties of the Majorana masses and the RH mixing angles of m_D , if we assume that m_D is similar to the u -quark mass matrix, i.e., m_D has small LH mixing angles and $m_i \approx (m_u, m_c, m_t)$. It was found that there must be a large hierarchy in M , proportional not only to the ratios of Dirac masses squared, but also

to the squares of the mixing angles. In addition, the RH mixing angles in m_D combined with M^{-1} must be very small, and must be equal to the mass ratios in m_D . Physically, the large hierarchy in M implies the existence of intermediate mass scales. It would be most interesting if these conclusions can be corroborated by other sources.

Using solar neutrino solutions as inputs, we calculated the individual Majorana masses. Because of their strong dependence on the physical neutrino mixing angles, a wide range of values was found. In particular, M_3 is so large ($\gg 4 \times 10^{17}$ GeV) for the VO solution which makes it highly disfavored. In this work we have not treated the issue of renor-

malization, although it can be shown that the RGE effects are small [11]. We have also not discussed the case when either or both of m_D and M are complex. Although the 2×2 problem can be solved, the 3×3 case does not seem to have a simple solution. Nevertheless, it can be shown that, with hierarchical masses and small angles, complex phases do not contribute significantly [4]. We hope to return to this problem in the future.

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