Noncommutative bion core

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We examine noncommutative solutions of the non-Abelian theory on the world-volume of N coincident D-strings. These solutions can be interpreted in terms of noncommutative geometry as funnels describing the non-Abelian D-string expanding out into an orthogonal D3-brane. These configurations are ''dual'' to the bion solutions in the Abelian world-volume theory of the D3-brane. In the latter, a charge N magnetic monopole describes N D-strings attached to the D3-brane with a spike deformation of the world volume. The noncommutative D-string solutions give a reliable account of physics at the core of the monopole, where the bion description is expected to break down. In the large N limit, we find good agreement between the two points of view, including the energy, couplings to background fields, and the shape of the funnel. We also study fluctuations traveling along the D-string, again obtaining agreement in the large N limit. At finite N, our results give a limit on the number of modes that can travel to infinity along the N D-strings attached to the D3-brane.

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I. INTRODUCTION

D-branes [1,2,3] have become important tools in the quest to develop a full understanding of string theory. The low energy action describing the dynamics of test D-branes consists of two parts: the Born-Infeld action [4] and the Chern-Simons action [5,6]. This nonlinear action reliably captures the physics of D-branes with great accuracy. One interesting aspect of this story is that one finds that the D-brane action supports solitonic configurations describing lower-dimensional branes protruding from the original D-brane [7,8,9].

For instance, in the case of a D3-brane, one finds spike solutions corresponding to fundamental strings and D-strings (as well as strings) attached to the D3-brane. These configurations have both the world-volume gauge fields and transverse scalar fields excited. The gauge field corresponds to that of a point charge arising from the end-point of the attached string, i.e., an electric charge for a fundamental string and a magnetic monopole charge for a D-string. The scalar field on the other hand, represents a deformation of the geometry of the D3-brane, caused by attaching the strings.

Naively, the range of validity of this analysis is limited to a range far from the core of the spike where the fields on the D3-brane world-volume are slowly varying. This range can be increased by increasing the number N of attached strings. (Although N cannot be too large if we are to ignore gravitational effects.) However, the results obtained seem to have a larger regime of validity, maybe even all the way to the center of the spike where it protrudes an infinite distance from the original position of the D3-brane. This can partly be understood in light of the fact that the basic BPS spike is a solution to the full derivative-corrected equations of motion following from string theory [10]. Even the dynamics of the spike, as probed through small fluctuations, agree with expected string behavior [11,12,13]. However, Kastor and Traschen [13] showed that certain fluctuation modes that are inherently three-dimensional also appear to propagate to infinity in this picture, and hence the spike seems to retain its three-brane character even at large distances.

The purpose of this paper is to study the ''dual'' description of a system of N D-strings attached to a D3-brane. This system has been analyzed previously in [14,15,16,17,18,19] using the connection [14] between the Nahm equations for Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles [20] and the BPS condition for the non-Abelian D-string theory. This theory contains noncommutative solutions describing the D-strings expanding out in a funnel-like geometry to become an orthogonal D3-brane. These solutions are valid in a regime complimentary to the bion spikes discussed above. That is, the solutions will accurately describe the physics very close to the center of the spike, or alternatively very far from the D3-brane. The two approaches, i.e., the D3-brane spikes and the D-string funnels, turn out to agree exactly in the large N limit, while we get new insights into the physics at finite N near the core of the spike from the D-string funnels.

In the next section we will quickly review the full non-Abelian D-brane action, followed by an outline of the remainder of this paper.

II. NON-ABELIAN BRANE ACTION

Our starting point is the non-Abelian world-volume action describing N coincident D-strings, whose complete form was recently discussed by Myers [21], as well as Taylor and Van Raamsdonk [22]. The action consists of two parts: the Born-Infeld action

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$$S_{BI}=-T_1\int d^2\sigma \mathrm{STr}[e^{-\phi}\sqrt{-\det(P[E_{ab}+E_{ai}(Q^{-1}-\delta)^{ij}E_{jb}]+\lambda F_{ab})\det(Q^i_j)}], \tag{1}$$

with

$$\lambda=2\pi\ell_s^2, \quad E_{\mu\nu}=G_{\mu\nu}+B_{\mu\nu}, \quad \text{and}$$

$$Q^i_j\equiv\delta^i_j+i\lambda[\Phi^i,\Phi^k]E_{kj}, \tag{2}$$

and the Chern-Simons action

$$S_{CS}=\mu_1\int \mathrm{STr}\left(P\left[e^{i\lambda i_\Phi i_\Phi}\left(\sum C^{(n)}e^B\right)\right]e^{\lambda F}\right). \tag{3}$$

Implicitly, Eqs. (1) and (3) employ static gauge where the two worldsheet coordinates are identified with two spacetime coordinates. We have chosen $\tau=t=x^0$ and $\sigma=x^9$. In these expressions, $P[\cdots]$ denotes the pullback of the enclosed spacetime tensors to the worldsheet. The Φ^i, $i=1,\ldots,8$, are the transverse scalars, which are $N\times N$ matrices in the adjoint representation of the $U(N)$ worldsheet gauge symmetry. The notation i_Φ denotes the interior product by Φ^i regarded as a vector in the transverse space, e.g., acting on a two-form $C^{(2)}=\frac{1}{2}C^{(2)}_{\mu\nu}dx^\mu dx^\nu$, we have

$$i_\Phi i_\Phi C^{(2)}=\frac{1}{2}[\Phi^i,\Phi^j]C^{(2)}_{ji}. \tag{4}$$

In both Eqs. (1) and (3), the gauge trace indicated by $\mathrm{STr}(\cdots)$ is a symmetrized trace. The precise prescription proposed in Ref. [21] was that inside the trace one takes a symmetrized average over all orderings of the F_{ab}, $D_a\Phi^i$, $i[\Phi^i,\Phi^j]$, and also the individual Φ^i appearing in the functional dependence of the background supergravity fields. We refer the reader to Ref. [21] for more details on these actions.

In Ref. [21], the D-particle version of this action was used to analyze the behavior of N D-particles when placed in a constant background Ramond-Ramond (RR) field $F^{(4)}$. This RR four-form is the field strength associated with D2-brane charge, and ordinarily D0-branes would be considered neutral with respect to this field. However, new couplings to the corresponding RR potential $C^{(3)}$ appear in the non-Abelian Chern-Simons action (3) of the D-particles. As a result, the D-particles are ''polarized'' by the external field into a noncommutative two-sphere, which can be interpreted as a spherical D2-D0 bound state. This analysis can readily be generalized to Dp-branes in a background of constant $F^{(p+4)}$. Starting with a flat Dp-brane with spatial geometry R^p, it will be energetically favorable for the brane to expand into a noncommutative $R^p\times S^2$ structure. For instance, in the case of D-strings, the Chern-Simons action (3) involves a coupling

$$i\lambda\mu_1\int \mathrm{Tr}P[i_\Phi i_\Phi C^{(4)}]. \tag{5}$$

Using the same manipulations as in Ref. [21], assuming $F^{(5)}$ constant (in space), this term produces a new contribution to the scalar potential of the form

$$\frac{i}{3}\lambda^2\mu_1\int dt d\sigma \mathrm{Tr}(\Phi^i\Phi^j\Phi^k)F^{(5)}_{t\sigma ijk}(t). \tag{6}$$

As in Ref. [21], one can find solutions for Φ^i in terms of N dimensional representations of the $SU(2)$ algebra, describing a D-string with an $R^1\times S^2$ structure. The energy and radius of this solution can also be calculated from a dual perspective from the D3-brane action with appropriate world volume gauge fields excited, corresponding to dissolved D-strings. The two approaches exactly agree in the large N limit. Similar calculations in the dual D3-brane theory appear in Ref. [23].

In this paper we consider similar solutions of the non-Abelian D-string theory. The scalar field configuration again has a similar interpretation in terms of noncommutative geometry such that spatial slices have the topology $R\times S^2$. However, now the Φ^i matrices depend on the worldsheet coordinate σ, and so the radius of the two-sphere can vary along the length of the D-string. In Sec. III A, we study possible solutions in a flat space background, and we obtain configurations corresponding to N D-strings attached to a D3-brane. We compare features of this solution—such as total energy, couplings to background fields, and the shape of the configuration—to the corresponding aspects in the D3-brane spike. In the large N limit, there is an exact correspondence. Note that there are no nontrivial background fields here, and so these constructions are quite distinct from the dielectric effect discussed above and in Ref. [21].

Following the initial papers on the D3-brane spike [7,8,9], there has been a considerable literature studying various generalizations. This includes the construction of dyonic spikes describing (p,q)-strings [15], analysis of double-funnel solutions [7,8,24], solutions in an additional D3-brane supergravity background [25,26], and solutions with fundamental strings dissolved in the D3-brane [27]. We give a sample of how these situations can be described from the dual D-string picture. In Sec. III B, we show the existence of the double-funnel solutions describing D-strings stretched between a D3-brane and an anti-D3-brane (or another D3-brane). In Sec. III C, we give a brief description of how to construct (p, q)-string configurations, while Sec. III D demonstrates how the BPS funnels survive even when the system is put in a supergravity background corresponding to a collection of D3-branes.

A D-string stretched between two D3-branes is represented as a non-Abelian BPS monopole in the D3-brane theory. The ''dual'' D-string description provides a physical realization [14] of the Nahm equations [20]. This interpretation has already received extensive attention in the literature

[14,15,16,17,18,19]. In this paper we consider the primarily the case of the D-string funnel, in which one of the D3-branes has moved off to infinity. Our focus is on the large N limit at the point in monopole moduli space where all the monopoles coincide. This allows us to make a direct comparison with the bion spikes in the $U(1)$ theory on the remaining D3-brane. In particular, using the new terms in the non-Abelian Chern-Simons action (3) [21,22], we can explicitly show that the funnel has couplings corresponding to a D3-brane.

The dynamics of the D3-brane spike has also been considered [11,12,13]. In Sec. IV, we analyze small fluctuations propagating along the D-string in the funnel configuration. We study both modes that are transverse and parallel to the D3-brane. Again, in the large N limit, we obtain exact agreement with the D3-brane analysis [11,12,13], in spite of the fact that the present calculation involves noncommuting matrices and looks rather different. At finite N, we find significant discrepancies with the D3-brane analysis [13] for the higher l modes. In particular, due to the noncommutative character of the funnel, the spectrum is truncated at $l_{\max}=N-1$. This suggests a resolution of the puzzle appearing in Ref. [13], which was mentioned above.

We conclude in Sec. V with some further discussion and comments on our results.

III. D3-BRANES FROM D-STRINGS

In this section we will describe various solutions in the non-Abelian world-volume theory of a D-string corresponding to the D-string opening up into a D3-brane.

A. The BPS funnel

In flat background, the Chern-Simons part (3) of the D-string action plays no role, while the Born-Infeld action (1) reduces to [28,21][1]

$$S=-T_1\int d^2\sigma \mathrm{STr}\sqrt{-\det(\eta_{\beta}+\lambda^2\partial_a\Phi^i Q_{ij}^{-1}\partial_b\Phi^j)\det(Q^{ij})}, \tag{7}$$

where

$$Q^{ij}=\delta^{ij}+i\lambda[\Phi^i,\Phi^j]. \tag{8}$$

Implicitly here, we have set the world-volume gauge field to zero. This is consistent with the equations of motions for the scalar field configurations considered here. However, the gauge field will play an essential role in Sec. III C below. Recall that we are using static gauge and have chosen the world-volume coordinates to be $\tau=t=x^0$ and $\sigma=x^9$. Expanding this action (7) to leading order (in λ), yields the usual non-Abelian scalar action

$$S\simeq-T_1\int d^2\sigma\left(N+\frac{\lambda^2}{2}\mathrm{Tr}\left(\partial^a\Phi^i\partial_a\Phi^i + \frac{1}{2}[\Phi^i,\Phi^j][\Phi^j,\Phi^i]\right)+\cdots\right). \tag{9}$$

Varying this action yields the following equation of motion:

$$\partial^a\partial_a\Phi^i=[\Phi^j,[\Phi^j,\Phi^i]]. \tag{10}$$

Now we are looking for solutions which represent the D-string expanding into a D3-brane, analogous or ''dual'' to the bion solutions of the D3-brane theory [7,8]. The corresponding geometry would be a long funnel where the cross section at fixed σ has the topology of a two-sphere. Hence motivated by the noncommutative two-sphere constructions of Refs. [29,21], we consider the spherically symmetric ansatz

$$\Phi^i=\hat{R}(\sigma)\alpha^i, \quad i=1,2,3, \tag{11}$$

where the α^i give some $N\times N$ matrix representation of the $SU(2)$ algebra

$$[\alpha^i,\alpha^j]=2i\varepsilon_{ijk}\alpha^k. \tag{12}$$

Now at fixed σ, this ansatz for non-Abelian scalars describes a noncommutative two-sphere with a physical radius given by

$$R(\sigma)^2=\frac{\lambda^2}{N}\sum_{i=1}^{3}\mathrm{Tr}[\Phi^i(\sigma)^2]=\lambda^2 C\hat{R}(\sigma)^2. \tag{13}$$

Here C is the quadratic Casimir of the particular representation of the generators under consideration, defined by the identity

$$\sum_{i=1}^{3}(\alpha^i)^2=CI_N, \tag{14}$$

where I_N is the $N\times N$ identity matrix. For example, $C=N^2-1$ for the irreducible $N\times N$ representation.

Now given the ansatz (11), the matrix equations of motion (10) reduce to a single scalar equation

$$\hat{R}''(\sigma)=8\hat{R}(\sigma)^3. \tag{15}$$

Considering a trial solution, $\hat{R}\propto\sigma^p$, yields

[1]In these expressions, the transverse space indices are raised and lowered with $g_{ij}=\delta_{ij}$ and $g^{ij}=\delta^{ij}$.

$$\hat{R}(\sigma)=\pm\frac{1}{2(\sigma-\sigma_\infty)}, \tag{16}$$

where we have used the translation invariance of Eq. (15) to introduce the integration constant σ_∞. Since the second order equation (15) should have a general solution with two integration constants, it is clear that Eq. (16) is not the most general solution—we will leave this solution to the next section. However, this solution (16) indeed describes the desired funnel, with the D-string opening up into a three-brane at $\sigma=\sigma_\infty$, where the radius of the funnel diverges. As it stands Eqs. (11) and (16) only represent a solution of the leading order equations of motion (10), and so naively one expects that it should only be valid for small $\hat{R}$ or small radius. However, we will find that this configuration also solves the full equations of motion extremizing the action (7).

Before plunging into the full equations of motion, let us investigate the supersymmetry of the funnel configuration above. Following the analysis of Callan and Maldacena [7], we investigate the linearized supersymmetry conditions, which strictly speaking would only apply for the leading order action (9).[2] We may write the linearized conditions as

$$\Gamma^{\mu\nu}F_{\mu\nu}\epsilon=0, \tag{17}$$

where μ,ν are ten-dimensional indices and ϵ is some constant spinor. The latter world-volume supersymmetry parameter also satisfies the usual D-string projection [3]: $\Gamma^{09}\epsilon=\epsilon$. Note that ϵ transforms a spinor under both the $SO(1,1)$ Lorentz transformations of the D-string world-volume theory, and the $SO(8)$ rotations of the transverse space. Hence it is reasonable to multiply ϵ by ten-dimensional Dirac matrices, such as $\Gamma^{\mu\nu}=[\Gamma^\mu,\Gamma^\nu]/2$. Following the standard notation (see, e.g., Refs. [31]), where F_{ab} denotes the world-volume gauge field strength which vanishes in the present case, one also has

$$F_{ai}=D_a\Phi^i,\quad F_{ij}=i[\Phi^i,\Phi^j]. \tag{18}$$

Hence Eq. (17) yields

$$(2\Gamma^{\sigma i}D_\sigma\Phi^i+i\Gamma^{jk}[\Phi^j,\Phi^k])\epsilon=0. \tag{19}$$

This condition can be solved by spinors satisfying the projection

$$\Gamma^{\sigma123}\epsilon=\pm\epsilon, \tag{20}$$

provided that the scalars satisfy the Nahm equations (14)

$$D_\sigma\Phi^i=\pm\frac{i}{2}\varepsilon^{ijk}[\Phi^j,\Phi^k]. \tag{21}$$

Now inserting our ansatz (11) this implies

$$\hat{R}'=\mp2\hat{R}^2. \tag{22}$$

However, the solution of this equation is precisely that given in Eq. (16). Hence, we conclude that the funnel configurations given by Eqs. (11) and (16) are in fact BPS solutions preserving 1/2 of the supersymmetry of the D-string theory (9). From Ref. [24], we can infer that BPS solutions of the leading order theory (9) are also BPS solutions of the full non-Abelian Born-Infeld action (7). That is, the funnel solutions will also solve the full equations of motion, as we will explicitly demonstrate below. In a related discussion, Ref. [15] showed, in the context of the full non-Abelian D-string theory (7), that supersymmetric configurations satisfying Eq. (21) minimize the energy of the system.

We begin by substituting our ansatz (11) directly into the action (7), and find that it becomes

$$S=-T_1\int d^2\sigma\,\mathrm{STr}\sqrt{(1+\lambda^2\alpha^i\alpha^i(\hat{R}')^2)(1+4\lambda^2\alpha^j\alpha^j\hat{R}^4)}, \tag{23}$$

where both i and j are summed over 1,2,3. In deriving this result, we have eliminated certain combinations of matrices from the determinants (and inverses) which will cancel under the symmetrized trace. In the remaining expression, symmetrization applies to each of the individual generators α^i appearing there. Now extremizing this action (23) with respect to variations of $\hat{R}$ yields an equation of motion which may be written as

$$\frac{1}{\hat{R}'}\frac{d}{d\sigma}\mathrm{STr}\sqrt{\frac{1+4\lambda^2\alpha^j\alpha^j\hat{R}^4}{1+\lambda^2\alpha^i\alpha^i(\hat{R}')^2}}=0. \tag{24}$$

If the radius profile satisfies the supersymmetry constraint (22), then the expression under the square root is simply the identity and it follows that the equation of motion is satisfied. Hence the supersymmetric funnel solutions are in fact solutions of the full non-Abelian equations of motion (24). Note that we were able to derive this result without making an expansion (in λ) of the matrix expression in Eq. (24) and explicitly implementing the symmetric trace on the individual terms in this expansion.

It is clear that the funnel solution, Eqs. (11) and (16), describes the non-Abelian D-string opening up into a three-brane on the (x^1,x^2,x^3) hypersurface at $\sigma=\sigma_\infty$. While the natural intuition is that the latter is actually a D3-brane, it remains to be demonstrated. We begin by comparing our funnel solution to the D3-brane monopole or spike [7]. For these purposes, we will focus on the funnel where the α^i are chosen as the irreducible $N\times N$ representation, with $C=N^2-1$. In this case, the radius (13) becomes (16)

$$R=\frac{N\pi l_s^2}{\sigma-\sigma_\infty}\sqrt{1-1/N^2}. \tag{25}$$

To leading order for large N, this yields precisely (including numerical coefficient) the corresponding formula for the height of D3-brane spike [7], i.e.,

[2] Supersymmetry conditions for the full non-Abelian Born-Infeld action (7) would be expected to be more complicated [28,30].

$$\sigma-\sigma_\infty=\frac{N\pi l_s^2}{R}. \tag{26}$$

This remarkable agreement is perhaps more than one should expect, since the D3-brane analysis is strictly speaking only valid for R large, while the current calculations will be reliable for small R. We will comment more on this in the discussion section.

To further corroborate the fact that our funnel yields a D3-brane, let us compare the energy and some couplings to those obtained from the dual D3-brane action. Given our static solution, the energy is easily derived from the D-string action (23). Note that using the supersymmetry condition (22), the two expressions under the square root are equal and hence the action is ''linearized'' [15]. We are then left with

$$\begin{aligned}E&=T_1\int d\sigma \mathrm{STr}|1+4\lambda^2\alpha^i\alpha^i\hat{R}^4|\\&=2NT_1\int d\sigma\hat{R}^2|\hat{R}'|\left[\left(\frac{d\sigma}{d\hat{R}}\right)^2+\lambda^2C\right],\end{aligned} \tag{27}$$

where we have repeatedly applied $\hat{R}'=\pm 2\hat{R}^2$ in producing the second expression. We can further manipulate this result by introducing the physical radius $R=\lambda\sqrt{C}|\hat{R}|$, as well as using $T_1=4\pi^2 l_s^2 T_3$ to put this expression in the form

$$E=T_3\frac{N}{\sqrt{C}}\int 4\pi R^2 dR\left[\left(\frac{d\sigma}{dR}\right)^2+1\right]. \tag{28}$$

In the dual D3-brane picture, the energy of any (spherically symmetric) BPS configuration[3] is simply given by [7,15]

$$E=T_3\int d^3x[1+(\nabla\sigma)^2]=T_3\int 4\pi R^2 dR\left[1+\left(\frac{d\sigma}{dR}\right)^2\right]. \tag{29}$$

If we chose the irreducible representation, we have $N/\sqrt{C}=(1-1/N^2)^{-1/2}$, and hence the energy calculated in these two formulations agrees up to N^{-2} corrections for large N.

If a D3-brane is emerging in the funnel solution, this configuration should act as a source for the RR four-form potential, $C^{(4)}$. Such a coupling arises in the Chern-Simons action (3) because of the non-Abelian expectation value of the scalars in this solution. To leading order, we can focus on the interaction given in Eq. (5), which yields

$$\begin{aligned}i\lambda\mu_1&\int \mathrm{Tr}P[i_\Phi i_\Phi C^{(4)}]\\&=\frac{i\lambda^2\mu_1}{2}\int d^2\sigma C^{(4)}_{tkji}(\tau,\sigma)\mathrm{Tr}(\partial_\sigma\Phi^k[\Phi^i,\Phi^j])+\cdots\\&=\mp i\mu_3\frac{N}{\sqrt{C}}\int dt4\pi R^2 dR C^{(4)}_{t123}(t,R).\end{aligned} \tag{30}$$

Here we have used the ansatz (11) and $R=\lambda\sqrt{C}|\hat{R}|$, as well as $\mu_1=4\pi^2 l_s^2\mu_3$ and $\mathrm{Tr}(\alpha^i\alpha^j)=(N/3)C\delta^{ij}$. In the dual D3-brane formulation, essentially the same expression arises in the standard coupling to the RR four-form

$$\begin{aligned}\mu_3\int P[C^{(4)}]&=\mu_3\int dtdRd\theta d\phi C^{(4)}_{tijk}\partial_\sigma x^i\partial_\theta x^j\partial_\phi x^k+\cdots\\&=\mu_3\int dt4\pi R^2 dR C^{(4)}_{t123}(t,R).\end{aligned} \tag{31}$$

So once again if we chose the irreducible representation, we would have $N/\sqrt{C}=(1-1/N^2)^{-1/2}$, and for large N Eqs. (30) and (31) agree up to N^{-2} corrections. It is interesting that in deriving this agreement for the RR coupling, we only used the basic ansatz (11), but not the details of the funnel solution (16). Hence this result will hold more generally, and in particular it still holds in the following sections. In Eq. (30), the minus (plus) sign arises if $\hat{R}$ is positive (negative). Hence this calculation shows that the minus solution in Eq. (16) corresponds to the D-string opening up into a D3-brane (assuming we approach from $\sigma>\sigma_\infty$), while the plus solution has the opposite orientation and corresponds to an anti-D3-brane.

To summarize this section, we have shown that by allowing for suitable boundary conditions in the non-Abelian D-string theory, the latter can ''grow'' into a D3-brane. This construction is a dual formulation of the BPS magnetic monopole in the Abelian D3-brane theory which describes a D-string spike growing out of the three-brane surface. In the present calculation, we see that the geometry at the core of the spike is noncommutative, with the level of discreteness set by N, the number of D-strings. In these last few calculations, we have focused on using the irreducible $N\times N$ representation of the $SU(2)$ generators (12), and we found good quantitative agreement at large N between the two formulations. These calculations indicate that the funnel solution describes the N D-strings expanding into a single fundamental D3-brane. Using reducible representations would correspond to creating several (independent) D3-branes from the same N D-strings. Paralleling the constructions in Ref. [21], one could then construct multicenter funnels located at different positions in the (x^1,x^2,x^3) hypersurface.

B. Double funnels

With the ansatz (11), the leading-order matrix equations became $\hat{R}''=8\hat{R}^3$ in Eq. (15). As a first step to generating the most general solution, we integrate this equation as

[3] Note that this expression holds regardless of whether electric fields, magnetic fields, or both are excited on the D3-brane. Hence the agreement found here is more generally applicable.

$$(\hat{R}')^2 = 4(\hat{R}^4 - \hat{R}_0^4), \tag{32}$$

where $\hat{R}_0^4$ is an arbitrary integration constant. In principle then, integrating once more yields the general solution

$$\sigma = \sigma_\infty \pm \frac{1}{2}\int_R^\infty \frac{d\tilde{R}}{\sqrt{\tilde{R}^4 - \tilde{R}_0^4}}. \tag{33}$$

This solution looks remarkably similar to those describing double funnels or wormholes in dual D3-brane framework [7,8].

However, before examining the details of these configurations, let us consider the analogous solutions of the full equation of motion (24). The symmetrized prescription [32,21] instructs us to expand the square root expression and symmetrize over all permutations of the generators α_i in the trace of each term in the expansion. For example: $\mathrm{STr}(\alpha^i\alpha^i) = NC$, $\mathrm{STr}(\alpha^i\alpha^i\alpha^j\alpha^j) = N(C^2 - 4C/3)$, and $\mathrm{STr}(\alpha^i\alpha^i\alpha^j\alpha^j\alpha^k\alpha^k) = N(C^3 - 4C^2 + 16C/3)$. Unfortunately, we have not been able to find a systematic construction for the general term in this expansion. However, observing that at leading order, $\mathrm{STr}(\alpha^i\alpha^i)^m \simeq NC^m$, we can construct an approximate equation by replacing the $\alpha^i\alpha^i$ by CI_N in Eq. (24). For large N, this keeps the leading order contribution at every order (in λ) in the expansion of the square roots. Within this approximation the equation of motion becomes

$$\frac{N}{\hat{R}'}\frac{d}{d\sigma}\sqrt{\frac{1 + 4\lambda^2 C\hat{R}^4}{1 + \lambda^2 C(\hat{R}')^2}} = 0. \tag{34}$$

Integrating this equation is trivial, and the result may be expressed as

$$(\hat{R}')^2 = 4\,\frac{\hat{R}^4 - \hat{R}_0^4}{1 + 4\lambda^2 C\hat{R}_0^4}. \tag{35}$$

In terms of the physical radius (13), we have

$$(R')^2 = 4\,\frac{R^4 - R_0^4}{\lambda^2 C + 4R_0^4}, \tag{36}$$

where we have also rescaled the integration constant in the obvious way. The solution of this equation is then implicitly given by

$$\sigma = \sigma_\infty + \frac{1}{2}\int_R^\infty d\tilde{R}\,\sqrt{\frac{\lambda^2 C + 4R_0^4}{\tilde{R}^4 - R_0^4}}. \tag{37}$$

With $C = N^2 - 1$ for the irreducible representation, Eq. (37) precisely reproduces the general solutions constructed in Refs. [7,8] for large N. For $R_0 = 0$ we recover the supersymmetric funnel solution (25). For large R, the general solution approximates this funnel and so given our previous discussion the nonabelian D-string is again expanding into a D3-brane at $\sigma = \sigma_\infty$. Assuming $R_0^4 > 0$, we see from Eq. (36) that the noncommutative funnel stops contracting when $R = R_0$. The obvious solution to continue past this point is [7,8]

$$\sigma = \sigma_\infty + 2\Delta\sigma - \frac{1}{2}\int_R^\infty d\tilde{R}\,\sqrt{\frac{\lambda^2 C + 4R_0^4}{\tilde{R}^4 - R_0^4}}, \tag{38}$$

where $\Delta\sigma = \sigma(R_0) - \sigma_\infty$. Hence beyond the minimum radius, the solution reexpands into an anti-D3-brane at $\sigma = \sigma_\infty + 2\Delta\sigma$.[4] Thus we have reproduced the wormhole solutions of Refs. [7,8] from the point of view of the non-Abelian D-string theory.

In integrating Eq. (34), one could also choose a negative integration constant, in which case it is natural to write

$$(R')^2 = (R_0')^2 + 4\,\frac{1 + (R_0')^2}{\lambda^2 C}R^4, \tag{39}$$

where R_0' is a new dimensionless integration constant. The general solution then becomes

$$\sigma = \sigma_\infty + \frac{1}{2}\int_R^\infty \frac{\lambda\sqrt{C}\,d\tilde{R}}{\sqrt{(1 + (R_0')^2)\tilde{R}^4 + \lambda^2 C(R_0')^2/4}}. \tag{40}$$

In this case, the funnel collapses all the way down to zero radius, which is approached with a finite slope, i.e., from Eq. (39), $R'(R=0) = R_0'$. The integrand has no singularity at $\tilde{R} = 0$, and so one can continue the solution beyond this point if one allows the radius to become negative. Alternatively, keeping the radius positive, we would match Eq. (40) onto

$$\sigma = \sigma_\infty + 2\Delta\sigma - \frac{1}{2}\int_R^\infty \frac{\lambda\sqrt{C}\,d\tilde{R}}{\sqrt{(1 + (R_0')^2)\tilde{R}^4 + \lambda^2 C(R_0')^2/4}}, \tag{41}$$

where now $\Delta\sigma = \sigma(R=0) - \sigma_\infty$. Hence in this solution, the funnel collapses down to zero size and then reexpands into another D3-brane[4] at $\sigma = \sigma_\infty + 2\Delta\sigma$. These general solutions again match, for large N, the analogous cusp configurations constructed in the D3-brane framework [24]. For comparison purposes, it may be simpler to think of these solutions in the form given in Eqs. (37) and (38), but with $R_0^4 < 0$. Note that in this case, we must choose $-\lambda^2 C/4 < R_0^4$ to produce a real solution. This lower bound $R_0^4 \to -\lambda^2 C/4$ corresponds to the singular limit $R_0' \to \infty$.

These cusp solutions describe N D-strings stretched between two parallel D3-branes (or anti-D3-branes), which should be a supersymmetric configuration. However, the supersymmetry condition (22) is only satisfied when $R_0' = 0$, in which case the D-string extends off to infinity before reaching zero size. In the dual D3-brane framework, Hashimoto

[4]Verifying that the emergence of a D3-brane or an anti-D3-brane at either end of these double funnel solutions actually requires examining the sign of $\hat{R}$ as $\sigma = \sigma_\infty$ and $\sigma_\infty + 2\Delta\sigma$, as per the discussion in Sec. III A.

[24] identified the correct supersymmetric solution for $N=1$ as a BPS monopole of the non-Abelian world-volume theory describing the two D3-branes. To find the corresponding BPS solutions in the D-string theory, one must begin with an ansatz more general than Eq. (11) when solving the Nahm equations (21). Such solutions are known, see for example Refs. [33,18]. We leave a discussion of these solutions for future work [34].

C. (*p*,*q*)-strings

The previous analysis is readily generalized to (p,q)-strings, i.e., bound states of D-strings and fundamental strings [35]. This is done by simply introducing a background $U(1)$ electric field on the D-strings, corresponding to fundamental strings dissolved on the worldsheet. Denoting the electric field as $F_{\tau\sigma}=\mathcal{E}I_N$, the D-string action (23) becomes

$$S=-T_1\int d^2\sigma \,\mathrm{STr}\sqrt{(1-\lambda^2\mathcal{E}^2+\lambda^2\alpha^i\alpha^i(\hat{R}')^2)(1+4\lambda^2\alpha^j\alpha^j\hat{R}^4)}, \tag{42}$$

where we have inserted the noncommutative ansatz (11). Extremizing with respect to variations of $\hat{R}$ yields

$$\frac{1}{\hat{R}'}\frac{d}{d\sigma}\mathrm{STr}\sqrt{\frac{1+4\lambda^2\alpha^j\alpha^j\hat{R}^4}{1-\lambda^2\mathcal{E}^2+\lambda^2\alpha^i\alpha^i(\hat{R}')^2}}=0. \tag{43}$$

Now assuming $\mathcal{E}$ constant, a simple rescaling of Eq. (22) yields an exact solution, i.e.,

$$\hat{R}'=\pm 2\sqrt{1-\lambda^2\mathcal{E}^2}\hat{R}^2. \tag{44}$$

Hence the funnel solution for the (p,q)-string becomes

$$\hat{R}(\sigma)=\frac{1}{2\sqrt{1-\lambda^2\mathcal{E}^2}}\frac{1}{\sigma-\sigma_\infty}. \tag{45}$$

[For simplicity, we will only consider the positive root of Eq. (44) in the following.] It is also useful to consider the electric displacement D, conjugate to E,

$$D\equiv\frac{1}{N}\frac{\delta S}{\delta\mathcal{E}}=\frac{1}{N}\mathrm{STr}\sqrt{\frac{1+4\lambda^2\alpha^i\alpha^i\hat{R}^4}{1-\lambda^2\mathcal{E}^2+\lambda^2\alpha^j\alpha^j(\hat{R}')^2}}\lambda^2T_1\mathcal{E}$$
$$=\frac{\lambda^2T_1\mathcal{E}}{\sqrt{1-\lambda^2\mathcal{E}^2}}, \tag{46}$$

where we have used Eq. (44) to derive the final result. One can verify that the equations of motion for the world-volume gauge field specify D to be a constant, and so our assumption of constant $\mathcal{E}$ is consistent for any solution $\hat{R}(\sigma)$ obeying Eq. (43). For N_f fundamental strings, one obtains the correct $(p,\ q)$-string tension by quantizing $D=N_f/N$, remembering that the fundamental string tension is simply $1/\lambda$—see below.

To determine the energy of the system, we must evaluate the Hamiltonian, $\int d\sigma(D\mathcal{E}-\mathcal{L})$, for the dyonic funnel solutions. Manipulating this expression in a manner similar to the analogous calculations in Eqs. (27),(28), the final result may be expressed as a sum of two terms

$$E=T_1\int d\sigma\sqrt{N^2+g^2N_f^2}+T_3\frac{N}{\sqrt{C}}\int 4\pi R^2dR, \tag{47}$$

where g is the string coupling, and we remind the reader that $T_1=(\lambda g)^{-1}$. Here the first term comes from collecting the contributions independent of $\hat{R}$, and correctly matches the energy of the (N,N_f)-string bound state [35]. The second contribution involves the terms containing $\hat{R}$ and as in Sec. III A, Eq. (44) is used to put these in the form $\hat{R}^2|\hat{R}'|$. The final result corresponds to the expected energy of an orthogonal D3-brane, at large N. Our expression (47) also matches the expectations (for large N) from the similar calculations in the D3-brane theory [15]. Equation (29) still applies in this case, and in this formulation the $(\nabla\sigma)^2$ term provides the contribution of the (N,N_f)-string. In the D3-brane formulation, it is straightforward to show that the dyonic spike is still supersymmetric. However, in the D-string formulation, introducing a constant background electric field moves the theory to a new superselection sector where the supersymmetry is nonlinearly realized.[5] By the $SL(2,Z)$ duality of the type IIb superstring theory, it is clear that a (N,N_f)-string has precisely the same amount of supersymmetry as an ordinary D-string. Similarly the dyonic funnel will be a BPS configuration preserving 1/2 of the world-volume supersymmetries. This supersymmetry is reflected in that these configurations (44) satisfy the full equations of motion (43), and that the corresponding energy (47) splits into a sum of string and three-brane contributions.

There are also various other solutions known in the literature involving (p,q)-strings (see, for example, Ref. [26]), and these should presumably also follow from the full equations of motion (43) in a straightforward manner. For related discussion on string junctions, see Refs. [18,19].

D. Embedding in a D3-brane background

We now consider constructing a noncommutative funnel for a non-Abelian D-string sitting in the background of a set

[5]We would like to thank Amanda Peet for a discussion on this point.

of orthogonal D3-branes. According to Refs. [25,26], when working with a test D3-brane sitting in such a supergravity background, the BPS Born-Infeld spike solutions are unchanged by the background fields. Hence one might expect that the funnel solutions (11),(16) will also appear unchanged in the modified world-volume theory of the D-strings.

An extremal D3-brane background can be written as [36]

$$\begin{aligned} ds^2 &= \frac{-dt^2+(dx^i)^2}{\sqrt{\mathcal{H}}} + \sqrt{\mathcal{H}}(dx^m)^2, \\ F^{(5)} &= \mp \mathcal{H}^{-2}\partial_m \mathcal{H} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^m \\ &\quad \pm \partial_m \mathcal{H} i_{\hat{x}^m}(dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9), \end{aligned} \tag{48}$$

where the x^i, $i=1 \ldots 3$, directions are parallel to the D3-brane, and the x^m, $m=4 \ldots 9$, directions are transverse. The function $\mathcal{H}$ satisfies the Laplace equation: $\partial^m \partial_m \mathcal{H}=0$. The single center harmonic function is

$$\mathcal{H} = 1 + 4\pi g N_3 \left(\frac{l_s}{r}\right)^4 \tag{49}$$

for N_3 D3-branes, where $r^2 = \Sigma_{m=4}^{9}(x^m)^2$. The potential for the five-form field strength has an electric component which may be written

$$C^{(4)}_{\rm elec} = \pm(\mathcal{H}^{-1}-1)dt dx^1 dx^2 dx^3. \tag{50}$$

The magnetic part of the potential will only involve indices in the transverse x^m directions, and we will argue below that it is irrelevant for the present calculation.

Now consider N D-strings extending into the transverse space along the x^9-axis. We again choose static gauge for the D-string action with $\tau=t$ and $\sigma=x^9$. In the non-Abelian Chern-Simons action (3), we have the interactions

$$\begin{aligned} \mu_1 \int & i\lambda \,\mathrm{STr} P[i_\Phi i_\Phi C^{(4)}] \\ &= \frac{i}{2}\lambda \mu_1 \int d^2\sigma \,\mathrm{STr}\Big(C^{(4)}_{t9ji}[\Phi^i,\Phi^j] \\ &\quad + \lambda C^{(4)}_{tkji} D_\sigma \Phi^k [\Phi^i,\Phi^j] - \lambda C^{(4)}_{9kji} D_t \Phi^k [\Phi^i,\Phi^j] \\ &\quad + \frac{\lambda^2}{2} C^{(4)}_{lkji} D_t \Phi^l D_\sigma \phi^k [\Phi^i,\Phi^j]\Big). \end{aligned} \tag{51}$$

In the solution which we will construct, we assume that (i) there are no background gauge fields so the covariant derivatives in the pullbacks are simply ordinary partial derivatives, (ii) the solution is static so that $D_t\Phi^l = \partial_t \Phi^l = 0$, and (iii) for later purposes, that only the Φ^i in the x^i directions are relevant, i.e., we only consider ''deformations'' of the D-string in its transverse directions which are parallel to the world-volume directions of the background D3-brane. Further, from Eqs. (48) or (50), we know that $C^{(4)}_{t9ji}=0$. Hence the only relevant interaction above is

$$\frac{i}{2}\lambda^2 \mu_1 \int d^2\sigma \,\mathrm{STr}(C^{(4)}_{tkji}\partial_\phi \Phi^k [\Phi^i,\Phi^j]). \tag{52}$$

In the background RR potential and the metric, we have the harmonic function (49) which is only a function of the non-Abelian radius

$$r^2 = \sigma^2 + \lambda^2[(\Phi^4)^2 + (\Phi^5)^2 + \cdots] \simeq \sigma^2. \tag{53}$$

In the last step, in keeping with the assumptions listed above, we have ignored the fluctuations of the D-string in the x^m directions. This simplifies the calculation since we have $\mathcal{H} = \mathcal{H}(\sigma)$, however, we might expect some smearing of the D-string in the x^m directions at higher order. With this simplification, the Chern-Simons interaction becomes

$$\frac{i}{2}\lambda^2 \mu_1 \int d^2\sigma (\mathcal{H}(\sigma)^{-1}-1)\varepsilon_{kji} \mathrm{Tr}(\partial_\sigma \Phi^k [\Phi^i,\Phi^j]), \tag{54}$$

where for definiteness, we have chosen the plus sign for the potential in Eq. (50). This choice corresponds to a background of D3-branes (as opposed to anti-D3-branes).

The Born-Infeld part of the action is only slightly modified by the background metric. For our usual ansatz (11), the Born-Infeld action now reads

$$S_{BI} = -T_1 \int d^2\sigma \,\mathrm{STr} \sqrt{\left(1+\frac{\lambda^2}{\mathcal{H}}\alpha^i\alpha^i(\hat{R}')^2\right)\left(1+\frac{4\lambda^2}{\mathcal{H}}\alpha^i\alpha^j\hat{R}^4\right)}. \tag{55}$$

Similarly inserting Eq. (11) into the Chern-Simons interaction (54) yields

$$S_{CS} = 2\lambda^2 NCT_1 \int d^2\sigma (\mathcal{H}(\sigma)^{-1}-1)\hat{R}^2\hat{R}' = -\frac{2}{3}\lambda^2 NCT_1 \int d^2\sigma \partial_\sigma(\mathcal{H}(\sigma)^{-1})\hat{R}^3. \tag{56}$$

Since the function $\mathcal{H}$ depends on σ, the full equations of motion following from this action are considerably more complicated than in the flat space case. The full equations of motion may be written as

$$-\frac{1}{\hat{R}'}\frac{d}{d\sigma}\mathrm{STr}\sqrt{\frac{1+\frac{4\lambda^2}{\mathcal{H}}\alpha^j\alpha^j\hat{R}^4}{1+\frac{\lambda^2}{\mathcal{H}}\alpha^i\alpha^i(\hat{R}')^2}}+\frac{1}{\hat{R}'}\mathrm{STr}\left[\left(1+\frac{\lambda^2}{\mathcal{H}}\alpha^i\alpha^i(\hat{R}')^2\right)\frac{\tilde{d}}{\tilde{d}\sigma}\sqrt{\frac{1+\frac{4\lambda^2}{\mathcal{H}}\alpha^j\alpha^j\hat{R}^4}{1+\frac{\lambda^2}{\mathcal{H}}\alpha^i\alpha^i(\hat{R}')^2}}\right]$$
$$+\mathrm{STr}\left[\alpha^k\alpha^k\sqrt{\frac{1+\frac{4\lambda^2}{\mathcal{H}}\alpha^j\alpha^j\hat{R}^4}{1+\frac{\lambda^2}{\mathcal{H}}\alpha^i\alpha^i(\hat{R}')^2}}\right]\lambda^2\partial_\sigma(\mathcal{H}(\sigma)^{-1})\hat{R}'=2\lambda^2NC\partial_\sigma(\mathcal{H}(\sigma)^{-1})\hat{R}^2. \tag{57}$$

In the second term on the left hand side, $\tilde{d}/\tilde{d}\sigma$ denotes that the σ derivative only acts on the harmonic function $\mathcal{H}$. The right hand side of this equation is the contribution from the Chern-Simons term in Eq. (56). Now if as in Eq. (22), we set $(\hat{R}')^2=4\hat{R}^4$, the first two terms vanish and the entire expression reduces to simply

$$\hat{R}'=2\hat{R}^2, \tag{58}$$

with the standard solution

$$\hat{R}=-\frac{1}{2(\sigma-\sigma_\infty)}. \tag{59}$$

Hence the background picks out the noncommutative funnel which corresponds to the D-string expanding into a D3-brane, but not the one where an anti-D3-brane emerges. This should have been expected because an anti-D3-brane would be unstable in the D3-brane background. Choosing the opposite sign of the RR potential in Eq. (54) would correspond to putting the D-string in the supergravity background generated by a collection of anti-D3-branes. This would also change the sign of the Chern-Simons contribution in the equation of motion (57) to produce $\hat{R}'=-2\hat{R}^2$ in place of Eq. (58). Hence in this case, the non-commutative funnel corresponding to an anti-D3-brane would be picked out. In any event, the BPS funnel solution consistent with the supersymmetry of the background survives unchanged, just as for the analogous solutions found in the D3-brane formulation. As a final note here, we observe that this configuration solves the full equations of motion (57) *regardless* of the detailed functional form of $\mathcal{H}$. In particular, the position of the end of the funnel, i.e., $\sigma=\sigma_\infty$, is still an independent parameter, not correlated to the position(s) of the background D3-branes. Further, one could consider multicenter solutions for $\mathcal{H}$.

IV. FLUCTUATIONS OF THE D-STRING FUNNEL

In this section, we analyze the dynamics of the BPS funnel solution (16) (in a flat background). That is, we examine the linearized equations of motion for small, time-dependent fluctuations of the transverse scalars Φ^r, around the exact background $\Phi^i=(1/2\sigma)\alpha^i$.[6] There are two types of fluctuations to consider: The first, in the language of Ref. [13], are the ''overall transverse'' excitations given by the scalars $\delta\Phi^m$ which are transverse to both the D-string and the noncommutative two-sphere (or D3-brane). The second are the ''relative transverse'' fluctuations of the coordinate fields Φ^i which lie in the two-sphere directions. Our notation in the following will be that indices $i,j=1\ldots3$ denote the directions parallel to the D3-brane, $m,n=4\ldots8$ represent directions transverse to both the D-string and the D3-brane, and finally $r,s=1\ldots8$ include all of these directions.

We start with the overall transverse fluctuations. The simplest type of fluctuation is just proportional to the identity matrix, say $\delta\Phi^m(\sigma,t)=f^m(\sigma,t)I_N$. For these modes it is straightforward to plug into the action (7), and we find

$$S=-T_1\int d^2\sigma\mathrm{STr}\sqrt{\left(1+\frac{\lambda^2}{4\sigma^4}\alpha^i\alpha^i\right)\left[\left(1+\frac{\lambda^2}{4\sigma^4}\alpha^j\alpha^j\right)(1-\lambda^2(\partial_t\delta\Phi^m)^2)+\lambda^2(\partial_\sigma\delta\Phi^m)^2\right]}$$
$$\simeq -NT_1\int d^2\sigma\left[H-\frac{\lambda^2}{2}H(\partial_t f^m)^2+\frac{\lambda^2}{2}(\partial_\sigma f^m)^2+\cdots\right], \tag{60}$$

[6] For simplicity, we have set $\sigma_\infty=0$. We will also choose the generators for the background solution to lie in $N\times N$ irreducible representation. Hence $C=N^2-1$.

where we introduced

$$H(\sigma)=1+\frac{\lambda^2 C}{4\sigma^4}. \tag{61}$$

In the final action, we have only kept the terms quadratic in the fluctuations as this is sufficient to determine the linearized equations of motion:

$$(H\partial_t^2-\partial_\sigma^2)f^m=0. \tag{62}$$

This is precisely the equation of motion found for the transverse fluctuations of the DBI spike soliton in Refs. [7,11,13]. The identification of the function H with the corresponding functions in Refs. [7,11,13] requires $\lambda^2 C/4=\pi^2 N^2 \ell_s^2$, which again holds up to $1/N^2$ corrections for large N. Note that this equation was also found to agree with the equation of motion for a fluctuating test string in the supergravity background of a D3-brane [11], after identifying parameters on both sides in a specific way.[7]

In the detailed analysis of Ref. [13], the fluctuations in Eq. (62) correspond to the $\ell=0$ modes, i.e., modes constant on the two-sphere. We will now show that, up to an important modification, similar agreement also holds for the higher ℓ modes. To describe the $\ell>0$ modes we first note, following Refs. [29,37], that the fluctuations $\delta\Phi^m$ can be expanded on the noncommutative two-sphere as a polynomial series in the matrices α^i as follows:

$$\delta\Phi^m(\sigma,t)=\sum_{\ell=0}^{N-1}\psi^m_{i_1 i_2\dots i_\ell}(\sigma,t)\,\alpha^{i_1}\alpha^{i_2}\cdots\alpha^{i_\ell}, \tag{63}$$

where the coefficients $\psi^m_{i_1\dots i_\ell}$ are completely symmetric and traceless in the lower indices. Also note that the series must terminate after $N-1$ terms since there are at most this many linearly independent matrices which can be formed from an $N\times N$ irreducible representation of the α^i. In the large N limit this expansion is analogous to expanding the fluctuations in spherical harmonics on a commutative two-sphere.

Substituting this form of the fluctuations into the action (7) is now slightly more involved, and it is more straightforward to use an alternative form of the action, given in terms of Eq. (26) in Ref. [21]. In flat space, this form of the action (7) reads $S=-T_1\int\sqrt{-\tilde{D}}$, with

$$\tilde{D}=\det\begin{pmatrix}\eta_{ab} & \lambda\,\partial_a\Phi^s\\ -\lambda\,\partial_b\Phi^r & \delta^{rs}+i\lambda[\Phi^r,\Phi^s]\end{pmatrix}, \tag{64}$$

where Φ^r includes the background Φ^i and the overall transverse fluctuations $\delta\Phi^m$. In general, this leads to a 10×10 determinant, which, however, is straightforward to evaluate keeping in mind the symmetrization procedure. We find that the resulting action is the same as in the first line of Eq. (60) up to the addition of an extra commutator term $\lambda^2[\Phi^i,\delta\Phi^m][\delta\Phi^m,\Phi^i]$ in the second factor under the square root. The quadratic action then becomes

$$\begin{aligned}S\simeq -T_1\int d^2\sigma\,\mathrm{Tr}\Big(&H-\frac{\lambda^2}{2}H(\partial_t\delta\Phi^m)^2+\frac{\lambda^2}{2}(\partial_\sigma\delta\Phi^m)^2\\&+\frac{\lambda^2}{2}[\Phi^i,\delta\Phi^m][\delta\Phi^m,\Phi^i]\\&+\frac{\lambda^4}{12}[\partial_\sigma\Phi^i,\partial_t\delta\Phi^m][\partial_t\delta\Phi^m,\partial_\sigma\Phi^i]\Big),\end{aligned} \tag{65}$$

where the last term arises from taking care to expand the symmetrized trace in the kinetic term. Now the linearized equation of motion becomes

$$\begin{aligned}(H\partial_t^2-\partial_\sigma^2)\delta\Phi^m+[\Phi^i,[\Phi^i,\delta\Phi^m]]&\\-\frac{\lambda^2}{6}[\partial_\sigma\Phi^i,[\partial_\sigma\Phi^i,\partial_t^2\delta\Phi^m]]&=0.\end{aligned} \tag{66}$$

In order to make contact with the discussion in Ref. [13], we must evaluate the commutator terms in this equation for the background solution $\Phi^i=(1/2\sigma)\alpha^i$. To facilitate this we will make use of the expansion in Eq. (63). Specifically, we evaluate

$$\begin{aligned}[\alpha^i,[\alpha^i,\delta\Phi^m]]&=\sum_{\ell<N}\psi^m_{i_1 i_2\dots i_\ell}[\alpha^i,[\alpha^i,\alpha^{i_1}\alpha^{i_2}\cdots\alpha^{i_\ell}]]\\&=\sum_{\ell<N}4\ell(\ell+1)\psi^m_{i_1 i_2\cdots i_\ell}\alpha^{i_1}\alpha^{i_2}\cdots\alpha^{i_\ell},\end{aligned} \tag{67}$$

making use of the fact that $\psi^m_{i_1 i_2\dots i_\ell}$ is completely symmetric and traceless. So we see that the double commutator above essentially acts like the Laplacian on the noncommutative two-sphere. Hence restricting the fluctuation to contain products with a fixed number of generators, i.e., to contain a specific spherical harmonic on the two-sphere, we have

$$[\Phi^i,[\Phi^i,\delta\Phi^m_\ell]]=\frac{\ell(\ell+1)}{\sigma^2}\delta\Phi^m_\ell$$

and (68)

$$[\partial_\sigma\Phi^i,[\partial_\sigma\Phi^i,\partial_t^2\delta\Phi^m_l]]=\frac{\ell(\ell+1)}{\sigma^4}\partial_t^2\delta\Phi^m_\ell.$$

Thus we see that the equation of motion for each mode becomes

$$(\tilde{H}_\ell\partial_t^2-\partial_\sigma^2)\delta\Phi^m_\ell+\frac{\ell(\ell+1)}{\sigma^2}\delta\Phi^m_\ell=0, \tag{69}$$

where

[7]More precisely, the functional forms of the equations agree, while the parameters undergo some form of renormalization between the two pictures. Exact matching occurs only at a specific point in a parameter space, on the border between the two regimes of validity.

$$\tilde{H}_\ell(\sigma)=1+\frac{\lambda^2}{4\sigma^4}\left(C-\frac{2}{3}\ell(\ell+1)\right). \tag{70}$$

Again for large N and $\ell \ll N$, this reproduces the equation of motion for overall transverse fluctuations found from the D3-brane spike [13]. However, for large ℓ the coefficient in $\tilde{H}_\ell$ is significantly modified compared to the D3-brane analysis. Another important difference is that in the noncommutative D-string analysis, the spectrum of modes is truncated at $\ell_{\max}=N-1$. Note that given this truncation, the coefficient of $1/\sigma^4$ in Eq. (70) cannot be negative—such a negative coefficient would have caused drastic changes in the mode propagation. Thus for finite N, there are a finite number of modes propagating in the core of the bion. Note that this number is $\Sigma_{\ell=0}^{N-1}(2\ell+1)=N^2$. Of course, this counting is precisely what is expected for the adjoint scalars in the $U(N)$ theory on the world-volume of the N D-strings. A puzzle was raised in the D3-brane analysis [13], where it appeared that modes with arbitrarily high ℓ would propagate out along the D-string spike. The present D-string analysis suggests that this is not the case, and that the propagation of high ℓ modes is significantly modified. We will comment more on this point in the discussion section.

We finally briefly discuss the case of relative transverse fluctuations, $\delta\Phi^i$. From the D3-brane point of view these are considerably more complicated to analyze, because of the interplay between the scalar field and the gauge field. In the D-string picture, an increased complication arises in evaluating the symmetrized trace. Unfortunately, we do not have an exact treatment of the quadratic action. Instead we use the same approximation as in Sec. III B, replacing in the action everywhere $\alpha^i\alpha^i$ by CI_N. For large N, this keeps the leading contribution at every order in λ in an expansion of the action. As above, first let us consider the $\ell=0$ mode, and for concreteness consider a fluctuation in the x^3 direction, $\delta\Phi^3(\sigma,t)=f(\sigma,t)I_N$. One can show that fluctuations in different directions decouple at linear order, using $\mathrm{Tr}(\alpha^i\alpha^j)=NC/3\delta^{ij}$. The determinant in Eq. (64) now involves a 5×5 matrix which again is straightforward to calculate. We find

$$S=-T_1\int d^2\sigma \mathrm{STr}\sqrt{H\left(H-\dot{f}^2+\frac{1}{H}[1+\alpha_3^2/(4\sigma^4)](f')^2+\alpha_3 f'/\sigma^2\right)}, \tag{71}$$

where H is given in Eq. (61). Using $\mathrm{Tr}\alpha_3=0$ in an expansion in the amplitude f, the terms nicely arrange into

$$S=-NT_1\int d^2\sigma\left[H-\frac{\dot{f}^2}{2}+\frac{(f')^2}{2H}+\mathcal{O}(f^4)\right]. \tag{72}$$

The equations of motion follow immediately, and again agree, for large N, with the results from the D3-brane spike and from supergravity [11,13].

Higher ℓ modes can be treated similarly. For instance, the $\ell=1$ ‘‘breathing’’ mode considered in Ref. [13] is implemented by the fluctuations $\delta\Phi^i(\sigma,t)=f(\sigma,t)\alpha^i$. The resulting action is found most easily by substituting $\hat{R}\to\hat{R}+f$ in the Born-Infeld action (23), adding the obvious term involving time-derivatives. The resulting equation of motion reads

$$\partial_t^2 f-\partial_\sigma\left(\frac{\partial_\sigma f}{H}\right)+\frac{2}{H^2\sigma^2}(4-H)f=0. \tag{73}$$

The corresponding equation is only given indirectly in Ref. [13], but through a bit of algebra one can verify that the two approaches agree, again up to $1/N^2$ corrections.

V. DISCUSSION

Making use of the recently proposed non-Abelian extension of the Born-Infeld action describing the world-volume physics of Dp-branes [21,22], we have found a description of a D-string ending on a D3-brane ‘‘dual’’ to that obtained from the Abelian D3-brane action. Specifically we have shown that the world-volume action for N coincident D-strings has *exact* BPS solutions which describe a D3-brane growing out of the noncommuting transverse coordinates of the collection of D-strings. We generalized this construction to describe (p,q)-strings by considering the case where there are a number of fundamental strings dissolved on the worldsheet of the D-string. We have also considered the case where the D-string is embedded in the supergravity background of a collection of D3-branes, and shown that the supersymmetric funnel remains a solution with its form unchanged. In all these cases we have found, in the large N limit, precise agreement with the earlier literature on the DBI spike soliton [7,8,9,15,25].

As commented before, in these constructions (except in Sec. III D), there are no nontrivial supergravity fields in the ambient spacetime. Hence these solutions are quite distinct from the configurations arising from the dielectric effect discussed in Ref. [21]. The latter involves a collection of Dp-branes being ‘‘polarized’’ into a noncommutative configuration by an external field. As well as using the non-Abelian character of the D-string theory, the essential new feature of the present constructions is the introduction of unusual (i.e., singular) boundary conditions in the world-volume theory [16,17]. For example, in the BPS funnel (16) the scalars diverge at $\sigma=\sigma_\infty$. To comment on these boundary conditions further, let us consider the solution for different representation of the generators α^i. Throughout the paper, we emphasized the irreducible $N\times N$ representation, for which we found that the funnel corresponded precisely to the N D-strings expanding into a single D3-brane. One could reconsider the analysis when the α^i are chosen as the direct sum of q copies of the $N/q\times N/q$ representation. In this case, one would find that the BPS funnel (16) describes an expansion into q coincident D3-branes at $\sigma=\sigma_\infty$. Using energy considerations as in Eq. (28), naively one might conclude that it is favorable for this configuration to decay into the original funnel. Of course, this is incorrect—the qD3-branes cannot ''decay'' into a single D3-brane. Rather one should think of the new solution as a different superselection sector, which is distinguished in our construction by imposing a distinct set of boundary conditions at $\sigma\to\sigma_\infty$ (or alternatively at $R\to\infty$).

Actually from the D-string point of view, the scalar fields start to vary extremely rapidly as $\sigma\to\sigma_\infty$ and so our description in terms of the low energy world-volume action (7) will break down before this point is reached. On the other hand, as $\sigma\to\infty$ the world-volume scalars are both slowly varying and small, and so our formulation should give a very reliable description of the physics. This behavior is complementary to the D3-brane analysis. From this point of view, the world-volume fields are slowly varying and small for large R, and rapidly varying for small R. Thus these two approaches give complementary descriptions for the DBI spike. We note that this complementarity arises because of the ''duality''[8]

$$R\simeq\frac{N\ell_s^2}{\sigma} \tag{74}$$

between the world-volume coordinates in the two different formulations.

Let us try to be more precise about the ranges of validity where we think we can trust the Born-Infeld analysis in each of these approaches. Essentially, we must determine when we can confidently ignore higher derivative corrections to the action arising from the usual α' expansion in string theory. Schematically we would require $\ell_s\partial^2\Phi\ll\partial\Phi$. For the spike soliton on the D3-brane, this translates into $R\gg\ell_s$, or using Eq. (74), $\sigma\ll N\ell_s$. From the D-string funnel point of view, we must require $\sigma\gg\ell_s$, which is equivalent to $R\ll N\ell_s$. So we see that in a large N limit, there is a significant overlap region, and this explains, at least partially, the good agreement we find between the two approaches in this regime.

Beyond the higher derivative corrections, the non-Abelian action (1),(3) requires additional higher order commutator corrections [38,39,40]—see also the discussion in Ref. [21]. Given this limitation, we might conclude that our non-Abelian D-string calculations are reliable only for small commutators. For the BPS funnel, we require $\ell_s|\hat{R}|\ll1$ since the commutators of the scalar fields are characterized by the dimensionless quantity $\ell_s\hat{R}$. In terms of the physical radius, this restriction becomes $R\ll N\ell_s$, which coincides with the restriction derived in the previous discussion. A better restriction for avoiding the higher commutator corrections is that the Taylor expansion of the square root in the action (23) should converge rapidly. This requirement leads to the more restrictive condition that $R\ll\sqrt{N}\ell_s$. However, for large N, there is still overlap with the D3-brane approach over a large region. We should add, however, that this discussion applies to generic field corrections. There are some indications that for supersymmetric configurations, the higher commutator corrections may vanish [38,39], and so the less conservative restriction above may be the correct one for the BPS funnel.

We should also remember that we have neglected gravitational effects, which is justified when $gN\ll1$. Since none of our analysis involves the string coupling g, this requirement is easily satisfied by going to very weak coupling.

In Sec. IV, we also found remarkable agreement for the dynamics of small fluctuations on the D-string funnel and those on the D3-brane spike, for large N and also $\ell\ll N$. Our analysis begins to show significant discrepancies for higher ℓ modes. In particular, the spectrum of modes on the noncommutative funnel is truncated at $\ell_{\rm max}=N-1$. This brings us to the puzzle arising from Ref. [13]. There the detailed analysis of the fluctuations on the D3-brane spike showed that there was no suppression of the higher ℓ modes near the core. Hence modes with arbitrarily large ℓ appeared to propagate out to infinity, and the spike would seem to retain its three-dimensional character arbitrarily far out rather making a transition to stringlike behavior. Since our D-string analysis provides a reliable description of physics at the core of the spike, we conclude that this result cannot be correct. We have found that only a finite number of modes propagate far from the D3-brane. Note, however, that for N D-strings, this number is N^2, not just N, due to the nonabelian character of the coincident D-strings.

Above we considered in detail in what regimes the D3-brane and D-string descriptions would be trustworthy. However, this analysis was only for the spike or funnel solution itself, which plays the role of a background in the calculations of the linearized fluctuations. Hence we should repeat this preceding analysis for the fluctuations themselves. In particular, a fluctuation on the D3-brane with angular momentum number ℓ oscillates on spheres of constant radius with an effective wavelength $\lambda=R/\ell$. Hence for higher derivative corrections to the D3-brane action to be negligible, we must require that $\lambda/\ell_s\gg1$. Hence $R\gg\ell\ell_s$ or from Eq. (74), $\sigma\ll(N/\ell)\ell_s$. Therefore even if we assume N is large, we can only trust the linearized equations of motion to accurately describe the propagation of fluctuations far out on the spike for $\ell\ll N$. Similarly the regime of validity of the analysis for fluctuations on the D-string funnel is more restrictive for the higher ℓ modes. In this case, we require that higher commutator corrections to the non-Abelian Born-Infeld action remain negligible. Given the commutators in Eq. (68), it appears the relevant quantity to characterize the commutators is $\ell\ell_s\hat{R}$. Thus our calculations would be trustworthy for $R\ll(N/\ell)\ell_s$ or $\sigma\gg\ell\ell_s$. Hence we conclude that we should not expect the two approaches to agree on the dynamics of the linearized fluctuations for $\ell\sim\sqrt{N}$ or higher.

[8]For simplicity, we set $\sigma_\infty=0$ here. Further we assume the irreducible $N\times N$ representation with large N, so that $C\simeq N^2$. Both of these assumptions will apply throughout the remainder of the discussion.

Therefore it seems that the resolution of the conflict between the results of Ref. [13] and the present paper is that the dynamics of the high ℓ modes is significantly altered in a transition region between regimes where either of the two formulations can be trusted. In particular, higher derivative corrections to the D3-brane action must play an important role near the core of the spike, and cause the very high ℓ modes to be reflected back out to the region of large radius. Unfortunately, beyond the observations made above, we cannot provide a detailed account of this suppression mechanism.

To summarize, we have again seen that the Born-Infeld action is a remarkably powerful tool in describing the low energy dynamics of Dp-branes. On the one hand, with the D3-brane action one can construct spike configurations corresponding to D-strings attached to the D3-brane, and the validity of these solutions seems to go far beyond naive expectations. In this paper, we have shown how these configurations also emerge from the D-string action in terms of noncommutative geometry. This formulation provides a reliable description of the central core of the DBI spike, but is also reliable to a very large radius when the number of D-strings is large. Hence we find surprising agreement with the original D3-brane theory point of view. Combining these two approaches presents an intriguing picture of D-strings attached to an orthogonal D3-brane. At large radius we have a continuous D3-brane being smoothly deformed into the spike geometry. However, near the core far out along the spike, there is a metamorphosis to a discrete structure, namely a noncommutative funnel geometry. One can begin to gain insight into this transition from the recent observations in Ref. [41]. In the D3-brane analysis, the spike is a magnetic monopole. Thus there is also a constant flux of magnetic field on the spheres of constant radius surrounding the spike. As the radius shrinks the local flux density becomes very large, and hence we can expect to enter a regime where noncommutative geometry provides an efficient description of the system.

Despite the striking agreement in the shape of the D3-brane spike and the D-string funnel, one may question whether or not the agreement should actually be complete. In Ref. [10], Thorlacius showed that the BPS spike solution found on the D3-brane was actually a solution of the full string action. This analysis was made for the electric spike describing a fundamental string, and it appears the proof would be more involved for the magnetic monopole describing the D-string [42]. Certainly the D3-brane and the D-string theories yield the same shape for the latter configuration, however they only agree on the overall coefficient for large N. For small N, the coefficients will differ significantly. It may still be that the monopole provides an exact boundary conformal field theory, but that there is a ''renormalization'' of the metric relevant to describing the geometry. This would be somewhat similar to the difference between the closed and open string metrics in situations where noncommutative geometry is relevant [41]. However, we will leave this question for future work.

It would be interesting to generalize the present discussion to other D-brane systems. The extension to Dp-branes ending on orthogonal D$(p+2)$-branes would follow trivially by the application of T duality. A more interesting extension would be to consider a D-string ending on an orthogonal Dp-brane with $p\neq 3$. From the lowest order equation of motion (10), a static configuration would still have to satisfy

$$\partial_\sigma^2\Phi^i=[\Phi^j,[\Phi^j,\Phi^i]]. \tag{75}$$

With the ansatz $\Phi^i=\hat{R}(\sigma)G^i$ for some constant matrices G^i, this equation can still only yield a differential equation of the form

$$\hat{R}''=a\hat{R}^3, \tag{76}$$

with some constant a, and one will again find solutions of the form $R\propto\sigma^{-1}$. Hence this type of profile is universal for all funnels on the D-string in any situation, and the only difference from the D3-brane funnel will be in the overall constant coefficient. This result is slightly surprising since from the dual Dp-brane formulation, one would generically expect that for large R, solutions will essentially be harmonic functions behaving like $\sigma\propto R^{-(p-2)}$ or $R\propto\sigma^{-1(p-2)}$. The resolution of this puzzle seems to be that the two profiles apply in distinct regimes, the first for small R and the second for large R. However, there is the possibility that solutions of the full Born-Infeld action will display a transition from one kind of behavior to another. In fact, we have begun analyzing the case of a D-string ending on an orthogonal D5-brane in detail, and we find that funnel solutions do indeed make this kind of transition [43].

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