Flat-space scattering and bulk locality in the AdS-CFT correspondence

Steven B. Giddings*
Department of Physics, University of California, Santa Barbara, California 93106-9530

(Received 14 October 1999; published 27 April 2000)

The large radius limit in the AdS-CFT correspondence is expected to provide a holographic derivation of flat-space scattering amplitudes. This suggests that questions of locality in the bulk should be addressed in terms of properties of the *S* matrix and their translation into the conformal field theory. There are, however, subtleties in this translation related to generic growth of amplitudes near the boundary of anti–de Sitter space. Flat space amplitudes are recovered after a delicate projection of CFT correlators onto normal-mode frequencies of AdS. Once such amplitudes are obtained from the CFT, possible criteria for approximate bulk locality include bounds on growth of amplitudes at high energies and reproduction of semiclassical gravitational scattering at long distances.

PACS number(s): 11.25.Sq

I. INTRODUCTION

Maldacena's proposed correspondence [1] between string (or M) theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ supersymmetric gauge theory in four dimensions has stimulated a great deal of recent excitement.[1] A particularly fascinating aspect of this correspondence is that it apparently serves as a concrete realization of holographic ideas [3,4]. Although a great deal of work has been done to deduce properties of the large-N Yang-Mills conformal field theory from this correspondence, an even more interesting question is how to deduce properties of string or M theory from the the boundary conformal field theory.

Several steps have recently been taken in this direction. In particular, in Refs. [5,6] the relationship between boundary correlators and an AdS analogue of the *S* matrix (called the boundary *S* matrix in Ref. [6]) was described. This work had been closely preceded by related work [7,8] which sketched a prescription to derive flat space *S*-matrix elements in the infinite radius R limit of AdS space; one naturally expects this *S*-matrix to be an appropriate $R\rightarrow\infty$ limit of the boundary *S* matrix of Refs. [5,6].

This paper will investigate this question: in particular, it will address the issue of how flat-space *S*-matrix elements can be obtained from conformal field theory data. As we will find, this is somewhat nontrivial.

A useful analogy to bear in mind is that between anti–de Sitter space and a resonant cavity. If one quantizes a free field in AdS, generic frequencies produce non-normalizable states, and the normalizable states correspond to a discrete set of frequencies and are analogous to cavity modes. Since there are no true asymptotic states among these modes, it is not *a priori* clear how to formulate scattering problems. Here the example of a resonant cavity serves as a guide. Consider for example two atoms at diametrically opposite ends of the cavity, and suppose one is in an excited state, and one in the ground state. Even when the transition energy is not a normal-mode frequency of the cavity, it is possible for atom one to decay to its ground state with atom two transitioning to the corresponding excited state. Or one may have more atoms, acting as emitters and detectors, and the emitted particles may mutually scatter before being detected. Such Gedanken experiments give a framework to discuss scattering.[2]

This analogy is not perfect. For one thing, AdS space has much more volume at infinity than flat space, and this together with growth at infinity of the wave functions at non-normalizable frequencies can have important consequences. In particular, as we will see, these combined effects can lead to growth of interaction strengths near the AdS boundary if we work with states at non-normalizable frequencies. This poses a difficulty for extracting flat-space scattering amplitudes in the center of a large radius anti–de Sitter space. An obvious retort is that one should project scattering amplitudes onto normalizable frequencies, with a corresponding projection on the correlators of the boundary conformal field theory (CFT). However, this is not made any easier by our lack of knowledge of the spectrum of normalizable states at the multiloop or nonperturbative levels, except in the case of protected states. We will discuss this problem and possible resolutions in more detail.

If we assume that it is indeed possible to extract flat-space *S*-matrices from the conformal field theory, another profound question arises. If the correspondence of Ref. [1] is correct, the theory in the bulk should exhibit approximate $(d+1)$-dimensional locality, in an appropriate low-energy limit. Although the bulk theory is conjectured to be string theory, which is manifestly nonlocal, this nonlocality is expected to only be apparent ''at the string scale'' (in some appropriate sense), or perhaps in black hole experiments. The theory should be *macroscopically* local, namely, it should give low-energy, weak field amplitudes that are derivable from a local low-energy effective field theory. An

*Email address: giddings@physics.ucsb.edu

[1]For a recent review and extensive set of references, see Ref. [2].

[2]Another possibility in the resonant cavity case is to cut small holes in the wall of the cavity through which particles may enter and exit. This clearly more difficult for AdS space, although an analogous construction exists for the case where an AdS bubble is embedded in a space with asymptotic particle states, and these states must penetrate a potential barrier to reach the interior of AdS, as described in Refs. [5,6].

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important question is what property of the boundary CFT implies macroscopic bulk locality in the large radius limit of anti–de Sitter space? Also, one would like to better characterize and understand the nonlocalities, and their possibly quite important implications.

To address this question we need a way to diagnose locality. From at least two view-points (gauge invariance, holography) it was expected that the AdS-CFT correspondence would only yield *S*-matrix elements—not off-shell amplitudes—and direct study has partially confirmed this [5–8]. Therefore we need a way of inferring from the *S*-matrix whether—or to what extent—the underlying theory is local. Various bounds exist for *S*-matrices derived from local theories, or even from generalizations to theories with nonlocal behavior above a definite energy scale. However, derivation of these bounds becomes problematic in the case of theories with massless particles, and particularly for theories with gravity. There is some related information about the high-energy structure of string scattering amplitudes, but at present nothing that serves as a definitive test of locality near the string scale. At macroscopic scales, one important test of locality is that the theory correctly produce semiclassical gravity amplitudes or other coulombic behavior, at long distances or equivalently small momentum transfer.

Yet another question that can be addressed is that of whether, or to what extent, the underlying bulk theory can be reconstructed from complete knowledge of the *S*-matrix elements. This is a difficult problem, and in general the solution is not unique. A few comments about this problem will also be made.

The paper will begin with a brief review of the Maldacena conjecture, as formulated by Gubser, Klebanov, Polyakov, and Witten. The next section will then describe the basic properties of the large radius limit for AdS. Section IV will then turn to the question of whether, and how, bulk flat-space *S*-matrix elements can be extracted from boundary conformal field theory correlators in the large radius limit. If the latter are written in frequency space, an important distinction occurs between frequencies corresponding to normalizable and non-normalizable modes; at frequencies corresponding to non-normalizable modes, boundary correlators receive important contributions from regions near the boundary of anti–de Sitter space. This behavior poses some difficulty for extracting the flat-space *S*-matrix, although outlines of a procedure will be given. Section V contains some discussion of the problem of investigating the bulk locality properties of the theory, as well as on the problem of reconstructing the bulk theory. Following the conclusion is a rather lengthy Appendix containing a number of useful properties of anti–de Sitter space, its propagators, and the large radius limit. Many of these appear previously in the literature, although there are some new results.

II. REVIEW OF THE GKPW CORRESPONDENCE

We begin by recalling the precise form of the correspondence for local correlators, as formulated by Gubser, Polyakov, Klebanov, and Witten [9,10], and extended to Lorentzian signature in Refs. [11,12]. This paper will use global coordinates $x=(\tau,\rho,\Omega)$ for the cover of anti–de Sitter space,

$$ds^2=\frac{R^2}{\cos^2\rho}(-d\tau^2+d\rho^2+\sin^2\rho d\Omega_{d-1}^2). \tag{2.1}$$

The boundary has topology $S^{d-1}\times\mathbb{R}$, or alternatively can be represented as the conformally equivalent infinite-sheeted cover of Minkowski space. A point on the boundary sphere can be specified by a d-dimensional unit vector $\hat{e}$; boundary points are parametrized as $b=(\tau,\hat{e})$.

The basic statement of this correspondence is

$$Z[\phi(\phi_0)]=\langle Te^{i\int_\partial db\,\phi_0(b)\mathcal{O}(b)}\rangle. \tag{2.2}$$

In this formula labels for different fields and their corresponding operators have been suppressed. On the left hand side is the bulk partition function for string theory on $AdS_5\times S_5$ with both radii set to R. This is evaluated with bulk fields ϕ constrained to obey the boundary condition

$$\phi \xrightarrow{\rho\to\pi/2} (\cos\rho)^{2h_+}\phi_0(b), \tag{2.3}$$

where the constants $h_\pm$ and ν, defined by

$$4h_\pm=d\pm\sqrt{d^2+4m^2R^2}=d\pm 2\nu, \tag{2.4}$$

for a field of mass m, govern the asymptotic behavior of the field. The right side is the corresponding generating functional for CFT correlation functions in $\mathcal{N}=4\,SU(N)$ Yang-Mills theory. The operator $\mathcal{O}$ corresponding to ϕ has conformal weight $\Delta=2h_+$. The parameters of the theories are related by $g_s=g_{YM}^2$ for the couplings, and $R=(g_{YM}^2N)^{1/4}$ for the $AdS_5\times S_5$ radius.

At the level of correlation functions, the correspondence (2.2) may be rewritten as

$$\langle T[\mathcal{O}(b_1)\mathcal{O}(b_2)\cdots\mathcal{O}(b_n)]\rangle = \int\prod_{i=1}^{n}[dx_i G_{B\partial}(b_i,x_i)]G_T(x_1,\ldots,x_n). \tag{2.5}$$

In this expression $G_{B\partial}(b_i,x_i)$ denotes the full (multiloop) bulk-boundary propagator, and $G_T(x_1,\ldots,x_n)$ denotes a bulk n-point Green function with its external legs truncated. Some properties of bulk and bulk-boundary propagators will be reviewed in the Appendix.

In particular, the basic issues will be illustrated using the relation between four-point functions,

$$\langle T\mathcal{O}_1(b_1)\cdots\mathcal{O}_4(b_4)\rangle = \int\prod_{i=1}^{4}[dx_i G_{B\partial}(b_i,x_i)]G_T(x_1,\ldots,x_4). \tag{2.6}$$

As a concrete example, consider truncating string theory down to a scalar sector with a three-point coupling, as in Ref. [13], and suppressing dependence on the S^5 coordinates:

$$S=-\int dV\left[\frac{1}{2}(\nabla\Phi)^2+\frac{m^2}{2}\Phi^2+g\Phi^3\right]. \tag{2.7}$$

To leading order in the coupling, the CFT correlator is given by a term of the form

$$\begin{aligned}\langle T\mathcal{O}_1(b_1)\cdots\mathcal{O}_4(b_4)\rangle_{\mathrm{tree},t}\\ &=-g^2\int dVdV'[K_{B\partial}(b_1,x)K_{B\partial}(b_3,x)K_B(x,x')\\ &\quad\times K_{B\partial}(b_2,x')K_{B\partial}(b_4,x')]\end{aligned} \tag{2.8}$$

plus s and u channel contributions, where K_B is the bulk Feynman propagator for the Φ field, and $K_{B\partial}$ is the tree-level bulk-boundary propagator.

III. THE LARGE-R LIMIT

Our goal is to recover flat-space S-matrix elements from boundary correlators. In order to extract these we must consider the regime of large R, where the AdS geometry has a large region [of size $\mathcal{O}(R)$] that is approximately flat.

The relation between the large-R geometry of AdS and flat space is easily exhibited in the global coordinates (2.1). An arbitrary point P of AdS space can be moved to the origin $(\tau,\rho)=(0,0)$ by an SO$(2,d)$ transformation. For large R, we can readily recover the nearly flat metric in the vicinity of P in spherical polar coordinates by defining

$$t=R\tau;\quad r=R\rho. \tag{3.1}$$

Then the metric (2.1) becomes

$$ds^2=\frac{1}{\cos^2(r/R)}\left[-dt^2+dr^2+R^2\sin^2\left(\frac{r}{R}\right)^2d\Omega^2\right]. \tag{3.2}$$

For $r\ll R$, this clearly reduces to the flat metric.

Since the spacetime in the vicinity of P is approximately flat, one expects to recover flat-space physics in this region. For example, consider the bulk Feynman propagator, which appeared in the amplitude (2.8). This has been given in closed form in terms of a hypergeometric function [14,15], and the large R limit of this expression is derived in the Appendix. The result is, for particles[3] of masses $\lesssim\mathcal{O}(1/R)$,

$$iK_B(x,x')=\frac{\tilde{C}}{[s^2(x,x')+i\epsilon]^{(d-1)/2}}, \tag{3.3}$$

where $\tilde{C}$ is a constant given in the Appendix, and $s(x,x')$ is the geodesic distance between the points x and x'. This is the standard flat-space propagator for a massless field.

[3]The case of massive particles, $m\gg\mathcal{O}(1/R)$, can also be treated.

IV. S-MATRICES AT LARGE R?

We would like to determine whether the flat-space S-matrix can be extracted from the conformal correlators, such as Eq. (2.8), in the limit of large R. To begin with, recall the form of the flat-space S-matrix for the process corresponding to that of Eq. (2.8). This is simply the tree-level, t-channel contribution to the flat-space S-matrix, which is given by a Fourier transform of the Feynman propagator:

$$\begin{aligned}S_{\mathrm{Mink},t}(s,t)&=-g^2\int d^{d+1}xd^{d+1}x'e^{i(k_1+k_3)x}e^{i(k_2+k_4)x'}\\ &\quad\times K_F(x,x')\\ &=-ig^2\delta^{(d+1)}(k_1+k_2+k_3+k_4)\frac{1}{t}.\end{aligned} \tag{4.1}$$

The general question at hand is how to extract the S-matrix, for example Eq. (4.1), directly from the corresponding contribution to the CFT correlation function, for example Eq. (2.8). Note that these formulas appear very similar. By the LSZ prescription, the flat S-matrix consists of truncated flat-space n-point functions, convoluted against on-shell wave functions; this should be compared to Eq. (2.8), which has the identical structure with bulk-boundary propagators replacing flat-space wave functions. This served as a central observation behind the interpretation of the conformal correlators as providing a ''boundary S-matrix'' for anti–de Sitter space [5,6].

In searching for a general prescription to extract the flat-space S-matrix from the AdS-CFT correspondence, we will begin by ''reverse engineering'' the expression (2.8) to see how the corresponding piece of the S-matrix could be extracted from this contribution to the conformal correlators. This will allow us to infer some lessons for the more general problem of the complete S-matrix.

We have just seen that at large R the contribution from K_B to the correlator reduces to the expected flat-space expression. Therefore, the remaining task is to understand whether the contributions from the factors $K_{B\partial}$ can be related to on-shell, flat-space wave functions in the Minkowski region. Some properties of $K_{B\partial}$ are reviewed in the Appendix. Given the relationship (3.1), it is convenient to work with the frequency conjugate to the global AdS time τ, as this also corresponds to definite Minkowski energy;

$$\omega=ER. \tag{4.2}$$

In frequency space, the relation (2.8) becomes

$$\begin{aligned}\langle T\mathcal{O}(\omega_1,\hat{e}_1)\cdots\mathcal{O}(\omega_4,\hat{x}_4)\rangle_{\mathrm{tree},t}\\ &=-g^2\int dVdV'[K_{B\partial}(\omega_1,\hat{e}_1;x)K_{B\partial}(\omega_3,\hat{e}_3;x)\\ &\quad\times K_B(x,x')K_{B\partial}(\omega_2,\hat{e}_2;x')K_{B\partial}(\omega_4,\hat{e}_4;x')].\end{aligned} \tag{4.3}$$

It will turn out that there is an important distinction between the cases where the frequency of the external state is generic, corresponding to a non-normalizable mode, and where it is that of one of the normalizable modes. We will consider these in turn.

A. Generic frequencies

In the case where the ω_i do not correspond to normal-mode frequencies, the bulk-boundary propagator must be a non-normalizable solution of the scalar wave equation. One can most easily explore the consequences of this by working in a partial-wave basis; as seen from Eq. (3.2) (and also shown in terms of generators in the Appendix), AdS angular momentum is directly identified with Minkowski angular momentum in the large-R limit. In an angular momentum basis and at arbitrary frequency, the bulk-boundary propagator has asymptotic behavior

$$K_{B\partial}(\omega,l,\vec{m};x)\to e^{i\omega\tau}Y^{*}_{l\vec{m}}(\cos\rho)^{2h_-}. \tag{4.4}$$

From this we immediately see a problem in extracting the flat-space S-matrix, (4.1), directly from Eq. (4.3). In order to do so, we would like K_B to be convolved with wave functions that have their support concentrated in the flat region $r\ll R$. However, the behavior (4.4) ensures that all linear combinations of the non-normalizable modes grow as $(\cos\rho)^{2h_-}$ at infinity, and thus are concentrated in the region $r\gg R$ instead. Indeed, consider the behavior of the integral over x in Eq. (4.3). The volume element is

$$dV=\frac{R^{d+1}(\sin\rho)^{d-1}}{(\cos\rho)^{d+1}}d\tau d\rho d^{d-1}\Omega, \tag{4.5}$$

and so near $\rho=\pi/2$ the ρ part of the integral takes the form

$$\begin{aligned}&\sim\int d\rho\frac{(\sin\rho)^{d-1}}{(\cos\rho)^{d+1}}(\cos\rho)^{4h_-}K_B(x,x')\\&\sim\int d\rho\frac{1}{(\cos\rho)^{2\nu+1}}K_B(x,x').\end{aligned} \tag{4.6}$$

Thus the wave function factor convolving the bulk Green function has its main support in the vicinity $\rho\approx\pi/2$.[4] For non-normalizable frequencies, there is no apparent way to obtain a limit in which one recovers the desired asymptotic falloff of flat-space wavefunctions for r large but $r\ll R$ without also encountering this growth at infinity.

Evidence for this behavior can also be seen directly in position space. As shown in the Appendix, the position space bulk-boundary propagator takes the form

$$K_{B\partial}(b,x')=C_{B\partial}\left[\frac{\cos\rho'}{\cos(\tau-\tau')-\sin\rho'\hat{e}\cdot\hat{e}'}\right]^{2h_+} \tag{4.7}$$

up to the $i\epsilon$ prescription (see the Appendix). This is singular for points on the boundary light cone: at points such that $\cos(\tau-\tau')-\hat{e}\cdot\hat{e}'=0$, it has behavior $\sim(\rho-\pi/2)^{-2h_+}$ at $\rho\to\pi/2$. In Eq. (2.8), the product of the first two bulk-boundary propagators will be singular on the intersection of their boundary light cones; this behavior is exacerbated by the growth of the volume element. These arguments, suggest that for generic frequencies AdS holography is in a sense only ‘‘skin deep.’’

B. Normalizable frequencies

We now turn to the case where all the external frequencies ω_i correspond to those of normalizable modes. To investigate this case, recall that the bulk-boundary propagator can be found as a limiting case of the bulk propagator,[5]

$$G_{B\partial}(b,x')=2\nu iR^{d-1}\lim_{\rho\to\pi/2}(\cos\rho)^{-2h_+}G_B(x,x'), \tag{4.8}$$

and that the Feynman propagator can be written in the form

$$iK_B(x,x')=\int\frac{d\omega}{2\pi}\sum_{nl\vec{m}}e^{i\omega(\tau-\tau')}\frac{\phi^{*}_{nl\vec{m}}(\rho,\hat{e})\phi_{nl\vec{m}}(\rho',\hat{e}')}{\omega_{nl}^2-\omega^2-i\epsilon}. \tag{4.9}$$

The normalizable wave functions $\phi_{nl\vec{m}}(\rho,\hat{e})$ have asymptotic behavior

$$\phi_{nl\vec{m}}(\rho,\hat{e})\xrightarrow{\rho\to\pi/2}k_{nl}(\cos\rho)^{2h_+}Y_{l\vec{m}}(\hat{e}) \tag{4.10}$$

for certain constants k_{nl} (see the Appendix). The frequency-space form of $K_{B\partial}$ is thus

$$K_{B\partial}(\omega,\hat{e};x')=2\nu R^{d-1}e^{i\omega r'}\sum_{n,l,\vec{m}}\frac{K_{nl}Y^{*}_{l\vec{m}}(\hat{e})\phi_{nl\vec{m}}(\rho',\hat{e}')}{\omega_{nl}^2-\omega^2-i\epsilon}. \tag{4.11}$$

From Eq. (4.11) we see that extracting the normalizable-frequency piece is rather delicate. We want only the contribution corresponding precisely to the normalizable-mode frequency, in order to eliminate the above non-normalizable behavior. The Green function is divergent at this frequency; one must extract the residue at the pole. Once one takes into account higher-loop corrections, these frequencies are not *a priori* known (though they should be determined by knowledge of the exact conformal weights), and the frequency-space behavior of the amplitude may be more complicated. These factors pose significant difficulties. Nevertheless, let us assume that we are able to perform these steps, and see where they lead.

Thus, we define a modified bulk-boundary propagator by

[4]This is directly connected to the fact that in the more general case of scalar fields of unequal masses, the integral (4.6) only converges for certain values of the scalar masses [13].

[5]For more details, see the Appendix.

$$\hat{K}(n,l,\vec{m},x)=\lim_{\omega\to\omega_{nl}}(\omega^2-\omega_{nl}^2)K_{B\partial}(\omega,l,\vec{m},x)$$
$$=2\omega_{nl}\oint\frac{d\omega}{2\pi i}K_{B\partial}(\omega,l,\vec{m},x)\tag{4.12}$$
$$=-2\nu R^{d-1}e^{i\omega_{nl}\tau}k_{nl}\phi^*_{nl\vec{m}}(x).$$

The corresponding operations can be performed directly on CFT correlators, where they are the operations needed to project onto a definite *state* of the CFT.

The modified propagator (4.12) does reproduce on-shell wave functions in flat space. Indeed, in the Appendix it is shown that in the $r\ll R$ limit, the modes $\phi_{nl\vec{m}}$ go over into the flat-space wave functions

$$\phi_{nl\vec{m}}(\vec{x})\xrightarrow[R\to\infty]{}\sqrt{2E}\frac{1}{r^{d/2-1}}J_{l+d/2-1}(Er)Y_{l\vec{m}}(\vec{e}).\tag{4.13}$$

Likewise, the coefficients k_{nl} can be worked out, with the result

$$\hat{K}(n,l,\vec{m},x)\xrightarrow[R\to\infty]{}C(E,R)i^l\frac{1}{(Er)^{d/2-1}}J_{l+d/2-1}(Er)\times Y^*_{l\vec{m}}(\hat{e})\tag{4.14}$$

with coefficient function

$$C(E,R)=-\frac{2^{2-\nu}}{\Gamma(\nu)}(-1)^{ER/2-h_+}(ER)^{2h_+}.\tag{4.15}$$

In fact, this can easily be transformed back to position space on the boundary, giving[6]

$$\hat{K}(E,\hat{e};x)=\sum_{l,\vec{m}}Y_{l\vec{m}}(\hat{e})\hat{K}(n,l,\vec{m},x)\xrightarrow[R\to\infty]{}\frac{C(E,R)}{(2\pi)^{d/2}}e^{i\vec{k}\cdot\vec{x}},\tag{4.16}$$

a plane wave with $\vec{k}=E\hat{e}$.

Putting all of this together suggests tree-level prescriptions for extracting flat-space S-matrix elements from boundary correlators

[6]Note that the following is valid to the extent that contributions from very large l do not contribute to the sum over all l. Such contributions give AdS corrections to the plane waves. These can be suppressed by consideration of wavepackets with spatial spread $<R$.

$$S[k_1,\cdots,k_n]=\prod_{i=1}^n\left[(2\pi)^{d/2}2E_iRC(E_i,R)^{-1}\oint_{E_iR}\frac{d\omega}{2\pi i}\right]\times\langle T\mathcal{O}(\omega_1,\hat{e}_1)\cdots\mathcal{O}(\omega_n,\hat{e}_n)\rangle$$
$$=\prod_{i=1}^n\left[(2\pi)^{d/2}C(E_i,R)^{-1}\lim_{\omega_i\to\omega_{n_il_i}}(\omega_i^2-\omega_{n_il_i}^2)\right]\times\langle T\mathcal{O}(\omega_1,\hat{e}_1)\cdots\mathcal{O}(\omega_n,\hat{e}_n)\rangle\tag{4.17}$$

where $\omega_{n_il_i}=E_iR$ is a normal mode frequency, and $k_i=E_i(1,\hat{e}_i)$.

However, at the multiloop level these expressions are potentially problematical. In particular, one does not *a priori* know the frequencies to tune ω_i to in order to sit on a normalizable mode, and the analytic structure in ω may contain more than just simple poles at these frequencies. One possible approach to this problem–given complete knowledge of the Yang-Mills correlators—would be to take a correlator and look for the frequencies where poles appear, extract the residues at these poles, and then use the result to construct the flat-space S-matrix along the above lines. These frequencies are approximately given by locating the poles in the boundary two-point function, which at tree level takes the form (see the Appendix)

$$K_\partial(b,b')\propto\lim_{\rho,\rho'\to\pi/2}(\cos\rho\cos\rho')^{-2h_+}K_B(x,x')$$
$$=k_\partial\int\frac{d\omega}{2\pi}e^{i\omega(\tau-\tau')}\sum_{nl\vec{m}}\frac{k_{nl}^2Y^*_{l\vec{m}}(\hat{e})Y_{l\vec{m}}(\hat{e}')}{\omega_{nl}^2-\omega^2}.\tag{4.18}$$

However, interactions will in general shift the energy of the two-particle state relative to twice the single-particle energy, causing added difficulty in precisely identifying the relevant frequencies.

Another approach, advocated in Refs. [7,8], is to convolve the boundary correlators with appropriately chosen wave packets. This appears to have some difficulty, as we can see from Eq. (4.11). If $f(\omega,\hat{e})$ is the wave packet profile for one of the external states, then it is connected to the rest of the diagram through a factor of the form

$$\int d\omega d^{d-1}\Omega f(\omega,\hat{e})K_{B\partial}(\omega,\hat{e},x')$$
$$=\int d\omega d^{d-1}\Omega f(\omega,\hat{e})\sum_{n,l\vec{m}}2\nu R^{d-1}\times\frac{k_{nl}Y^*_{l\vec{m}}(\hat{e})\phi_{nl\vec{m}}(\rho',\hat{e}')}{\omega_{nl}^2-\omega^2-i\epsilon}e^{i\omega\tau'}.\tag{4.19}$$

For any regular f this receives contributions from non-normalizable frequencies even if f is sharply peaked near a specific normalizable frequency. This in turn implies sensitivity to contributions from interactions at $r\gg R$, as described in Sec. IV A. It is not clear how to make wave packets focussed in the Minkowski region from these modes.

A modification of this procedure would be to *first* extract the amplitudes restricted to normalizable frequencies, as outlined above. Then wave packets can be built by taking linear combinations of those with different normal mode frequencies. These results suggest that while it may be possible to derive flat-space *S*-matrix elements from the AdS-CFT correspondence, the procedure is not so simple as it first appeared.

V. LOCALITY

Let us assume that it is possible to infer the full flat-space *S*-matrix from the conformal field theory correlators, through some variant of the above procedure or some other procedure. This would provide an even more concrete realization of the holographic proposal. A profound question underlying this proposal is how it is possible for the boundary theory to produce a bulk theory that is an approximately local theory in one higher dimension.

One piece of the answer appears to be that the boundary theory is a conformal field theory. In a conformal field theory, there is no mass-shell condition on the states, since there are no masses. Correspondingly, one can have states with fixed boundary momenta and a spectrum (discrete in global coordinates, continuous in Poincare coordinates) of frequencies. This statement is made manifest in the Kallen-Lehman representation for the boundary two-point function, which can be found from that of AdS given in Ref. [16]. In the bulk theory this spectrum is interpreted as arising from the different values of the momentum in the extra radial direction.

Thus, in this sense the boundary theory contains enough states to represent a bulk theory. However, this is no guarantee that the interactions have the correct properties to produce a sensible and approximately local bulk theory. One other key property is bulk momentum conservation, which has been the subject of one recent discussion [17]. This seems assured by the correspondence between the symmetries of the two theories: the conformal group SO$(2,d)$ is also the group of isometries of anti–de Sitter space, and in the large-R limit reduces to the Poincare group. However, this does not imply locality—there is an infinite variety of momentum conserving but nonlocal interactions.

One would like to investigate the possible presence of such nonlocality in the bulk theory. Of course, the bulk theory is not expected to be a local theory, but rather to have nonlocalities present on a scale of order the string scale. But this is to be contrasted with the situation where there are macroscopic nonlocalities, for example on scales of order the (large) AdS radius. One could imagine an observer living in an approximately Minkowski region in a very large AdS space without ever knowing about the large-scale curvature, and to such observers the only nonlocalities present should be very subtle and difficult to measure effects not easily seen at long distances. Such an AdS observer should not be able to exploit macroscopic nonlocality to win the lottery. What property is it of the boundary conformal theory that ensures preservation of locality at the macroscopic level? How does one characterize the amount of nonlocality present? What experiments could be performed to measure it? And is it sufficient to resolve the black hole information paradox or solve the cosmological constant problem? These are all questions of considerable importance.

As expected on grounds of both holography and gauge invariance, it appears that one can at best compute the *S*-matrix of the bulk theory from full knowledge of the CFT correlators. Therefore conventional tests of locality—such as the commutativity of field operators at spacelike separations—are not available. One must find ways of diagnosing locality directly from the *S*-matrix.

This is a difficult problem.[7] First consider theories with a mass gap. Here one test of locality for the *S*-matrix is that it respect various bounds that can be derived as a result of locality. Amplitudes must satisfy both upper and lower bounds. For example, the Froissart bound [18] states that for four-point scattering at arbitrary angle, the amplitude must obey

$$|A(s,\theta)|<Cs\log^2 s, \tag{5.1}$$

where C is a constant; there are more stringent bounds for fixed angle, $0<\theta<\pi$. Polynomial boundedness also implies the Cerulus-Martin bound [19,20], which states that amplitudes at arbitrary angles cannot fall too rapidly at fixed angle:

$$|A(s,\theta)|\geqslant s^{-c\sqrt{s}}. \tag{5.2}$$

Similar bounds have also been found in nonlocal theories. One way of producing a nonlocal theory is to construct a theory with an exponentially increasing density of states, such as string theory. This leads to the definition of quasilocalizable theories,[8] which are theories with nonlocality occurring below a definite length scale l. These theories are characterized for example by densities of states that grow as

$$\rho(m)\sim e^{(lm)^{\gamma}}, \tag{5.3}$$

for some power γ, and satisfy bounds of the form

$$|A(s,\theta)|<C's^2\log\rho(s^{1/2}). \tag{5.4}$$

This suggests the possibility of reading off the scale of nonlocality from the behavior of the four-point scattering amplitude.

However, derivation of these bounds assumes the existence of a gap. The requisite analyticity properties are spoiled by massless particles; gravity is particularly problematical. Here of course the usual IR divergences imply that in four dimensions one must study inclusive cross sections, summing over soft particles below some energy resolution relevant to the experiment in question. The *S*-matrix can be defined in higher dimensions.[9] One might hope that similar

[7] I thank K. Bardakci for several conversations on this issue.

[8] For more discussion of these and their bounds, see Ref. [21], and references therein.

[9] I thank T. Banks for a conversation on this point.

bounds could be derived for either the four-dimensional (4D) inclusive rates or for higher-dimensional *S*-matrices, providing a way of quantifying the degree of nonlocality of the theory.[10] This is an important problem for the future.

We have some partial information about high-energy string scattering [22,23] that might be considered as a standard of comparison. For example, Ref. [22] investigated the large *s*, fixed angle regime, and found perturbative amplitudes of the form

$$|A_G(s,\theta)| \sim e^{-(s\ \ln s + t\ \ln t + u\ \ln u)/4G} \tag{5.5}$$

at genus *G*. It would be very interesting to see whether the CFT reproduces this and other stringy behavior. Moreover, there are a number of open questions about the large order and nonperturbative completion of results such as Eq. (5.5) for high-energy string scattering, and one hope is that the boundary CFT could teach us something new about this.

So far the discussion has focussed on locality in the large *s* regime. However, it is not even *a priori* clear how the AdS/CFT correspondence produces bulk locality at the macroscopic level. We would like to find appropriate criteria for this.

One possibility is to for example consider fixed energy scattering of physical particles[11] at large distances, or equivalently small *t*. One very rough criterion is that we have interactions falling at least as fast as $1/r^{d-2}$ at long distances, since the only long range forces are expected to be gravity and other coulombic interactions. This corresponds to scattering amplitudes that grow like $1/t$ as t decreases. In the AdS context, one of course expects this growth to be truncated at the AdS radius $t \sim (1/R)^2$, but for $R \gg 1$ there is a clean separation of scales and one can study growth of amplitudes in the regime $1 \gg t \gg (1/R)^2$ to see if they satisfy this crude criterion. Indeed, taking this one step further, one could also ask whether the boundary theory reproduces the correct structure for gravitational (or other Coulombic) scattering for a wide variety of semiclassical states. This would be an important test, sensitive to bulk locality and more, of any independent calculation of the CFT correlators. It would be very interesting to go beyond to understand what property of the boundary theory ensures recovery of the correct semiclassical limit.

One might inquire whether locality properties are encoded nicely in the operator product expansion of the boundary theory. To investigate this, one first needs a translation of the kinematical variables—the Mandelstam invariants—into the AdS and CFT contexts. This is provided by considering the quadratic Casimir of $SO(2,d)$, which in the large-R limit reduces to the Poincare invariant P^2 as shown in the Appendix. Thus if we wish to combine two states in representations of $SO(2,d)$, the analogue of s is now provided by the conformal weight Δ of the resulting states in the product representation.

First consider the problem of large-*s* scattering. This for example should be governed by fusion of two high energy states into a third state with large Δ. The operator product expansion (OPE) takes the general form

$$\phi_{\Delta_i}(x)\phi_{\Delta_j}(y) \sim \sum_k c_{ijk} \frac{\phi_{\Delta_k}(y)}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}}. \tag{5.6}$$

Thus large Δ_k corresponds to very high order terms in the OPE, rather than the leading behavior. One can see a similar effect by considering scattering of two particles with momenta $E(1,\hat{e}_1)$ and $E(1,\hat{e}_2)$. For these, $s = E^2(1-\hat{e}_1\cdot\hat{e}_2)$. In the OPE limit, $\hat{e}_1 \to \hat{e}_2, s \sim E^2\theta^2$ where θ is the angle between the unit vectors. Small boundary distance only corresponds to large *s* if we simultaneously take very large *E*.

However, this suggests that behavior of amplitudes at $t \to 0$ could be explored in the OPE limit in the *t* channel. I hope to return to the implications for CFT in future work.

Finally, related to this discussion is the question of whether one can reconstruct the entire bulk theory given complete data in the boundary theory. In general reconstruction of a theory given the *S*-matrix is not unique, but one may try to reconstruct even *one* bulk theory that reproduces the correct amplitudes. Consider first three-point functions. A problem here is that the CFT three-point function is uniquely determined by conformal symmetry, up to a constant; in the example of scalars, both interactions

$$g_1 \int dV \phi^3 \quad \text{and} \quad g_2 \int dV \phi^2 \partial^2 \phi \tag{5.7}$$

give identical three-point functions up to this constant [13]. Of course the dependence of this constant on *R* for the two different cases will be different; this will help in decoding the different interactions. But in general the program will involve considering four- and higher-point functions, with their nontrivial dynamics. The problem of reconstruction is an interesting one for the future.

VI. CONCLUSION

This paper has attempted a modest beginning of an investigation of bulk locality in the AdS-CFT correspondence. A first problem is to extract the flat space *S*-matrix from the boundary correlators. Ideas for how to do this have been previously presented in Refs. [7,8,5,6], but there are subtleties. In particular, it was showed that at arbitrary *non-normalizable* frequencies, the boundary correlators are sensitive to interactions in the "skin" of AdS, at radii $r > R$, due to growth of the wave functions and of the volume of anti–de Sitter space. It appears that only by a delicate procedure of extracting the residues of poles at the normalizable frequencies may we find the flat *S*-matrix. There are bound to be wrinkles in this procedure when all-order perturbative or nonperturbative amplitudes are considered.

We would like to know what form locality takes in the

[10]Also, in AdS the radius supplies an intrinsic IR cutoff, which may be useful in circumventing the usual IR problems.

[11]These must in particular be neutral under any non-Abelian gauge groups.

bulk theory. There is no known procedure to extract off-shell data about this theory—in accord with holography as well as gauge invariance—and so information about locality properties must be derived directly from the *S*-matrix. In theories with a gap, locality or even nonlocality on a definite scale implies certain bounds for the *S*-matrix, but the presence of massless particles and particularly gravity complicates the story. Nonetheless, some information is known about the behavior of high-energy string scattering amplitudes, and one might as a first test try to investigate this behavior from the CFT and even to go beyond to new results. Furthermore, the bulk theory should exhibit macroscopic locality, namely bulk observers should find an approximately local theory at long distances. One very rough criterion for this is falloff of potentials (thus growth of amplitudes) bounded by Coulomb behavior. Indeed, another important test is reproduction of semiclassical gravitational scattering. We would like to know whether the CFT reproduces such behavior, and in particular what properties of the boundary theory allow the surprising result that an approximately local higher-dimensional theory emerges from it.

ACKNOWLEDGMENTS

I have greatly benefited from conversations with K. Bardakci, D. Gross, G. Horowitz, M. Kugler, D. Jackson, H. Ooguri, S. Shenker, and J. Polchinski who has been working on closely related ideas. I would like to particularly thank the theory group, and especially my host S. Kachru, at Berkeley and Lawrence Berkeley Labs, where much of this work was carried out, for their kind hospitality, and the Institute for Theoretical Physics and Center for Mathematical Physics at the University of Amsterdam, where this work was completed. This work was partially supported by the U.S. DOE Contract No. DE-FG-03-91ER40618.

APPENDIX: BASIC AdS TOOLS

This appendix will review some basic properties of the geometry of anti–de Sitter space and its boundary, and of wave functions and propagators on AdS, as well as deriving some new and useful results. In particular, an explicit treatment of the large R limit will be given.

1. Geometry

AdS_{d+1} can be represented as the solution of the equation

$$(X^M)^2 = -R^2 \tag{A1}$$

in flat $(2,d)$ signature Minkowski space with coordinates (X^{-1},X^0,X^i) and with metric

$$dS^2 = \eta_{MN} dX^M dX^N = -(dX^{-1})^2 - (dX^0)^2 + (dX^i)^2. \tag{A2}$$

Global coordinates $(\tau,\rho,\hat{e})$, where $\hat{e}$ is a d-dimensional unit vector, are defined by

$$X^{-1} = R\frac{\cos\tau}{\cos\rho},\quad X^0 = R\frac{\sin\tau}{\cos\rho},\quad X^i = R\tan\rho\,\hat{e}^i, \tag{A3}$$

and in these coordinates the metric takes the form

$$ds^2 = \frac{R^2}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{d-1}^2). \tag{A4}$$

Typically we will work on the universal cover of anti–de Sitter space, which in an abuse of notation will also be denoted AdS_{d+1}. The $\mathbb{R}\times S^{d-1}$ boundary of this cover corresponds to $\rho = \pi/2$ and is parametrized as $b = (\tau,\hat{e})$.

There are two related notions of invariant distance on AdS_{d+1}. The first is *geodetic* distance, defined with respect to the embedding space metric:

$$\sigma(X_1,X_2) = \frac{1}{2}\,\eta_{MN}\Delta X^M \Delta X^N; \tag{A5}$$

in global coordinates it can be shown that

$$\sigma(x_1,x_2) = -R^2 + \frac{R^2}{\cos\rho_1\cos\rho_2}[\cos(\tau_1-\tau_2) - \sin\rho_1\sin\rho_2\,\hat{e}_1\cdot\hat{e}_2]. \tag{A6}$$

The second is *geodesic* distance, as measured in the AdS metric (A4). This can be shown to be given by

$$s(x_1,x_2) = R\cosh^{-1}\left[\frac{\cos(\tau_1-\tau_2) - \sin\rho_1\sin\rho_2\,\hat{e}_1\cdot\hat{e}_2}{\cos\rho_1\cos\rho_2}\right] \tag{A7}$$

in global coordinates. Geodesic and geodetic distances are related by

$$\cosh\left(\frac{s}{R}\right) = 1 + \frac{\sigma}{R^2}. \tag{A8}$$

There is of course no conformally-invariant notion of interval on the boundary. When working in Poincaré coordinates for AdS_{d+1},

$$ds^2 = R^2\left(\frac{dU^2}{U^2} + U^2 dx^2\right), \tag{A9}$$

where

$$U = \frac{X^{-1} - X^d}{R};\quad x^\alpha = X^\alpha/RU,\quad \alpha = 0,1,\ldots,d-1, \tag{A10}$$

one frequently uses the nonconformally invariant interval $x_{12}^2 = |x_1 - x_2|^2$. In global coordinates this becomes

$$x_{12}^2 = \frac{2[\cos(\tau_1-\tau_2) - \hat{e}_1\cdot\hat{e}_2]}{(\cos\tau_1 - \hat{e}_1^d)(\cos\tau_2 - \hat{e}_2^d)}; \tag{A11}$$

thus the definition

$$s_{12}^2 = \cos(\tau_1-\tau_2) - \hat{e}_1\cdot\hat{e}_2 \tag{A12}$$

gives an analogous nonconformally invariant interval in the global frame. In order to form conformal invariants, one must have four or more points, allowing the definition of cross ratios such as

$$\frac{s_{12}^2 s_{34}^2}{s_{13}^2 s_{24}^2}=\frac{[\cos(\tau_1-\tau_2)-\hat{e}_1\cdot\hat{e}_2][\cos(\tau_3-\tau_4)-\hat{e}_3\cdot\hat{e}_4]}{[\cos(\tau_1-\tau_3)-\hat{e}_1\cdot\hat{e}_3][\cos(\tau_2-\tau_4)-\hat{e}_2\cdot\hat{e}_4]}. \tag{A13}$$

2. Wave functions

In many respects anti–de Sitter space behaves as a resonant cavity. In particular, solutions of the scalar wave equation

$$(\Box-m^2)\phi=0 \tag{A14}$$

are for generic frequencies non-normalizable, having asymptotic behavior

$$\phi\propto(\cos\rho)^{2h_-}, \tag{A15}$$

where

$$4h_\pm=d\pm\sqrt{d^2\pm 4m^2R^2}=d\pm 2\nu, \tag{A16}$$

near the boundary at $\rho=\pi/2$.

Only for special frequencies,

$$\omega_{nl}=2h_++2n+l \tag{A17}$$

do normalizable solutions exist. Explicit forms for these solutions are given in Refs. [14,11],

$$\phi_{nl\vec{m}}(\vec{x},\tau)=\chi_{nl}(\rho)Y_{l\vec{m}}(\hat{e})\frac{e^{-i\omega_{nl}\tau}}{\sqrt{2\omega_{nl}}}, \tag{A18}$$

where the radial wave functions $\chi_{nl}(\rho)$ are written in terms of the Jacobi polynomials[12]

$$\chi_{nl}(\rho)=A_{nl}(\cos\rho)^{2h_+}(\sin\rho)^l P_n^{l+d/2-1,\nu}(\cos 2\rho). \tag{A19}$$

Here A_{nl} is a normalization constant. If the $\phi_{nl\vec{m}}$ are given the conventional Klein-Gordon normalization

$$(\phi_{nl\vec{m}},\phi_{n'l'\vec{m}'})=\int d\Sigma^\mu \phi^*_{nl\vec{m}} i\overleftrightarrow{\partial}_\mu \phi_{n'l'\vec{m}'}=\delta_{nn'}\delta_{ll'}\delta_{\vec{m}\vec{m}'} \tag{A20}$$

with respect to surfaces of constant τ, then these constants are

$$A_{nl}^2=\frac{2\omega_{nl}}{R^{d-1}}\frac{n!\Gamma(n+2h_++l)}{\Gamma\left(n+l+\frac{d}{2}\right)\Gamma(n+\nu+1)}. \tag{A21}$$

The normalizable solutions have asymptotic behavior given by

$$\chi_{nl}(\rho)\xrightarrow{\rho\to\pi/2}k_{nl}(\cos\rho)^{2h_+}, \tag{A22}$$

with constants k_{nl} given by

$$k_{nl}=(-1)^n A_{nl}\frac{\Gamma(n+\nu+1)}{n!\Gamma(\nu+1)}. \tag{A23}$$

3. Green functions

The bulk Feynman Green function is defined by solving

$$(\Box-m^2)iK_B(x,x')=-\delta(x,x') \tag{A24}$$

with Feynman boundary conditions. It can be represented as an infinite sum over the normalizable modes,

$$iK_B(x,x')=\int\frac{d\omega}{2\pi}\sum_{nl\vec{m}} e^{i\omega(\tau-\tau')}\frac{\phi^*_{nl\vec{m}}(\vec{x})\phi_{nl\vec{m}}(\vec{x}')}{\omega_{nl}^2-\omega^2-i\epsilon}. \tag{A25}$$

The sum has been explicitly performed to yield [14,15]

$$iK_B(x,x')=\frac{C_B}{[\cosh^2(s/R)]^{h_+}}F\left(h_+,h_++\frac{1}{2};\nu+1;\frac{1}{\cosh^2(s/R)}-i\epsilon\right), \tag{A26}$$

where C_B is a constant,

$$C_B=\frac{1}{R^{d-1}}\frac{\Gamma(2h_+)}{2^{2h_++1}\pi^{d/2}\Gamma(\nu+1)}, \tag{A27}$$

and F is the hypergeometric function.

The bulk-boundary propagator $K_{B\partial}$ is designed to provide a solution to the free wave equation (A14) satisfying the boundary condition

$$\phi\xrightarrow{\rho\to\pi/2}(\cos\rho)^{2h_-}f(b). \tag{A28}$$

This solution is given by

$$\phi(x)=\int db f(b)K_{B\partial}(b,x). \tag{A29}$$

[12]The conventions of Ref. [24] will be followed.

A simple Green's theorem-type argument [6] shows that

$$K_{B\partial}(b,x')=2\nu R^{d-1}\lim_{\rho\to\pi/2}(\cos\rho)^{-2h_+}iK_B(x,x'). \tag{A30}$$

We can therefore find two equivalent expressions for $K_{B\partial}$, the first from the limit of Eq. (A25),

$$K_{B\partial}(b,x')=2\nu R^{d-1}\int\frac{d\omega}{2\pi}\sum_{nl\vec{m}}e^{i\omega(\tau-\tau')}\times\frac{k_{nl}Y^*_{l\vec{m}}(\hat{e})\phi_{nl\vec{m}}(\vec{x}')}{\omega_{nl}^2-\omega^2-i\epsilon}, \tag{A31}$$

and the second from the limit of Eq. (A26),

$$K_{B\partial}(b,x')=C_{B\partial}\left[\frac{\cos^2\rho'}{[\cos(\tau-\tau')-\sin\rho'\hat{e}\cdot\hat{e}']^2+i\epsilon}\right]^{h_+}, \tag{A32}$$

where

$$C_{B\partial}=\frac{\Gamma(2h_+)}{2^{2h_+}\pi^{d/2}\Gamma(\nu)}. \tag{A33}$$

Finally, the boundary two-point function can be obtained by taking the second point to the boundary,

$$G_\partial(b,b')\propto\lim_{\rho'\to\pi/2}(\cos\rho')^{-2h_+}G_{B\partial}(b,x'). \tag{A34}$$

This gives

$$K_\partial(b,b')=k_\partial\int\frac{d\omega}{2\pi}\sum_{nl\vec{m}}e^{i\omega(\tau-\tau')}\frac{k_{nl}^2Y^*_{l\vec{m}}(\hat{e})Y_{l\vec{m}}(\hat{e}')}{\omega_{nl}^2-\omega^2-i\epsilon} \tag{A35}$$

and

$$K_\partial(b,b')=C_\partial\frac{1}{([\cos(\tau-\tau')-\hat{e}\cdot\hat{e}']^2+i\epsilon)^{h_+}}, \tag{A36}$$

where k_∂ and C_∂ are constants. Note that K_∂ is naturally written in terms of the boundary interval (A12).

4. Large-*R* limits

It was shown in Sec. III that for large R, in a patch of proper size $\mathcal{O}(R)$, anti–de Sitter space may be approximated by Minkowski space. This can be explicitly seen in the coordinates

$$t=R\tau;\quad r=R\rho, \tag{A37}$$

where the metric takes the form

$$ds^2=\frac{1}{\cos^2(r/R)}\left[-dt^2+dr^2+R^2\sin^2\left(\frac{r}{R}\right)^2d\Omega^2\right]. \tag{A38}$$

This subsection will discuss other aspects of the relationship between AdS_{d+1} and M_{d+1} at large R.

First consider the relation between the symmetry generators J_{MN} of SO(2,d) and those of the Poincáre group. In the vicinity of $(\tau,\rho)=(0,0)$, the connection can be made through the identification

$$p_\mu=\frac{J_{-1\mu}}{R},\quad M_{\mu\nu}=J_{\mu\nu}; \tag{A39}$$

the SO(2,d) algebra clearly goes over to the Poincare algebra in the limit $R\to\infty$. Also useful is the quadratic Casimir, which is important for the classification of the representations of the conformal group. It is given by

$$C_2=\frac{1}{2}J_{MN}J^{MN}=\frac{1}{2}M_{\mu\nu}M^{\mu\nu}-R^2P_\mu P^\mu \tag{A40}$$

and takes value $C_2=-\Delta(d-\Delta)$ in a representation of conformal weight Δ. From the relationship $\Delta=2h_+$ and Eq. (A16), we see that

$$C_2=m^2R^2, \tag{A41}$$

which combined with Eq. (A40) gives the correct mass-shell relation in the large-R limit. The quadratic Casimir may be used to find analogues of the Mandelstam invariants.

Next consider wave functions. The relations (4.2),(A17) imply that

$$ER=2h_++2n+l, \tag{A42}$$

so for fixed Minkowski energy and angular momentum, and $m\lesssim\mathcal{O}(1/R)$, large R corresponds to large n. A useful relation for large order Jacobi polynomials is

$$\lim_{n\to\infty}\frac{1}{n^\alpha}P_n^{\alpha\beta}[\cos(x/n)]=\left(\frac{2}{x}\right)^\alpha J_\alpha(x). \tag{A43}$$

For $r\ll R$ this gives

$$\lim_{R\to\infty}\cos^{2h_+}(r/R)\sin^l(r/R)P_n^{(l+d/2-1,\nu)}[\cos(2r/R)]=(ER)^{d/2-1}\frac{1}{(Er)^{d/2-1}}J_{l+d/2-1}(Er), \tag{A44}$$

which is, up to the overall power of ER, the standard flat space radial wave function. We also need A_{nl}, which from Eq. (A21) via Sterling's approximation is

$$A_{nl}\xrightarrow[R\to\infty]{}\sqrt{\frac{2\omega_{nl}}{R^{d-1}}}. \tag{A45}$$

Combining this with Eq. (A44) then gives the desired wave functions at large R:

$$\phi_{nl\vec{m}}(\vec{x})\xrightarrow[R\to\infty]{}\sqrt{2E^{d-1}}\frac{1}{(Er)^{d/2-1}}J_{l+d/2-1}(Er)Y_{l\vec{m}}(\hat{e}), \tag{A46}$$

where $r \ll R$ is understood. At $\rho \to \pi/2$, corresponding to large r, one still finds behavior $\propto (\cos\rho)^{2h_+}$, and in fact the large-$R$ limit of the coefficients k_{nl} of Eq. (A22) is

$$k_{nl} \xrightarrow[R\to\infty]{} \frac{(-1)^n}{\Gamma(\nu+1)} \frac{\sqrt{2E}}{R^{d/2-1}} \left(\frac{ER}{2}\right)^{\nu}. \tag{A47}$$

Finally consider the large-R behavior of the Green functions. The asymptotics of the bulk propagator immediately follows from Eq. (A26). Geodesic distance on AdS_{d+1} trivially becomes the Minkowski interval. The hypergeometric function must therefore be evaluated with argument near one, which is done via the formula

$$\begin{aligned} F\left(h_+, h_+ + \frac{1}{2}; \nu+1; z\right) &= \frac{\Gamma(\nu+1)\Gamma\left(\frac{1-d}{2}\right)}{\Gamma(1-h_-)\Gamma\left(\frac{1}{2}-h_-\right)} F\left(h_+, h_+ + \frac{1}{2}; \frac{d+1}{2}; 1-z\right) \\ &\quad + (1-z)^{(1-d)/2} \frac{\Gamma(\nu+1)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(h_+)\Gamma\left(h_+ + \frac{1}{2}\right)} \\ &\quad \times F\left(1-h_-, \frac{1}{2}-h_-; \frac{3-d}{2}; 1-z\right). \end{aligned} \tag{A48}$$

Taking

$$z = \frac{1}{\cosh^2(s/R)} \approx 1 - \frac{s^2}{R^2}, \tag{A49}$$

and assuming that ν stays finite as $R \to \infty$, we find

$$iK_B(x,x') \xrightarrow[R\to\infty]{} \frac{\tilde{C}}{[s^2(x,x') + i\epsilon]^{(d-1)/2}}, \tag{A50}$$

where

$$\tilde{C} = \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma(2h_+)}{\pi^{d/2} 2^{2h_+ + 1} \Gamma(h_+) \Gamma\left(h_+ + \frac{1}{2}\right)}. \tag{A51}$$

This is the expected massless flat-space propagator.

As we saw in Sec. IV, for generic frequencies the bulk-boundary propagator $K_{B\partial}$ grows as $(\cos\rho)^{2h_-}$ at the boundary, and so in the large-R limit is not concentrated in the Minkowski region. This can be remedied by restricting to normalizable frequencies, as in Eq. (4.12). The large-R behavior of the resulting function $\hat{K}$ is readily inferred from Eqs. (A46) and (A47), and gives

$$\hat{K}(n,l,\vec{m},x) \xrightarrow[R\to\infty]{} C(E,R)(-1)^{l/2} \times \frac{1}{(Er)^{d/2-1}} J_{l+d/2-1}(Er) Y^*_{l\vec{m}}(\hat{e}), \tag{A52}$$

where the coefficient function $C(E,R)$ is given by

$$C(E,R) = \frac{2^{2-\nu}}{\Gamma(\nu)} (-1)^{ER/2 - h_+} (ER)^{2h_+}. \tag{A53}$$

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