Compactified little string theories and compact moduli spaces of vacua

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It is emphasized that compactified little string theories have compact moduli spaces of vacua, which globally probe compact string geometry. Compactifying various little string theories on T^3 leads to three-dimensional (3D) theories with an exact, quantum Coulomb branch given by an arbitrary T^4 of volume M_s^2, an arbitrary $K3$ of volume M_s^2, and moduli spaces of $G=\mathrm{SU}(N)$, $\mathrm{SO}(2N)$, or E_6, E_7, E_8 instantons on an arbitrary T^4 or $K3$ of fixed volume. Compactifying instead on a T^2 leads to 4D theories with a compact Coulomb branch base which, when combined with the exact photon gauge coupling fiber, is a compact, elliptically fibered space related to the above spaces.

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I. INTRODUCTION

Over the past few years there have been a variety of connections between the moduli spaces of supersymmetric gauge theories and stringy geometry. For example, singular background geometry or gauge bundles can lead to enhanced, nonperturbative, gauge theories, whose moduli spaces reproduce the local singularity [1,2]. Another connection is via branes, whose world-volume supersymmetric gauge theory has a moduli space which ''probes'' [3,4] the geometry in which the branes live. In an extreme form of this connection [5], we perhaps actually live in the moduli space of a supersymmetric theory. It is thus interesting to consider, generally, what types of geometry can be reproduced via moduli spaces of vacua.

A basic issue is whether moduli spaces of vacua can be compact. Moduli spaces of vacua of standard gauge theories are generally noncompact cones: if a given set of scalar expectation values $\langle \phi_i \rangle$ is a D-flat vacuum, so is $\lambda\langle \phi_i \rangle$ for arbitrary scaling factor $\lambda \to \infty$. [This is slightly modified by Fayet-Iliopoulos terms for U(1) factors.] An exception is the Coulomb branch moduli, associated with the Wilson lines, of gauge theories which are compactified on tori; these moduli live on dual tori, modded out by the Weyl group.

The present note is devoted to emphasizing that toroidally compactified ''little string theories'' [6,7][1] can have a variety of interesting, compact, moduli spaces of vacua. The present discussion is an elaboration of a footnote which appeared in [11]. The basic message is that, while world-volume gauge theories only *locally* probe the geometry transverse to the brane, little string extensions can *globally* probe *compact* geometry. While this fact is perhaps well known to some experts, it is hoped that some readers will find it of interest.

For example, the basic $\mathcal{N}=(1,0)$ heterotic little string theory, when compactified on T^3, is argued to have a Coulomb branch moduli space of vacua which is a $K3$ of volume M_s^2. (Since a 3D scalar has mass dimension 1/2, this has the correct dimensions.) The Coulomb branch is a nonlinear σ model with an exact, quantum metric equal to the Ricci-flat metric of $K3$. The $K3$ is an arbitrary metric of fixed volume, whose parameter space coincides with that of the T^3 compactified heterotic little string theory; the map between these parameter spaces is the same as enters in the duality between the 10D heterotic string on T^3 and M theory on $K3$. Geometric symmetries of the $K3$ Coulomb branch map to nontrivial T dualities of the T^3 compactified little string theory.

More generally, it will be argued that the little string theories obtained in [11] from K heterotic (or type-II) Neveu-Schwarz (NS) branes at a transverse $\mathbb{C}^2/\Gamma_G$ singularity, when compactified on T^3, have a compact Coulomb branch moduli space of vacua given by the moduli space of K G instantons on $K3$ (or T^4). Here G is an arbitrary A, D, E group and Γ_G is the corresponding SU(2) subgroup. The $K3$ or T^4 appearing here is precisely that of M theory duality, which the compactified little string theory globally probes. The volume of the compact Coulomb branch is again set by M_s. In each case, the Coulomb branch σ model metric must be the unique one which is Ricci flat.

Similarly, it will be argued that little string theories, when compactified to 4D on a T^2, have Coulomb branches which globally probe F theory. The Coulomb branch is the base space of F theory, and the photon kinetic terms are the elliptic fibration. For example, compactifying the basic $\mathcal{N}=(1,0)$ little string theory on T^2 leads to a 4D theory whose total space of Coulomb branch base and Seiberg-Witten curve is an elliptically fibered $K3$ of volume M_s^2. The map between the T^2 compactification data and the parameter space of fixed volume, elliptically fibered, $K3$ spaces is the same as in the duality between the 10D heterotic theory on T^2 and F theory on an elliptically fibered $K3$ [12].

The next section will review little string theories and their compactification, with several new minor comments included. Section III outlines classical T^3 dimensional reduction of ordinary 6D U(1) and SU(2) gauge fields. This al-

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[1]The extent to which these 6D theories decouple from the 10D bulk, for energies above some gap value, is subtle [8,9,10]; we will ignore these issues and only discuss the vacuum manifold.

ready leads to compact Coulomb branches, of fixed volume g_6^{-2}; the Coulomb branch for U(1) is T^4, while that of SU(2) is $K3$. Section IV extends the probe argument of [13] to argue for our main message: that compactifying little string theories on T^3 leads to theories whose exact Coulomb branch is compact and globally probes the T^4 or $K3$ of M theory duality. Section IV discusses T^2 compactification of little string theories, which similarly have compact Coulomb branches that globally probe compactification of F theory, e.g., on elliptically fibered $K3$'s.

The proposed relation of [11] between compactified type-II little string theories and moduli spaces of instantons on T^4 also entered in [14,15,16], where it was extended to moduli spaces of instantons on a noncommutative T^4 by introducing R-symmetry twists in the compactification. A relation between the twisted, compactified (2,0) theory and $K3$ was proposed in [15]; this appears to be unrelated to the presently discussed appearance of $K3$ in the context of the untwisted, compactified, heterotic little string theories. Much as in [15,16], it should also be possible to introduce R-symmetry twists for the compactified heterotic little string theories, perhaps leading to moduli spaces of instantons on a noncommutative $K3$, though this will not be done here.

II. REVIEW OF LITTLE STRING THEORIES AND THEIR COMPACTIFICATION

Four classes of 6D little string theories were obtained in [7] via the world-volume of five branes in the limit $g_s \to 0$ with M_s held fixed:

(iia) $\mathcal{N}=(1,1)$ supersymmetric, via K IIB NS five-branes [7] or via type-IIA or M theory with a $\mathbb{C}^2/T_G$ asymptotically locally euclidion singularity [17,18].

(iib) $\mathcal{N}=(2,0)$ supersymmetric, via K-type-IIA five branes [7] or type-IIB with a $\mathbb{C}^2/\Gamma_G$ singularity [17].

(o) $\mathcal{N}=(1,0)$ supersymmetric, via K SO(32) heterotic small instantons.

(e) $\mathcal{N}=(1,0)$ supersymmetric, via K heterotic $E_8 \times E_8$ small instantons.

Cases (iia) and (o) contain gauge fields with coupling $g_6^{-2} = M_s^2$ and are IR free. Instantons in the 6D gauge theories are fundamental strings, with tension $g_6^{-2} = M_s^2$. Cases (iib) and (e) instead contain tensor multiplet two-form gauge fields, with self-dual field strength, and lead to interacting renormalization-group fixed point field theories in the IR.

$\mathcal{N}=(1,0)$ tensor multiplet theories [of which $\mathcal{N}=(2,0)$ is a special case] always have an associated group G. For cases (iib) it is SU(K) or the ADE singularity group G, while for (e) it is Sp(K). There is a $r = \mathrm{rank}(G)$ dimensional, compact Coulomb branch moduli space, with the real scalars in the $\mathcal{N}=(1,0)$ tensor multiplets taking values $\langle \mathbf{\Phi} \rangle$ in the "G-Coxeter box" $(S^1)^{\otimes r}/W_G$, where the S_1 is of radius M_s^2 and W_G is the Weyl group of G. The theory is interacting at the boundaries of the Coxeter box but, in the bulk, behaves in the IR as r free self-dual tensor multiplets. Strings are charged under the r two-form gauge fields of these tensor multiplets, with charge vectors $\boldsymbol{\alpha}$ in the G root lattice.[2] Via a Bogomol'nyi-Prasad-Sommerfield (BPS) formula, a string with charges $\boldsymbol{\alpha}$ has tension $Z = \boldsymbol{\alpha} \cdot \mathbf{\Phi}$, becoming tensionless at the origin of the Coulomb branch. Reducing to 5D leads to a gauge theory with non-Abelian gauge group G at the origin, so the 6D theory can be regarded as a non-Abelian, self-dual, two-form gauge theory with group G (whatever that means).

Each of the four above classes has either vector or tensor multiplets, but not both. Theories containing both vector and tensor multiplets were discussed in [11] by combining five branes with $\mathbb{C}^2/\Gamma_G$ orbifold singularities in the transverse dimensions, using results obtained in [19–22]. In this way, new theories can be obtained for each of the four classes of branes, type-IIA, IIB, SO(32) heterotic, and $E_8 \times E_8$ heterotic, at $\mathbb{C}^2/\Gamma_G$ singularities. All of these theories generally have $\mathcal{N}=(1,0)$ supersymmetry.

For example, K-type-IIB NS five branes at a $\mathbb{C}^2/\Gamma_G$ singularity [21] has a quiver gauge theory, based on the extended Dynkin diagram of the ADE singularity group G, with gauge group $\mathrm{U}(1)_D \times \Pi_{\mu=0}^r \mathrm{SU}(Kn_\mu)$ and bifundamental matter. n_μ are the G Dynkin indices and $r = \mathrm{rank}(G)$. There are r $\mathcal{N}=(1,0)$ tensor multiplets, which are associated, as described above, with the singularity group G. Via an anomaly cancellation mechanism, SU(Kn_μ) has gauge coupling $g_{\mu,\mathrm{eff}}^{-2} = M_s^2 \delta_{\mu,0} + \boldsymbol{\alpha}_\mu \cdot \mathbf{\Phi}$ and an SU(Kn_μ) instantion, which is a string in 6D, has tensor-multiplet charges $\boldsymbol{\alpha}_\mu$ and BPS tension $Z_\mu = g_{\mu,\mathrm{eff}}^{-2}$. Here the $\boldsymbol{\alpha}_\mu$ are the G root vectors ($\boldsymbol{\alpha}_0$ is the extending root) and the condition that all $g_{\mu,\mathrm{eff}}^{-2} \geqslant 0$ is precisely that the Coulomb branch $\langle \mathbf{\Phi} \rangle$ is the G Coxeter box, of side length M_s^2. The SU(Kn_μ) instanton string charges span the G root lattice.

The instanton string for a diagonal $\mathrm{SU}(K)_D \subset \Pi_{\mu=0}^r \mathrm{SU}(Kn_\mu)$, with index of embedding n_μ in SU(Kn_μ), has tension $n^\mu Z_\mu = M_s^2$, and is identified with the fundamental IIB string. The other r-independent instanton strings in $\Pi_{\mu=0}^r \mathrm{SU}(Kn_\mu)$ are to be identified with the strings obtained [23] by wrapping the type-IIB three brane on the r-independent, fully collapsed, two-cycles of the $\mathbb{C}^2/\Gamma_G$ singularity; $m = 1...r$ of these strings become tensionless for $\langle \mathbf{\Phi} \rangle$ at a codimension m boundary of the Coulomb branch Coxeter box.

The simplest heterotic case is K SO(32) five branes at a $\mathbb{C}^2/\Gamma_G$ singularity [19–22]. The theories are associated [11] with a subgroup H of the singularity group G, with $G \to H$ as $\mathrm{SU}(2P) \to \mathrm{Sp}(P)$, $\mathrm{SO}(4P+2) \to \mathrm{SO}(4P+1)$, $\mathrm{SO}(4P) \to \mathrm{SO}(4P)$, $E_6 \to F_4$, $E_7 \to E_7$, $E_8 \to E_8$. The gauge group

[2] Because of the self-duality, these strings can be regarded as either "electrically" or "magnetically" charged. The Dirac quantization condition thus implies that the lattice Λ must be an *integer lattice*, i.e., the dot product of any two lattice vectors is an integer, so $\Lambda \subset \tilde{\Lambda}$, where $\tilde{\Lambda}$ is the dual lattice. This is, of course, a weaker condition than self-duality of the lattice. For example, the root lattice of a simple group G is generally not self-dual but, rather, a subgroup, of degree given by the center of G, in the dual lattice, which is the weight lattice.

and matter content is given by a quiver diagram, which is the extended H Dynkin diagram, with SO, Sp, and SU groups at various nodes, e.g., the group at the $\mu=0$ node is Sp(K). There are $r=\mathrm{rank}(H)$ $\mathcal{N}=(1,0)$ tensor multiplets, which are associated with the group H. Via an anomaly cancellation mechanism, the gauge group at node $\mu=0\ldots r$ of the quiver diagram has coupling $g_{\mu,\mathrm{eff}}^{-2}=M_s^2\delta_{\mu,0}+\boldsymbol{\alpha}_\mu\cdot\boldsymbol{\Phi}$, and an instantons string in this group has tensor multiplet charges $\boldsymbol{\alpha}_\mu$ and BPS tension $Z_\mu=g_{\mu,\mathrm{eff}}^{-2}$. Here $\boldsymbol{\alpha}_\mu$ are the simple and extending roots of H, so the instanton strings span the H root lattice. Instantons in a diagonal $\mathrm{Sp}(K)_D$ are identified with the fundamental heterotic string, of tension M_s^2. The other r-independent instanton strings can again be identified with three branes wrapped on collapsed two cycles; $m=1...r$ of these become massless at a codimension m boundary of the Coulomb branch (the H Coexter box).

The other heterotic case, K $E_8\times E_8$ five branes at a $\mathbb{C}^2/\Gamma_G$ singularity, leads to little string theories with a more involved spectrum of tensor multiplets, gauge groups, and matter content [22,11].

Compactifying on a circle, 6D vector and tensor multiplets both lead to 5D vector multiplets. A 6D $\mathcal{N}=(1,0)$ theory with a gauge group of rank r_V and n_T tensor multiplets, when compactified, leads to a 5D theory with a Coulomb branch moduli space of vacua of dimension $d_C=r_V+n_T$. Compactifying to 4D, the Coulomb branch has real dimension $2(r_V+n_T)$ and in 3D, upon dualizing the d_C photons, there is a Coulomb branch of real dimension $4(r_V+n_T)$.

Little string theories exhibit T duality when compactified on a circle [7], with the (iia) theory on a circle of radius R identical to the (iib) theory on a circle of radius $1/M_s^2R$. Similarly, the (o) heterotic theory, on a circle of radius R, and with a Wilson line around the circle breaking SO(32) to SO(16)×SO(16), is identical to the (e) heterotic theory on a circle of radius $1/M_s^2R$, again with a Wilson line breaking $E_8\times E_8$ to SO(16)×SO(16). (See [24] for the heterotic T duality with general Wilson lines.) In these cases, T-duality exchanges 6D tensor and vector multiplets, $r_V\leftrightarrow n_T$. This is nice because the 5D classical kinetic terms for the scalars coming from 6D tensor multiplets, $M_s^4R(d\Phi)^2$, is indeed exchanged with the kinetic term, $g_6^{-2}R(R^{-1}d\Phi)^2$, of a vector multiplet on a circle of radius R. In both cases Φ is a compact scalar, normalized so $\Phi\in[0,1]$, and the two kinetic terms are exchanged by $R\leftrightarrow(M_s^2R)^{-1}$ upon setting $g_6^{-2}=M_s^2$.

More generally, there is an expected T duality, with $R\leftrightarrow(M_s^2R)^{-1}$ exchanging the theories coming from type-IIA and type-IIB or SO(32) and $E_8\times E_8$ heterotic branes at $\mathbb{C}^2/\Gamma_G$ singularities. T-dual theories must have $r_V+n_T=\tilde{r}_V+\tilde{n}_T$. As was noted in [25,11], this is the case for the SO(32) and $E_8\times E_8$ branes at singularities: both cases have $r_V+n_T=C_2(G)K-|G|$, where $C_2(G)$ is the dual Coxeter number of the singularity group G and $|G|$ is its dimension. This formula will be important in what follows. A point of concern mentioned in [11] is that a stronger condition, $r_V=\tilde{n}_T$ and $n_T=\tilde{n}_V$, needed for T duality to exchange the classical kinetic terms as above, is not satisfied. The present situation is, in fact, closely connected to that of [26], where it was argued that T duality can fail. Here, however, there is a simpler resolution: the Coulomb branch metric can get quantum corrections and, while the quantum-corrected metrics are expected to agree, the classical metrics need not. The stronger condition is thus unnecessary.

All little string theories, when compactified on T^D, have the parameter space [7]

$$O(D+y,D;\mathbb{Z})\backslash O(D+y,D)/O(D+y)\times O(D), \quad (2.1)$$

where $y=0$ for the type-II cases and $y=16$ for the heterotic cases. These are the T^D metric and B_{NS} fields (D^2 real parameters), and also the SO(32) or $E_8\times E_8$ Wilson lines in the heterotic cases (16D real parameters). $O(D+y,D;\mathbb{Z})$ is the full T-duality group.

III. COMPACTIFICATION PRELIMINARIES

We first consider the classical dimensional reduction of a 6D U(1) gauge field,

$$\int d^6x\left(-\frac{1}{4g_6^2}F_{\mu\nu}F^{\mu\nu}+B_{\mathrm{NS}}\wedge F\wedge F\right), \quad (3.1)$$

on a T^3 to three dimensions. B_{NS} is an external, background, two-form gauge field. We take the space to be $\mathbb{R}^3\times T^3$, with $\mathbb{R}^3$ coordinates x^i, $i=1,2,3$, and periodic coordinates $\rho^a\in[0,1]$, $a=1,2,3$, for the T^3; the metric is $ds^2=\delta_{ij}dx^idx^j+h_{ab}d\rho^ad\rho^b$. Taking all fields to be independent of the T^3 coordinates ρ^a, Eq. (3.1) becomes

$$S=\int d^3x\left[\frac{\sqrt{\det h}}{g_6^2}\left[-\tfrac{1}{4}F_{ij}F^{ij}+\tfrac{1}{2}(h^{-1})^{ab}\partial_i\phi_a\partial^i\phi_b\right]\right.$$
$$\left.+\theta^a\epsilon^{ijk}F_{ij}\partial_k\phi_a\right], \quad (3.2)$$

where $B_{\mathrm{NS}}=\epsilon_{abc}\theta^ad\rho^b\wedge d\rho^c$ for some constants θ^a, $a=1,2,3$. The three real scalars ϕ_a are associated with the Wilson lines of the gauge field around the cycles $d\rho^a$ of the T^3 and are periodic, normalized so that $\phi_a\in[0,1]$.

The 3D U(1) gauge field can be dualized to another real scalar, which also lives on a circle. This is done as in [27]: we replace $F_{ij}\to F_{ij}-H_{ij}$ in Eq. (3.2) and introduce an additional term $\epsilon^{ijk}H_{ij}\partial_k\phi_4$, with the scalar ϕ_4 periodic, normalized so that $\phi_4\in[0,1]$. First integrating out ϕ_4 leads back to the original theory. First integrating out H sets $F_{ij}=0$ and leads to ϕ_4 kinetic terms. Combining with the ϕ_a kinetic terms in Eq. (3.2), the upshot is a T^4 Coulomb branch moduli space of vacua $\langle\phi_A\rangle$, $A=1,\ldots 4$, with metric

$$ds^2=\frac{\sqrt{\det h}}{g_6^2}(h^{-1})^{ab}d\phi_ad\phi_b+\frac{g_6^2}{\sqrt{\det h}}(d\phi_4-\theta^ad\phi_a)^2$$
$$\equiv G^{AB}d\phi_Ad\phi_B, \quad (3.3)$$

where a runs over 1, 2, 3 and $A=1,\ldots,4$. This metric G^{AB} has ten real components, which depend on the nine real parameters h^{ab} and θ^a, and thus satisfies one constraint. The relation is that the T^4 of Eq. (3.3) has fixed volume, independent of the h_{ab} and the θ^a:

$$\text{Volume}(T^4)=\sqrt{\det(G^{AB})}=\frac{1}{g_6^2}=M_s^2. \tag{3.4}$$

Although the above discussion was purely classical, the map (3.3) between the T^3 metric h_{ab} and B fields θ^a is exactly the relevant one for relating type-IIB string theory on T^3 to M theory on T^4, to be discussed in the next section. Indeed, the map (3.3) was also obtained in [16] in the context of the compactified (2,0) theory via a chain of string duality gymnastics.

We pause to note that the metric (3.3) nicely exhibits properties to be expected based on its connection to M theory. In particular, the obvious, geometric $SL(4;\mathbb{Z})$ discrete symmetries of the T^4 correspond to nontrivial T dualities, in a subgroup of the T-duality group appearing in Eq. (2.1). For example, consider the obvious requirement that the T^4 be invariant under the relabeling exchange $\phi_3\leftrightarrow-\phi_4$. Taking, for simplicity, T^3 with $h_{ab}=L_a^2\delta_{ab}$ and $\theta_a=0$, it follows from Eq. (3.3) that this operation corresponds to the operation

$$L_1\rightarrow(M_s^2L_2)^{-1},\quad L_2\rightarrow(M_s^2L_1)^{-1},\quad L_3\rightarrow L_3, \tag{3.5}$$

where we set $g_6^{-2}=M_s^2$. This is a T duality in two circles, which is nontrivial but, nevertheless, a symmetry taking the IIA or IIB theory back to itself. The generalization of the T duality (3.5) for general h_{ab} and θ_a is quite complicated, see, e.g., [28]; remarkably, it is indeed reproduced from Eq. (3.3) by simply requiring the $\phi_3\leftrightarrow-\phi_4$ symmetry.

On the other hand, T duality in an odd number of cycles, such as the $O(3,3;\mathbb{Z})$ element taking all $L_i\rightarrow(M_s^2L_i)^{-1}$, for $i=1,2,3$, is not a geometric $SL(4;\mathbb{Z})$ symmetry of Eq. (3.3). This is sensible, since such operations are not symmetries of type-IIA or type-IIB string compactifications but, rather, exchange types IIA and IIB.

In particular, starting instead from a 6D tensor multiplet, dimensional reduction on a T^3 leads to a T^4 Coulomb branch moduli space, with metric related to Eq. (3.3) by T duality in an odd number of the T^3 cycles, corresponding to the exchange of types IIA and IIB.

Now consider T^3 reduction of a 6D SU(2) gauge theory. The above discussion for U(1) carries over to this case with almost no changes. The only difference is that the real scalars ϕ_A must be modded out by the Weyl group action $\phi_A\sim-\phi_A$. Modding out the T^4 by this $\mathbb{Z}_2$ action leads to a $K3$. Thus the Coulomb branch of a 6D SU(2) gauge theory reduced to 3D on a T^3 is given by $\langle\phi_A\rangle$ in a compact $K3$. The volume of the $K3$ is again set by g_6^{-2}, and equal to M_s^2. The full parameter space of $K3$ metrics of fixed volume is 57 dimensional and given by Eq. (2.1) with $D=3$ and $y=16$, while that obtained here only depends on the nine-dimensional subspace given by Eq. (2.1) with $D=3$ and $y=0$. The remaining parameters will come from three real masses for each of 16 SU(2) fundamental matter flavors; these enter as the Wilson loop parameters in Eq. (2.1).

IV. THE PROBE ARGUMENT, CHECKS, AND COMMENTS

The parameter space (2.1) for T^3-compactified heterotic (or type-II) little string theories coincides with the geometric parameter space of a $K3$ (or T^4) of fixed volume. These are, of course, the standard miracles which enter in the duality between the 10D heterotic (or type-II) string on T^3 and M theory on $Y=K3$ (or T^4). The fundamental string arises as the $M5$ brane wrapped on Y, so $M_p^6\,\text{Vol}(Y)=M_s^2$. We will here extend the probe argument of [13] to argue that these M-theory dualities provide the solution for the exact, quantum, Coulomb branch metric of the T^3 compactified little string theories.

Recall that the argument of [13] started with 3D $\mathcal{N}=4$ supersymmetric (eight supercharges) SU(2) gauge theory with fundamental matter, which is the world-volume field theory in a D2 brane in type-I' string theory on T^3. This maps to a $M2$ brane in M theory of $K3$, which can be at an arbitrary point in the transverse $\mathbb{R}^4\times K3$. The $\mathbb{R}^4$ corresponds to a decoupled hypermultiplet in the world-volume theory. The $K3$ factor is more interesting: it was thus argued in [13] that the full, quantum-corrected metric on the Coulomb branch of the D2 brane world-volume field theory must be a local piece of the corresponding $K3$; this was confirmed in [27] purely in the context of 3D field theory.

The D2 brane world-volume field theory only *locally* probes the $K3$ because of the particular limit taken to decouple the bulk dynamics: $g_s\rightarrow0$ and $M_s\rightarrow\infty$. On the other hand, we can take $g_s\rightarrow0$, but with M_s held fixed. This theory is precisely the 6D heterotic little string theory (o), compactified to 3D on the same T^3 as the 10D heterotic or type-I' bulk theory. The T^3-compactified little string theory (o) globally probes the fixed volume $K3$ of M theory, and must thus have a Coulomb branch moduli space of vacua which is the same $K3$. The geometric $K3$ has volume $M_s^2M_p^{-6}$ and, taking into account how the properly normalized Coulomb moduli scalars probe geometry, the volume of the Coulomb branch $K3$ is M_s^2. This matches with the result of the previous section. This compact Coulomb branch properly becomes noncompact in the field theory limit $M_s^2\rightarrow\infty$.

The $K3$ Coulomb branch can have singularities, depending on the choice of parameters in Eq. (2.1). As in [13], these singularities mark the intersection of the Coulomb branch with a Higgs branch, with an interacting 3D infrared conformal field theory at the intersection.

Unfortunately, both sides in the present equivalence, between the quantum Coulomb branch of the T^3 compactified little string theory on the one hand, and the metric of $K3$ on the other, are presently not well understood. Perhaps the present equivalence will eventually be useful for using one of the two sides to learn about the other.

A direct generalization of the above is to consider a T^3 compactification of the little string theory (o) associated with the K SO(32) heterotic small instanton. This maps to K $M2$ branes at points on $\mathbb{R}^4\times K3$. The Coulomb branch is, corresponding, the symmetric product $(K3)^{\otimes K}/S_K$, where each $K3$ is again of fixed volume M_s^2.

The geometric symmetries (see, e.g., [29]) of the Coulomb branch $K3$ correspond to nontrivial T dualities in Eq. (2.1) though, as in Eq. (3.5), only the subgroup which takes the SO(32) heterotic theory back to itself. An additional $\mathbb{Z}_2$ component of T dualities in $O(19,3;\mathbb{Z})$ reflects the fact that, instead compactifying the $E_3\times E_8$ heterotic little string (e), with T-dual T^3 compactification data, also yields the same 3D theory, with the *same* $K3$ compact Coulomb branch as described above.

Each of the little string theories reviewed in Sec. II can be compactified to 3D on a T^3, and each has an exact quantum Coulomb branch which globally probes the dual M theory compactification. In each case, there is a compact Coulomb branch component, with unit volume in units of M_s. The 3D field theory limit is recovered by taking $M_s\rightarrow\infty$.

The $\mathcal{N}=(1,1)$ little string theories with group U(K), when compactified on T^3, have a Coulomb branch which is $(\mathbb{R}^4\times T^4)^{\otimes K}/S_K$ [more generally, $(\mathbb{R}^4\times T^4)^{\mathrm{rank}(G)}/Weyl(G)$], which probes the duality between type-II strings on T^3 and M theory on T^4. The Coulomb branch T^4 has metric G^{AB} which is given exactly in terms of the T^3 compactification date by Eq. (3.3), with volume M_s^2. There is a similar statement for the $\mathcal{N}=(2,0)$ little string theory on T^3, differing from the $\mathcal{N}=(1,1)$ case by a T duality in one of the T^3 cycles; the fixed volume T^4 in this context was also discussed in [15,16].

The $\mathcal{N}=(1,0)$ little string theories associated with K type-II or heterotic five branes at an $X_G\equiv\mathbb{C}^2/\Gamma_G$ singularity, when compactified on T^3, similarly probe M theory geometry. In the heterotic (or type-II) cases, the M theory dual is given by K $M2$ branes with a transverse space $X_G\times K3$ (or $X_G\times T^4$). In both cases, M theory with a X_G singularity has an enhanced G gauge symmetry and $M2$ branes, when sitting directly on top of the G singularity of X_G, can be interpreted as small G instantons. In the heterotic (or type-II) cases, these K G instantons have the fixed volume $K3$ (or T^4) as their four spatial coordinates. There is a moduli space for these instantons given by their positions in these four spatial coordinates, as well as their moduli for fattening up and rotating in G.

Thus, by the probe argument, the little string theory associated with K heterotic (or type-II) branes at a $\mathbb{C}^2/\Gamma_G$ singularity, when compactified on a T^3, has a compact Coulomb branch moduli space of vacua which is exactly given by the moduli space of K G instantons on a $K3$ (or T^4) of volume M_s^2. A quick check is that the dimension of the Coulomb branch of the T^3 compactified little string theories indeed agrees with the dimension of the moduli space of K G instantons[3] on T^4 or $K3$: the type-II cases indeed have $4(r_V+n_T)=4KC_2(G)$ and the heterotic cases indeed have $4(r_V+n_T)=4(KC_2(G)-|G|)$. This latter fact also played a role in the mirror symmetry of [25].

Another check is to consider the limit $M_s^2\rightarrow\infty$, where $T^4\rightarrow\mathbb{R}^4$ or $K3$ becomes a noncompact piece of $K3$, and where the compactified little string theory goes over to its 3D field theory limit. In the type-II cases, the resulting 3D field theory has the quiver gauge group $\Pi_{\mu=0}^{r}U(Kn_\mu)$, based on the extended G-Dynkin diagram, which was indeed argued in [30,31] to have a quantum Coulomb branch which is the moduli space of K G instantons on $\mathbb{R}^4$. This theory was argued in [30] to have a hidden, global G symmetry. Because M theory has G gauge symmetry even for finite M_s, the full compactified little string theory is expected to also have this hidden global symmetry. Similar statements should hold in the heterotic cases.

Moduli spaces of instantons on T^4 or $K3$ have made a variety of appearances in physics and mathematics, though usually with $G=\mathrm{U}(N)$ as the gauge group. In that case, the moduli space also depends on $v_a=\int_{\Sigma_a}\mathrm{Tr}F$, where Σ_a is a basis for the two cycles of T^4 or $K3$. In the present case, G is a simple A, D, E group so $\mathrm{Tr}F=0$. (B fields can possibly still contribute to $v_a\neq 0$, e.g., as in [21].)

The moduli spaces of the instantons obtained above have many interesting singularities. At these Coulomb branch singularities, there is an attached Higgs branch, with an interacting 3D IR conformal field theory (CFT) at the intersection.

All of the above compact Coulomb branches are hyper-Kähler type, with $c_1=0$, and the σ model metric is the unique one which is Ricci flat.

V. T^2 COMPACTIFICATION: PROBING F THEORY

Compactifying the heterotic (or type-II) little string theories to 4D on a T^2 leads to quantum Coulomb branches which globally probe F theory compactifications on a fixed volume, elliptically fibered $K3$ (or T^4). For example, consider the $K=1$ case of the heterotic little string theory (o), whose low-energy field theory content is that of the worldvolume of $D3$ branes in type I' on T^2. This latter theory has a noncompact, quantum Coulomb branch which was argued [32,4] to locally probe the duality to F theory on an elliptically fibered $K3$. The Coulomb branch in the 4D field theory is the noncompact complex u plane, over which the photon coupling $\tau_{\mathrm{eff}}(u)$ is fibered according to the Seiberg-Witten curve [33]; the total space of the u plane base and $\tau_{\mathrm{eff}}(u)$ fiber is a local, noncompact piece of $K3$. This is the $M_s\rightarrow\infty$ limit of the T^2 compactified little string theory.

Considering now the T^2 compactified little string theory for finite M_s^2, the u-plane base is a compact box of volume M_s^2 (this is the correct mass dimension for 4D scalars). As in Sec. III, reducing a 6D U(1) gauge field on a T^2 with metric $h_{ab}d\rho^a d\rho^b$ leads to scalars living on a dual T^2, with metric $g_6^{-2}\sqrt{\det h}(h^{-1})^{ab}d\phi_a d\phi_b$, which has volume $g_6^{-2}=M_s^2$ for all h_{ab}. For SU(2) rather than U(1), we mod out by the Weyl group $\phi_a\sim-\phi_a$, yielding a 2D box of volume M_s^2. Considering the elliptic fiber $\tau(u)$ over the compact base as a dimensionless coordinate, the total space of base and fiber is an elliptically fibered $K3$ of volume M_s^2. This elliptically fibered $K3$ of fixed volume is that of the F theory dual to the 10D heterotic string on T^2. As was the case there, the parameter space (2.1) of data in the T^2 compactification of the

[3]For a general four manifold with Euler character χ and signature σ, the dimension is $4KC_2(G)-\frac{1}{2}|G|(\chi+\sigma)$. For T^4, $\chi=\sigma=0$ and, for $K3$, $\chi=24$ and $\sigma=-16$.

heteroticstring matches that of the fixed volume, elliptically fibered $K3$'s. This can be regarded as a special case of the T^3 compactification considered in the previous sections, where one of the radii is taken to infinity. It is thus good that we again get a $K3$ of volume M_s^2, since that was the case in the previous sections for all radii.

More generally, T^2 compactifying the little string theories associated with K type-II five branes at a $\mathbb{C}^2/\Gamma_G$ singularity leads to a compact Coulomb branch which is a $2(r_V+n_T)=2KC_2(G)$ dimensional torus of unit volume in units of M_s. Including the $KC_2(G)$ complex dimensional elliptic fiber, associated with the kinetic terms of the $KC_2(G)$ photons, the total space is the moduli space of K G instantons on a T^4 of volume M_s^2, where both the T^4 and the resulting instanton moduli space are regarded as an elliptic fibration.

Compactifying on a T^2 the little string theories associated with K heterotic five branes at a $\mathbb{C}^2/\Gamma_G$ singularity leads to a compact Coulomb branch which is a $2(r_V+n_T)=2[KC_2(G)-|G|]$ dimensional box, of unit volume in units of M_s. Including the fiber associated with the photons, the total space is an elliptically fibered space which is exactly the moduli space of K G instantons on an elliptically fibered $K3$ of volume M_s^2.

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