$N=2$ supersymmetric model with Dirac-Kähler fermions from generalized gauge theory in two dimensions

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We investigate the generalized gauge theory which has been proposed previously and show that in two dimensions the instanton gauge fixing of the generalized topological Yang-Mills action leads to a twisted $N=2$ supersymmetric action. We have found that the R symmetry of $N=2$ supersymmetry can be identified with the flavor symmetry of the Dirac-Kähler fermion formulation. Thus the procedure of twist allows topological ghost fields to be interpreted as the Dirac-Kähler matter fermions.

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I. INTRODUCTION

In formulating a unified theory it is the general consensus that supersymmetry may play a crucial role. It is important to understand the origin of supersymmetry and fermion and boson correspondence. There is an interesting example of a topological field theory analysis by Witten [1] which suggests the possible origin of $N=2$ supersymmetry and the generation of fermionic fields from ghosts via a twisting procedure. Later it has been pointed out that this theory can be derived from the ''partially'' Becchi-Rouet-Stora-Tyutin (BRST) gauge-fixed action of the topological Yang-Mills action with instanton gauge fixing [2,3]. This subject has been intensively investigated [4], particularly in connection with supersymmetric field theories [5]. In this paper we claim that the topological twist generating the matter fermions from ghosts is essentially related to the Dirac-Kähler fermion formulation.

In the 1960s Kähler [6] showed that the Dirac equation is constructed from inhomogeneous differential forms which are called Dirac-Kähler fields [7]. Moreover the Dirac-Kähler fermion is a curved spacetime version of the Kogut-Susskind fermion [8] or staggered fermion [9] and thus a natural framework of the lattice fermion formulation [10].

About ten years ago one of the authors (N.K.) and Watabiki proposed a generalization of the ordinary three-dimensional Chern-Simons theory into arbitrary dimensions by introducing all the degrees of differential forms as gauge fields and parameters together with a quaternion structure [11]. Later the quantization of the even-dimensional version of the generalized Chern-Simons actions was completed by the Batalin-Vilkovisky formulation [12]. This formulation can be, however, generalized to the topological Yang-Mills and ordinary Yang-Mills actions.

Since the generalized gauge theory is formulated by differential forms it has a close connection with the Dirac-Kähler fermion formulation. We believe that the generalized gauge theory may play a crucial role in formulating the unified model including quantum gravity on the simplicial lattice manifold [13].

In this paper we investigate the generalized topological Yang-Mills theory from the topological field theory point of view. An enlarged algebraic structure of BRST transformations in the manner of Baulieu and Singer [3] is naturally constructed in a unified way by the generalized gauge theory. As the simplest example towards more realistic case, we quantize the two-dimensional version of the generalized topological Yang-Mills action and show that the ''partially'' gauge-fixed action with instanton gauge fixing leads to a twisted $N=2$ supersymmetric Abelian-Higgs action [14,15] without a symmetry-breaking potential term. It is interesting to recognize that our instanton relations coincide with dimensionally reduced Seiberg-Witten equations [16] from four into two dimensions [17]. We point out that the fermionic ghost fields can be interpreted as Dirac-Kähler fermion fields and thus the twisting procedure is nothing but the Dirac-Kähler fermion formulation.

This paper is organized as follows. In Sec. II we summarize the generalized gauge theory in arbitrary dimensions. In Sec. III we analyze the generalized two-dimensional topological Yang-Mills theory as a topological field theory. In Sec. IV we explicitly verify the twisted $N=2$ supersymmetric algebra for the gauge-fixed action. In Sec. V we explain the twisting mechanism via Dirac-Kähler formulation. Conclusions and discussions are given in the final section.

II. GENERALIZED GAUGE THEORIES IN ARBITRARY DIMENSIONS

In this section we summarize the formulation of the generalized gauge theory with an emphasis on their algebraic structures.

The essential point of the generalization is to extend a one-form gauge field and zero-form gauge parameter to a quaternion valued generalized gauge field and gauge parameter which contain forms of all possible degrees.

In the most general form, a generalized gauge field $\mathcal{A}$ and a gauge parameter $\mathcal{V}$ are defined by the following component form:

$$\mathcal{A}=\mathbf{1}\psi+\mathbf{i}\hat{\psi}+\mathbf{j}A+\mathbf{k}\hat{A}, \tag{2.1}$$

$$\mathcal{V}=\mathbf{1}\hat{a}+\mathbf{i}a+\mathbf{j}\hat{\alpha}+\mathbf{k}\alpha, \tag{2.2}$$

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where (ψ,α), $(\hat{\psi},\ \hat{\alpha})$, (A,a), and $(\hat{A},\hat{a})$ are direct sums of fermionic odd forms, fermionic even forms, bosonic odd forms, and bosonic even forms, respectively, and they take values on a gauge algebra. The symbols **1**, **i**, **j**, and **k** satisfy the following quaternion algebra:

$$\mathbf{1}^2=\mathbf{1},\quad \mathbf{i}^2=-\mathbf{1},\quad \mathbf{j}^2=-\mathbf{1},\quad \mathbf{k}^2=-\mathbf{1},$$
$$\mathbf{ij}=-\mathbf{ji}=\mathbf{k},\quad \mathbf{jk}=-\mathbf{kj}=\mathbf{i},\quad \mathbf{ki}=-\mathbf{ik}=\mathbf{j}. \tag{2.3}$$

The following graded Lie algebra can be adopted as a gauge algebra:

$$\begin{aligned}[T_a,T_b]&=f_{ab}^{c}T_c,\\ [T_a,\Sigma_\beta]&=g_{a\beta}^{\gamma}\Sigma_\gamma,\\ \{\Sigma_\alpha,\Sigma_\beta\}&=h_{\alpha\beta}^{c}T_c,\end{aligned} \tag{2.4}$$

where all the structure constants are subject to consistency conditions which follow from the graded Jacobi identities. The components of the gauge field $\mathcal{A}$ and the gauge parameter $\mathcal{V}$ are particularly assigned as elements of the gauge algebra

$$A=T^aA_a,\quad \hat{\psi}=T^a\hat{\psi}_a,\quad \psi=\Sigma^\alpha\psi_\alpha,\quad \hat{A}=\Sigma^\alpha\hat{A}_\alpha,$$
$$\hat{a}=T^a\hat{a}_a,\quad \alpha=T^a\alpha_a,\quad \hat{\alpha}=\Sigma^\alpha\hat{\alpha}_\alpha,\quad a=\Sigma^\alpha a_\alpha. \tag{2.5}$$

The component expansion of the same type as $\mathcal{A}$ and $\mathcal{V}$ are classified as elements of the Λ_- class and Λ_+ class, respectively. These elements fulfill the Z_2-grading structure

$$[\lambda_+,\lambda_+]\in\Lambda_+,\quad [\lambda_+,\lambda_-]\in\Lambda_-,\quad \{\lambda_+,\lambda_+\}\in\Lambda_+, \tag{2.6}$$

where $\lambda_+\in\Lambda_+$ and $\lambda_-\in\Lambda_-$. In particular the exterior derivative belongs to Λ_- class

$$Q=\mathbf{j}d, \tag{2.7}$$

and the following relations similar to the ordinary exterior derivative operator hold:

$$Q(\lambda_1\lambda_2)=(Q\lambda_1)\lambda_2+(-)^{|\lambda_1|}\lambda_1(Q\lambda_2),\quad Q^2=0, \tag{2.8}$$

where $|\lambda_1|=0$ for $\lambda_1\in\Lambda_+$ and $|\lambda_1|=1$ for $\lambda_1\in\Lambda_-$. To construct the generalized actions, the two types of traces for the gauge algebra should be introduced,

$$\mathrm{Tr}[T^a,\ldots]=0,\quad \mathrm{Tr}[\Sigma^\alpha,\ldots]=0, \tag{2.9}$$

$$\mathrm{Str}[T^a,\ldots]=0,\quad \mathrm{Str}\{\Sigma^\alpha,\ldots\}=0, \tag{2.10}$$

where $(\cdots)$ in the commutators and the anticommutator denote a product of the generators. These definitions of the traces are crucial so that the generalized actions are invariant under the generalized gauge transformations.

We can then construct generalized actions in terms of these generalized quantities. The generalized Chern-Simons actions which have been previously proposed [11] are given on even- and odd-dimensional manifolds M,

$$S_{\mathrm{even}}=\int_M \mathrm{Tr}_{\mathbf{k}}\left(\mathcal{A}Q\mathcal{A}+\frac{2}{3}\mathcal{A}^3\right), \tag{2.11}$$

$$S_{\mathrm{odd}}=\int_M \mathrm{Str}_{\mathbf{j}}\left(\mathcal{A}Q\mathcal{A}+\frac{2}{3}\mathcal{A}^3\right), \tag{2.12}$$

where $\mathrm{Tr}_{\mathbf{k}}(\cdots)$ and $\mathrm{Str}_{\mathbf{j}}(\cdots)$ are defined so as to pick up only the coefficient of **k** and **j** from $(\cdots)$ and take the traces. We then need to pick up d-form terms corresponding to d-dimensional manifolds M. These actions are invariant up to surface terms under the following generalized gauge transformation:

$$\delta\mathcal{A}=[Q+\mathcal{A},\mathcal{V}], \tag{2.13}$$

where $\mathcal{V}$ is the generalized gauge parameter. It should be noted that this symmetry is much larger than the usual gauge symmetry since the gauge parameter $\mathcal{V}$ contains many parameters of various forms.

There is another suggestive topological nature due to the parallel construction to the standard gauge theory. In the generalized gauge theory it is possible to define the generalized Chern character which is expected to have topological nature

$$\mathrm{Str}_{\mathbf{1}}(\mathcal{F}^n)=\mathrm{Str}_{\mathbf{1}}(Q\Omega_{2n-1}), \tag{2.14}$$

$$\mathrm{Tr}_{\mathbf{i}}(\mathcal{F}^n)=\mathrm{Tr}_{\mathbf{i}}(Q\Omega_{2n-1}), \tag{2.15}$$

where $\mathcal{F}$ is a generalized curvature

$$\mathcal{F}=Q\mathcal{A}+\mathcal{A}^2, \tag{2.16}$$

and Ω_{2n-1} are the "generalized" Chern-Simons form. Equations (2.14) and (2.15) are bosonic even form and bosonic odd form, respectively. Especially, for the $n=2$ case in Eq. (2.14), we obtain a topological Yang-Mills type action related to a one dimension lower generalized Chern-Simons action on an even-dimensional manifold M,

$$\int_M \mathrm{Str}_{\mathbf{1}}\mathcal{F}^2=\int_M \mathrm{Str}_{\mathbf{1}}\left(Q\left(\mathcal{A}Q\mathcal{A}+\frac{2}{3}\mathcal{A}^3\right)\right), \tag{2.17}$$

which has the same forms of the standard relation.

III. GENERALIZED TOPOLOGICAL YANG-MILLS THEORY IN TWO DIMENSIONS

In this section we analyze the two-dimensional version of the generalized topological Yang-Mills action. Our formulation of this section is the two-dimensional realization of the known four-dimensional scenario [1–3] and can be extended to arbitrary dimensions.

As we have already mentioned, the action we consider satisfies the following well known relation:

$$\int_M \mathrm{Str}_{\mathbf{1}} \mathcal{F}_0^2 = \int_M \mathrm{Str}_{\mathbf{1}} \left(Q \left(\mathcal{A}_0 Q \mathcal{A}_0 + \frac{2}{3} \mathcal{A}_0^3 \right) \right), \tag{3.1}$$

where $\mathcal{A}_0$ and $\mathcal{F}_0$ are the two-dimensional counter part of the classical gauge field and curvature. More explicitly, they are given by

$$\mathcal{A}_0 = \mathbf{j}\omega + \mathbf{k}(\phi + B), \quad \in \Lambda_-, \tag{3.2}$$

$$\begin{aligned} \mathcal{F}_0 &= Q\mathcal{A}_0 + \mathcal{A}_0^2 \\ &= -\mathbf{1}(d\omega + \omega^2 + \{\phi, B\} + \phi^2) + \mathbf{i}(d\phi + [\omega, \phi]), \quad \in \Lambda_+, \end{aligned} \tag{3.3}$$

where ϕ, ω, and B are graded Lie algebra valued zero-, one- and two-form gauge fields, respectively. Due to the topological nature of the action, the action has so-called shift symmetry. In other words, the action is invariant under the arbitrary deformation of the gauge field $\mathcal{A}_0$, which we denote $\mathcal{E}_0$. Thus the gauge transformation of the generalized topological Yang-Mills action has the following form:

$$\delta \mathcal{A}_0 = [Q + \mathcal{A}_0, \mathcal{V}_0] + \mathcal{E}_0, \tag{3.4}$$

where $\mathcal{V}_0$ is the generalized gauge parameter

$$\mathcal{V}_0 = \mathbf{1}(v + b) + \mathbf{i}u, \quad \in \Lambda_+, \tag{3.5}$$

while $\mathcal{E}_0$ is a new gauge parameter of the shift symmetry and is given by

$$\mathcal{E}_0 = \mathbf{j}\xi_{(1)} + \mathbf{k}(\xi_{(0)} + \xi_{(2)}), \quad \in \Lambda_-, \tag{3.6}$$

where the suffix (n) with $n = 0,1,2$ denotes the form degree. Hereafter, we use the same notation to the form degree. The field strength is transformed under the gauge transformation (3.4),

$$\delta \mathcal{F}_0 = [\mathcal{F}_0, \mathcal{V}_0] + \{Q + \mathcal{A}_0, \mathcal{E}_0\}. \tag{3.7}$$

The first term is transformed covariantly, and the second term is the inhomogeneous gauge transformation of $\mathcal{F}_0$ by the gauge parameter which belongs to the Λ_--class.

The topological shift symmetry of $\mathcal{E}_0$, however, can absorb the usual gauge transformation, so that this is a reducible system with the following obvious reducibility conditions:

$$\begin{aligned} \mathcal{V}_0 &= \mathcal{V}_1, \\ \mathcal{E}_0 &= -[Q + \mathcal{A}_0, \mathcal{V}_1]. \end{aligned} \tag{3.8}$$

Correspondingly we need to introduce ghost fields with respect to the generalized gauge symmetry and the topological shift symmetry, and the ghost for ghost fields with respect to the additional gauge symmetry of the gauge parameter (3.8).

Although we can construct the nilpotent BRST algebra of the above reducible system by the procedure of cohomological perturbation [18], we can treat it in an algebraically unified way by using the characteristic of the generalized gauge system. We redefine the generalized gauge field by introducing the generalized ghost fields $C_{(0)}$, $C_{(1)}$, and $C_{(2)}$:

$$\mathcal{A} = \mathbf{1}C_{(1)} + \mathbf{i}(C_{(0)} + C_{(2)}) + \mathbf{j}\omega + \mathbf{k}(\phi + B), \quad \in \Lambda_-. \tag{3.9}$$

We need to introduce a generalized field which belongs to another class of $\mathcal{A}$ to accomodate the topological ghost fields $\tilde{C}_{(0)}$, $\tilde{C}_{(1)}$, and $\tilde{C}_{(2)}$ and the ghost for ghost fields $\eta_{(0)}$, $\eta_{(1)}$, and $\eta_{(2)}$:

$$\mathcal{C} = \mathbf{1}(\eta_{(0)} + \eta_{(2)}) + \mathbf{i}\,\eta_{(1)} + \mathbf{j}(\tilde{C}_{(0)} + \tilde{C}_{(2)}) + \mathbf{k}\tilde{C}_{(1)}, \quad \in \Lambda_+. \tag{3.10}$$

Here $\mathcal{C}$ belongs to Λ_+ and could be identified as a part of generalized curvature later.

Furthermore, we extend the concept of the differential operator by introducing the BRST operator s as a fermionic zero form:[1]

$$\mathcal{Q} \equiv Q + Q_{\mathrm{B}} = \mathbf{j}d + \mathbf{i}s, \quad \in \Lambda_-. \tag{3.11}$$

It should be noted that s commutes with d, i.e., $[d,s]=0$ and $s^2=0$. This operator satisfies the nilpotency property due to the quaternion structures:

$$\mathcal{Q}^2 = 0. \tag{3.12}$$

The following graded Leibnitz rule acting on generalized gauge fields can be derived:

$$\mathcal{Q}(\lambda_1\lambda_2) = (\mathcal{Q}\lambda_1)\lambda_2 + (-)^{|\lambda_1|}\lambda_1(\mathcal{Q}\lambda_2), \tag{3.13}$$

where $|\lambda_1| = 0$ for $\lambda_1 \in \Lambda_+$ and $|\lambda_1| = 1$ for $\lambda_1 \in \Lambda_-$.

We can now construct the BRST transformation algebraically in a unified way. We define the generalized curvature by using the redefined gauge field

$$\mathcal{F} = \mathcal{Q}\mathcal{A} + \mathcal{A}^2 = \mathcal{F}_0 + \mathcal{C}, \tag{3.14}$$

where the second relation is imposed to relate the BRST transformation with respect to classical and generalized ghost fields. $\mathcal{C}=0$ corresponds to imposing the usual horizontal conditions. The transformations with respect to the topological ghost and the ghost for ghost fields are derived by the following Bianchi identity of the generalized field:

$$\mathcal{Q}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0. \tag{3.15}$$

The component wise expressions of the BRST transformation can be read from (3.14) and (3.15):

$$\begin{aligned} s\phi &= -[C_{(0)}, \phi] - \tilde{C}_{(0)}, \\ s\omega &= dC_{(0)} + [\omega, C_{(0)}] - \{C_{(1)}, \phi\} + \tilde{C}_{(1)}, \\ sB &= dC_{(1)} + \{\omega, C_{(1)}\} - [C_{(0)}, B] - [C_{(2)}, \phi] - \tilde{C}_{(2)}, \end{aligned}$$

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[1] The fermionic operator s acts as a left derivative on fields in the same way as the operation of the exterior derivative d.

$$sC_{(0)}=-C_{(0)}^2-\eta_{(0)},$$
$$sC_{(1)}=-\{C_{(0)},C_{(1)}\}+\eta_{(1)},$$
$$sC_{(2)}=C_{(1)}^2-\{C_{(0)},C_{(2)}\}-\eta_{(2)},$$
$$s\tilde{C}_{(0)}=-\{C_{(0)},\tilde{C}_{(0)}\}-[\phi,\eta_{(0)}], \tag{3.16}$$
$$s\tilde{C}_{(1)}=d\eta_{(0)}+[\omega,\eta_{(0)}]+[C_{(1)},\tilde{C}_{(0)}]-\{C_{(0)},\tilde{C}_{(1)}\}+\{\phi,\eta_{(1)}\},$$
$$s\tilde{C}_{(2)}=d\eta_{(1)}+\{\omega,\eta_{(1)}\}-[C_{(1)},\tilde{C}_{(1)}]-\{C_{(0)},\tilde{C}_{(2)}\}-\{C_{(2)},\tilde{C}_{(0)}\}-[\phi,\eta_{(2)}]-[B,\eta_{(0)}],$$
$$s\eta_{(0)}=-[C_{(0)},\eta_{(0)}],$$
$$s\eta_{(1)}=[C_{(1)},\eta_{(0)}]-[C_{(0)},\eta_{(1)}],$$
$$s\eta_{(2)}=-[C_{(1)},\eta_{(1)}]-[C_{(0)},\eta_{(2)}]-[C_{(2)},\eta_{(0)}].$$

These algebraic and geometric constructions of the BRST transformation were emphasized by Baulieu-Singer [3] for the four-dimensional topological Yang-Mills model. We here propose the natural extension of their approach in the framework of the generalized gauge theory. Moreover, we do not have to introduce the ghost number for fields and the BRST operator which played an important role in the above authors' formulations. In deriving BRST transformations (3.16) we only compare the terms expanded in the form degrees and the coefficients of quaternions in Eqs. (3.14) and (3.15). The conventional ghost number for particular fields and the BRST charge are automatically assigned by the quaternionic classifications.

Next we can consider the physical observable. We can construct BRST invariant polynomials because of the nilpotency property of the extended differential operator. Bianchi identity leads to the following algebraic relation,

$$\mathcal{Q}\mathcal{F}^n=-[\mathcal{A},\mathcal{F}^n]. \tag{3.17}$$

Taking a trace of the gauge algebra and particular quaternion sector, we obtain the following relations due to the vanishing nature of the right-hand side of Eq. (3.17),

$$\mathrm{Str}_{\mathbf{j}}(\mathcal{Q}\mathcal{F}^n)=\mathrm{Str}_{\mathbf{j}}[(Q+Q_{\mathrm{B}})\mathcal{F}^n]=0,$$
$$\mathrm{Str}_{\mathbf{i}}(\mathcal{Q}\mathcal{F}^n)=\mathrm{Str}_{\mathbf{i}}[(Q+Q_{\mathrm{B}})\mathcal{F}^n]=0,$$

which lead to the following descendent equations:

$$s\,\mathrm{Str}_{\mathbf{k}}(\mathcal{F}^n)=-d\,\mathrm{Str}_{\mathbf{1}}(\mathcal{F}^n), \tag{3.18}$$
$$s\,\mathrm{Str}_{\mathbf{1}}(\mathcal{F}^n)=-d\,\mathrm{Str}_{\mathbf{k}}(\mathcal{F}^n). \tag{3.19}$$

We can then find a series of gauge invariant physical observables:

$$\mathcal{O}_{\mathrm{o}}^{\mathrm{f}}=\int_\gamma \mathrm{Str}_{\mathbf{k}}\mathcal{F}^n, \tag{3.20}$$
$$\mathcal{O}_{\mathrm{e}}^{\mathrm{b}}=\int_\gamma \mathrm{Str}_{\mathbf{1}}\mathcal{F}^n, \tag{3.21}$$

where γ is a homology cycle on the submanifold in M and $\mathcal{O}_{\mathrm{o}}^{\mathrm{f}}$ and $\mathcal{O}_{\mathrm{e}}^{\mathrm{b}}$ denote odd-dimensional fermionic and even-dimensional bosonic observables, respectively.

Although we consider the above BRST algebra in the two-dimensional case, we will see that the algebra in arbitrary dimensions can be treated in a similar way.

We next introduce a particular model to carry out explicit analyses. To make the formulation concrete and simpler we specify to the two-dimensional flat Euclidean case and take the following two-dimensional anti-Hermitian Euclidean Clifford algebra as the graded algebra, which closes under the multiplication and is the simplest example:

$$T^a: 1, \quad \gamma_5,$$
$$\Sigma^\alpha: \gamma^a, \tag{3.22}$$

where $\gamma^a=(i\sigma^1,i\sigma^2)$, which satisfy $\{\gamma^a,\gamma^b\}=-2\delta^{ab}$ and $\gamma_5=\frac{1}{2}\epsilon_{ab}\gamma^a\gamma^b=-i\sigma^3$ with $\epsilon_{12}=1$. A grading generator can be identified as γ_5 and then we define the supertrace

$$\mathrm{Str}(\cdots)=\mathrm{Tr}(\gamma_5\cdots).$$

The two-dimensional topological Yang-Mills action leads to

$$S_0=\frac{1}{2}\int \mathrm{Str}_{\mathbf{1}}\mathcal{F}_0\wedge\mathcal{F}_0$$
$$=\int d^2x(\epsilon^{\mu\nu}F_{\mu\nu}|\phi|^2+\epsilon^{\mu\nu}\epsilon^{ab}(D_\mu\phi)_a(D_\nu\phi)_b)$$
$$=\int d^2x\,\epsilon^{\mu\nu}\partial_\mu(2\omega_\nu|\phi|^2+\epsilon^{ab}\phi_a\partial_\nu\phi_b), \tag{3.23}$$

where $F_{\mu\nu}=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu$ and $(D_\mu\phi)_a=\partial_\mu\phi_a-2\epsilon_a{}^b\omega_\mu\phi_b$. In the action (3.23) the scalar part of the one-form field $\omega_{\mu s}$ and two-form field $B_{a\mu\nu}$ in the generalized field (3.2) drop out because of the reducible structures of the gauge transformations. Then the generalized gauge transformations are consistently truncated to the following SO(2) invariance:

$$\delta_{\mathrm{gauge}}\phi_a=2v\,\epsilon_a{}^b\phi_b,$$
$$\delta_{\mathrm{gauge}}\omega_\mu=\partial_\mu v, \tag{3.24}$$

where v is a zero-form gauge parameter. As we have discussed we impose the topological shift symmetry, then BRST transformations (3.16) lead to the following truncated forms:

$$s\phi_a=2\epsilon_a{}^b\phi_bC-\tilde{C}_a,$$
$$s\omega_\mu=\partial_\mu C+\tilde{C}_\mu,$$
$$sC=-\eta,$$

$$s\tilde{C}_a = 2\epsilon_a{}^b C\tilde{C}_b - 2\epsilon_a{}^b \phi_b \eta,$$
$$s\tilde{C}_\mu = \partial_\mu \eta,$$
$$s\eta = 0, \tag{3.25}$$

where C, $(\tilde{C}_a, \tilde{C}_\mu)$, and η are the ghost fields associated with SO(2) gauge symmetry and topological shift symmetries, and the ghost for the ghost field with the reducible symmetry, respectively. These BRST transformations indeed satisfy the nilpotency property.

We can now find a two-dimensional instanton relation of our generalized gauge system by imposing the self- (anti-self-) dual condition

$$*\mathcal{F}_0 = \pm \mathcal{F}_0. \tag{3.26}$$

Since a repeated application of * on the generalized field strength must yield the identity map, we define the following duality relation for the gauge operators and quaternions in addition to the usual Hodge dual operation on the differential forms:

$$*1 = -\gamma_5, \quad *\gamma^a = -\epsilon^a{}_b \gamma^b, \quad *\gamma_5 = -1, \tag{3.27}$$

$$*\mathbf{1} = \mathbf{1}, \quad *\mathbf{i} = -\mathbf{i}. \tag{3.28}$$

We can then find the following minimal condition of the action leading to instanton relations:

$$\pm \frac{1}{2}\int \mathrm{Str}_{\mathbf{1}} \mathcal{F}_0 \wedge \mathcal{F}_0 + \frac{1}{2}\int \mathrm{Str}_{\mathbf{1}} \mathcal{F}_0 \wedge * \mathcal{F}_0$$
$$= \int d^2x \Bigg(\left(\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} \pm |\phi|^2\right)\left(\frac{1}{2}\epsilon^{\rho\sigma}F_{\rho\sigma} \pm |\phi|^2\right)$$
$$+ \frac{1}{2}[(D_\mu\phi)_a \pm \epsilon_\mu{}^\nu \epsilon_a{}^b (D_\nu\phi)_b][(D^\mu\phi)^a$$
$$\pm \epsilon^\mu{}_\rho \epsilon^a{}_c (D^\rho\phi)^c]\Bigg). \tag{3.29}$$

Then the instanton relations are obtained from the conditions for the absolute minima of the generalized Yang-Mills action

$$\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} - |\phi|^2 = 0, \tag{3.30}$$

$$(D_\mu\phi)_a^{(-)} \equiv \frac{1}{2}[(D_\mu\phi)_a - \epsilon_\mu{}^\nu \epsilon_a{}^b (D_\nu\phi)_b] = 0. \tag{3.31}$$

These instanton relations are a natural consequence of the formulation of the generalized topological Yang-Mills action as we have seen above.

It has come to our attention (the recent paper [17]) that the dimensionally reduced Seiberg-Witten equation [16] from four into two dimensions coincide with Eqs. (3.30) and (3.31). It should be noted that the Weyl spinor in the Seiberg-Witten equation corresponds to the Higgs scalar in our formulation. The explicit solutions have been obtained as the Liouville vortex solution by Nergiz and Saçlıoğlu [17] for the solution of the Seiberg-Witten equation. The solutions are

$$\phi = \phi_1 + i\phi_2 = \sqrt{2}\,\frac{dg/dz}{1-\bar{g}g},$$
$$\omega = \omega_\mu dx^\mu = \frac{i}{2}\left(\frac{g d\bar{g} - \bar{g} dg}{1-\bar{g}g}\right), \tag{3.32}$$

where $g=g(z)$ is an arbitrary holomorphic function and $\bar{g}$ is the complex conjugate of g with $z = x_1 + ix_2$.

The topological nature of the solutions is explicitly verified by calculating a flux

$$\Phi = \int F d^2x = 4\pi n, \tag{3.33}$$

where $F = \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} = |\phi|^2$. Here we have chosen the holomorphic function as $g(z) = z^n$. Due to the singular nature of the solutions (3.32), we need particular regularization to obtain the explicit topological relation (3.33).

It is worth to mentioning at this stage that there is another kind of solution to the modified instanton relation or equivalently Bogomol'nyi equation,

$$\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu} - |\phi|^2 + |v|^2 = 0, \tag{3.34}$$

while the second relation (3.31) is the same. These relations yield the Nielsen-Olesen vortex solution [19] which has again a topological nature [14]. It is important to recognize that our formulation leading to the instanton relations (3.30) and (3.31) by the generalized topological Yang-Mills formulation will never lead to the relation (3.34). Instead it may lead to the relations where ϕ can get a constant shift: $\phi \rightarrow \phi + v$, which is different from Eq. (3.34). Therefore, the instanton solutions obtained from the generalized topological Yang-Mills formulation are not Nielsen-Olesen vortex-type solution but the dimensionally reduced one derived from four-dimensional Seiberg-Witten equations.

We now derive the gauge-fixed action with instanton relations (3.30) and (3.31) as gauge fixing conditions of topological (shift)symmetry together with the following Landau-type gauge fixing conditions to fix the usual gauge symmetry and the reducible symmetry:

$$\partial_\mu \omega^\mu = 0, \quad \partial_\mu \tilde{C}^\mu = 0. \tag{3.35}$$

Correspondingly we introduce a set of antighost fields λ, $\chi_{\mu a}$, $\bar{\eta}$, and $\bar{C}$, and associated Lagrange multipliers, $\tilde{\pi}$, $\pi_{\mu a}$, ρ, and π. These fields obey the following closed BRST subalgebra:

$$s\lambda = \tilde{\pi}, \quad s\tilde{\pi} = 0,$$
$$s\chi_{\mu a} = \pi_{\mu a}, \quad s\pi_{\mu a} = 0,$$

$$s\bar{\eta}=\rho, \quad s\rho=0,$$

$$s\bar{C}=\pi, \quad s\pi=0, \tag{3.36}$$

where the anti-self-dual field $\chi_{\mu a}$ obeys the condition $\epsilon_\mu{}^\nu\epsilon_a{}^b\chi_{\nu b}=-\chi_{\mu a}$ and $\pi_{\mu a}$ also obeys the same condition.

We then obtain the following ''completely'' gauge-fixed action by adding the BRST-exact terms:

$$S_{\text{g-f}}=S_0+s\int d^2x\biggl\{+\lambda\left(\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}-|\phi|^2-\beta\tilde{\pi}\right) - \chi_{\mu a}[(D^\mu\phi)^{a(-)}-\alpha\pi^{\mu a}]+\bar{\eta}\partial_\mu\tilde{C}^\mu+\bar{C}\partial_\mu\omega^\mu\biggr\}, \tag{3.37}$$

where α and β are arbitrary parameters. Using the equations of motions for auxiliary fields $\pi_{\mu a}$ and $\tilde{\pi}$ and choosing the parameters $\alpha=-\frac{1}{8}$ and $\beta=\frac{1}{4}$, we can eliminate topological sectors and then we obtain the physical Yang-Mills action plus fermion interaction terms.

IV. TWISTED $N=2$ SUPERSYMMETRIC ACTION

We first summarize the twisting procedure of $N=2$ superalgebra. The algebra of $N=2$ supersymmetry without a central extension is constructed by the following relations:[2]

$$\begin{aligned}
\{Q_{\alpha,i},Q_{\beta,j}\}&=\delta_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu,\\
[J,P_\mu]&=i\epsilon_\mu{}^\nu P_\nu,\\
[J,Q_{\alpha,i}]&=\frac{i}{2}(\gamma_5)_\alpha{}^\beta Q_{\beta,i},\\
[R,Q_{\alpha,i}]&=\frac{i}{2}(\gamma_5)_i{}^j Q_{\alpha,j},\\
[P_\mu,Q_{\alpha,i}]&=[R,J]=[R,P_\mu]=0.
\end{aligned} \tag{4.1}$$

Here $Q_{\alpha,i}$ are the generators of supersymmetry, where the indices $\alpha(=1,2)$ and $i(=1,2)$ are Lorentz spinor and internal spinor indices labeling two different $N=2$ generators, respectively. We can take these operators to be Majorana. P_μ, J, and R are generators of translation, SO(2) Lorentz rotation, and internal $SO(2)_I$ rotation called R symmetry, representing spin and isospin rotation, respectively.

The above $N=2$ superalgebra is transformed into the twisted $N=2$ superalgebra by the following procedure. The essential meaning of the topological twist is to identify the isospinor indices as spinor indices. Then the isospinor indices should then transform as spinors under the Lorentz transformation. This will then lead to the redefinition of the energy momentum tensor and the Lorentz rotation generator.

We consider the energy momentum tensor $T_{\mu\nu}$ and the conserved current R_μ associated with R symmetry. We then define a new energy momentum tensor $T'_{\mu\nu}$ without spoiling the conservation law by the following relation:

$$T'_{\mu\nu}=T_{\mu\nu}+\epsilon_{\mu\rho}\partial^\rho R_\nu+\epsilon_{\nu\rho}\partial^\rho R_\mu. \tag{4.2}$$

This modification of the energy momentum tensor leads to a redefinition of the Lorentz rotation generator J,

$$J'=J+R. \tag{4.3}$$

This rotation group is interpreted as the diagonal subgroup of $SO(2)\times SO(2)_I$. Now the supercharges have double spinor indices and thus can be decomposed into the following scalar, pseudoscalar, and vector components:

$$Q_\alpha{}^\beta=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\delta_\alpha{}^\beta Q_{\rm B}+\sqrt{2}(\gamma^\mu)_\alpha{}^\beta Q_\mu+\frac{1}{\sqrt{2}}(\gamma_5)_\alpha{}^\beta\tilde{Q}\right). \tag{4.4}$$

Solving conversely, we obtain

$$\begin{aligned}
Q_{\rm B}&=\sqrt{2}\ {\rm Tr}\,Q,\\
\tilde{Q}&=-\sqrt{2}\ {\rm Tr}(\gamma_5 Q),\\
Q_\mu&=\frac{1}{\sqrt{2}}\ {\rm Tr}(\gamma_\mu Q).
\end{aligned} \tag{4.5}$$

The essence of the twisting procedure is reflected in the fact that the spin-1/2 charge having the first spinor suffix turns into the spin 0 or spin 1 charge by adding the isospin-1/2 charge, which can be understood by the above relations (4.3) and (4.4).

The part of the algebra including Lorentz generator J in Eq. (4.1) can be rewritten in terms of the new Lorentz generators J' in the following form:

$$\begin{aligned}
[J',Q_{\rm B}]&=[J',\tilde{Q}]=0,\\
[J',Q_\mu]&=i\epsilon_\mu{}^\nu Q_\nu,\\
[J',P_\mu]&=i\epsilon_\mu{}^\nu P_\nu,
\end{aligned} \tag{4.6}$$

where the scalar and vector nature of the fermionic charges measured by the new Lorentz generator J' after the twist is obvious from these relations.

The following algebra together with the algebra (4.6) construct the twisted version of $N=2$ superalgebra,

$$Q_{\rm B}^2=\tilde{Q}^2=0, \quad [R,Q_{\rm B}]=\frac{i}{2}\tilde{Q},$$

[2]Our convention of Hermite Euclidean γ matrices is $(\gamma^\mu)_\alpha{}^\beta=\{\sigma^1,\sigma^3\}$, where σ^i are Pauli matrices, and $\gamma_5=\gamma^1\gamma^2$. Majorana fermion is a two-dimensional real representation of SO(2), and the Lorentz spinor indices are lowered and raised by the charge conjugation matrix $C_{\alpha\beta}=\delta_{\alpha\beta}$.

$$\{Q_B,\tilde{Q}\}=\{Q_\mu,Q_\nu\}=0,\quad [R,\tilde{Q}]=-\frac{i}{2}Q_B,$$
$$\{Q_B,Q_\mu\}=2P_\mu,\quad [R,Q_\mu]=\frac{i}{2}\epsilon_\mu{}^\nu Q_\nu,$$
$$\{\tilde{Q},Q_\mu\}=-2\epsilon_\mu{}^\nu P_\nu,\quad [R,J']=[R,P_\mu]=0. \tag{4.7}$$

Here we identify the scalar charge Q_B as the BRST charge since it has a nilpotency property. It should be noted that the momentum operator is BRST exact, which reflects the characteristic of topological field theory.

Here we explicitly show that the ''partially'' gauge-fixed action possesses twisted $N=2$ supersymmetry [1]. ''Partially'' we mean to fix the gauge of topological symmetry only and recover the SO(2) gauge symmetry. We first modify the gauge-fixed action (3.37) by adding another BRST-exact term

$$-2is\int d^2x\,\bar{\eta}\epsilon_{ab}\phi^a\tilde{C}^b,$$

and make all fields Hermitian to assure the hermicity property of the action:

$$\begin{aligned}S=\int d^2x\bigg(&+\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+(D_\mu\phi)_a(D^\mu\phi)^a+|\phi|^4+i\rho\partial_\mu\tilde{C}^\mu\\&-i\lambda\epsilon^{\mu\nu}\partial_\mu\tilde{C}_\nu-i\chi_{\mu a}(D^\mu\tilde{C})^a+\partial_\mu\bar{\eta}\partial^\mu\eta-2i\rho\epsilon^{ab}\phi_a\tilde{C}_b\\&-2i\lambda\phi^a\tilde{C}_a-2i\chi_{\mu a}\epsilon^{ab}\tilde{C}^\mu\phi_b-\frac{i}{4}\epsilon^{\mu\nu}\chi_{\mu a}\chi^a_\nu\eta\\&+2i\bar{\eta}\epsilon_{ab}\tilde{C}^a\tilde{C}^b+4\bar{\eta}\eta|\phi|^2\bigg).\end{aligned} \tag{4.8}$$

It is easy to see that kinetic terms of ϕ_a, $\tilde{C}_\mu$, ρ, λ, $\chi_{\mu a}$, and $\tilde{C}_a$ are nondegenerate, while that of ω_μ is degenerate. Indeed this action is invariant under the following SO(2) gauge transformations with a gauge parameter v:

$$\begin{aligned}&\delta_{\text{gauge}}(\phi_a,\tilde{C}_a,\chi_{\mu a})=2v\epsilon_a{}^b(\phi_b,\tilde{C}_b,\chi_{\mu b}),\\&\delta_{\text{gauge}}\omega_\mu=\partial_\mu v,\\&\delta_{\text{gauge}}(\tilde{C}_\mu,\eta,\lambda,\rho,\bar{\eta})=0.\end{aligned} \tag{4.9}$$

Corresponding to the Lagrangian given in Eq. (4.8), we can find explicit transformations of fields by the supercharges:

$$\theta_A s^A\varphi=[i\theta_A Q^A,\varphi], \tag{4.10}$$

where $s^A=\{s,\tilde{s},s_\mu\}$ and $Q_A=\{Q_B,\tilde{Q},Q_\mu\}$.

We first point out that the action (4.8) is invariant under the following BRST-like fermionic transformations:

$$\begin{aligned}&s\phi_a=-\tilde{C}_a,\quad s\chi_{\mu a}=4i(D_\mu\phi)_a^{(-)},\\&s\omega_\mu=\tilde{C}_\mu,\quad s\lambda=-2i\left(\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}-|\phi|^2\right),\\&s\tilde{C}_a=-2i\epsilon_a{}^b\phi_b\eta,\quad s\bar{\eta}=\rho,\\&s\tilde{C}_\mu=i\partial_\mu\eta,\quad s\rho=0,\\&s\eta=0.\end{aligned} \tag{4.11}$$

These transformations are only on-shell nilpotent up to corresponding gauge transformations in Eq. (4.9):

$$s^2=i\delta_{\text{gauge}_\eta}, \tag{4.12}$$

where $\delta_{\text{gauge}_\eta}$ denotes a gauge transformation associated with a gauge parameter η.

Furthermore we find that the action possesses the following fermionic vector symmetry:

$$\begin{aligned}&s_\mu\phi_a=\frac{1}{2}\chi_{\mu a},\quad s_\mu\chi_{\nu a}=-4i(\delta_{\mu\nu}\epsilon_{ab}-\epsilon_{\mu\nu}\delta_{ab})\bar{\eta}\phi^b,\\&s_\mu\omega_\nu=-\frac{1}{2}(\epsilon_{\mu\nu}\lambda+\delta_{\mu\nu}\rho),\quad s_\mu\lambda=2i\epsilon_\mu{}^\nu\partial_\nu\bar{\eta},\\&s_\mu\tilde{C}_a=-2i(D_\mu\phi)_a^{(+)},\quad s_\mu\bar{\eta}=0,\\&s_\mu\tilde{C}_\nu=i(F_{\mu\nu}+\epsilon_{\mu\nu}|\phi|^2),\quad s_\mu\rho=2i\partial_\mu\bar{\eta},\\&s_\mu\eta=2\tilde{C}_\mu.\end{aligned} \tag{4.13}$$

These transformations satisfy the following anticommutation relations:

$$\begin{aligned}&\{s,s_\mu\}=2i\partial_\mu-2i\delta_{\text{gauge}_{\omega_\mu}},\\&\{s_\mu,s_\nu\}=-2i\delta_{\mu\nu}\delta_{\text{gauge}_{\bar{\eta}}},\end{aligned} \tag{4.14}$$

where these algebras are also satisfied on shell.

Lastly we can introduce the fermionic pseudoscalar symmetry which are the partner of the BRST-like symmetry:

$$\begin{aligned}&\tilde{s}\phi_a=\epsilon_a{}^b s\phi_b=-\epsilon_a{}^b\tilde{C}_b,\\&\tilde{s}\chi_{\mu a}=-\epsilon_\mu{}^\nu s\chi_{\nu a}=-4i\epsilon_\mu{}^\nu(D_\nu\phi)_a^{(-)},\\&\tilde{s}\omega_\mu=\epsilon_\mu{}^\nu s\omega_\nu=\epsilon_\mu{}^\nu\tilde{C}_\nu,\quad \tilde{s}\lambda=0,\\&\tilde{s}\tilde{C}_a=-\epsilon_a{}^b s\tilde{C}_b=-2i\phi_a\eta,\quad \tilde{s}\bar{\eta}=-\lambda,\\&\tilde{s}\tilde{C}_\mu=-\epsilon_\mu{}^\nu s\tilde{C}_\nu=-i\epsilon_\mu{}^\nu\partial_\nu\eta,\\&\tilde{s}\rho=-2i\left(\frac{1}{2}\epsilon^{\mu\nu}F_{\mu\nu}-|\phi|^2\right),\quad \tilde{s}\eta=0.\end{aligned} \tag{4.15}$$

These transformations lead to the following anticommutation relations:

$$\{\tilde{s},s_\mu\} = -2i\epsilon_\mu{}^\nu\partial_\nu + 2i\delta_{\text{gauge}_{\epsilon_\mu{}^\nu\omega_\nu}},$$

$$\{\tilde{s},\tilde{s}\} = 2i\delta_{\text{gauge}_\eta}, \tag{4.16}$$

$$\{\tilde{s},s\} = 0.$$

It is apparent from Eqs. (4.12), (4.14), and (4.16), that the operators s, s_μ, and $\tilde{s}$ obey the twisted $N=2$ supersymmetric algebra.

V. TOPOLOGICAL TWIST AND DIRAC-KÄHLER FERMION

In this formulation we discover a very interesting correspondence. We point out that two multiplets for the ghost fields and the multiplier fields $(\rho,\tilde{C}_\mu,\lambda)$ and $(\tilde{C}_a,\chi_{\mu a})$ can be interpreted as Dirac-Kähler multiplets, as we shall see below.

In two-dimensional flat Euclidean spacetime, we introduce the following Dirac-Kähler field [7,10]:

$$\Psi = \psi + \psi_\mu dx^\mu + \frac{1}{2}\psi_{\mu\nu}dx^\mu\wedge dx^\nu \equiv \sum_{\alpha,(\beta)} \psi_\alpha{}^{(\beta)} Z^\alpha{}_{(\beta)}, \tag{5.1}$$

where ψ, ψ_μ, and $\psi_{\mu\nu}$ are the Hermitian fermionic scalar, vector, and antisymmetric tensor fields, respectively. The base $Z^\alpha{}_{(\beta)}$ is a 2×2 matrix and can be expanded into the following inhomogeneous forms:

$$Z = 1 + \gamma_\mu^T dx^\mu + \frac{1}{2}\gamma_\mu^T\gamma_\nu^T dx^\mu\wedge dx^\nu. \tag{5.2}$$

The coefficient $\psi_\alpha{}^{(\beta)}$ can be equivalently rewritten as

$$\psi_\alpha{}^{(\beta)} = \frac{1}{2}\left(\psi + \psi_\mu\gamma^\mu + \frac{1}{2}\epsilon^{\mu\nu}\psi_{\mu\nu}\gamma_5\right)_{\alpha^{(\beta)}}. \tag{5.3}$$

It is interesting to note the remembrance of the expansion relations of the fermionic charge in Eq. (4.4) and the coefficients of the Dirac-Kähler field in Eq. (5.3). This could be understood as the origin of the Dirac-Kähler interpretation of ghost fields. We then find that massless Dirac equations are expressed as the following set of equations by the use of antisymmetric tensor fields:

$$(d+\delta)\Psi = (\gamma^\mu\partial_\mu\psi)_\alpha{}^{(\beta)} Z^\alpha{}_{(\beta)} = 0, \tag{5.4}$$

where δ is an adjoint operator $\delta = *d*$ and the index α is a spinor one, while the index (β) is regarded as a ''flavor'' one for two degenerate Dirac fermions. The Dirac-Kähler action which leads to the above equation of motion is defined by

$$\begin{aligned} S &= \frac{1}{2}\int i\Psi^*\wedge *(d+\delta)\Psi \\ &= \int d^2x \sum_{(\beta)} i(\psi^\dagger)_{(\beta)}{}^\alpha(\gamma^\mu\partial_\mu\psi)_\alpha{}^{(\beta)} \\ &= \int d^2x\, \mathrm{Tr}(i\psi^\dagger\gamma^\mu\partial_\mu\psi). \end{aligned} \tag{5.5}$$

We now turn to describe ghost fields in terms of Dirac-Kähler fields. The kinetic terms of these multiplets in the action (4.8) can be expressed as

$$\begin{aligned} &\int d^2x(i\rho\partial_\mu\tilde{C}^\mu - i\lambda\epsilon^{\mu\nu}\partial_\mu\tilde{C}_\nu - i\chi^{\mu a}\partial_\mu\tilde{C}_a) \\ &= \int d^2x\, \mathrm{Tr}(i\psi^\dagger\gamma^\mu\partial_\mu\psi + i\chi^\dagger\gamma^\mu\partial_\mu\chi), \end{aligned} \tag{5.6}$$

where Dirac-Kähler fields ψ and χ are given by

$$\begin{aligned} \psi &= \frac{1}{2}(\rho + \tilde{C}_\mu\gamma^\mu - \lambda\gamma_5), \\ \chi &= \frac{1}{2}(-\tilde{C}_{a=1} + \chi_{\mu a=1}\gamma^\mu - \tilde{C}_{a=2}\gamma_5). \end{aligned} \tag{5.7}$$

Here we impose anti-self-dual conditions for $\chi_{\mu a}$. It is easy to see that the action possess SO(2) ''flavor'' symmetry.

The final expression of the twisted $N=2$ supersymmetric action with Dirac-Kähler fermions is

$$\begin{aligned} S = \int d^2x\bigg(&+\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)_a(D^\mu\phi)^a + |\phi|^4 \\ &+\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B + \mathrm{Tr}(i\psi^\dagger\gamma^\mu\partial_\mu\psi) \\ &+\mathrm{Tr}(i\chi^\dagger\gamma^\mu D_\mu\chi) - 4i\phi_1\,\mathrm{Tr}(\psi^\dagger\gamma_5\chi) \\ &+4i\phi_2\,\mathrm{Tr}(\psi^\dagger\gamma_5\chi\gamma_5) - i\sqrt{2}A\,\mathrm{Tr}(\chi^\dagger\gamma_5\chi) \\ &+i\sqrt{2}B\,\mathrm{Tr}(\chi^\dagger\chi\gamma_5) + 2(A^2-B^2)|\phi|^2\bigg), \end{aligned} \tag{5.8}$$

where we denote $\eta \equiv 2/\sqrt{2}(A+B)$ and $\bar{\eta} \equiv 1/2\sqrt{2}(A-B)$ and the covariant derivative with respect to the flavor group on the Dirac-Kähler field χ is given by $D_\mu\chi \equiv \partial_\mu\chi + 2\omega_\mu\chi\gamma_5$. It is worth to mentioning that this action is equivalent to the extended supersymmetric Abelin Higgs system [14] and topological Bogomol'nyi theory [15] except for the symmetry-breaking potential.

As we have seen in the formulation, the fermionic fields appearing in the quantization procedure such as ghost fields turns into the Dirac-Kähler matter fermion. It would be important to confirm algebraically that the Dirac-Kähler fermions tranform as spinor fields and possess half-integral spin, unlike the ghost fields.

Redefining the Lorentz generator, we will perform a change of the spin of the operators. Indeed we will assign

R-quantum number integrals and half-integrals for boson fields and fermion fields, respectively. The twisted $N=2$ theory defined by J' is the topological field theory, whose superalgebra corresponds to Eqs. (4.6) and (4.7), while the theory defined by J is the $N=2$ supersymmetric field theory.

It is important to recognize that in the present model we can identify the R symmetry as the flavor symmetry of the Dirac-Kähler fields

$$\delta_R \psi = \psi\left(\frac{1}{2}\gamma_5\right), \quad \delta_R \chi = \chi\left(\frac{1}{2}\gamma_5\right), \tag{5.9}$$

which should be compatible with the algebra (4.7). The origin of this identification is again due to the resemblance of the relations between Eqs. (4.4) and (5.3). In other words this identification is originated from the observation that the second flavor suffix of the Dirac-Kähler field in Eq. (5.3) has faithful correspondence with the second spinor suffix of the fermionic charge in Eq. (4.4), which originally corresponds to the isospinor suffix of the R generator before the twist. Then Lorentz transformation on the Dirac-Kähler field ψ induced by J' is

$$\delta_{J'}\psi = \frac{1}{2}(-\epsilon_\mu{}^\nu \tilde{C}_\nu \gamma^\mu) = -\frac{1}{2}[\gamma_5, \psi]. \tag{5.10}$$

On the other hand, the Lorentz transformation induced by $J = J' - R$ is

$$\delta_J \psi = \delta_{J'}\psi - \delta_R \psi = -\frac{1}{2}\gamma_5 \psi, \tag{5.11}$$

which precisely coincides with the Lorentz transformation of the spinor field. This implies that the Dirac-Kähler field is exactly transformed as spinors. We can obtain the same relation for χ. Consequently, we have found that the twisting mechanism in the two-dimensional $N=2$ theory has been understood from the Dirac-Kähler fermion formulation and the R symmetry is nothing but the flavor symmetry of the Dirac-Kähler fermion.

VI. CONCLUSIONS AND DISCUSSIONS

We have investigated the generalized gauge theory from the topological field theory point of view. First, we have extended the algebraic structure of the BRST transformation in the manner of Baulieu-Singer and derived sets of BRST-invariant physical operators. This extension fits naturally in the framework of the generalized gauge theory. The classical fields, the ghost fields, ghost for ghost fields, the differential operator, and the BRST operator are treated in a unified way by the quaternion algebra. In particular, commutator and anticommutator difference in the algebra is automatically accomodated in the generalized gauge theory formulation while this point is treated in an *ad hoc* way in the previous treatments.

As a concrete example we have quantized the generalized topological Yang-Mills action in two-dimensional flat Euclidean spacetime with the two-dimensional Clifford algebra as the simplest graded Lie algebra. We have shown that the generalized topological Yang-Mills action is BRST equivalent to the standard Yang-Mills action plus fermionic ghost and Lagrange multiplier terms by imposing the instanton relations as the gauge fixing conditions. It turns out that the instanton relations coincide with the two-dimensional counterpart of the Seiberg-Witten relations dimensionally reduced from four into two dimensions. The explicit topological solutions of the instanton relations have been obtained [17]. The full twisted $N=2$ supersymmetric algebra has been examined for the gauge-fixed action and explicit transformations of fields for the fermionic charge family including BRST charge has been obtained.

We found that the fermionic sector including ghost fields in the gauge-fixed action can be identified with the Dirac-Kähler fermions. The crucial observation is that the R symmetry of the $N=2$ supersymmetric action can be identified with the ‘‘flavor’’ symmetry of the Dirac-Kähler fermion action. Then the ghost fields together with the fermionic multiplier fields turn into matter fermions via the twisting mechanism. On the other hand, the twisting mechanism is equivalent to the Dirac-Kähler fermion formulation when we identify the R symmetry and the ‘‘flavor’’ symmetry. In this sense we have found that the twisting mechanism is essentially equivalent to the generation of matter fermions from fermionic ghosts via Dirac-Kähler fermion formulation. It is interesting to see if this mechanism works even in higher dimensions.

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[1] E. Witten, Commun. Math. Phys. **117**, 353 (1988); Int. J. Mod. Phys. A **6**, 2775 (1991); J. Math. Phys. **35**, 5101 (1994).

[2] J. M. F. Labastida and M. Pernici, Phys. Lett. B **212**, 56 (1988); R. Brooks, D. Montano, and J. Sonnenschein, *ibid.* **214**, 91 (1988).

[3] L. Baulieu and I. Singer, Nucl. Phys. B (Proc. Suppl.) **5**, 12 (1988); H. Kanno, Z. Phys. C **43**, 477 (1989).

[4] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, Phys. Rep. **209**, 129 (1991); S. Cordes, G. Moore, and S. Rangoolam, in Les Houches Session LXII, edited by F. David, P. Ginsparg, and J. Zinn-Justin (Elsevier, Amsterdam, 1996), and references therein.

[5] D. Birmingham, M. Rakowski, and G. Thompson, Phys. Lett. B **212**, 187 (1988); Nucl. Phys. **B315**, 577 (1989); F. Delduc, F. Gieres, and S. P. Sorella, Phys. Lett. B **225**, 367 (1989); D. Birmingham and M. Rakowski, *ibid.* **269**, 103 (1991); J. M. F. Labastida and P. M. Llatas, Nucl. Phys. **B372**, 220 (1992); R. Brooks, J.-G. Demers, and C. Lucchesi, *ibid.* **B415**, 353 (1994); M. Alvarez and J. M. F. Labastida, *ibid.* **B437**, 356 (1995).

[6] E. Kähler, Rend. Math. Appl. **21**, 425 (1962).
[7] W. Graf, Ann. Inst. Henri Poincare, Sect. A **29**, 85 (1978); I. M. Benn and R. W. Tucker, Commun. Math. Phys. **89**, 341 (1983); J. A. Bullinaria, Ann. Phys. (N.Y.) **168**, 301 (1986).
[8] J. Kogut and L. Susskind, Phys. Rev. D **11**, 395 (1975); L. Susskind, *ibid.* **16**, 3031 (1977).
[9] N. Kawamoto and J. Smit, Nucl. Phys. **B192**, 100 (1981).
[10] P. Becher and H. Joos, Z. Phys. C **15**, 343 (1982); J. M. Rabin, Nucl. Phys. **B201**, 315 (1982); T. Banks, Y. Dothan, and D. Horn, Phys. Lett. **117B**, 413 (1982).
[11] N. Kawamoto and Y. Watabiki, Commun. Math. Phys. **144**, 641 (1992); Mod. Phys. Lett. A **7**, 1137 (1992).
[12] N. Kawamoto, E. Ozawa, and K. Suehiro, Mod. Phys. Lett. A **12**, 219 (1997); N. Kawamoto, K. Suehiro, T. Tsukioka, and H. Umetsu, Commun. Math. Phys. **195**, 233 (1998); Nucl. Phys. **B532**, 429 (1998).
[13] N. Kawamoto and Y. Watabiki, Phys. Rev. D **45**, 605 (1992); Nucl. Phys. **B396**, 326 (1993); N. Kawamoto, H. B. Nielsen, and N. Sato, *ibid.* **B555**, 629 (1999); N. Kawamoto, Suppl. Prog. Theor. Phys. **134**, 84 (1999).
[14] P. Di Vecchia and S. Ferrara, Nucl. Phys. **B130**, 93 (1977).
[15] F. S. Schaposnik and G. Thompson, Phys. Lett. B **224**, 379 (1989).
[16] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**, 485(E) (1994); **B431**, 484 (1994); E. Witten, Math. Res. Lett. **1**, 769 (1994).
[17] S. Nergiz and C. Saçhoğlu, J. Math. Phys. **37**, 3753 (1996); Nucl. Phys. **B503**, 675 (1997).
[18] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. **69B**, 309 (1977); **102B**, 27 (1981); Phys. Rev. D **28**, 2567 (1983); **30**, 508 (1984); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, NJ, 1992).
[19] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).