Charged rotating black hole in three spacetime dimensions

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The generalization of the black hole in three-dimensional spacetime to include an electric charge Q in addition to the mass M and the angular momentum J is given. The field equations are first solved explicitly when Q is small and the general form of the field at large distances is established. The total ''hairs'' M, J and Q are exhibited as boundary terms at infinity. It is found that the inner horizon of the rotating uncharged black hole is unstable under the addition of a small electric charge. Next it is shown that when $Q=0$ the spinning black hole may be obtained from the one with $J=0$ by a Lorentz boost in the $\varphi-t$ plane. This boost is an ''illegitimate coordinate transformation'' because it changes the physical parameters of the solution. The extreme black hole appears as the analog of a particle moving with the speed of light. The same boost may be used when $Q\neq 0$ to generate a solution with angular momentum from that with $J=0$, although the geometrical meaning of the transformation is much less transparent since in the charged case the black holes are not obtained by identifying points in anti–de Sitter space. The metric is given explicitly in terms of three parameters, $\tilde{M}$, $\tilde{Q}$ and ω which are the ''rest mass'' and ''rest charge'' and the angular velocity of the boost. These parameters are related to M, J and Q through the solution of an algebraic cubic equation. Altogether, even without angular momentum, the electrically charged $2+1$ black hole is somewhat pathological since (i) it exists for arbitrarily negative values of the mass, and (ii) there is no upper bound on the electric charge.

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I. INTRODUCTION

The black hole in three-dimensional spacetime [1,2] has been the object of considerable study [3]. However—as pointed out by several authors [3,4]—if one includes electric charge, the solution reported in the original article [1] only applies when the angular momentum vanishes. In this article we analyze the case when all three ''hairs'' M, J and Q are different from zero.

The work is organized as follows. Section II presents the Einstein-Maxwell action in Hamiltonian form. The Hamiltonian form is well suited for the present analysis because it permits us to identify directly the physical meaning of the integration constants as the total charges and their conjugates. The requirements of circular symmetry and time independence (stationarity) are then imposed and the simplified equations applying in that case are written down. The equations are a coupled set of first order differential equations. If the electric charge is small the equations can be integrated directly. This is done in Sec. III. A noteworthy fact is that the inner horizon of the rotating uncharged black hole is unstable under the addition of a small electric charge.

Section IV begins by showing that when $Q=0$ the spinning black hole may be obtained from the one with $J=0$ by a Lorentz boost in the $\varphi-t$ plane. This boost is an ''illegitimate coordinate transformation'' because it changes the physical parameters of the solution. The extreme black hole appears as the analogue of a particle moving with the speed of light. The same boost may be used when $Q\neq 0$ to generate a solution with angular momentum from that with $J=0$, although the geometrical meaning of the transformation is much less transparent since in that case the black holes are not obtained by identifying points in anti–de Sitter space. The metric is given explicitly in terms of three parameters, $\tilde{M}$, $\tilde{Q}$ and ω which are the ''rest mass'' and ''rest charge'' and the angular velocity of the boost. These parameters are related to M, J and Q through the solution of an algebraic cubic equation. Altogether, even without angular momentum, the electrically charged $2+1$ black hole is somewhat pathological since (i) it exists for arbitrarily negative values of the mass, and (ii) there is no upper bound on the electric charge.[1]

II. EINSTEIN-MAXWELL EQUATIONS

A. Hamiltonian form

The Hilbert action coupled to electromagnetism

$$I_{\mathrm{H}}=\int d^3x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{1}$$

[1]As it was observed in [5], the lack of a lower bound for the mass prevents charged black holes from being supersymmetric.

can be cast in Hamiltonian form,

$$I=\int dtd^2x(\pi^{ij}\dot{g}_{ij}+\mathcal{E}^i\dot{A}_i-N^\perp\mathcal{H}_\perp-N^i\mathcal{H}_i-A_0G), \quad (2)$$

by adding appropriate surface terms.

The canonical variables are the spatial metric g_{ij} and the vector potential A_i together with their conjugate momenta π^{ij} and $\mathcal{E}^i$. The generators of normal deformations ($\mathcal{H}_\perp$), tangential deformations ($\mathcal{H}_i$) and gauge transformations (G) are given by

$$\mathcal{H}_\perp=2\kappa g^{-1/2}(\pi^{ij}\pi_{ij}-(\pi^i_i)^2)-(2\kappa)^{-1}g^{1/2}(R-2\Lambda)+\frac{1}{2}g^{-1/2}(g_{ij}\mathcal{E}^i\mathcal{E}^j+\mathcal{B}^2) \quad (3)$$

$$\mathcal{H}_i=-2\pi^j_{i|j}-\epsilon_{ji}\mathcal{E}^j\mathcal{B} \quad (4)$$

$$G=-\mathcal{E}^i_{,i}. \quad (5)$$

In terms of the lapse ($N^\perp$) and shift (N^i) Lagrange multipliers and the spatial two-dimensional metric g_{ij}, the spacetime line element is

$$ds^2=-(N^\perp)^2dt^2+g_{ij}(dx^i+N^idt)(dx^j+N^jdt), \quad (6)$$

and the magnetic density $\mathcal{B}$ is given by

$$F_{ij}=\epsilon_{ij}\mathcal{B}. \quad (7)$$

We will use units such that the gravitational constant $\kappa=8\pi G$, which has dimensions of an inverse mass, is set equal to $\frac{1}{2}$. The cosmological constant is negative and is related to the cosmological length l by $\Lambda=-l^{-2}$. This length will be set equal to unity.

B. Rotational symmetry and time independence

If one requires invariance under spatial rotations and time translations, the metric may be cast in the form

$$ds^2=-N^2(r)f^2(r)dt^2+f^{-2}(r)dr^2+r^2(d\varphi+N^\varphi(r)dt)^2,$$
$$0\leqslant r<\infty, \quad 0\leqslant\varphi<2\pi, \quad t_1\leqslant t\leqslant t_2. \quad (8)$$

This implies that the only non-vanishing component of the gravitational momentum is given by

$$\pi^r_\varphi=p(r). \quad (9)$$

The only non-vanishing electromagnetic strengths are

$$\mathcal{E}^r=Q, \quad (10)$$

the total electric charge, and

$$\mathcal{B}=\partial_rA_\varphi. \quad (11)$$

Equation (10) follows from solving the constraint $G=0$, whereas $\mathcal{E}^\varphi=0$ follows from $\mathcal{H}_r=0$. To arrive at Eq. (11) one uses the radial gauge $A_r=0$.

In terms of the remaining fields, which are only functions of r, the action reads

$$I=-(t_2-t_1)2\pi\int(N\mathcal{H}+N^\varphi\mathcal{H}_\varphi)dr, \quad (12)$$

with

$$\mathcal{H}=f\mathcal{H}_\perp=(f^2)'-2r+\frac{2p^2}{r^3}+\frac{Q^2}{2r}+\frac{f^2\mathcal{B}^2}{2r}, \quad (13)$$

and

$$\mathcal{H}_\varphi=-2p'-Q\mathcal{B}. \quad (14)$$

Extremization of the reduced action (12) with respect to N, N^φ, f^2, p and A_φ gives, respectively,

$$(f^2)'-2r+\frac{2p^2}{r^3}+\frac{Q^2}{2r}+\frac{f^2\mathcal{B}^2}{2r}=0 \quad (15)$$

$$2p'+Q\mathcal{B}=0 \quad (16)$$

$$N'-\frac{N\mathcal{B}^2}{2r}=0 \quad (17)$$

$$(N^\varphi)'+\frac{2Np}{r^3}=0 \quad (18)$$

$$\left(\frac{Nf^2\mathcal{B}}{r}-N^\varphi Q\right)'=0. \quad (19)$$

III. SOLUTIONS IN SPECIAL CASES AND PHYSICAL MEANING OF INTEGRATION CONSTANTS

Equation (19) can be solved for $\mathcal{B}$ with an appropriate integration constant which is fixed by regularity and is not an independent free parameter [see Eq. (49) below]. The integration constants for f^2 and p can be expressed in terms of the mass M and the angular momentum J. Thus, together with the electromagnetic field, the solution is characterized by the charges M, J, Q and their conjugates $N(\infty)$, $N^\varphi(\infty)$ and $A_0(\infty)$ (''chemical potentials'' in the Euclidean formulation).

The system of first order nonlinear differential equations (15)–(18) is not straightforward to solve for f^2, p, N and N^φ, when $Q\neq0$. It is therefore useful to analyze special cases and approximations in order to gain insight into the properties of the solution.

We first turn our attention to the two limiting cases when the solution is known in closed form. These are $Q=0$, $J\neq0$, $M\neq0$ (the standard 2+1 black hole) and $Q\neq0$, $J=0$, $M\neq0$ (charged black hole without angular momentum).

A. $J\neq 0,\ Q=0$

The solution of the equations is

$$f^2=r^2-M+\frac{J^2}{4r^2} \tag{20}$$

$$p=p(\infty)=-\frac{J}{2} \tag{21}$$

$$N=N(\infty) \tag{22}$$

$$N^\varphi=-\frac{J}{2r^2}+N^\varphi(\infty). \tag{23}$$

If one varies the reduced action (12) one picks up a surface term [10]

$$(t_2-t_1)2\pi[-\delta f^2(\infty)N(\infty)-2\,\delta p(\infty)N^\varphi(\infty)], \tag{24}$$

which identifies $\delta f^2(\infty)=\delta M$ and $\delta p(\infty)=-\delta J/2$. In order to have a black hole for $M\geqslant 0$ one adjusts the zero of the energy so that M is indeed the mass. In the same manner, in order to have a static black hole for $J=0$ one takes J as the total angular momentum. Similarly with the electric charge, which is identified with the integration constant Q appearing in Eq. (12) because its variation in the action (2) multiplies $A_0(\infty)$.

The function f^2 has zeros at r_+ and r_-, the outer and inner horizons, given by

$$r_+=\left[\frac{M+(M^2-J^2)^{1/2}}{2}\right]^{1/2}, \tag{25}$$

$$r_-=\left[\frac{M-(M^2-J^2)^{1/2}}{2}\right]^{1/2}, \tag{26}$$

which exist provided

$$J^2\leqslant M^2, \tag{27}$$

and coalesce if $J=M$ (extreme case).

Note that if $J\to 0$ (M fixed) the root r_- tends to $r=0$ which is considered as a (''chronological'') singularity [2]. Note also that when $J=0$, $f^2=0$ has only one root $r=r_+$. Thus the limit of the root r_- (which is zero) is not a root of the limit of f^2. In this sense the limit is discontinuous. This discontinuous behavior shows itself in the graph of f^2 as a function of r. The minimum of f^2 occurs at $r^4_{\rm min}=(J/2)^2$ whereas the value of f^2 at the minimum is $f^2_{\rm min}=|J|-M$ for all J. Thus when $J\to 0$ the whole branch of f^2 to the left of the minimum disappears. We shall see similar behaviors for $J=0,\ Q\neq 0$ and $J\neq 0,\ Q\neq 0$.

B. $J=0,\ Q\neq 0$

Now one has

$$f^2=r^2-M-\frac{1}{4}Q^2\ln r^2 \tag{28}$$

$$p=0 \tag{29}$$

$$N=N(\infty) \tag{30}$$

$$N^\varphi=0. \tag{31}$$

If one varies the action, one picks up a surface term

$$\left(-\delta M-\frac{1}{2}\,\delta Q^2\ln r\right)N(r), \tag{32}$$

which diverges if $r\to\infty$.

This divergence may be handled as follows. One encloses the system in a large circle of radius r_0 and rewrites Eq. (28) as

$$f^2=r^2-M(r_0)-\frac{1}{2}Q^2\ln\frac{r}{r_0}, \tag{33}$$

with

$$M=M(r_0)-\frac{1}{2}Q^2\ln r_0. \tag{34}$$

Then Eq. (32) is replaced by

$$\left(-\delta M(r_0)-\frac{1}{2}\,\delta Q^2\ln\frac{r}{r_0}\right)N(r), \tag{35}$$

and the second term vanishes when $r\to r_0$.

One might call $M(r_0)$ ''the energy within the radius r_0.'' It differs from M by $-\frac{1}{2}Q^2\ln r_0$ which may be thought of as the electrostatic energy outside r_0 up to an (infinite) constant which is absorbed in $M(r_0)$. The sum (34) is then independent of r_0, finite and equal to the total mass.

Thus in practice one does not bring in r_0 and writes

$$\text{``}\lim_{r\to\infty}\delta(Q^2\ln r)=0.\text{''} \tag{36}$$

A similar situation occurs in four spacetime dimensions. One may write the Reissner-Nordström g_{00} as $1-2M(r_0)/r+Q^2(1/r^2-1/r_0^2)$. Then $M=M(r_0)+Q^2/2r_0$ is the total mass, independent of r_0, and $Q^2/2r_0$ is the electrostatic energy outside a sphere of radius r_0. In this case it is not necessary to include an infinite constant in $M(r_0)$ since $Q^2/2r_0$ vanishes when $r_0\to\infty$.

The function f^2 given by Eq. (28) tends to positive infinity if $r\to 0$ or if $r\to\infty$. It has a minimum at $r=|Q|/2$. The value of f^2 at the minimum is

$$-M+\left(\frac{Q}{2}\right)^2\left[1-\ln\left(\frac{Q}{2}\right)^2\right]. \tag{37}$$

If Eq. (37) is negative there are two zeros of f^2, r_+ and r_-. If Eq. (37) vanishes the two roots coincide and one has an extreme black hole. The situation is illustrated in Fig. 1.

Now, the function $(Q/2)^2[1-\ln(Q/2)^2]$ vanishes at $Q=0$, has a maximum at $(Q/2)^2=1$ with value 1, vanishes at $(Q/2)^2=e$ and tends to negative infinity for large $(Q/2)^2$.

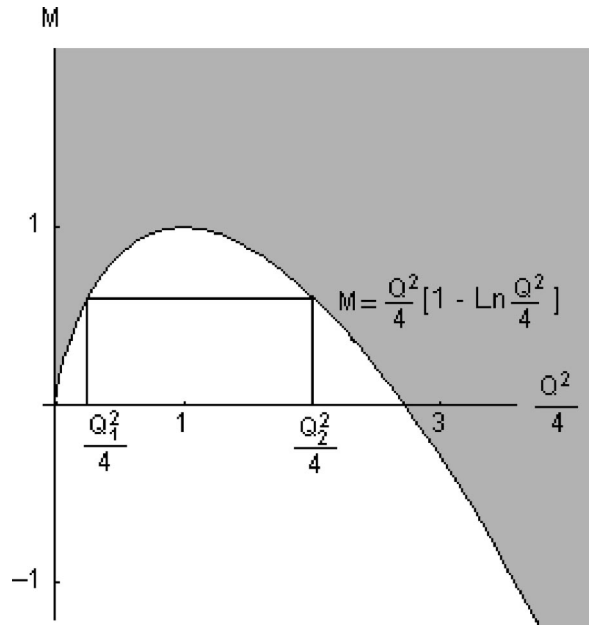

FIG. 1. Region in the mass-charge plane for which there are black holes. Black holes exist whenever $M-(Q/2)^2[1-\ln(Q/2)^2] \geqslant 0$. This corresponds to the shaded area in the diagram. Extremal black holes are at the boundary between the shaded and unshaded areas. Note that if the electric charge is large enough black hole solutions exist for arbitrarily negative values of the mass.

This means that if $M>1$ there are always two roots $r_\pm$ which are different. When $M=1$ the two roots coincide for $(Q/2)^2=1$. If $0<M<1$ there are two branches $Q^2 \leqslant Q_1^2$ and $Q^2 \geqslant Q_2^2$, where $Q_1^2<Q_2^2$ are the two roots of Eq. (37).

One sees that if the electric charge is large enough black hole solutions exist even for negative values of the mass. This feature is in sharp contrast with what happens in $3+1$ dimensions and makes the electrically charged $2+1$ black hole somewhat pathological.

Lastly, consider the limit $Q \to 0$. The situation is analogous to the limit $J \to 0$ discussed at the end of Sec. III A. Now one has $r_{\min}^2=(Q/2)^2$ and $f_{\min}^2 \to -M$ so that, again, the whole ascending branch to the left of the minimum disappears in the limit $Q \to 0$. Note that r_- approaches zero as $r_- \sim e^{-2M/Q^2}$.

C. $M \neq 0$, $J \neq 0$, $Q \neq 0$ small

One can solve Eqs. (15)–(19) perturbatively for small Q. This is of interest because the solution for small Q captures the behavior at large distances for a generic Q, and also because it gives insight into the properties of horizons.

We start with expressions (20)–(23) which we will denote with a subscript zero (''unperturbed'') and add to them corrections of order Q^2. One has $\mathcal{B}_0=0$. The equations for the perturbation then read

$$(\Delta f^2)' = \frac{2J}{r^3}\Delta p - \frac{Q^2}{2r} - \frac{f_0^2 \Delta \mathcal{B}^2}{2r} \tag{38}$$

$$\Delta p' = -\frac{Q \Delta \mathcal{B}}{2} \tag{39}$$

$$\Delta N' = \frac{\Delta \mathcal{B}^2}{2r} \tag{40}$$

$$(\Delta N^\varphi)' = \frac{J}{r^3}\Delta N - \frac{2}{r^3}\Delta p \tag{41}$$

$$\left(\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^3}\Delta \mathcal{B} + \frac{JQ}{2r^2} - N^\varphi(\infty)Q\right)' = 0. \tag{42}$$

These equations must be integrated demanding that $N(\infty)$, $N^\varphi(\infty)$, M and J be unchanged. Thus we should ask

$$\Delta N(\infty)=0, \quad \Delta N^\varphi(\infty)=0$$

and also demand that $\Delta p(r)$ and Δf^2 should vanish as $r \to \infty$, up to logarithmic terms multiplied by Q^2, which are not considered when identifying J and M by the reasoning of Sec. III B. The integration constant in Eq. (42) is fixed by a regularity requirement as discussed below.

One obtains

$$\Delta f^2 = -\frac{Q^2}{4}\left(1 + \frac{r_-^2}{r_+^2} - 2\frac{r_-^2}{r^2}\right)\ln(r^2-r_-^2) \tag{43}$$

$$\Delta p = -\frac{JQ^2}{8r_+^2}\ln(r^2-r_-^2) \tag{44}$$

$$\Delta N = -\frac{Q^2 r_-^2}{4r_+^2}\frac{1}{r^2-r_-^2} \tag{45}$$

$$\Delta N^\varphi = -\frac{JQ^2}{8r_+^2 r^2}(1+\ln(r^2-r_-^2)) \tag{46}$$

$$\Delta \mathcal{B} = \frac{JQ}{2r_+^2}\frac{r}{r^2-r_-^2}. \tag{47}$$

Here r_+ and r_- are those of the unperturbed solution. Note that when $J=0$ $(r_-=0)$ the results go over into the exact expressions found in the previous subsection.

The integration constant in Eq. (42) has been chosen so that the magnetic field referred to an orthonormal basis,

$$\hat{B} = F_{\hat{r}\hat{\varphi}} = g^{-1/2}\mathcal{B} = \frac{f\mathcal{B}}{r}, \tag{48}$$

is regular at the (outer) horizon r_+. Indeed, Eq. (47) is equivalent to writing

$$\mathcal{B}(r)=\frac{Q[N^{\varphi}(r)-N^{\varphi}(r_+)]}{N(r)f^2(r)}r. \tag{49}$$

Note that, as a consequence, $\hat{B}(r_+)=0$.

As a result of the perturbation the horizon is now at $r_+ + \Delta r_+$ where Δr_+ may be obtained from

$$\Delta r_+^2=-\left.\frac{\Delta f^2}{(df_0^2/dr_+^2)}\right|_{r_+}=-\frac{Q^2}{4}\ln[r_+^2-r_-^2]$$

$$=-\frac{Q^2}{2}\ln[M^2-J^2]. \tag{50}$$

The solution given by Eqs. (43)–(47) is singular at r_-. This means that the inner horizon of the rotating uncharged black hole is unstable under the addition of a small electric charge.

IV. EXACT SOLUTION WITH $M\neq 0$, $J\neq 0$, $Q\neq 0$

The system of ordinary, coupled, non-linear, first order differential equations (15)–(19) appears not to be tractable directly. However one may find the solution by means of an ''illegitimate coordinate transformation.'' In this case ''illegitimate'' means a transformation that does not preserve the periodicity of the angular variable and also changes the range of the radial variable. As a consequence of the change in periodicity, the Casimir invariants of the symmetry group at infinity are changed.

Generating new solutions from old ones by means of coordinate transformations which are illegitimate in some intermediate step, but lead to a sensible answer, is a procedure that has been quite useful in general relativity although there appears to be no general rationale behind it.

For a conical singularity in three spacetime dimensions, the idea of bringing in angular momentum through an illegitimate coordinate transformation was used in Refs. [6–8]. In [9] the procedure was used to generate the uncharged rotating black hole metric (with a transformation not quite the same as the one used here), but the parameters appearing in the construction were not related to the mass and the angular momentum.

A. Rotating solution with $Q=0$ revisited

In this case one knows that the black holes with $J=0$ and $J\neq 0$ are both obtained by identifications of anti–de Sitter space as discussed in [2]. Therefore, if one forgets the identifications there exists a coordinate transformation that takes the standard form of the black hole line element with $J=0$ onto the one with $J\neq 0$. It is given by

$$\tilde{t}=\frac{R_+}{r_+}\left(t-\frac{R_-}{R_+}\varphi\right) \tag{51}$$

$$\tilde{\varphi}=\frac{R_+}{r_+}\left(\varphi-\frac{R_-}{R_+}t\right) \tag{52}$$

$$r^2=R^2-R_-^2. \tag{53}$$

This transformation may be obtained by composing the transformations that relate each of these black holes to the standard form of the anti–de Sitter line element and which are given by Eq. (3.29) of [2]. The individual transformations and also their composition change the functional form of the anti–de Sitter line element and therefore do not belong to the $SO(2,2)$ symmetry group of anti–de Sitter space.

Here we have denoted by $(r,\tilde{\varphi},\tilde{t})$ and (R,φ,t) the standard coordinates for the non-rotating and rotating cases, respectively. To obtain the rotating black hole one must demand that φ be periodic with period 2π and also that $0\leqslant R^2<\infty$. As already anticipated, these requirements are incompatible with $\tilde{\varphi}$ having period 2π and $0\leqslant r^2<\infty$. It is important to realize that the outer horizon is the image R_+ of the unique horizon that the non rotating black hole has at $r_+^2=\tilde{M}$. On the other hand, the inner horizon comes in because the Jacobian factor dR/dr vanishes at $r=0$, whose image is R_-.

Now, the transformation (51),(52),(53) has more freedom than what we need because it takes a non-rotating black hole with a given mass onto the most general black hole. It suffices for our needs to be able to endow a given black hole with angular momentum, much as one endows a particle with linear momentum by means of a boost. This analogy is quite appropriate, indeed if we write

$$\left(\frac{R_-}{R_+}\right)^2=\omega^2\leqslant 1 \tag{54}$$

$$\frac{R_+}{r_+}=\alpha\frac{1}{\sqrt{1-\omega^2}}, \tag{55}$$

we see that Eqs. (51),(52) is the product of a Lorentz boost and a conformal rescaling of φ, t. One may set

$$\alpha=1 \tag{56}$$

by rescaling the radial variable R and the parameters R_- and R_+ by factor α, and we shall do so. Therefore we will take as our ''rotation boost'' the transformation

$$\tilde{t}=\frac{t-\omega\varphi}{\sqrt{1-\omega^2}}, \tag{57}$$

$$\tilde{\varphi}=\frac{\varphi-\omega t}{\sqrt{1-\omega^2}}, \tag{58}$$

supplemented by the change of the radial coordinate

$$r^2=R^2-\frac{\omega^2 r_+^2}{1-\omega^2}. \tag{59}$$

The reason for adopting this particular way of eliminating one of the redundant parameters in Eqs. (51),(52) (rather than, say, setting $R_+/r_+=1$) is, basically, that it simplifies considerably the analysis for the charged case. On less pragmatic grounds, taking Eqs. (51),(52) to be just a Lorentz

boost is appealing because it captures distinctly the limiting nature of the extreme black hole which appears as the analogue of a zero mass particle.

Indeed, one may rewrite Eqs. (54) and (55) as

$$R_+ = \frac{r_+}{\sqrt{(1-\omega^2)}}$$

$$R_- = \frac{r_+ \omega}{\sqrt{(1-\omega^2)}}.$$

Thus we see that (R_+, R_-) transforms as a vector under the boost and $r_+ = \tilde{M}^{1/2}$ plays the role of the rest mass. The extreme black hole corresponds to the limit $\omega^2 \to 1$, $\tilde{M} \to 0$, keeping $R_+^2 = \tilde{M}/(1-\omega^2)$ finite.

The corresponding transformation formulas for the mass and the angular momentum read

$$M = \tilde{M}(1+\omega^2)/(1-\omega^2), \tag{60}$$

$$J = 2\omega\tilde{M}/(1-\omega^2). \tag{61}$$

Recall that M and J are precisely the Casimir invariants associated with the identification which produces the black hole out of anti–de Sitter space [2]. It follows from Eqs. (60) and (61) that each of these Casimir invariants separately changes under the boost; however, the combination $M^2 - J^2 = r_+^4$ remains unchanged.

B. Charged case

When $Q \neq 0$, the black hole is not obtained through identifications of points in anti–de Sitter space or in another universal space. Therefore the reasoning accompanying equations (51)–(53) no longer applies. Nevertheless, we will again bring in angular momentum through the rotation boost (58) supplemented by a change of the radial coordinate which will be a generalization of Eq. (59) and which we will derive below. The procedure will of course produce a solution of the field equations since one is performing a change of coordinates. That the solution is a charged rotating black hole will follow from verifying that the asymptotic conditions (43)–(47) are met and because it will have the appropriate horizon structure.

We start from a line element of the form

$$ds^2 = -f^2(r)d\tilde{t}^2 + f^{-2}(r)dr^2 + r^2 d\tilde{\varphi}^2, \tag{62}$$

and apply the boost (58) to it to obtain

$$ds^2 = -N^2F^2dt^2 + F^{-2}dR^2 + R^2(d\varphi + N^\varphi dt)^2 \tag{63}$$

with

$$R^2 = \frac{r^2 - \omega^2 f^2}{1-\omega^2} \tag{64}$$

$$F^2 = \left(\frac{dR}{dr}\right)^2 f^2 \tag{65}$$

$$N = \frac{r}{R}\left(\frac{dr}{dR}\right) = \left(\frac{dr^2}{dR^2}\right) \tag{66}$$

$$N^\varphi = \frac{\omega(f^2 - r^2)}{(1-\omega^2)R^2}. \tag{67}$$

To boost to rotation the uncharged black hole one substitutes $r^2 - r_+^2$ for f^2 in Eqs. (64)–(67) and obtains the uncharged rotating black hole metric, with Eq. (64) giving the transformation (59) of the radial coordinate. If one starts instead with the charged nonrotating black hole, one takes

$$f^2 = r^2 - \tilde{M} - \frac{1}{4}\tilde{Q}^2 \ln r^2 \tag{68}$$

and obtains, after the boost,

$$R^2 = r^2 + \frac{\omega^2}{(1-\omega^2)}\left(\tilde{M} + \frac{\tilde{Q}^2}{4}\ln r^2\right) \tag{69}$$

$$F^2 = \frac{\left(r^2 + \dfrac{\omega^2\tilde{Q}^2}{4(1-\omega^2)}\right)^2}{R^2 r^2}\left(r^2 - \tilde{M} - \frac{1}{4}\tilde{Q}^2\ln r^2\right) \tag{70}$$

$$N = \frac{r^2}{r^2 + \dfrac{\omega^2\tilde{Q}^2}{4(1-\omega^2)}} \tag{71}$$

$$N^\varphi = -\omega\frac{\tilde{M} + \frac{1}{4}\tilde{Q}^2\ln r^2}{(1-\omega^2)R^2} \tag{72}$$

$$\frac{dR}{dr} = \frac{r^2 + \dfrac{\omega^2\tilde{Q}^2}{4(1-\omega^2)}}{Rr}. \tag{73}$$

The potential $A = -\tilde{Q}\ln r d\tilde{t}$ transforms into

$$A = -\frac{\tilde{Q}}{\sqrt{1-\omega^2}}\ln r(dt - \omega d\varphi) \tag{74}$$

yielding a magnetic density

$$\mathcal{B} = \frac{\tilde{Q}\omega R}{\sqrt{1-\omega^2}\left(r^2 + \dfrac{\omega^2\tilde{Q}^2}{4(1-\omega^2)}\right)}. \tag{75}$$

The momentum p is determined using Eq. (18)

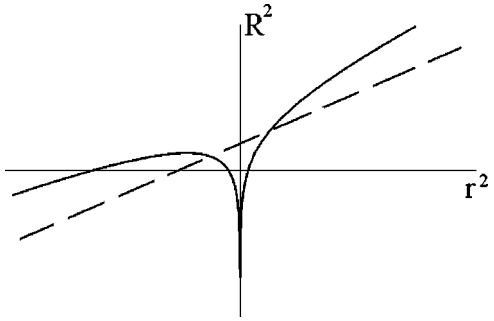

FIG. 2. R^2 as a function of r^2. The black hole space corresponds to the piece to the right of the throat for which $R^2 \geqslant 0$. In the limit $Q \to 0$ the left and right sides of the throat annihilate each other and the two branches merge on the segmented straight line. The separation between the solid and the segmented line behaves asymptotically as $[\omega^2/(1-\omega^2)](\tilde{Q}^2/4)\ln r^2$. The inner horizon of the uncharged black hole corresponds to the intersection of the straight line and the vertical axis. When the electric charge is turned on, the throat comes in and the old inner horizon becomes thus disconnected from the black hole space, while a new inner horizon appears in the black hole space. This is the origin of the perturbative instability of the inner horizon.

$$p = -\frac{\omega}{1-\omega^2}\left(\tilde{M} + \frac{\tilde{Q}^2}{4}(\ln r^2 - 1)\right). \tag{76}$$

The graph of R^2 as a function of r^2 given by Eq. (69) is shown in Fig. 2. One sees that the function has two branches separated by a singularity in the form of an infinite throat at $r^2 = 0$. The black hole space corresponds to the piece to the right of the throat for which $R^2 \geqslant 0$. In the limit $Q \to 0$ the left and right sides of the throat annihilate each other and the two branches merge on the segmented straight line. The inner horizon of the uncharged black hole corresponds to the intersection of the straight line and the horizontal axis, which occurs at a negative value of r^2. When the electric charge is turned on, the throat comes in and the old inner horizon becomes thus disconnected from the black hole space, while a new inner horizon appears in the black hole space. This is the origin of the perturbative instability of the inner horizon found in Sec. III. Both the outer and inner horizons of the charged rotating black hole come from the corresponding ones of the original charged black hole with $J = 0$. They are the images under the boost of the zeros r_+ r_- of f^2. Unlike what happens in the uncharged case, the Jacobian dR/dr in Eq. (73) is different from zero and does not bring in new zeros of F^2.

C. Asymptotic form

We now verify that the rotating solution has the correct asymptotic form and relate the parameters $\tilde{M}$, $\tilde{Q}$ to M, Q and J, the actual mass, charge and angular momentum of the charged rotating solution.

For $R \to \infty$, Eq. (69) takes the form

$$r^2 = R^2 - \frac{\omega^2}{1-\omega^2}\left(\tilde{M} + \frac{\tilde{Q}^2}{4}\ln R^2\right) + O(1/R^2). \tag{77}$$

Using this relation one obtains the asymptotic forms of F^2 and p

$$F^2 = R^2 - \frac{1}{1-\omega^2}\left(\tilde{M}(1+\omega^2) - \frac{\omega^2\tilde{Q}^2}{2}\right) - \frac{(1+\omega^2)\tilde{Q}^2\ln R^2}{4(1-\omega^2)} + O(1/R^2), \tag{78}$$

$$p = -\frac{\omega}{1-\omega^2}\left(\tilde{M} - \frac{\tilde{Q}^2}{4}\right) - \frac{\omega\tilde{Q}^2}{1-\omega^2}\ln R^2 + O(1/R^2), \tag{79}$$

whereas from Eq. (74) one has

$$\mathcal{E}^R = \frac{\tilde{Q}}{\sqrt{1-\omega^2}}. \tag{80}$$

From Eqs. (78), (79) and (80) one finds that the symptotic forms are the desired ones provided that

$$M = \frac{1}{1-\omega^2}\left(\tilde{M}(1+\omega^2) - \frac{\omega^2\tilde{Q}^2}{2}\right), \tag{81}$$

$$J = \frac{2\omega}{1-\omega^2}\left(\tilde{M} - \frac{\tilde{Q}^2}{4}\right), \tag{82}$$

and

$$Q = \frac{\tilde{Q}}{\sqrt{1-\omega^2}}. \tag{83}$$

It may seem surprising that Q differs from $\tilde{Q}$. This comes from the change in the periodicity of φ under the boost, which changes the region of integration of the charge density.

Now, for a black hole one wants to express the metric in terms of the charges at infinity (hairs). To this end, one should solve Eqs. (81),(82),(83) to express $\tilde{M}$, $\tilde{Q}$ and ω as functions of M, Q,and J. If one can do this, the ''rest mass'' $\tilde{M}(M,Q,J)$ and ''rest charge'' $\tilde{Q}(M,Q,J)$ are by construction invariants under the boost, and generalize the $\tilde{M}$ of the case without charge, whereas $\omega(M,Q,J)$ is the ''velocity expressed in terms of the momentum.''

The difficult part in inverting Eqs. (81),(82),(83) is to solve the cubic equation that they imply, namely,

$$\frac{Q^2}{4}\omega^3 - \frac{J\omega^2}{2} + \omega\left(M - \frac{Q^2}{4}\right) - \frac{J}{2} = 0. \tag{84}$$

To gain insight into this equation it is useful to consider the case $Q = 0$ which may be rewritten as

$$\omega^2 - \frac{2M}{J}\omega + 1 = 0. \tag{85}$$

The solution of this equation, which satisfies $\omega^2 < 1$, is

$$\omega = \mathrm{sgn}\left(\frac{M}{J}\right)\left(\left|\frac{M}{J}\right| - \sqrt{\left(\frac{M}{J}\right)^2 - 1}\right), \tag{86}$$

and exists if and only if

$$M^2 > J^2. \tag{87}$$

In the limiting case $M^2 = J^2$, one has $\omega \to \pm 1$.

If the solution (86) is inserted back in the original equations (60), (61), one finds

$$\tilde{M} = \mathrm{sgn}M\sqrt{M^2 - J^2}. \tag{88}$$

We see, therefore, two points of interest. They are: (i) A real ω, with $\omega^2 < 1$, exists only for a range of the parameters of the algebraic equation (85), namely $M^2 > J^2$. (ii) Even when $\omega^2 < 1$ exists, the solution is not necessarily a black hole. One needs, in addition to Eq. (87), $\tilde{M} \geqslant 0$, which in this case is equivalent to $M > 0$.

When $Q \neq 0$, one expects to face the same situation. There will be a certain region in the space of parameters M, J and Q (actually, only the two independent combinations M/J and $Q^2/4J$ enter) for which $\omega^2 < 1$ exists. Within that region one will be able to express $\tilde{M}$ and $\tilde{Q}$ as functions of M, J and Q and the solution will be a black hole when $\tilde{M}$ and $\tilde{Q}$ obey the additional restrictions described in Fig. 1. It does not seem to be possible to exhibit explicitly the necessary and sufficient condition for M, J and Q to be such that the cubic equation (84) admits a unique real solution $\omega^2 < 1$. The best we have been able to do is to show by analyzing the graph of the cubic, that sufficient, but not necessary conditions are $M^2 > J^2$ and $M - Q^2/4 > 0$. It is also straightforward to show that under these conditions J and ω have the same sign.

Note added in proof. Professor G. Clément informs us that he derived a rotating black hole metric in $2+1$ dimensions in [11], and that he also pointed out there that when $Q \neq 0$ the original solution in Ref. [1] was only valid in the absence of rotation. He also informs us that in [9] he chose not to relate the integration constants to the mass and angular momentum because ''in this case there is no scale-invariant definition of M and J.''

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