Low-energy expansion of the one-loop type-II superstring amplitude

Michael B. Green* and Pierre Vanhove†
DAMTP, Silver Street, Cambridge CB3 9EW, United Kingdom
(Received 7 October 1999; published 24 April 2000)

The one-loop four-graviton amplitude in either of the type-II superstring theories is expanded in powers of the external momenta up to and including terms of order $s^4 \ln s\mathcal{R}^4$, where $\mathcal{R}^4$ denotes a specific contraction of four linearized Weyl tensors and s is a Mandelstam invariant. Terms in this series are obtained by integrating powers of the two-dimensional scalar field theory propagator over the toroidal world sheet as well as the moduli of the torus. The values of these coefficients match expectations based on duality relations between string theory and eleven-dimensional supergravity.

PACS number(s): 04.50.+h

I. INTRODUCTION

The wealth of duality symmetries relating different parameterizations of nonperturbative string theory, or M theory, provide severe constraints on its structure. One striking manifestation of this is the relationship between the low-energy expansion of the type-II string theory action and one-loop effects in compactified eleven-dimensional supergravity [1]. Although the systematics of this relationship becomes very murky at higher loops, the leading behavior of the two-loop contribution of the eleven-dimensional theory is amenable to a detailed analysis (see the preceding paper [2]).

This detailed comparison between string theory and eleven-dimensional supergravity requires, among other things, detailed knowledge of the low-energy expansion of the effective action of the type-IIA and type-IIB superstring perturbation theories. Surprisingly, this has scarcely been considered in the literature beyond the most elementary tree-level terms. In this paper we will obtain terms in the effective action that arise from the momentum expansion of the one-loop type-II superstring theory contribution to the four graviton amplitude. Since the four-graviton tree and one-loop amplitudes in the type-IIA and -IIB theories are equal we need not distinguish between the two theories in the following.[1]

The tree-level amplitude for the scattering of four gravitons with polarization tensors $\zeta^{(r)}_{\mu\nu}$ and momenta k^μ_r ($r=1,2,3,4$, $\mu=0,1,\ldots,9$, and $k_r^2=0$) has the very simple form [3,4]

$$A_4^{\rm tree}=-\hat{K}\kappa_{10}^2 e^{-2\phi}T, \tag{1.1}$$

where ϕ is the constant dilaton field so that $g=\kappa_{10}^{-1}e^{\phi}$ is the string coupling and

$$T=\frac{64}{\alpha'^3 stu}\frac{\Gamma\left(1-\frac{\alpha'}{4}s\right)\Gamma\left(1-\frac{\alpha'}{4}t\right)\Gamma\left(1-\frac{\alpha'}{4}u\right)}{\Gamma\left(1+\frac{\alpha'}{4}s\right)\Gamma\left(1+\frac{\alpha'}{4}t\right)\Gamma\left(1+\frac{\alpha'}{4}u\right)}, \tag{1.2}$$

where the Mandelstam invariants are defined by $s=-(k_1+k_2)^2$, $t=-(k_1+k_4)^2$, and $u=-(k_1+k_3)^2$. The overall kinematic factor $\hat{K}$ is given by

$$\hat{K}=t^{\mu_1\cdots\mu_8}t^{\nu_1\cdots\nu_8}\prod_{r=1}^{4}\zeta^{(r)}_{\mu_r\nu_r}k^{(r)}_{\mu_{r+4}}k^{(r)}_{\nu_{r+4}}, \tag{1.3}$$

which is the linearized approximation to the standard contraction between four curvature tensors

$$\mathcal{R}^4=t_8t_8R^4\equiv t^{\mu_1\cdots\mu_8}t_{\nu_1\cdots\nu_8}R^{\nu_1\nu_2}_{\mu_1\mu_2}\cdots R^{\nu_7\nu_8}_{\mu_7\mu_8}, \tag{1.4}$$

where the tensor $t^{\mu_1\cdots\mu_8}$ is defined in Ref. [5] and in Appendix 9 of Ref. [6].[2] The value of the constant κ_{10} in Eq. (1.1) is arbitrary since it can be changed by shifting the dilaton field. It is convenient to set it to the value

$$\kappa_{10}^2=\frac{1}{2}(2\pi)^7\alpha'^4, \tag{1.5}$$

which normalizes the D-string tension to the value $T_{D_1}=e^{-\phi}T_F$, where $T_F=1/2\pi\alpha'$ is the fundamental string tension [7].

The one-loop type-II superstring four-graviton scattering amplitude in ten dimensions is also very simple and is given by [5]

$$A_4^{\text{one-loop}}=\frac{\kappa_{10}^4}{2^5\pi^6\alpha'^4}\hat{K}I=\kappa_{10}^2 2\pi I\hat{K}, \tag{1.6}$$

where I is the integral of a modular function

$$I=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}F(\tau,\bar{\tau}) \tag{1.7}$$

(where $\tau=\tau_1+i\tau_2$ and $d^2\tau\equiv d\tau_1 d\tau_2=d\tau d\bar{\tau}/2$), and $\mathcal{F}$ denotes the fundamental domain of Sl(2,**Z**),

$$\mathcal{F}=\{|\tau_1|\leqslant\tfrac{1}{2},|\tau|^2\leqslant 1\}. \tag{1.8}$$

*Email address: m.b.green@damtp.cam.ac.uk
†Email address: p.vanhove@damtp.cam.ac.uk

[1]The two type-II string perturbation theories are equal up to and including two loops [2].

[2]This contraction projects onto the purely traceless components of the curvature, which constitute the Weyl tensor.

The dynamical factor in Eq. (1.7) is given by an integral over the positions $\nu^{(i)}=\nu_1^{(i)}+i\nu_2^{(i)}$ of the four vertex operators on the torus

$$F(\tau,\overline{\tau})=\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}(\chi_{12}\chi_{34})^{\alpha' s}(\chi_{14}\chi_{23})^{\alpha' t}(\chi_{13}\chi_{24})^{\alpha' u}$$
$$=\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}e^{\mathcal{D}}$$
$$=\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}\exp(\alpha' s\Delta_s+\alpha' t\Delta_g+\alpha' u\Delta_u), \tag{1.9}$$

where $d^2\nu^{(i)}\equiv d\nu_1^{(i)}d\nu_2^{(i)}$, $\nu^{(4)}=\tau$, and

$$\mathcal{D}=\alpha' s\Delta_s+\alpha' t\Delta_t+\alpha' u\Delta_u, \tag{1.10}$$

with

$$\Delta_s=\ln(\chi_{12}\chi_{34}),\quad \Delta_t=\ln(\chi_{14}\chi_{23}),\quad \Delta_u=\ln(\chi_{13}\chi_{24}) \tag{1.11}$$

and $\ln\chi_{ij}(\nu^{(1)}-\nu^{(j)};\tau)$ is the scalar Green function between the vertices labeled i and j on the toroidal world sheet. These Green functions are integrated over the domain $\mathcal{T}$ defined by

$$\mathcal{T}=\left\{-\frac{1}{2}\leqslant\nu_1<\frac{1}{2},0\leqslant\nu_2<\tau_2\right\}. \tag{1.12}$$

It is understood that the mass shell condition

$$s+t+u=0 \tag{1.13}$$

is enforced in all expressions which ensures that only conformally invariant ratios of χ_{ij}'s arise in Eq. (1.9). For example, substituting $u=-s-t$ the exponent of Eq. (1.9) contains

$$\frac{\chi_{12}\chi_{34}}{\chi_{13}\chi_{24}},\quad \frac{\chi_{14}\chi_{23}}{\chi_{13}\chi_{24}}. \tag{1.14}$$

This also ensures that the integrand is modular invariant. Many of the following formulas will be expressed in a symmetric form in terms of s, t, and u even though these variables are related by the condition (1.13). The relative normalization between the two terms in Eqs. (1.1) and (1.7) can be determined by unitarity as in Refs. [8], [4].

The tree-level string amplitude (1.1) is sufficiently simple that it is easily expanded to all orders in powers of the momentum. Successive terms in this expansion lead to terms in the effective action that are polynomials in derivatives acting on $\mathcal{R}^4$. The expansion of T begins with the terms

$$T=\frac{64}{\alpha'^3 stu}+2\zeta(3)+\cdots. \tag{1.15}$$

Substitution of the first term in Eq. (1.1) reproduces the tree diagrams of classical ten-dimensional $\mathcal{N}=2$ supergravity which have poles in the s, t, and u channels. The second term gives the leading correction to the supersymmetric Einstein-Hilbert theory and determines a term in the effective action proportional to $\mathcal{R}^4$. Subsequent terms give information on higher derivative interactions. The complete tree-level expansion will be reviewed in Sec. II.

The one-loop string amplitude (1.7) also has a remarkably simple form—the overall kinematic factor multiplies an integral over the moduli space of the toroidal world sheet that is constructed entirely from the scalar world-sheet propagator. The leading contribution is proportional to $\mathcal{R}^4$ but the nonleading terms in the momentum expansion have not been calculated up to now. From general principles we can anticipate that the momentum expansion of the one-loop amplitude has the structure

$$I(s,t)=a+\frac{\alpha'}{4}I_{\text{nonan1}}(s,t,u)+b\,\frac{\alpha'^2}{16}(s^2+t^2+u^2)$$
$$+c\,\frac{\alpha'^3}{64}(s^3+t^3+u^3)+d\,\frac{\alpha'^4}{256}(s^4+t^4+u^4)$$
$$+\frac{\alpha'^4}{256}I_{\text{nonan2}}(s,t,u)+\cdots$$
$$=I_{\text{an}}(s,t,u)+I_{\text{nonan}}(s,t,u), \tag{1.16}$$

where a, b, c, $d,\ldots,$ are constant coefficients. Up to this order the polynomials in the Mandelstam invariants are the unique expressions that are s, t, u symmetric. These make up the analytic part of the amplitude $I_{\text{an}}(s,t,u)$, whereas the nonanalytic threshold terms

$$I_{\text{nonan}}(s,t,u)=\frac{\alpha'}{4}I_{\text{nonan1}}(s,t,u)+\frac{\alpha'^4}{256}I_{\text{nonan2}}(s,t,u)$$
$$+o(\alpha'^4), \tag{1.17}$$

have logarithmic singularities. The presence of such singularities follows very simply as a consequence of perturbative unitarity due to the phase space available for massless two-particle intermediate states. For energies such that $s<4\alpha'^{-1}$ (i.e., below the first massive string threshold) the amplitude $A_4^{\text{one-loop}}(s,t)$ satisfies the unitarity relation

$$\text{Disc}\,A_4^{\text{one-loop}}(s,t)=\frac{1}{(2\pi)^2}\int d^{10}p_1d^{10}p_2A_4^{\text{tree}}(k_1,k_2,$$
$$-p_1,p_2)[A_4^{\text{tree}}(k_3,k_4,p_1,-p_2)]^\dagger$$
$$\times\delta^{(10)}(p_1+p_2-k_1-k_2)\,\theta(p_1^0)$$
$$\times\delta^{(10)}(p_1^2)\,\theta(p_2^0)\,\delta^{(10)}(p_2^2). \tag{1.18}$$

Substituting the lowest-order (Einstein-Hilbert) tree-level term from Eq. (1.15) into both factors of A_4^{tree} on the right-hand side of Eq. (1.18) leads immediately to the I_{nonan1} term in Eq. (1.16). Substitution of the term with coefficient $\zeta(3)$ from Eq. (1.15) into one of the factors of A_4^{tree} and the Einstein-Hilbert term into the other leads to the I_{nonan2} term in Eq. (1.16), which has three extra powers of α'. These terms will be discussed in more detail in Secs. III and IV (see also Refs. [9–11]).

The main purpose of this paper is to evaluate a number of terms in the expansion (1.16). This exercise involves integrating modular invariant combinations products of the scalar field theory propagators $\ln\chi ij$ over the toroidal world sheet as well as integration over the moduli space of the torus. Although the integration of combinations of *derivatives* of world-sheet scalar propagators has arisen in the literature, for example, in connection with the elegant calculation of the elliptic genus [12], in order to perform the integrals that arise in this paper we will need to use some tricks that that will be presented in Sec. IV. This will allow us to determine all the terms in Eq. (1.16) up to order $\alpha'^4 I_{\text{nonan2}}$ (although the value of the coefficient d will be left as a quadruple sum). The values of these coefficients are compared in Ref. [2] with the values that emerge by considering two-loop eleven-dimensional supergravity compactified on a two-torus.

II. OVERVIEW OF THE TREE AMPLITUDE

The tree amplitude for the scattering of four gravitons of momenta k_1^μ, k_2^μ, k_3^μ, and k_4^μ in either of the type-II superstring theories is given by Eqs. (1.1) and (1.2) where T can be written as [5]

$$T=\frac{64}{\alpha'^3 stu}\exp\left(\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}\left(\frac{\alpha'}{4}\right)^{2n+1}\times(s^{2n+1}+t^{2n+1}+u^{2n+1})\right), \tag{2.1}$$

where we have used the elementary identity $\ln\Gamma(1-z)=\gamma z+\Sigma_{n>1}\zeta(n)z^n/n$. It is convenient to introduce the notation $\sigma_k=(\alpha'/4)^k(s^k+t^k+u^k)$ $(\sigma_1=0)$, which satisfies the recursion relation

$$\sigma_{3+j}=\frac{1}{2}\sigma_2\sigma_{j+1}+\frac{1}{3}\sigma_3\sigma_j,\quad \forall_j>0. \tag{2.2}$$

The solution of these conditions can be expressed by the generating function

$$\begin{aligned}\sum_{j=1}^{\infty}x^j\sigma_j&=\frac{x^2\sigma_2+x^3\sigma_3}{1-\frac{1}{2}\sigma_2x^2-\frac{1}{3}\sigma_3x^3}\\&=(x^2\sigma_2+x^3\sigma_3)\sum_{k\geqslant 0}x^k\\&\quad\times\left[\sum_{2p+3q=k}\frac{(p+q)!}{p!q!}\left(\frac{\sigma_2}{2}\right)^p\left(\frac{\sigma_3}{3}\right)^q\right].\end{aligned} \tag{2.3}$$

Therefore

$$\sigma_k=k\sum_{2p+3q=k}\frac{(p+q-1)!}{p!q!}\left(\frac{\sigma_2}{2}\right)^p\left(\frac{\sigma_3}{3}\right)^q. \tag{2.4}$$

Since $\alpha'^3 stu/64=\sigma_3/3$ and every σ_{2n+1} is divisible by σ_3, the expansion of the exponential in Eq. (1.1) can be expressed entirely in terms of polynomials of σ_2 and σ_3,

$$\begin{aligned}T=&\frac{3}{\sigma_3}+2\zeta(3)+\zeta(5)\sigma_2+\frac{2}{3}\zeta(3)^2\sigma_3+\frac{1}{2}\zeta(7)(\sigma_2)^2\\&+\frac{2}{3}\zeta(3)\zeta(5)\sigma_2\sigma_3+\cdots.\end{aligned} \tag{2.5}$$

It will be significant for the later discussion of unitarity that the series of powers of s, t, and u has gaps of three powers of the Mandelstam invariants between the first two terms and two powers between the second and third terms. Each term translates into a term in the effective action of the type-IIB string theory which is the linearized version of a number of covariant derivatives acting on $\mathcal{R}^4$. These higher derivative terms are part of the full duality-invariant effective action for the type-IIB string.

III. EXPANSION OF THE ONE-LOOP AMPLITUDE

In this section and the next we will consider the low energy expansion of the one-loop integral (1.7) in powers of s, t and u. Formally, this involves expanding the integrand $F(\tau,\overline{\tau})$ (1.9) in powers of the scalar Feynman propagator which are then integrated over the toroidal world sheet

$$I=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}F(\tau,\overline{\tau})=\sum_{n=0}^{\infty}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int_{T}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}\frac{1}{n!}\mathcal{D}^n, \tag{3.1}$$

where the exponent is given by

$$\mathcal{D}=\alpha' s\ln(\chi_{12}\chi_{34})+\alpha' t\ln(\chi_{14}\chi_{23})+\alpha' u\ln(\chi_{13}\chi_{24}). \tag{3.2}$$

This expansion is only formal since we already know that the amplitude is not analytic at $s=0$, $t=0$, or $u=0$. This lack of analyticity is manifested by divergent coefficients in the series (3.1). One way of dealing with this problem would be to consider the expansion in a power series around a nonzero value of s, t and $u\sim\epsilon$. The terms that are singular in the $\epsilon\to 0$ limit can then be resummed to give the logarithmic singularities.

A more straightforward procedure is to evaluate the coefficients of the derivatives of I in the small s, t, and u limit. We will therefore consider the general term

$$\begin{aligned}I_{an}^{(m,n)}&=\lim_{s,t\to 0}(I^{(m,n)}-I_{\text{nonan}}^{(m,n)})\\&\equiv\lim_{s,t\to 0}(4\alpha'^{-1})^{m+n}\partial_s^m\partial_t^n(I-I_{\text{nonan}})\\&=\lim_{s,t\to 0}\Bigg(\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}(4\Delta_s-4\Delta_u)^m\\&\quad\times(4\Delta_t-4\Delta_u)^n\exp(\alpha' s\Delta_s\\&\quad+\alpha' t\Delta_t+\alpha' u\Delta_u)-I_{\text{nonan}}^{(m,n)}\Bigg),\end{aligned}\tag{3.3}$$

where Δ_s, Δ_t, and Δ_u are defined in Eq. (1.11) and $I_{\text{nonan}}^{(m,n)}=(4\alpha'^{-1})^{m+n}\partial_s^m\partial_t^n I_{\text{nonan}}$.

Since I_{nonan} has logarithmic branch points the $I_{\text{nonan}}^{(m,n)}$ terms are singular functions of s and t which must be extracted from the complete expression (3.3) before the analytic terms can be determined. Since the nonanalytic terms originate, via unitarity, from the logarithmic normal thresholds due to on-shell intermediate states we can anticipate that they arise from the region of moduli space in which $\tau_2\to\infty$, which is the degeneration limit of the torus. Our strategy in calculating $I^{(m,n)}$ will therefore be to introduce a cutoff L at a finite but large value of τ_2. The region $\tau_2\leqslant L$ gives a finite contribution to $I^{(m,n)}$ which includes $I_{an}^{(m,n)}$ together with a L-dependent term. In this region the exponential factor in the integrand on the right-hand side of Eq. (3.3) can be replaced by unity. However, for $\tau_2\geqslant L$ the exponential factor plays a crucial role in regulating the integral, resulting in the terms in $I_{\text{nonan}}^{(m,n)}$ together with another finite L-dependent piece. Dependence on L cancels out in the full expression. These nonanalytic terms will be considered in detail in Secs. II C, IV C, and the Appendix.

Differentiating the analytic terms in Eq. (1.16) an appropriate number of times with respect to s and t we see that the coefficients that will be extracted from Eq. (3.3) have the form (up to fourth order)

$$\begin{aligned}&I_{\text{an}}^{(0,0)}=a,\quad I_{\text{an}}^{(1,0)}=0,\quad I_{\text{an}}^{(2,0)}=4b=2I_{an}^{(1,1)},\\&I_{\text{an}}^{(2,1)}=-6c,\quad I_{\text{an}}^{(3,0)}=0,\quad I_{\text{an}}^{(3,1)}=24d=I_{\text{an}}^{(2,2)},\quad I_{\text{an}}^{(4,0)}=48d,\end{aligned}\tag{3.4}$$

together with the terms obtained by interchanging s with t. The numerical values of the coefficients a, b and c will be determined in Sec. IV, although d will be left in the form of a multiple sum that will not be evaluated.

A. The scalar propagator on a torus

The exponent $\mathcal{D}=\alpha'(s\Delta_s+t\Delta_t+u\Delta_u)$, in the expression (1.9) is a linear combination of scalar world-sheet propagators joining the locations of the four vertex operators. The scalar propagator between two complex points $\nu^{(i)}=\nu_1^{(i)}+i\nu_2^{(i)}$ and $\nu^{(j)}=\nu_1^{(j)}+i\nu_2^{(j)}$ on a torus of modulus τ is the doubly periodic function of $\nu^{(ij)}=\nu^{(i)}-\nu^{(j)}$ in the domain τ that has a logarithmic short distance singularity. Thus, the propagator

$$\mathcal{P}(\nu^{(ij)}|\tau)=\ln\chi_{ij}(\nu^{(ij)},\tau),\tag{3.5}$$

satisfying toroidal boundary conditions can be written as a sum over image propagators as

$$\begin{aligned}\mathcal{P}(\nu|\tau)=-\frac{1}{2}\Bigg(&\sum_{n,m\in\mathbf{Z}}\ln|\nu+m+n\tau|\\&-\sum_{(m,n)\neq(0,0)}\ln|m+n\tau|\Bigg)+\frac{\pi\nu_2^2}{2\tau_2},\end{aligned}\tag{3.6}$$

where the last term is the zero mode of the Laplacian. The propagator can also be expressed as

$$\begin{aligned}\mathcal{P}(\nu|\tau)&=-\frac{1}{4}\ln\left|\frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)}\right|^2+\frac{\pi\nu_2^2}{2\tau_2}\\&=\frac{\pi\nu_2^2}{2\tau_2}-\frac{1}{4}\ln\left|\frac{\sin(\pi\nu)}{\pi}\right|^2\\&\quad-\sum_{m\geqslant 1}\left(\frac{q^m}{1-q^m}\frac{\sin^2(m\pi\nu)}{m}+\text{c.c.}\right),\end{aligned}\tag{3.7}$$

where $q=\exp(2i\pi\tau)$ and $\theta_1(\nu|\tau)$ is a standard Jacobi theta function.

Another representation of the propagator that we will use is obtained by Fourier transforming with respect to ν, which leads to an expression in terms of the sum over the discretized momentum $m\tau+n$,

$$\begin{aligned}\mathcal{P}(\nu|\tau)&=\frac{1}{4\pi}\sum_{(m,n)\neq(0,0)}\frac{\tau_2}{|m\tau+n|^2}\\&\quad\times\exp\left[2\pi im\left(\nu_1-\tau_1\frac{\nu_2}{\tau_2}\right)-2\pi in\frac{\nu_2}{\tau_2}\right]+C(\tau,\bar\tau)\\&=\frac{1}{4\pi}\sum_{(m,n)\neq(0,0)}\frac{\tau_2}{|m\tau+n|^2}\\&\quad\times\exp\left[\frac{\pi}{\tau_2}[\overline{\nu}(m\tau+n)-\nu(m\bar\tau+n)]\right]+C(\tau,\bar\tau).\end{aligned}\tag{3.8}$$

The zero mode is given by

$$C(\tau,\bar\tau)=\frac{1}{2}\ln|(2\pi)^{1/2}\eta(\tau)|^2,\tag{3.9}$$

where $\eta(\tau)$ is the standard Dedekind function.

The combination of propagators that enters the amplitude is one for which the zero mode C cancels out. This is a crucial point in considering the modular invariance of the integrand. The group Sl(2,**Z**) is generated by the two elements T: $\tau\to\tau+1$, $\nu\to\nu$ and S: $\tau\to-1/\tau$, $\nu\to\nu/\tau$. Under these transformations the propagator transforms as

$$T:\ \ \mathcal{P}(\nu|\tau+1)=\mathcal{P}(\nu|\tau),$$

$$S:\ \ \mathcal{P}\left(\frac{\nu}{\tau}\bigg|-\frac{1}{\tau}\right)=\mathcal{P}(\nu|\tau)+\frac{1}{2}\ln|\tau|, \tag{3.10}$$

so the propagator has a modular anomaly which comes from the zero mode C in Eq. (3.8). However, the sum over propagators in the exponent $\mathcal{D}$ is modular invariant since the zero modes cancel after using the on-shell condition $s+t+u=0$. Therefore, it is very convenient to use the subtracted propagator

$$\hat{\mathcal{P}}=\ln\hat{\chi}_{ij}(\nu^{(ij)},\tau)=\mathcal{P}-C, \tag{3.11}$$

which is modular invariant. The expression (3.8) can be written as a Poincaré series:[3]

$$\hat{\mathcal{P}}(\nu|\tau)=\sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi[\gamma(\nu),\gamma(\tau)],$$

$$\text{with}\ \ \psi(\nu,\tau)=\frac{\tau_2}{2\pi}e^{-2i\pi p\hat{\nu}_2}, \tag{3.12}$$

where $\hat{\nu}_2=\nu_2/\tau_2$ and the Sl(2,**Z**) transformation acts on ν and τ by

$$\tau\to\frac{a\tau+b}{c\tau+d},\ \ \nu\to\frac{\nu}{c\tau+d}, \tag{3.13}$$

where a, b, c, and d are integers and $ad-bc=1$.

We will also need to express the propagator as a Fourier series in powers of $e^{2i\pi\tau_1}$, which has the form

$$\hat{\mathcal{P}}(\nu|\tau)=\frac{\tau_2}{4\pi}\sum_{n\neq0}\frac{1}{n^2}e^{2i\pi n\hat{\nu}_2}+\frac{1}{4}\sum_{\substack{m\neq0\\ k\in\mathbf{Z}}}\frac{1}{|m|}e^{2i\pi m(k\tau_1+\nu_1)}e^{-2\pi\tau_2|m||k-\hat{\nu}_2|}. \tag{3.14}$$

In analyzing the singular terms in the amplitude it will be important to make use of the leading contribution to this expression for the propagator at large values of τ_2,

$$\hat{\mathcal{P}}(\nu|\tau)=\frac{\tau_2}{4\pi}\sum_{n\neq0}\frac{1}{n^2}e^{2i\pi n\bar{\nu}_2}=\frac{\pi\tau_2}{2}\left(\hat{\nu}_2^2-|\hat{\nu}_2|+\frac{1}{6}\right). \tag{3.15}$$

B. The diagrammatic rules

The calculation of $I^{(m,n)}$ in Eq. (3.3) involves integration of powers of the propagators, $\hat{\mathcal{P}}(\nu^{(ij)}|\tau)$, contracted between various combinations of the points i,j ($i,j\in\{1,2,3,4\}$) which are the locations of the vertex operators. It is easy to deduce a set of diagrammatic rules at any given order. A term of order Δ^n (where each power of Δ may be any of the three Δ_r's with $r=s,t,u$) contains a product of n propagators which join pairs of points (which we will call ''vertices'') $i,j(=1,\dots,4)$ with positions $\nu^{(i)}$ that are to be integrated over the torus. We will represent each vertex of a diagram by a dot and each propagator linking two vertices by a line. The complete nth order contribution requires a sum over all ways in which the propagators can join the vertices. For any term in this sum every vertex that is not connected to any propagator contributes a factor of $\int_{\mathcal{T}}d^2\nu_i/\tau_2=1$.

More generally, we need to isolate divergent contributions by dividing the τ integration domain into two regions

$$\mathcal{F}=\mathcal{F}_L+\mathcal{R}_L. \tag{3.16}$$

The domain $\mathcal{F}_L$ defines the ''restricted'' fundamental domain of the τ plane in which $\tau_2\leqslant L$, whereas the domain $\mathcal{R}_L$ defines a semi-infinite rectangle in the τ plane, in which $\tau_2\geqslant L$. As stated earlier, the terms $I_{\rm nonan}^{(m,n)}$ that have threshold singularities at vanishing Mandelstam invariants arise from the domain $\mathcal{R}_L$ and will be dealt with separately by integrating over this large-τ_2 region.

For the finite contributions that come from the domain $\mathcal{F}_L$ the integrations over the positions, $\nu^{(i)}$ enforce overall conservation of the discrete momentum $p=m\tau+n$ in any diagram. This means, for example, that any propagator with a free end point gives a vanishing contribution since it has been normalized to have a vanishing zero mode. Therefore, nonzero contributions only come from diagrams in which two or more propagators end on every vertex. Various combinatorial factors are associated with each diagram and will be described for each case separately.

C. The threshold term $I_{\rm nonan}$

The lack of analyticity of the low energy expansion of the one-loop amplitude (1.7) due to the logarithmic thresholds makes the integral representation (1.7) ill defined. Since there is no region of the Mandelstam invariants in which the amplitude is real the only way of making sense of the integral is to decompose the integration domain into three domains $\mathcal{T}_{st}$, $\mathcal{T}_{tu}$, and $\mathcal{T}_{us}$, so that the amplitude is separated into real analytic terms that have thresholds in the (s,t), (t,u), and (u,s) channels, respectively. The integral representation for each of these terms can then be defined in the region of physical scattering $s>0$; $t,u<0$, by analytic continuation. For example, the (s,t) term is defined by continuation from the region $s,t<0$ (with $u=-s-t>0$) where it is real. This decomposition follows very naturally in an operator construction of the loop amplitude but does not manifestly preserve modular invariance [9].

The leading logarithmic singularity in I is the leading term in $\lim_{s,t\to0}\int(e^{\mathcal{D}}-1)$. This can be extracted by first differentiating the integral representation with respect to s:

$$\partial_s I_{\rm nonan1}=\lim_{s,t\to0}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}4\partial_s\mathcal{D}e^{\mathcal{D}}$$

$$=\lim_{s,t\to0}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}(4\Delta_s-4\Delta_u)e^{\mathcal{D}}. \tag{3.17}$$

[3] Recall that the Poincaré series associated with a function ψ defined over $\mathcal{F}$ is $T\psi(\tau)=\Sigma_{\gamma\in\Gamma_\infty\backslash\Gamma}\psi(\mathrm{Im}\,\gamma z)$ for $\tau=\tau_1+i\tau_2\in\mathcal{H}=\{\tau_2=\mathrm{Im}\,\tau>0\}$ and $\Gamma_\infty=\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix},n\in\mathbf{Z}\}$.

The contribution from the domain $\mathcal{F}_L$ vanishes due to the integration over the $\nu^{(ij)}$. However, the region $\mathcal{R}_L$ leads to a nonzero result. In this region we can approximate $\mathcal{D}$ by using the asymptotic expression for the propagator (3.15) which is proportional to τ_2. In the term with thresholds in the $(s,\,t)$ channels the variables $\nu_2^{(i)}$ are ordered in such a manner that the rescaled variables

$$\omega_i=\frac{\nu_2^{(i)}}{\tau_2}, \tag{3.18}$$

span the range

$$\mathcal{T}_{st}:\quad 0\leqslant\omega_1\leqslant\omega_2\leqslant\omega_3\leqslant\omega_4=1 \tag{3.19}$$

(where we have used the conformal symmetry to fix $\nu^{(4)}=\tau$). The various permutations of this ordering are relevant in the (t,u) and (u,s) regions so that the whole range, $0\leqslant\omega_i\leqslant 1$ is covered by adding the three regions $\mathcal{T}_{st}$, $\mathcal{T}_{tu}$, and $\mathcal{T}_{us}$, together. In terms of these variables we have, in the region $\mathcal{T}_{st}$,

$$\begin{aligned}\mathcal{D}=\mathcal{D}(s,t)&=\lim_{r_2\to\infty}\alpha' s(\Delta_s-\Delta_u)+\alpha' t(\Delta_t-\Delta_u)\\&=\pi\tau_2\alpha'[s\omega_1(\omega_3-\omega_2)+t(\omega_2-\omega_1)(1-\omega_3)]\\&=\pi\tau_2\alpha'\mathcal{Q}(s,t),\end{aligned} \tag{3.20}$$

where

$$\mathcal{Q}(s,t)=s\omega_1(\omega_3-\omega_2)+t(\omega_2-\omega_1)(1-\omega_3). \tag{3.21}$$

Similar expressions for the functions $\mathcal{D}(t,u)=t(\Delta_t-\Delta_s)+u(\Delta_u-\Delta_s)$ and $\mathcal{D}(u,s)=u(\Delta_u-\Delta_t)+s(\Delta_s-\Delta_t)$, define $\mathcal{D}$ when expressed in the $\mathcal{T}_{tu}$ and $\mathcal{T}_{us}$ regions. In the $\mathcal{R}_L$ domain of Eq. (3.17) the τ_1 integration is trivial since the integrand has no τ_1 dependence. The τ_2 integration (from L to ∞) simply gives

$$\begin{aligned}\partial_s I_{\mathrm{nonan1}}&=4\pi\int_L^\infty\frac{d\tau_2}{\tau_2}\prod_{0\leqslant\omega_1\leqslant\omega_2\leqslant\omega_3\leqslant 1}d\omega_i\,\omega_1(\omega_3-\omega_2)\\&\quad\times e^{\alpha'\pi\tau_2\mathcal{Q}(s,t)}\\&=4\pi\int\prod_{0\leqslant\omega_1\leqslant\omega_2\leqslant\omega_3\leqslant 1}d\omega_i\,\omega_1(\omega_3-\omega_2)\\&\quad\times\{-\gamma-\ln[-\alpha'\pi\mathcal{Q}(s,t)]-\ln L\}+o(s).\end{aligned} \tag{3.22}$$

The $\ln L$ terms cancel out in the complete contribution to $I^{(1,0)}=\partial_s I-\partial_u I$. It it is easy to integrate Eq. (3.22) together with the corresponding expression for $\partial_t I_{\mathrm{nonan1}}$, giving

$$\begin{aligned}\frac{1}{4\pi}I_{\mathrm{nonan1}}&=\int\prod_{0\leqslant\omega_1\leqslant\omega_2\leqslant\omega_3\leqslant 1}d\omega_i\,\mathcal{Q}(s,t)\ln\mathcal{Q}(s,t)\\&\quad+\int\prod_{0\leqslant\omega_3\leqslant\omega_2\leqslant\omega_1\leqslant 1}d\omega_i\,\mathcal{Q}(t,u)\ln\mathcal{Q}(t,u)\\&\quad+\int\prod_{0\leqslant\omega_2\leqslant\omega_1\leqslant\omega_3\leqslant 1}d\omega_i\,\mathcal{Q}(u,s)\ln\mathcal{Q}(u,s).\end{aligned} \tag{3.23}$$

The scale of the logarithm cancels out of the sum of terms in the full expression. This threshold term is exactly the same as that obtained from the one-loop calculation of the four-graviton amplitude in either of the type-II supergravity theories in ten dimensions [11,10]. The corresponding discussion of the higher-order threshold term I_{nonan2} which is intrinsically stringy since it involves higher powers of α', will be given in the Appendix.

Such threshold terms are contained in the large-τ_2 region of the integration over moduli space, which means that they are contained in the coefficients $I^{(m,n)}_{\mathcal{R}_L}$ that are defined by integration over the domain $\mathcal{R}_L$. So long as $1<m+n<4$ it will be sufficient to substitute the asymptotic form of the propagator which will produce contributions of the form

$$\begin{aligned}\lim_{s,t\to 0}I^{(m,n)}_{\mathcal{R}_L}(s,t)&=\lim_{s,t\to 0}\int_{\mathcal{R}_L}\frac{d^2\tau}{\tau_2^2}\int_{\mathcal{T}}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\tau_2}(4\Delta_s-4\Delta_u)^m(4\Delta_t-4\Delta_u)^n e^{\mathcal{D}}\\&=\lim_{s,t\to 0}\int_L^\infty d\tau_2\,\tau_2^{m+n-2}\int_{\mathcal{T}}\prod_{i=1}^{3}d\omega_i(4\pi\partial_s\mathcal{Q})^m(4\pi\partial_t\mathcal{Q})^n e^{\alpha'\pi\tau_2\mathcal{Q}}\\&=\lim_{s,t\to 0}\left(\int_0^\infty d\tau_2-\int_0^L d\tau_2\right)\tau_2^{m+n-2}\int_{\mathcal{T}}\prod_{i=1}^{3}d\omega_i(4\pi\partial_s\mathcal{Q})^m(4\pi\partial_t\mathcal{Q})^n e^{\alpha'\pi\tau_2\mathcal{Q}}\\&=\lim_{s,t\to 0}I^{(m,n)}_{\mathrm{nonan}}(s,t)-2\times(4\pi)^{m+n}\frac{L^{m+n-1}}{m+n-1}\\&\quad\times\partial_s^m\partial_t^n\sum_{p+q=m+n}\frac{p!q!}{(2m+2n+3)!}[s^pt^q+(-1)^q s^p(s+t)^q+(-1)^p(s+t)^p t_q]\bigg|_{(s,t)=(0,0)}\end{aligned} \tag{3.24}$$

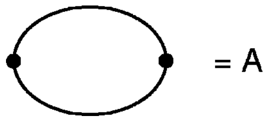

FIG. 1. The diagram that contributes to $I_{\rm an}^{(m,n)}$ with $m+n=2$.

with $p,q\geqslant 0$. The L-dependent term in this expression will cancel with a term that arises from the integration over the domain $\mathcal{F}_L$. When $m+n=4$ it is no longer adequate to use the leading τ_2 contribution to the propagators and a subleading contribution of the form $\ln L$ arises. This leads to the term $I_{\rm nonan2}$, as will be seen in more detail in Sec. IV C and the Appendix.

IV. THE ANALYTIC TERMS $I_{\rm an}^{(m,n)}$

The analytic terms are extracted from the integration over $\mathcal{F}_L$ which is finite. In this domain we can first perform the $\nu^{(i)}$ integrals to obtain a density on the moduli space and then integrate this over τ and $\bar{\tau}$.

The first term in the expansion of Eq. (1.9) using Eq. (3.3) is the trivial constant term. The result of the integrations is simply the finite volume of $\mathcal{F}$. This defines the first constant in Eq. (1.16),

$$I^{(0,0)}=a=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}=\frac{\pi}{3}, \tag{4.1}$$

which is the well-known coefficient of the loop contribution to the $\mathcal{R}^4$ term.

The next terms in the expansion are $I^{(1,0)}$ and $I^{(0,1)}$ which are given by Eq. (3.17). As remarked earlier, the $\nu^{(i)}$ integrations in Eq. (3.17) cause the integrand to vanish in the domain $\mathcal{F}_L$ and the integral only contributes to $I_{\rm nonan}^{(1,0)}$.

A. Terms of order s^2

The only nonvanishing contribution to the integrand of $I_{\rm an}$ at order α'^2 is the bubble diagram of Fig. 1. This term multiplies $(s^2+t^2+u^2)$ in the expansion (3.1) and therefore contributes to $I_{\rm an}^{(2,0)}$, $I_{\rm an}^{(1,1)}$, and $I_{\rm an}^{(0,2)}$. The density on moduli space arising from Fig. 1 is

$$\begin{aligned}A(\tau,\bar{\tau})&=\int_{\mathcal{T}}\frac{d^2\nu^{(i)}d^2\nu^{(j)}}{\tau_2^2}[\ln\hat{\chi}(\nu^{(ij)}|\tau)]^2\\&=\frac{1}{16\pi^2}\sum_{(m,n)\neq(0,0)}\frac{\tau_2^2}{|m\tau+n|^2}\\&=\frac{1}{16\pi^2}Z_2(\tau,\bar{\tau}).\end{aligned} \tag{4.2}$$

The function Z_2 an Epstein zeta function which is an example of a nonholomorphic Eisenstein series.[4] More generally these are nonholomorphic modular functions that are also eigenfunctions of the Laplace operator in the fundamental domain of Sl(2,Z),

$$\nabla^2 Z_s\equiv 4\tau_2^2\frac{\partial^2}{\partial\tau\partial\bar{\tau}}Z_s=s(s-1)Z_s. \tag{4.3}$$

These functions have τ_2 expansions in which there are two power-behaved terms together with an infinite set of exponentially suppressed, nonperturbative, terms

$$\begin{aligned}Z_s&=\sum_{(m,n)\neq(0,0)}\frac{\tau_2^s}{|m\tau+n|^{2s}}\\&=2\zeta(2s)\tau_2^s+2\pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)\tau_2^{1-s}\\&\quad+O(e^{-2\pi\tau_2}).\end{aligned} \tag{4.4}$$

Diagrams of this type with vertices at (1,2), (3,4), (1,3), and (2,4) contribute equally to $I_{\rm an}^{(2,0)}$. Integrating Eq. (4.2) over the restricted fundamental domain $\mathcal{F}_L$ and summing over these four contributions gives

$$I_{\mathcal{F}_L}^{(2,0)}=4\times\frac{1}{\pi^2}\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}Z_2(\tau,\bar{\tau}), \tag{4.5}$$

where the factor of 4 comes from $\partial_s^2[s^2+t^2+(s+t)^2]$. This expression is easily integrated by substituting $Z_2=\nabla^2 Z_2/2$ using Eq. (4.3) so that the integrand is a total derivative and Eq. (4.5) reduces to an integral over the boundary. The restricted fundamental domain has a single boundary which is at $\tau_2=L$ and the result is

$$I_{\mathcal{F}_L}^{(2,0)}=\frac{2^6\pi^2}{6!}L+O(L^{-2}). \tag{4.6}$$

This L-dependent term cancels the corresponding L dependence arising from $I_{\mathcal{R}_L}^{(2,0)}$ which is given by Eq. (3.24). Since there is no residual L-independent piece we conclude that $I_{\rm an}^{(2,0)}=0$, which implies that

$$b=0. \tag{4.7}$$

This means that there is no s^2 term in the expansion (1.16). In a similar manner it is easy to verify that the cross term $I_{\rm an}^{(1,1)}$ also vanishes, which is consistent with Eq. (3.4).

B. Terms of order s^3

The diagrams in Fig. 2 are the ones that survive the $\nu^{(ij)}$ integrations. These contribute to terms in the expansion (3.1) with coefficient $\alpha'^3(s^3+t^3+u^3)$. The first diagram is the

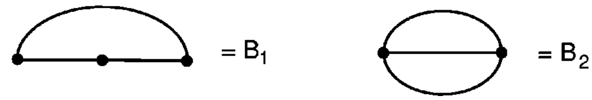

FIG. 2. Diagrams that contribute to $I_{an}^{(m,n)}$ with $m+n=3$.

[4]This is related to the function E_s in Ref. [13] by $Z_s=2\zeta(2s)E_s$.

product of three propagators joining three distinct vertices and gives contributions to the integrand of $I^{(2,1)}_{\mathcal{F}_L}$ (and $I^{(1,2)}_{\mathcal{F}_L}$) of the form $(k>j>i)$

$$B_1(\tau,\overline{\tau})=\frac{1}{(4\pi)^3}\sum_{m,n}\frac{\tau_2^3}{|m\tau+n|^6}=\frac{1}{(4\pi)^3}Z_3(\tau,\overline{\tau}). \tag{4.8}$$

This is again a nonholomorphic Eisenstein series satisfying Eq. (4.3), here with $s=3$, so it is an eigenfunction of the Laplace equation with $s(s-1)=6$. The integration of the density (4.8) over the the restricted fundamental domain can again be performed using Gauss' law. This gives,

$$I^{(2,1)}_{\mathcal{F}_L}(B_1)=-\frac{2^{10}\pi^3}{3\times 8!}L^2+O(L^{-3}), \tag{4.9}$$

where the factor of -6 arises from $\partial_s^2\partial_t[s^3+t^3-(s+t)^3]$.

The second diagram of Fig. 2 involves only two distinct vertices and potentially gives a contribution to the integrand of both $I^{(3,0)}_{\mathcal{F}_L}$ and $I^{(2,1)}_{\mathcal{F}_L}$ (as well as $I^{(0,3)}_{\mathcal{F}_L}$ and $I^{(1,2)}_{\mathcal{F}_L}$) of the form

$$\begin{aligned}B_2(\tau,\overline{\tau})&=\int\frac{d^2\nu^{(i)}d^2\nu^{(j)}}{\tau_2^2}[\hat{\mathcal{P}}(\nu^{(ij)}|\tau)]^3\\&=\frac{1}{(4\pi)^3}\sum_{(m,n),(k,l),(p,q)\neq(0,0)}\\&\quad\times\tau_2^3\frac{\delta(m+k+p)\,\delta(n+l+q)}{|m\tau+n|^2|k\tau+l|^2|p\tau+q|^2}.\end{aligned} \tag{4.10}$$

In fact, $I^{(3,0)}$ involves the combination $\partial_s^3[s^3+t^3-(s+t)^3]=0$, so it automatically vanishes as in Eq. (1.16). However, the integrand of $I^{(2,1)}$ is proportional to B_2.

Unlike the earlier examples, this expression is not an eigenfunction of the Laplacian on the fundamental domain so a new idea is needed in order to perform the integration over $\mathcal{F}_L$. We will make use of the well-known 'unfolding procedure' by using the representation of the propagator by a Poincaré series (3.12). This relates the integral of $\psi\times f$ over $\mathcal{F}$ (where ψ is any Poincaré series) to an integral over the semi-infinite line

$$\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\psi(\tau)f(\tau)=\int_{t=0}^{\infty}\frac{dt}{t^2}\psi(t)(Cf)(t), \tag{4.11}$$

where the expression (Cf) is the zero τ_1 mode of the function $f(\tau)$,

$$Cf(\tau_2)=\int_{-1/2}^{1/2}d\tau_1 f(\tau). \tag{4.12}$$

The relationship (4.11) is derived by making use of the identities

$$\int_{\Gamma\backslash\mathcal{H}}\sum_{\Gamma_\infty\backslash\Gamma}=\int_{\Gamma_\infty\backslash\mathcal{H}}=\int_{\tau_2>0}. \tag{4.13}$$

Since the integration in Eq. (4.10) is over the restricted fundamental domain $\mathcal{F}_L$ some care has to be taken in using the unfolding procedure. For an integral such as Eq. (4.10), which diverges as a power of L as $L\to\infty$, it turns out to be consistent to simply set $f(\tau)=0$ for $\tau_2\geqslant L$, which cuts off the divergence at $\tau_2\to\infty$.[5] Using this procedure we can express the contribution of Eq. (4.10) to $I^{(2,1)}_{\mathcal{F}_L}$ in the form

$$\begin{aligned}I^{(2,1)}_{\mathcal{F}_L}(B_2)&=-6\times\frac{64}{3}\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}B_2(\tau,\overline{\tau})\\&=-128\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2}\int\frac{d^2\nu^{(1)}d^2\nu^{(2)}}{\tau_2^2}[\hat{\mathcal{P}}(\nu^{(12)}|\tau)]^3\\&=-128\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^L\frac{dt}{t^2}\int\frac{d^2\nu^{(1)}d^2\nu^{(2)}}{t^2}\frac{t}{2\pi}\\&\quad\times e^{-2i\pi p\hat{\nu}_2}C(\hat{\mathcal{P}}^2),\end{aligned} \tag{4.14}$$

where the overall factor of -6 comes from $\partial_s^2\partial_t[s^3+t^3-(s+t)^3]$. The expression for the integral over $\nu_1^{(1)}$ and $\nu_1^{(2)}$ of the zero τ_1 Fourier mode is

$$\begin{aligned}\int_{-1/2}^{1/2}d\nu_1^{(1)}d\nu_1^{(2)}C(\mathcal{P}^2)&=\frac{t^2}{16\pi^2}\left(\sum_{n\neq 0}\frac{1}{n^2}e^{2\pi in\hat{\nu}_2^{(12)}}\right)^2\\&\quad+\frac{1}{16}\sum_{\substack{m\neq 0\\k}}\frac{1}{m^2}e^{-4\pi t|m||k-\hat{\nu}_2^{(12)}|}.\end{aligned} \tag{4.15}$$

Substituting the first term on the right-hand side into Eq. (4.14) leads to the L-dependent term

$$\begin{aligned}I^{(2,1)}_{1\mathcal{F}_L}(B_2)&=-128\times\frac{L^2}{2}\int_{-1/2}^{1/2}d\hat{\nu}_2^{(1)}d\hat{\nu}_2^{(2)}\\&\quad\times\left(\frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{n^2}e^{2i\pi n\hat{\nu}_2^{(12)}}\right)^3\\&=-\frac{L^2}{\pi^3}\sum_{n_i\in Z\backslash\{0\}}\frac{\partial(n_1+n_2+n_3)}{n_1^2n_2^2n_3^2}\\&=-\frac{2^8\pi^3}{3\times 8!}L^2.\end{aligned} \tag{4.16}$$

Substitution of the second term on the right-hand side of Eq. (4.15) into Eq. (4.14) gives the L-independent term

---

[5] Although this cutoff leads to the correct answer when the integral grows as a power of L, more care is needed in regularizing logarithmic growth of the kind we will meet in Sec IV C. In that case the integral diverges at the points on the $\tau_2=0$ axis that are the images under SL(2,**Z**) transformations of the point $\tau_2\to\infty$.

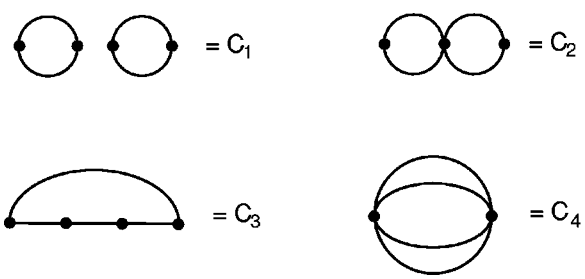

FIG. 3. The set of diagrams that contribute to $I^{(m,n)}$ with $m+n=4$.

$$I^{(2,1)}_{2\mathcal{F}_L}(B_2)=-\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^\infty\frac{dt}{t}\int_0^1 d\hat{\nu}_2^{(1)}d\hat{\nu}_2^{(2)}\sum_{\substack{m\neq 0\\ k}}\frac{1}{m^2}\times e^{-2i\pi p\hat{\nu}_2}e^{-4\pi t|m||k-\hat{\nu}_2|}$$
$$=-\frac{4}{\pi^2}\sum_{p=1}^{\infty}\sum_{m\neq 0}\frac{1}{p^2}\frac{1}{m^2}\int_0^\infty\frac{dt}{p^2+t^2}$$
$$=-6\times\frac{2}{3\pi}\zeta(2)\zeta(3). \tag{4.17}$$

The sum of the L-dependent terms $I^{(2,1)}_{1\mathcal{F}_L}(B_2)+I^{(2,1)}_{\mathcal{F}_L}(B_1)$ [see Eqs. (4.16) and (4.9)], again cancels with the corresponding term in the integration over the domain $\mathcal{R}_L$ [$I^{(2,1)}_{\mathcal{R}_L}$, in Eq. (3.24)]. However, in this case there is also a finite contribution to $I^{(2,1)}_{2\mathcal{F}_L}$, which determines the coefficient of the $\alpha'^3(s^3+t^3+u^3)$ term in Eq. (1.16) to be

$$c=\frac{2}{3\pi}\zeta(2)\zeta(3). \tag{4.18}$$

C. Terms of order s^4

The four kinds of diagrams that give nonzero contributions proportional to $\alpha'^4(s^4+t^4+u^4)$ in the expansion (3.1) are shown in Fig. 3. These each have four propagators and contribute to $I^{(m,n)}_{an}$ with $m+n=4$. Upon evaluating the $\nu^{(r)}$ integrations these give the following densities for the moduli space integrals

$$C_1(\tau,\overline{\tau})=\frac{1}{(4\pi)^4}(Z_2)^2, \tag{4.19}$$

$$C_2(\tau,\overline{\tau})=\frac{1}{(4\pi)^4}(Z_2)^2, \tag{4.20}$$

$$C_3(\tau,\overline{\tau})=\frac{1}{(4\pi)^4}Z_4, \tag{4.21}$$

$$C_4(\tau,\overline{\tau})=\int\frac{d^2\nu^{(i)}d^2\nu^{(j)}}{\tau_2^2}(\hat{\mathcal{P}}(\nu^{(ij)}|\tau))^4 \quad (j>i)$$
$$=\frac{1}{(4\pi)^4}\sum_{\substack{(m,n)\neq(0,0)\\(p,q)\neq(0,0)}}\sum_{\substack{(r,s)\neq(0,0)\\(v,w)\neq(0,0)}}\times\tau_2^4\frac{\delta(m+p+r+v)\,\delta(n+q+s+w)}{|m\tau+n|^2|p\tau+q|^2|r\tau+s|^2|v\tau+w|^2}. \tag{4.22}$$

The term $C_3(\tau,\overline{\tau})$ is once again a non-holomorphic Eisenstein series which can be integrated over the restricted fundamental domain using

$$\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}Z_4=\frac{2}{3}\zeta(8)L^3+O(L^{-4}). \tag{4.23}$$

Inserting the appropriate combinatoric factors gives rise to the L-dependent contributions

$$I^{(3,1)}_{\rm an}(C_3)=0,\quad I^{(2,2)}_{\rm an}(C_3)=\frac{2^{10}\times 3\pi^4}{10!}L^3,$$
$$I^{(4,0)}_{\rm an}(C_3)=\frac{2^{10}\times 3\pi^4}{10!}L^3. \tag{4.24}$$

The integration of the expressions C_2 and C_3 over the restricted fundamental domain involves the integration of the square of an Eisenstein series $(Z_2)^2$. This can be evaluated by using Green's theorem in the fundamental domain. For general real values of $s,s'>1/2$ this states that

$$\frac{1}{4\zeta(2s)\zeta(2s')}\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}Z_sZ_{s'}$$
$$=\frac{L^{s+s'-1}}{s+s'-1}-\frac{L^{1-s-s'}}{s+s'-1}\phi(s)\phi(s')+\frac{L^{s-s'}}{s-s'}\phi(s')-\frac{L^{s'-s}}{s-s'}\phi(s)+o(1), \tag{4.25}$$

where

$$\phi(s)=\frac{\zeta(2s-1)\Gamma(s-1/2)}{\pi^{s-1/2}}\frac{\pi^s}{\zeta(2s)\Gamma(s)}. \tag{4.26}$$

The symbol $o(1)$ means that the remainder goes to zero when L becomes infinite (for a more general statement see exercise 12, p. 216 of Vol. I of Ref. [13]). It follows from this that

$$\frac{1}{4\zeta(4)^2}\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}(Z_2)^2=\frac{1}{3}L^3+\pi\frac{\zeta(3)}{\zeta(4)}\ln L-\phi'(2). \tag{4.27}$$

The last term gives a L-independent contribution to the coefficient d in Eq. (1.16). The other terms in Eq. (4.27) give another contribution that behaves as L^3 as well as a new

L-dependent term proportional to $\ln L$. Such a term is implied by the presence of a new logarithmic threshold of order $\alpha'^4 s^4 \ln s$ which is contained in $L_{\rm nonan2}$ that is evaluated in the Appendix. Taking into account the combinatorial factors, the contribution of these L-dependent terms is

$$I_{\rm an}^{(3,1)}(C_1)=I_{\rm an}^{(3,1)}(C_2)=0,$$

$$I_{\rm an}^{(2,2)}(C_2)=2I_{\rm an}^{(2,2)}(C_1)=\frac{2^9\times 7\pi^4}{10!}L^3+\frac{48}{\pi^3}\zeta(3)\zeta(4)\ln L, \tag{4.28}$$

$$I_{\rm an}^{(4,0)}(C_2)=2I_{\rm an}^{(4,0)}(C_1)=\frac{2^{10}\times 7\pi^4}{10!}L^3+\frac{96}{\pi^3}\zeta(3)\zeta(4)\ln L.$$

The last remaining term to consider is C_4. As with the two-loop term (4.10) this gives an expression which is not an eigenfunction of the Laplacian on the fundamental domain. Once again the $d^2\tau$ integration may be performed by using the unfolding procedure as in the previous subsection. However, in this case we have to take greater care of the divergence of the integrand at $\tau_2=L\to\infty$ (as pointed out in the footnote in Sec. IV B). The integral can be rendered finite by subtracting a suitable linear combination of Z_4 and Z_2^2 from the integrand. Consider, for example,

$$I_{\mathcal{F}_L}^{(4,0)}(C_4)=48\times\frac{64}{3}\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}\int\frac{d^2\nu^{(1)}d^2\nu^{(2)}}{\tau_2^2}(\hat{\mathcal{P}}(\nu^{(12)}|\tau))^4, \tag{4.29}$$

where the factor of 48 comes from $\partial_s^4[s^4+t^4+(s+t)^4]$. It is easy to extract the terms in the integrand that are divergent in the limit $L\to\infty$ from explicit from of $\hat{\mathcal{P}}(\nu^{(12)}|\tau)$ given in Eqs. (3.14) and (3.15). This gives explicit L^3 and ln L terms that can be subtracted in a modular invariant manner by defining a regularized value of $I^{(4,0)}(C_4)$,

$$I_{\rm reg}^{(4,0)}(C_4)=I_{\mathcal{F}_L}^{(4,0)}(C_4)-I_{\rm div}^{(4,0)}(C_4), \tag{4.30}$$

where

$$I_{\rm div}^{(4,0)}(C_4)=1024\int_{\mathcal{F}_L}\frac{d^2\tau}{\tau_2^2}\int\frac{d^2\nu^{(1)}d^2\nu^{(2)}}{\tau_2^2}\left(-\frac{2}{(4\pi)^4}Z_4+\frac{3}{(4\pi)^4}(Z_2)^2\right), \tag{4.31}$$

and the integrals of Z_4 and Z_2^2 are given in Eqs. (4.23) and (4.27), respectively.

Since the expression (4.30) is finite and its integrand is modular invariant it is straightforward to evaluate using the unfolding procedure. This gives

$$I_{\rm reg}^{(4,0)}(C_4)=1024\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^L\frac{dt}{t^2}\int\frac{d^2\nu^{(1)}d^2\nu^{(2)}}{t^2}\frac{t}{2\pi}\times e^{-2i\pi p\hat{\nu}_2^{(12)}}\Bigg[C[\hat{\mathcal{P}}(\nu^{(12)})^3]+2\int\frac{d^2\nu^{(3)}d^2\nu^{(4)}}{t^2}C[\hat{\mathcal{P}}(\nu^{(23)})\hat{\mathcal{P}}(\nu^{(34)})\hat{\mathcal{P}}(\nu^{(41)})]-\frac{3}{(4\pi)^2}\int\frac{d^2\nu^{(3)}d^2\nu^{(4)}}{t^2}C[\hat{\mathcal{P}}(\nu^{(12)})Z_2]\Bigg]. \tag{4.32}$$

This term can be evaluated by using the explicit definitions of $\hat{\mathcal{P}}$ and Z_2, giving the L-independent result

$$I_{\rm reg}^{(4,0)}(C_4)=-\frac{24}{\pi^3}\zeta(3)\sum_{\substack{p\neq 0,n\neq 0\\ p\neq n}}\frac{\ln|p-n|}{p^2n^2}+\frac{8}{\pi}\sum_{\substack{m_i\neq 0,m_1+m_2+m_3=0\\ k_i\in\mathbf{Z},m_1k_1+m_2k_2+m_3k_3=0}}\frac{1}{|m_1m_2m_3|}\times\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^{\infty}\frac{dt}{t}\int_0^1 d\hat{\nu}_2^{(1)}d\hat{\nu}_2^{(2)}\times\exp\left(-2i\pi p\hat{\nu}_2^{(12)}-2\pi t\sum_{i=1}^{3}|m_i||k_i-\hat{\nu}_2^{(12)}|\right). \tag{4.33}$$

The L-dependent terms are contained in

$$I_{\rm div}^{(4,0)}(C_4)=2I_{\rm an}^{(2,2)}(C_4)=\frac{2^9\times 5\pi^4}{10!}L^3+\frac{48}{\pi^3}\zeta(3)\zeta(4)\ln L-\frac{48}{\pi^4}\zeta(4)^2\phi'(2). \tag{4.34}$$

The L^3 term is connected, as expected, to the presence of $I_{\rm nonan1}^{(4,0)}$ while the $\ln L$ term is again connected to the appearance of $I_{\rm nonan2}^{(4,0)}$.

The sum of the L^3 contributions arising in $I_{\rm an}^{(4,0)}(C_1)$ and $I_{\rm div}^{(4,0)}(C_4)$ indeed cancels the contributions from the integration of $I_{\rm nonan1}^{(4,0)}$ over the $\mathcal{R}_L$ domain in Eq. (3.24). Similarly, the total coefficient of $\ln L$ arising in the sum of $I_{\rm an}^{(2,2)}(C_1)$, $I_{\rm an}^{(2,2)}(C_2)$, and $I_{\rm an}^{(2,2)}(C_4)$ is

$$I_{\rm an\,ln}^{(2,2)}=\frac{96}{\pi^3}\zeta(3)\zeta(4)\ln L, \tag{4.35}$$

which will be cancelled by the presence of the new threshold term $I_{\rm nonan2}^{(2,2)}$. The general expression for $I_{\rm nonan2}$ is fairly complicated but we see from the Appendix that at $t=0$ it reduces to

$$I_{\text{nonan2}}(s,t=0) = \frac{4}{\pi^3}\zeta(3)\zeta(4)(\alpha' s)^4\left[\ln\left(\frac{\alpha' s}{L}\right)+\ln\left(-\frac{\alpha' s}{L}\right)\right]. \tag{4.36}$$

Taking four s derivatives leads to the same coefficient of $\ln L$ as that in Eq. (4.35).

The finite term $I_{\text{reg}}^{(4,0)}(C_4)$ (4.33), together with the L-independent parts of Eq. (4.34) and $I^{(4,0)}(C_2)$ and $I^{(4,0)}(C_3)$ [which come from the finite last term of Eq. (4.27)], determine the value of the coefficient d in the expansion of the loop amplitude in the form

$$\begin{aligned} d = & -\frac{4}{\pi^4}\zeta(4)^2\phi'(2) - \frac{1}{2\pi^3}\zeta(3)\sum\frac{\ln|p-n|}{p^2n^2} \\ & + \frac{1}{6\pi}\sum_{\substack{m_i\neq 0, m_1+m_2+m_3=0 \\ k_i\in\mathbf{Z}, m_1k_1+m_2k_2+m_3k_3=0}}\frac{1}{|m_1m_2m_3|} \\ & \times\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^\infty\frac{dt}{t}\int_0^1 d\hat{\nu}_2^{(1)}d\hat{\nu}_2^{(2)} \\ & \times\exp\left(-2i\pi p\hat{\nu}_2^{(12)} - 2\pi t\sum_{i=1}^{3}|m_i||k_i-\hat{\nu}_2^{(12)}|\right). \end{aligned} \tag{4.37}$$

We have not extracted the numerical value of this complicated looking expression.

V. SUMMARY AND CONCLUSION

In summary, we have determined the first few coefficients in the expansion (1.16) of the four-graviton one-loop amplitude in either of the ten-dimensional type-II string theories. After explicitly subtracting the nonanalytic threshold terms I_{nonan1} and I_{nonan2}, we found that

$$a=\frac{\pi}{3},\quad b=0,\quad c=\frac{2}{3\pi}\zeta(2)\zeta(3), \tag{5.1}$$

and d is given by the expression (4.37) that we have not evaluated.

These coefficients give a little more insight into the structure of the low-energy expansion of four-graviton interactions in the M-theory effective action. The leading term of this type is the $\mathcal{R}^4$ term about which a great deal is known [14–18,1]. For example, in the ten-dimensional limit corresponding to the type-IIB string theory, it has dependence on the complex coupling $\Omega = C^{(0)} + ie^{-\phi^B}$ (where $C^{(0)}$ is the $R\otimes R$ scalar and ϕ^B is the type-IIB dilation), that enters by an overall factor of $E_{3/2}(\Omega,\bar{\Omega})$, where E_s is the modular invariant Eisenstein series that is proportional to Z_s (see the footnote in Sec. IV A). This function has an expansion for large Ω_2 (weak coupling) that begins with the tree-level term with coefficient $\zeta(3)$ in Eq. (2.5) and is followed by a one-loop term with a coefficient that is precisely the value of a in Eq. (5.1). There are no further perturbative terms in the expansion but there is a precisely defined sequence of D-instanton contributions.

One method by which the exact form of the the $\mathcal{R}^4$ interaction was determined [1] by calculating the one-loop contribution to four-graviton scattering in eleven-dimensional supergravity compactified on a two-torus. Recently this method has been generalized to evaluate the two-loop contribution in eleven-dimensional supergravity which contributes at leading order in the low-energy expansion to the $D^4\mathcal{R}^4$ interaction, where the notation symbolically indicates four derivatives acting on four powers of the curvature. In the limit that gives the ten-dimensional type-IIB theory the interaction is given by a term in the effective action density of the form [2]

$$\zeta(5)\mathcal{V}^{-5/2}E_{5/2}\Omega,\overline{\Omega})(s^2+t^2+u^2)\mathcal{R}^4 \tag{5.2}$$

(where the factors of s^2, t^2, and u^2 represent appropriate derivatives acting on the curvature tensors). In this case the modular function $E_{5/2}$ has an expansion for large Ω_2 (weak coupling) that begins with the tree-level term with coefficient $\zeta(5)$ in Eq. (2.5) and is followed by a two-loop term—the one-loop contribution is absent. Again there are no further perturbative string theory contributions but there is an infinite series of D-instanton contributions. The vanishing of the one-loop contribution in Eq. (5.2) is confirmed by our statement that the coefficient b in Eq. (5.1) vanishes.

The value of c in Eq. (5.1) is the coefficient of the one-loop contribution to the $(s^3+t^3+u^3)\mathcal{R}^4$ interaction. This is not a term which has yet been motivated from any argument based on duality or supersymmetry. In particular, it is not yet clear how this term packages with the tree-level $\zeta(3)^2$ term in Eq. (2.5) to make a modular invariant expression in the type-IIB limit.

More generally, one might ask whether there is a simple modular invariant expression for the complete four-graviton amplitude that generalizes the tree amplitude (2.1). An obvious candidate is obtained by replacing the coefficients $2\zeta(2n+1)$ in the tree amplitude (2.1) by $\tau_2^{-2n-1}2\zeta(2n+1)E_{n+1/2}$ [19,20]. The resulting amplitude has s-, t-, and u-channel poles at values corresponding to the mass of every excited state of all the (p, q) D strings. This expression has been conjectured [19] to be some sort of approximation to the exact four-graviton amplitude of the type-IIB theory. It does indeed reproduce the first few of the known coefficients in the low-energy expansion: by definition, it contains the exact tree-level amplitude and it also contains the correct ratio of the $E_{3/2}\mathcal{R}^4$ term and the $E_{5/2}\alpha'^2(s^2+t^2+u^2)\mathcal{R}^4$ term. However, it produces a value for the coefficient of the one-loop part of the $\alpha'^3(s^3+t^3+u^3)\mathcal{R}^4$ interaction that is twice the value of c in Eq. (5.1). It is not surprising that the naive modular invariant conjecture of Ref. [19] fails since there is no obvious sense in which it can approximate the exact amplitude. After all it purports to describe an infinite number of highly unstable non-BPS states in a nonperturbative manner but lacks all of the (massless and massive) threshold cuts that are required by unitarity.

ACKNOWLEDGMENTS

P.V. is grateful to PPARC for financial support and the Service de Physique théorique de Saclay for hospitality. Both authors are also grateful to the organizers of the Amsterdam Workshop and to the Theory Division at CERN where this work was completed.

APPENDIX: MASSLESS NORMAL THRESHOLDS

The thresholds that arise from massless on-shell intermediate states come from the region of integration over near the boundary of moduli space at which the toroidal world-sheet pinches in such a manner that the four vertex operators are separated into two bunches. At this degeneration point the world sheet is the product of the two tree-level world sheets that enter in the right-hand side of Eq. (1.18).

In order to extract these thresholds from the expression (1.9) for the loop amplitude it is very useful to change the definition of the moduli from $\nu^{(r)}$ and τ to $\eta^{(r)}$ by defining

$$\nu^{(1)}=\eta^{(1)},\quad \nu^{(2)}=\eta^{(1)}+\eta^{(2)},\quad \nu^{(3)}=\eta^{(1)}+\eta^{(2)}+\eta^{(3)}, \tag{A1}$$
$$\nu^{(4)}=\tau=\eta^{(1)}+\eta^{(2)}+\eta^{(3)}+\eta^{(4)},$$

where we have used the conformal invariance of the loop amplitude to fix $\nu^{(4)}=\tau$. The η variables are the ones that arise naturally in the operator construction of the loop amplitude as a trace over a string tree. In such a construction the propagator describing each leg of the loop is written as

$$\Delta_i=\frac{\alpha'}{2\pi}\int_{|z|<1}\frac{dzd\bar{z}}{|z|^2}z^{L_0}\bar{z}^{\bar{L}_0}, \tag{A2}$$

where $z=e^{2\pi i\eta}$.

The degeneration limit of relevance to the s-channel thresholds is the one in which $\eta_2^{(1)}\to\infty$ and $\eta_2^{(3)}\to\infty$, which puts the two s-channel propagators in the loop on shell. This corresponds to the region of integration $\mathcal{T}_{st}$:

$$\nu_2^{(1)}\leqslant\nu_2^{(2)}\leqslant\nu_2^{(3)}\leqslant\nu_2^{(4)}=\tau_2 \tag{A3}$$

with $\tau_2\to\infty$. In this limit we may substitute the asymptotic values

$$\Delta_t\sim\tilde{\Delta}_t=\hat{\mathcal{P}}^\infty(\nu^{(14)})+\hat{\mathcal{P}}^\infty(\nu^{(23)}),$$
$$\Delta_u\sim\tilde{\Delta}_u=\hat{\mathcal{P}}^\infty(\nu^{(13)})+\hat{\mathcal{P}}^\infty(\nu^{(24)}), \tag{A4}$$

and

$$\Delta_s\sim\frac{\pi(\nu_2^{(12)})^2}{2\tau_2}-\frac{1}{4}\ln\left|\frac{\sin(\pi\nu^{(12)})}{\pi}\right|^2+\frac{\pi(\nu_2^{(34)})^2}{2\tau_2}-\frac{1}{4}\ln\left|\frac{\sin(\pi\nu^{(34)})}{\pi}\right|^2$$
$$=\tilde{\Delta}_s+\delta_s, \tag{A5}$$

where

$$\tilde{\Delta}_s=\hat{\mathcal{P}}^\infty(\nu^{(12)})+\hat{\mathcal{P}}^\infty(\nu^{(34)}) \tag{A6}$$

and

$$\delta_s=\sum_{m\neq0}\frac{1}{4|m|}(e^{2i\pi(m\nu_1^{(12)}+i|m|\nu_2^{(12)})}+e^{2i\pi(m\nu_1^{(34)}+i|m|\nu_2^{(34)})}). \tag{A7}$$

The sum over m in Eqs. (A5) and (A7) gives the effect of the massive string states that propagate between the vertices for the particles 1 and 2 or the vertices for the particles 3 and 4, i.e., in the legs of the loop that are not degenerating. These terms are the ones that give rise to the stringy corrections to the low-energy field theory thresholds.

The contribution to the one-loop amplitude in the $\mathcal{T}_{st}$ region can be rewritten as

$$I_{\mathcal{T}_{st}}=\int_{\mathcal{R}_L}^\infty\frac{d^2\tau}{\tau_2^2}\int_{\mathcal{T}_{st}}\prod_{i=1}^3\frac{d^2\nu^{(i)}}{\tau_2}\exp[\alpha's(\tilde{\Delta}_s-\tilde{\Delta}_u)$$
$$+\alpha't(\tilde{\Delta}_t-\tilde{\Delta}_u)]\exp(\alpha's\delta_s). \tag{A8}$$

The α' expansion is obtained by expanding the last exponential in powers of δ_s. The leading term reproduces the field theory s-channel threshold given by the first term in Eq. (3.23). The next contribution, linear in δ_s, vanishes due to the integration over $v_1^{(2)}$ or $v_1^{(4)}$. The next term has a factor of $(\alpha's\delta)^2$ and gives a nonzero contribution to the logarithmic behavior at order $(\alpha's)^4$. After a little algebra (and adding the contributions of the $\mathcal{T}_{tu}$ and $\mathcal{T}_{us}$ domains) this gives the threshold contribution

$$I_{\text{nonan2}}(s,t,-s-t)=\sum_{m\neq0}\frac{(\alpha's)^2}{32m^2}\int_L^\infty\frac{d\tau_2}{\tau_2^2}\int_{\mathcal{T}_{st}}\prod_{i=1}^3d\omega_i$$
$$\times e^{\pi\alpha'\tau_2Q(s,t)}(e^{-4\pi m\tau_2(\omega_2-\omega_1)}$$
$$+e^{-4\pi m\tau_2(1-\omega_3)})+tu\text{ term}+us\text{ term}, \tag{A9}$$

where the integration variables ω_i are defined in Eq. (3.18). This integral is complicated but for the special case $t=0$ it reduces to the simple expression

$$I_{\text{nonan2}}(s,0,-s)=\frac{4}{\pi^3}\zeta(3)\zeta(4)\left(\frac{\alpha's}{4}\right)^4\left[\ln\left(\frac{\alpha's}{4L}\right)+\ln\left(-\frac{\alpha's}{4L}\right)\right]. \tag{A10}$$

[1] M. B. Green, M. Gutperle, and P. Vanhove, Phys. Lett. B **409**, 177 (1997).
[2] M. B. Green, H. Kwon, and P. Vanhove, preceding paper, Phys. Rev. D **61**, 104010 (2000).
[3] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 441 (1982).
[4] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60**, 917 (1988).
[5] M. B. Green and J. H. Schwarz, Nucl. Phys. **B198**, 252 (1982).
[6] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).
[7] J. Polchinski, ''TASI lectures on D-branes,'' hep-th/9611050.
[8] N. Sakai and Y. Tanii, Nucl. Phys. **B287**, 457 (1987).
[9] E. D'Hoker and D. H. Phong, Nucl. Phys. **B440**, 24 (1995); E. D'Hoker and D. H. Phong, Phys. Rev. Lett. **70**, 3692 (1993).
[10] J. G. Russo and A. A. Tseytlin, Nucl. Phys. **B508**, 245 (1997).
[11] M. B. Green, in *Proceedings of the 1997 Advanced Study Institute on Strings, Branes and Dualities*, Cargese, France [Nucl. Phys. B (Proc. Suppl.) **68**, 242 (1998).]
[12] W. Lerche, B. E. W. Nilsson, A. N. Schellekens, and N. P. Warner, Nucl. Phys. **B299**, 91 (1988).
[13] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications I & II* (Springer, Verlag, Berlin, 1985).
[14] M. B. Green and M. Gutperle, Nucl. Phys. **B498**, 195 (1997).
[15] M. B. Green and P. Vanhove, Phys. Lett. B **408**, 122 (1997).
[16] M. B. Green and S. Sethi, Phys. Rev. D **59**, 046006 (1999).
[17] B. Pioline, Phys. Lett. B **431**, 73 (1998).
[18] H. Partouche and A. Kehagias, Phys. Lett. B **422**, 109 (1998); Int. J. Mod. Phys. A **13**, 5075 (1998).
[19] J. G. Russo, Nucl. Phys. **B535**, 116 (1998).
[20] M. B. Green and G. Moore (unpublished).