Two loops in eleven dimensions

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The two-loop Feynman diagram contribution to the four-graviton amplitude of eleven-dimensional supergravity compactified on a two-torus $\mathcal{T}^2$ is analyzed in detail. The Schwinger parameter integrations are reexpressed as an integration over the moduli space of a second torus $\hat{\mathcal{T}}^2$, which enables the leading low-momentum contribution to be evaluated in terms of maps of $\hat{\mathcal{T}}^2$ into $\mathcal{T}^2$. The ultraviolet divergences associated with boundaries of moduli space are regularized in a manner that is consistent with the expected duality symmetries of string theory. This leads to an exact expression for terms of order $\partial^4\mathcal{R}^4$ in the effective M-theory action (where $\mathcal{R}^4$ denotes a contraction of four Weyl tensors), thereby extending earlier results for the $\mathcal{R}^4$ term that were based on the one-loop eleven-dimensional amplitude. Precise agreement is found with terms in type-IIA and -IIB superstring theory that arise from the low-energy expansion of the tree-level and one-loop string amplitudes and predictions are made for the coefficients of certain two-loop string theory terms as well as for an infinite set of *D*-instanton contributions. The contribution at the next order in the derivative expansion $\partial^6\mathcal{R}^4$ is problematic, which may indicate that it mixes with higher-loop effects in eleven-dimensional supergravity.

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I. INTRODUCTION

This paper continues the study of the interconnections between quantum supergravity in eleven dimensions [1] compactified on $\mathcal{T}^2$ and properties of perturbative and nonperturbative string theory [2,3]. In earlier papers [4–6] it was shown that the one-loop diagrams of eleven-dimensional supergravity that contribute to certain special amplitudes reproduce terms in the effective type-II superstring actions that may be described by integrals over sixteen Grassmann components, which is half the dimension of the type-II superspace. These terms include the $\mathcal{R}^4$ term, which is a specific contraction of four Weyl tensors that arises from the leading behavior in the low-energy expansion of the four-graviton amplitude.

The main objective of this paper is to extend this analysis to the evaluation of higher-derivative terms in the effective action by considering the low-energy expansion of the two-loop contribution to eleven-dimensional supergravity compactified on $\mathcal{T}^2$. This seemingly awesome calculation is greatly facilitated by the observation in Ref. [7] that the two-loop amplitude has a surprisingly simple expression as a kinematic factor multiplying a subset of the two-loop amplitudes of φ^3 *scalar* field theory. This is a generalization of the well-known structure of the one-loop amplitude.

Eleven-dimensional supergravity is only the long wavelength approximation to M theory and does not by itself define the short distance physics that is necessary for a consistent quantum theory. This is evident from the fact that the quantum theory has nonrenormalizable ultraviolet behavior that can only be consistently interpreted with additional microscopic input that is not contained in the supergravity theory but should be built into a detailed microscopic theory, such as the matrix model. However, it was seen in Ref. [4] that if some mild extra information is fed in from string theory the regularized value of the one-loop divergence in the four-graviton scattering amplitude is uniquely specified. This mild information is the fact that the type-IIA and II-B superstring theories have identical one-loop four-graviton amplitudes. Similar statements hold for other interactions of the same dimension that are related to the four-graviton interaction by supersymmetry [5]. We will see that requiring the various string duality symmetries to hold will also severely restrict the form of special higher-dimension interactions that arise at two loops in eleven-dimensional supergravity and contribute higher-derivative terms in the effective action.

In Sec. II we will give a schematic overview of the loop amplitudes of eleven-dimensional supergravity compactified on a circle and on a two-dimensional torus, and their correspondence with terms in the string theory effective action. The purpose of this section is to show how simple dimensional arguments can hint at connections between these quantum loop amplitudes and the structure of higher-order terms in the effective action of type-II string theory in nine and ten dimensions. An important point to be discussed at the end of Sec. II is that the four-graviton amplitudes in the ten-dimensional type-IIA and type-IIB theories can be shown to be *equal* up to two loops, even though the two-loop amplitudes are notoriously difficult to evaluate in closed form. This rather non-obvious consequence of supersymmetry follows from careful consideration of the effect of the insertion of world-sheet supermoduli. In order to compare our results obtained from one and two-loop diagrams of eleven-dimensional supergravity on $\mathcal{T}^2$ with the corresponding string theory results, we include an appendix which contains a brief review of the expansion of the four-graviton tree-level

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and one-loop string theory amplitudes in a series of derivatives.

Section III will review the detailed calculation of the one-loop four-graviton amplitude compactified on $\mathcal{T}^2$ and its contributions to the effective M-theory action, developing the arguments in Refs. [8], [9] concerning the momentum dependence. The lowest order term in the momentum expansion determines the interaction of the form [10]

$$\int d^9x\sqrt{-G^{(9)}}\mathcal{V}\mathcal{R}^4\left(\frac{4\pi}{3}\Lambda^3 l_{11}^3+\mathcal{V}^{-3/2}f_1(\Omega,\bar{\Omega})\right), \tag{1.1}$$

where $G^{(9)}$ is the nine-dimensional M-theory metric, $\mathcal{V}$ is the dimensionless volume of $\mathcal{T}^2$, $\Omega=\Omega_1+i\Omega_2$ is its complex structure, and $f_1(\Omega,\bar{\Omega})$ is a modular-function invariant. When translated into the nine-dimensional type-IIB string theory parameters the complex structure is identified with the complex coupling constant where Ω_1 is the Ramond-Ramond ($R\otimes R$) scalar field and $\Omega_2=e^{-\phi^B}$, with ϕ^B the type-IIB dilaton. The function $f_1(\Omega,\bar{\Omega})$ has a large-Ω_2 expansion that begins with two power-behaved terms. These are interpreted in string theory as terms that arise from the tree-level string amplitude and from the one-loop string amplitude. The remainder of $f_1(\Omega,\bar{\Omega})$ consists of an infinite sequence of exponentially suppressed contributions of the form $e^{-2\pi(|K|\Omega_2-iK\Omega_1)}$ which correspond to D-instanton contributions. The one-loop ultraviolet divergence is cubic in the loop momentum and has been cut off in Eq. (1.1) at a momentum scale Λ measured in units of l_{11}^{-1}, where l_{11} is the eleven-dimensional Planck length. It was shown in Ref. [10] that in order for Eq. (1.1) to be consistent with the equality of the one-loop four-graviton amplitudes in the type-IIA and -IIB string theories the cutoff must be set to the value $(\Lambda l_{11})^3=\pi/2$. Alternatively, a local $\mathcal{R}^4$ counterterm should be added to the one-loop action with a coefficient chosen to cancel the Λ dependence and give the appropriate finite value.

The one-loop amplitude compactified on $\mathcal{T}^2$ also contains an infinite set of higher-derivative terms. Among these are the nonanalytic terms containing the nine-dimensional massless threshold singularities implied by unitarity which have the symbolic form $(-l_s^2 s)^{1/2}$ (where s represents any of the Mandelstam invariants[1]). After subtracting this term the loop amplitude can be expanded in a series of powers of the momenta corresponding to higher derivative terms in the effective action [8,9]. These higher-derivative terms translate into terms in the type-IIA and -IIB string theory effective actions that have a dependence on the coupling constant that implies that they should be identified with multiloop string theory effects. Among these terms are contributions of order $s^2\mathcal{R}^4$ that have the dilaton dependence of string theory one-loop and two-loop contributions. These terms apparently violate the equality of type-IIA and -IIB four-graviton amplitudes at two string loops. However, we will see that the expected equality is restored when two-loop supergravity effects are added.

Section IV will be concerned with the two-loop supergravity four-graviton amplitude compactified on $\mathcal{T}^2$, making use of its expression in terms of scalar field theory [7]. An important feature of the two-loop and higher-loop contributions is that they have overall kinematic factors of the form $D^4\mathcal{R}^4$, so that they do not give extra contributions to the one-loop $\mathcal{R}^4$ term.[2] However, it is not known if the $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ terms, which get contributions from both one and two loops in eleven dimensions, are protected from corrections arising from higher-loop diagrams. In a sense, the results of this paper indicate that the $D^4\mathcal{R}^4$ terms are completely accounted for by the two-loop contributions and should therefore not receive higher-order corrections.

We will be interested in the expression for the two-loop amplitude compactified on $\mathcal{T}^2$ so that each loop is associated with an independent nine-dimensional momentum integral and a sum over two Kaluza-Klein momentum components. After performing the integration over the continuous loop momenta the leading term in the low-energy expansion of the two-loop supergravity amplitude of $D^4\mathcal{R}^4$ will be expressed as an integral over three Schwinger parameters and a sum over the Kaluza-Klein charges. This needs to be regularized in order to suppress the ultraviolet divergences which are of two kinds. One of these is the two-loop primitive divergence while the second comprises the three independent subdivergences that come from the divergences of one-loop subdiagrams.

In order to describe these divergences in a systematic manner we first perform Poisson resummations over the Kaluza-Klein momenta to rewrite the amplitude as a sum over the windings of the internal lines around $\mathcal{T}^2$ as well as an integral over three loop parameters. The leading divergence arises, as expected, in the sector of zero winding number while the one-loop subdivergences arise in sectors in which a subset of winding numbers vanish. In order to analyze these subdivergences we have found it very helpful to make use of a hidden SL(2,**Z**) symmetry of the two-loop supergravity integrand. This is made explicit by redefining the three loop integration variables to be the volume and complex structure of a second two torus $\hat{\mathcal{T}}^2$. The ultraviolet divergences are regularized in a natural manner that respects the SL(2,**Z**) symmetry by introducing a cutoff at the boundaries of the moduli space of this torus. The evaluation of the loop amplitude then involves mappings of $\hat{\mathcal{T}}^2$ into $\mathcal{T}^2$.

In this manner we will be able to evaluate contributions to the effective action that have the form of a prefactor, which is a function of the moduli, multiplying $D^4\mathcal{R}^4$. There is a

[1]We use lower-case letters *s, t, u* to denote the Mandelstam invariants in the ten-dimensional theory in the string frame and upper-case letters *S, T, U* for the corresponding invariants in the eleven-dimensional theory.

[2]This symbolic notation indicates a term in which there are four (covariant) derivatives and four factors of the Riemann curvature. The precise pattern of index contractions will be specified later.

finite (cutoff-independent) contribution to this prefactor that is independent of the string coupling and is interpreted as a string one-loop contribution. The dependence on the complex structure of the torus is encoded in a contribution to the prefactor that is again a modular invariant nonholomorphic Eisenstein series. This enters in the sectors that have one-loop subdivergences proportional to Λ^3, where Λ is a momentum cutoff. These subdivergences are cancelled by additional one-loop four-graviton diagrams in which the one-loop $\mathcal{R}^4$ counterterm (and its supersymmetric partners) is inserted as one of the vertices. When translated into string theory coordinates the renormalized value of this prefactor contains equal tree-level, one-loop and two-loop perturbative contributions to the type-IIA and -IIB string theory four-graviton amplitudes. The agreement between these type-IIA and -IIB perturbative terms follows from detailed comparison between the one-loop expressions of Sec. III and the two-loop expressions of Sec. IV. The coefficients of these terms are also in precise agreement with the corresponding terms in the expansion of the string tree and one loop amplitudes given in the Appendix.

We will also argue on the basis of string dualities that the leading two-loop divergence can make no contribution to the $D^4\mathcal{R}^4$ interaction, which means that its renormalized value must be set equal to zero. However, it can contribute to the $D^6\mathcal{R}^4$ interaction at string tree level. The analysis of the eleven-dimensional two-loop contribution to this interaction indeed appears to be a mess. This suggests that this interaction may also receive contributions from higher-loop effects in eleven-dimensional supergravity. Section V contains a summary and some concluding comments.

II. HIGHER ORDER TERMS IN ELEVEN DIMENSIONS

The derivative expansion of the M-theory action for the eleven-dimensional theory compactified on $\mathcal{T}^2$ starts with the classical Einstein-Hilbert term

$$S_{\rm EH}=\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G^{(11)}}R, \tag{2.1}$$

where $2\kappa_{11}^2=(2\pi)^8 l_{11}^9$ and l_{11} is the eleven-dimensional Planck length.[3] There is no coupling constant that can be tuned to a small value in the eleven-dimensional theory so there is no meaningful perturbative expansion. Therefore, we will only be able to make sense of ''protected'' quantities that receive only a finite number of perturbative contributions. The dimensional ultraviolet cutoff is determined in units of the eleven-dimensional Planck scale l_{11}. Upon compactification it will often be convenient to change to the string theory parameters, which are given in units of the string scale l_s. Compactification on a circle of radius R_{11} gives rise to the type-IIA string theory where the string coupling constant, $g^A=e^{\phi^A}$ (where ϕ^A is the type-IIA dilaton), is given by $l_{11}=(g^A)^{1/3}l_s$ and $R_{11}^3=e^{2\phi^A}=(g^A)^2$. Masses are measured with the metric [12]

$$\begin{aligned} ds^2&=G_{MN}^{(11)}dx^M dx^N \\ &=\frac{l_{11}^2}{l_s^2 R_{11}}g_{\mu\nu}dx^\mu dx^\nu+R_{11}^2 l_{11}^2(dx^{11}-C_\mu dx^\mu)^2, \end{aligned} \tag{2.2}$$

where $g_{\mu\nu}$ is the string frame metric. Since the compactification radius R_{11} depends of the string coupling constant the Kaluza-Klein modes are mapped to the massless fundamental string states and the nonperturbative $D0$-brane states. When expressed in terms of the type-IIA string theory parameters the compactified classical action becomes

$$S_{\rm EH}=\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\phi^A}R, \tag{2.3}$$

where $2\kappa_{10}^2=(2\pi)^7 l_5^8$ and l_5 is the string length scale.[4]

More generally, we will be concerned with the compactification of eleven-dimensional supergravity on $\mathcal{T}^2$. The dictionary that relates ν and Ω to the nine-dimensional type-IIA and type-IIB string theory parameters is [2,3]

$$\begin{aligned} &\mathcal{V}=R_{10}R_{11}=\exp\left(\frac{1}{3}\phi^B\right)r_B^{-1/3},\quad r_B=\frac{1}{R_{10}\sqrt{R_{11}}}=r_A^{-1}, \\ &\Omega_1=C^{(0)}=C_9^{(1)},\quad \Omega_2=\frac{R_{10}}{R_{11}}=\exp(-\phi^B)=r_A\exp(-\phi^A), \end{aligned} \tag{2.4}$$

where r_A and r_B are the dimensionless radii of the tenth dimension as measured in the type-IIA and -IIB string frames, respectively. The one-form $C^{(1)}$ and the zero-form $C^{(0)}$ are the respective $R\otimes R$ potentials and ϕ^A,ϕ^B are the type-IIA and -IIB dilatons.

The higher order corrections to the four-graviton interaction in the M-theory effective action compactified on $\mathcal{T}^2$ can be schematically represented by the expression,

$$S_4\sim\frac{1}{l_{11}}\int d^9x\sqrt{-G^{(9)}}\mathcal{V}h(\mathcal{V},\Omega;l_{11}^2\partial^2)\mathcal{R}^4, \tag{2.5}$$

where the expansion of the function h summarizes general features of the higher order corrections to the action. The lowest-order contribution of this type is the $\mathcal{R}^4$ term [10,13–15,4], which denotes the familiar contraction between four Weyl tensors

$$\mathcal{R}^4\sim t^{\mu_1\cdots\mu_8}t^{\nu_1\cdots\nu_8}R^{\mu_1\nu_1\mu_2\nu_2}\cdots R^{\mu_7\nu_7\mu_8\nu_8}, \tag{2.6}$$

where the tensor $t^{\mu_1\cdots\mu_8}$ ($\mu_r=0,1,\ldots,9$) is defined in Ref. [16]. In the following we will be evaluating the scattering amplitude for four on-shell gravitons that contributes to ef-

[3]With this convention the value of the tension of the fundamental string is equal to the tension of the $M2$-brane wrapped on a circle of radius $2\pi R_{11}$, i.e., $T_F=2\pi R_{11}l_{11}(2\pi^2/\kappa_{11}^2)^{1/3}$ [3,11].

[4]In this convention the fundamental string tension is related to the string scale by $T_F^2=\pi(2\pi l_s)^4/\kappa_{10}^2$.

fective interactions of this type. Instead of specifying the precise normalization constant in Eq. (2.6), it will therefore be more useful to define the linearized version of this interaction in momentum space, which is given by

$$\hat{K}=t^{\mu_1\cdots\mu_8}t^{\nu_1\cdots\nu_8}\prod_{r=1}^{4}\zeta^{(r)}_{\mu_r\nu_r}k^{(r)}_{\mu_{r+4}}k^{(r)}_{\nu_{r+4}}, \tag{2.7}$$

where $\zeta^{(r)}_{\mu_r\nu_r}$ $(r=1,2,3,4)$ are the polarization vectors for the gravitons with momenta $k^{(r)}_{\mu_r}$ satisfying the conditions $k^{(r)2}=0$ and $\Sigma^4_{r=1}k_r=0$.

In writing the effective action in the form (2.5) it is necessary to first subtract the nonlocal threshold terms that arise from integration over the massless intermediate states in the loop amplitudes. In the nine-dimensional compactification to be considered later these thresholds generate square root branch points of the form $(-s)^{1/2}$. Having subtracted this behavior the amplitude has a power series expansion in powers of s, t, and u. This translates into an expansion of the function h in powers of ∂^2, beginning with terms that we will write symbolically $D^4\mathcal{R}^4$, which have the linearized form

$$\partial^4\mathcal{R}^4\sim t^{\mu_1\cdots\mu_8}t^{\nu_1\cdots\nu_8}(\partial_{\mu_2}\partial_{\nu_2}h_{\mu_1\nu_1}) \times(\partial_{\mu_4}\partial_{\nu_4}h_{\mu_3\nu_3})\partial^4[(\partial_{\mu_6}\partial_{\nu_6}h_{\mu_5\nu_5})(\partial_{\mu_8}\partial_{\nu_8}h_{\mu_7\nu_7})] \tag{2.8}$$

The precise normalization of this term will be relevant later when its contribution to the four-graviton amplitude will be discussed. In that case we will define $D^4\mathcal{R}^4$ in such a manner that it gives a four-graviton contact term that is equal to

$$(S^2+T^2+U^2)\hat{K}. \tag{2.9}$$

The possible term of order ∂^2 vanishes by use of the equations of motion (the mass shell condition $S+T+U=0$). There are expected to be higher order nonanalytic terms of the form $S^3(-S)^{1/2}$ which will also need to be subtracted before powers of S^4 and higher can be considered. However, the considerations of this paper will cover only the terms of order $S^2\mathcal{R}^4$ (together with a few comments about terms of order $S^3\mathcal{R}^4$) so the higher-order thresholds will not be relevant.

A. One-loop contributions

Some of the systematics of the correspondence between the loop calculations of eleven-dimensional supergravity compactified on a circle or on a two torus can be understood from dimensional arguments. For example, the one-loop four-graviton diagram of Fig. 1 has dimension (momentum) [11] but it actually only diverges cubically with momentum. This follows from the fact that an overall factor of the linearized approximation to $\mathcal{R}^4$ factors out of the amplitude and this prefactor contains eight powers of the external momenta. After accounting for this prefactor the dynamical part of the loop calculation is identical to the box diagram of φ^3 field theory. Importantly, no other diagrams contribute. In particular, there are no diagrams with vertices corresponding to

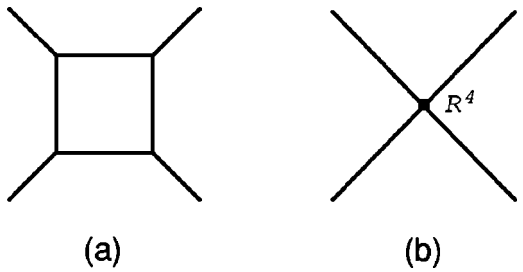

FIG. 1. (a) The scalar field theory one-loop diagram contributing to the four-graviton amplitude of compactified eleven-dimensional supergravity. (b) The one-loop $\mathcal{R}^4$ counterterm that cancels the cubic ultraviolet divergence.

gravitational contact interactions. This simplification is a very special feature of the four-graviton amplitude and other related processes that are protected by supersymmetry [6].

The box diagram can be expressed as a sum over the windings of the worldline of the particle circulating in the loop, which gives an expression that is the sum of integer winding numbers around the circle or the two torus. The term with zero winding number, which is ultraviolet divergent, does not depend on the geometry of the torus. This divergence will be regulated by introducing a cutoff Λ^{-2} on the Schwinger parameter conjugate to the loop momentum which suppresses the ultraviolet domain. This gives a contribution to the amplitude proportional to Λ^3. The dependence on the cutoff can be cancelled by adding a local $\mathcal{R}^4$ counterterm to the action. The sum over nonzero windings gives a finite contribution which is necessarily proportional to R_{11}^{-3} which has the dimensions (momentum) [3] where, for simplicity, we are considering compactification on a circle. Comparing this to the expected result in the type-IIA theory, in which $R_{11}^3=e^{2\phi^A}$, we see that the finite term is interpreted as a string tree-level effect while the regularized term (which is independent of e^{ϕ^A}) is a string one-loop effect. Compactification of the loop on a two-torus gives a dependence on the modulus of the torus as well as its volume. In the limit of zero volume the ultraviolet divergent zero winding number term vanishes and the finite sum over nonzero windings gives the finite result that corresponds to the type-IIB string theory.

As will be explained in Sec. II C the four-graviton amplitudes of the type-IIA and -IIB string perturbation expansions are equal up to and including two loops. This is not an automatic property of the eleven-dimensional field theory calculation but it is true if the coefficient of the counterterm is chosen to have an appropriate value. Furthermore, this is the same value that is required by supersymmetry (based on an indirect argument given in Ref. [10]).

In the next section we will consider terms of higher order in the derivative expansion that come from the expansion of the one-loop supergravity amplitude in powers of S, T, and U. When compactified on a circle this gives terms in the effective action of the symbolic form $R_{11}^{3n}D^{2n}\mathcal{R}^4$, which contribute to the n-loop string action since $R_{11}^3=e^{2\phi^A}$.

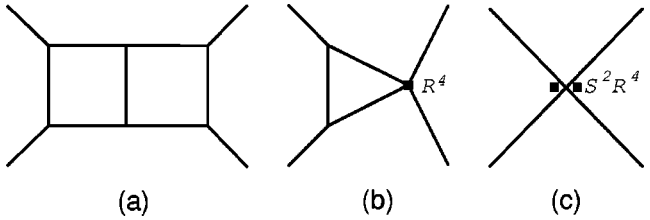

FIG. 2. (a) A scalar field theory two-loop diagram that contributes to four-graviton scattering. (b) One-loop diagrams in which one vertex is the $\mathcal{R}^4$ counterterm cancel the subdivergences of the two-loop diagrams. (c) A two-loop counterterm proportional to $S^2\mathcal{R}^4$.

B. Two-loop contribution

New primitive divergences arise at each order in perturbation theory. For example, the two-loop Feynman diagrams contributing to the four-graviton amplitude in eleven-dimensional supergravity have the superficial degree of divergence Λ^{20}. However, the amplitude has an expansion in powers of derivatives acting on $\mathcal{R}^4$ so there are at least eight powers of the external momenta, reducing the naive divergence to Λ^{12}, or less (depending on the number of derivatives). According to Ref. [7] at two loops there is also an additional factor of S^2 (or T^2 or U^2) so that the naive two-loop divergence is Λ^8, which is the same as that of φ^3 scalar field theory. More generally, at n loops there is a new primitive divergences of the form $\Lambda^{9n-10}S^2\mathcal{R}^4$. From the work of Ref. [7] it is not yet clear whether extra overall powers of S, T, and U arise beyond two loops which would reduce the naive degree of divergence still further (although it seems unlikely that there will be a simple expression for higher loops in terms of φ^3 field theory). These ultraviolet divergences come from the sector in which all winding numbers vanish and are independent of the geometry of the compactified dimensions. Their cutoff dependence can therefore be subtracted by the inclusion of local counterterms proportional to powers of derivatives acting on four powers of the curvature [as in Fig. 2(c)]. In Ref. [7] the two-loop amplitude was evaluated by dimensional regularization, which picks out the logarithmically divergent term. This arises from the finite part of a term of the (symbolic) form $S^{6-2\epsilon}\mathcal{R}^4/\epsilon$ in $11-\epsilon$ dimensions. Likewise, the diagram will contribute nonanalytic threshold terms at order S^6. Dimensional regularization discards the power divergences that have lower powers of S, which are precisely the terms we are interested in this paper.

In translating to string theory we must use the relations between the string theory Mandelstam invariants s, t, and u, and those of eleven-dimensional supergravity

$$s=S\frac{l_{11}^2}{l_s^2R_{11}},\quad t=T\frac{l_{11}^2}{l_s^2R_{11}},\quad u=U\frac{l_{11}^2}{l_s^2R_{11}}, \tag{2.10}$$

where the presence of the inverse powers of R_{11} results from the inverse metric in the definition of the invariants $[S=-G^{\mu\nu}(k_1+k_2)_\mu(k_1+k_2)_\nu,\quad T=-G^{\mu\nu}(k_1+k_4)_\mu(k_1+k_4)_\nu,\ U=-G^{\mu\nu}(k_2+k_4)_\mu(k_2+k_4)_\nu]$.

As in the case of one-loop diagrams the effects of the internal propagators winding around the compact direction(s) leads to dependence on the geometry of the compact dimensions. In this case these effects arise both in finite terms as well as in terms that contain subdivergences. For example, Fig. 2(a) shows an example of a two-loop supergravity diagram which has dimension (momentum) [20]. After accounting for the twelve powers of the external momenta in the overall $S^2\mathcal{R}^4$ factor eight powers of momenta remain that must be replaced either by powers of Λ or appropriate powers of the dimensional parameters $(R_{11})^{-1}$ and $s=S/R_{11}$, $t=T/R_{11}$, or $u=U/R_{11}$. When compactified on a circle of radius R_{11} this will contribute to the string tree level amplitude if it is proportional to $(R_{11})^{-3}$. There are therefore two possible kinds of term that contribute at the tree level, namely, terms of the form

$$\frac{K_1}{R_{11}^3}\Lambda^3 l_{11}^4\left(\frac{S^2+T^2+U^2}{R_{11}^2}\right)=g_A^{-2}\Lambda^3 l_s^4(s^2+t^2+u^2) \tag{2.11}$$

and

$$\frac{K_2}{R_{11}^3}l_{11}^6\left(\frac{S^3+T^3+U^3}{R_{11}^3}\right)=g_A^{-2}l_s^6(s^3+t^3+u^3), \tag{2.12}$$

where K_1 and K_2 are constants. The second of these terms does not depend on the cutoff and is a finite contribution whereas the first term results from the one-loop subdivergences.

These subdivergences are cancelled by including the one-loop diagram of Fig. 2(b), in which the vertex indicated by the dot is the local $\mathcal{R}^4$ counterterm that was extracted from the one-loop diagram and has a coefficient that depends on the cutoff. Since the particles circulating in the loop include all components of the supermultiplet, the supersymmetric partners of the $\mathcal{R}^4$ vertices are also involved. These couple the two external gravitons to two internal third-rank antisymmetric tensors, or two gravitini, in addition to two internal gravitons. In practice, this complication will be avoided since we will find that the consistency of the renormalization procedure requires Fig. 2(b) to be given in terms of scalar field theory in the same manner as the other one-loop and two-loop diagrams. This makes the diagram very easy to evaluate. Its dimension of (momentum) [17] is accounted for by the cutoff-independent factor $R_{11}^{-3}(S/R_{11})^2\mathcal{R}^4$ that has the same form as Eq. (2.11). The sum of this diagram and Eq. (2.11) should give a specific overall coefficient that is independent of the cutoff. In fact, we will see from the explicit calculations in Sec. IV that the coefficient of the $D^4\mathcal{R}^4$ term is proportional, in the type-IIB limit, to $E_{5/2}(\Omega,\overline{\Omega})$, which is the natural modular invariant completion of the tree-level term (2.11). As anticipated, the overall coefficient is precisely determined by requiring that the type-IIA and type-IIB string loop amplitudes are equal (up to two loops).

The status of the $S^3\mathcal{R}^4$ term (2.12) will not be resolved in this paper. It seems likely that a complete understanding will have to take into account higher-loop supergravity contributions. This is one of many complications in understanding in

detail the systematics of the correspondence between the higher-loop supergravity diagrams and string diagrams. Whereas the $\mathcal{R}^4$ and related terms of the same dimension are integrals over half the superspace, terms with more derivatives are formally integrals over a higher fraction of the superspace. Each power of momentum is equivalent to two powers of θ so that terms with less than eight powers of momentum acting on $\mathcal{R}^4$ should be protected and may be determined in this manner. This includes the $S^3\mathcal{R}^4 \sim D^6\mathcal{R}^4$ term which should therefore also be determined by similar considerations. Whether it is possible to go beyond this and relate terms in string perturbation theory at higher order in the momentum expansion to eleven-dimensional supergravity is much less obvious.

C. Comparison of type-IIA and -IIB perturbation expansions

An important constraint on the structure of the results of the eleven-dimensional calculations that we will make use of is a relationship between the type-IIA and -IIB four-graviton scattering amplitudes that holds up to and including two loops. It is well known that the tree-level and one-loop four-graviton amplitudes of the type-IIA and type-IIB superstring theories are identical (ignoring the parity-violating part of the loop amplitude, which vanishes in topologically trivial backgrounds). This property, which is also true for compactifications, is not an obvious consequence of the duality symmetries. For example, T duality (which applies to all orders in perturbation theory as well as nonperturbatively) only identifies the two theories when one is compactified on a circle and the other on the inverse circle, whereas we are comparing the theories on circles of the same radius (which may, for example, be infinite). The question is how far this generalizes to higher genus diagrams, which have not been explicitly evaluated? Such equality can be seen by considering the explicit construction of the four-graviton loops in the two theories.[5]

Recall that the type-IIB theory differs from that of the type-IIA by a flip of sign in the Gliozzi-Scherk-Olive (GSO) projection for the odd spin structure of the left-moving fermions while the right-moving fermions have identical GSO projections. Therefore, loop amplitudes with external gravitons, or any other massless states in the Neveu-Schwarz $\otimes$Neveu-Schwarz (NS$\otimes$NS) sector, differ only in the sign of the odd-odd spin structures—those spin structures that are odd both in the left-moving and in the right-moving sectors (we will again ignore the odd-even spin structures which vanish in the topologically trivial backgrounds that we are considering). Consider the scattering of gravitons with momenta $k_{\mu_r}^{(r)}(r=1,2,3,4)$, where $\Sigma_{r=1}^4 k_\mu^{(r)}=0$, and polarization vectors $h_{\mu_r\nu_r}^{(r)}$ which can be written in terms of left-moving and right-moving vectors a $h_{\mu_r\nu_r}^{(r)}=\Sigma_i h_{\mu_r}^{(r;i)}\tilde{h}_{\nu_r}^{(r;i)}$. For genus $l\geqslant 1$ these terms are associated with the product of two epsilon tensors $\epsilon^{\mu_0\cdots\mu_9}\tilde{\epsilon}^{\nu_0\cdots\nu_9}$. The tensor indices can contract with the three independent external momenta, the left-moving and right-moving vectors in the polarization tensors or with each other. At one loop there are no contractions between the two epsilon tensors so the odd-odd spin structures vanish and the two theories are identical. At higher loops there are insertions of $2l-2$ supermoduli associated with picture changing. Each one of these inserts a factor of $\partial X^\mu \bar{\partial} X^\nu$ which leads to a total of two contractions of the form $\eta^{\mu\nu}$ between the two epsilon tensors. This is still not enough to allow the sixteen remaining indices of these tensors to be saturated by the external momenta and polarizations. When $l>2$ there are more contractions between the epsilon tensors due to the higher number of picture changing operators, in which case the odd-odd spin structures give a nonzero contribution. *We conclude, therefore, that the four-graviton amplitudes in the type-IIA and -IIB superstring theories are equal up to two loops, but not beyond*. A corollary is that amplitudes with five external gravitons are not equal at two loops while those with more gravitons are not equal at one or two loops.

[5]This subsection is based on conversations with Nathan Berkovits.

III. MOMENTUM DEPENDENCE OF THE ONE-LOOP SUPERGRAVITY AMPLITUDE

The one-loop amplitude describing the elastic scattering of two gravitons in eleven-dimensional Minkowski space is given by [4]

$$A_4^{(1)}=\frac{\kappa_{11}^4}{(2\pi)^{11}}\hat{K}[I(S,T)+I(S,U)+I(U,T)], \qquad (3.1)$$

where the function I has the form of a Feynman integral for a box diagram in massless scalar φ^3 field theory

$$I(S,T)=\int d^{11}q \frac{1}{q^2}\frac{1}{(q+k_1)^2}\frac{1}{(q+k_1+k_2)^2}\frac{1}{(q-k_4)^2} \qquad (3.2)$$

and $q_\mu(\mu=0,\ldots,10)$ is the eleven-dimensional loop momentum. The numerical coefficient in Eq. (3.1) follows the conventions of Ref. [7] [with a slight reshuffling of the powers of $(2\pi)^{11}$] which will be convenient for later consideration of two-loop diagrams.

We want to consider this amplitude compactified on $M^9\times\mathcal{T}^2$ so that two components of the loop momentum are proportional to integer Kaluza-Klein charges (l_1 and l_2). For simplicity, we will choose a kinematic configuration in which the external gravitons have their polarizations and momenta oriented in directions transverse to the two torus. After representing the propagators as integrals of Schwinger parameters in the usual manner the compactified version of Eq. (3.2) can be written as

$$I(S,T)=\frac{1}{l_{11}^2\mathcal{V}}\int\prod_{r=1}^4 d\sigma_r\int d^9q\sum_{\{l_1l_2\}} e^{-G^{IJ}l_1l_J\sigma-\Sigma_{r=1}^4 p_r^2\sigma_r}, \qquad (3.3)$$

where $\sigma=\Sigma_{r=1}^{4}\sigma_r$ and $p_r=q+\Sigma_{s=1}^{r}k_s$ are the momenta in the legs of the loop. The Schwinger parameters σ_r have dimension (length).[2]

After a few manipulations [8,9,4] each of the three terms in the scalar integral (3.3) can be rewritten as

$$I(S,T)=\frac{2\pi^{9/2}}{l_{11}^2\mathcal{V}}\int_0^\infty d\sigma\ \ \sigma^{-3/2}\int_{\mathcal{T}_{ST}}\prod_{r=1}^{3}d\omega_r \times\sum_{\{l_1,l_2\}}e^{-G^{IJ}l_1I_J\sigma-Q(S,T;\omega_r)\sigma}, \tag{3.4}$$

where $Q(S,T;\omega_r)=-S\omega_1(\omega_3-\omega_2)-T(\omega_2-\omega_1)(1-\omega_3)$. The domain of integration indicated by $\mathcal{T}_{ST}$ is defined by $0\leqslant\omega_1\leqslant\omega_2\leqslant\omega_3\leqslant1$. The other two terms in Eq. (3.3) come from integration over the two remaining regions $\mathcal{T}_{TU}$: $0\leqslant\omega_3\leqslant\omega_2\leqslant\omega_1\leqslant1$ and $\mathcal{T}_{SU}$: $0\leqslant\omega_2\leqslant\omega_1\leqslant\omega_3\leqslant1$. The integral (3.4) is to be evaluated with $S,T<0$ where it converges and then analytically continued to the physical region. The amplitude can be split into a momentum-dependent and a momentum-independent part

$$I(S,T)=I_0+I'(S,T), \tag{3.5}$$

where

$$I_0\equiv I(0,0)=\frac{\pi^4}{l_{11}^2\mathcal{V}}\int_0^\infty d\sigma\sigma^{-3/2}\sum_{\{l_1,l_2\}}e^{-\pi G^{IJ}l_Il_J\sigma}, \tag{3.6}$$

where σ has been rescaled by a factor of π in passing from Eq. (3.4) to Eq. (3.6). This expression diverges for small σ, which is the cubic ultraviolet divergence of the scalar box diagram in eleven dimensions. We will regularize this divergence by introducing a cutoff on σ so that $1/\Lambda^2\leqslant\sigma$. It is convenient to carry out Poisson summations on l_1 and l_2 which replaces the Kaluza-Klein charges by winding numbers $\hat{l}_1$ and $\hat{l}_2$. The divergence is now isolated in the zero winding number term $(\hat{l}_1,\hat{l}_2)=(0,0)$. The result is

$$I_0=\frac{\pi^3}{2l_{11}^3}\left(\frac{4\pi}{3}(\Lambda l_{11})^3+\mathcal{V}^{-3/2}f_1(\Omega,\bar{\Omega})\right), \tag{3.7}$$

where Λ^3 is the regularized value of the zero winding number term. In balancing the dimensions in this and other equations it is important to note that we have defined all distances as dimensionless multiples of the Planck distance l_{11}. In this convention $\mathcal{V}$ is dimensionless while the one-loop cutoff Λ has dimension $(\text{length})^{-1}$. In addition to the one-loop contribution there is the freedom to add the local counterterm $\delta^{(1)}S\sim l_{11}^{-3}c_1\int d^9x\sqrt{-G^{(9)}}\mathcal{V}\mathcal{R}^4$, which adds a term

$$\delta A_4^{(1)}=\frac{\kappa_{11}^4}{(2\pi)^{11}}\hat{K}\delta I_0 \tag{3.8}$$

to the amplitude, where

$$\delta I_0=\frac{\pi^3}{2l_{11}^3}c_1, \tag{3.9}$$

and c_1 is an arbitrary coefficient that will shortly be given a Λ-dependent value.

The Λ-independent term in Eq. (3.7) has coefficient $f_1(\Omega,\bar{\Omega})=2\zeta(3)E_{3/2}$, where the Eisenstein series E_s is defined by

$$2\zeta(2s)E_s=\sum_{(m,n)\neq(0,0)}\frac{\Omega_2^s}{|m+n\Omega|^{2s}}. \tag{3.10}$$

The volume dependence $\mathcal{V}^{-3/2}$ of this term means that it vanishes in the eleven-dimensional limit $\mathcal{V}\to\infty$. However, this is the only term that survives in the limit that corresponds to the decompactified type-IIB theory, $r_B\to\infty$ with fixed e^{ϕ^B}, while the cutoff dependent term in Eq. (3.7) gives vanishing contribution. The complex structure of $\mathcal{T}^2$ is to be identified with the complex type-IIB scalar field, $\Omega=\Omega_1+i\Omega_2$. Expanding $E_{3/2}$ for large Ω_2 (small type-IIB coupling e^{ϕ^B}) gives

$$2\zeta(3)E_{3/2}=2\zeta(3)e^{-3\phi^B/2}+\frac{2\pi^2}{3}e^{\phi^B/2}+\text{nonperturbative}. \tag{3.11}$$

The successive terms in this expansion can be identified with tree-level, one-loop, and nonperturbative terms in the coefficient of the type-IIB string theory $\mathcal{R}^4$ interaction. The nonperturbative terms have the form of an infinite series of D-instanton terms where each charge-K D-instanton contribution has an infinite series of perturbative fluctuations.

The total one-loop contribution to the amplitude comes from the combination $I_0+\delta I_0$ [the sum of Eqs. (3.9) and (3.7)]. The dominant term in the large $\mathcal{V}$ limit is proportional to $[4\pi(\Lambda l_{11})^3/3+c_1]$ and is independent of $\mathcal{V}$ so in the string theory parametrization this term is independent of the dilaton and arises from one string loop in the type-IIA theory. Although this coefficient is not determined by the physics of quantized eleven-dimensional supergravity, it is determined by insisting that the four-graviton interactions in the type-IIA and type-IIB effective string actions should be equal when the radii r_A and r_B are equal. As argued in Sec. II C this is known property of string perturbation theory up to and including two loops. More explicitly, the nine-dimensional effective actions that give rise to the momentum independent part of $I+\delta I_0$ have $\mathcal{R}^4$ terms that are expressed as

$$S_{\mathcal{R}^4}=\frac{1}{3\times(4\pi)^7l_{11}}\int d^9x\sqrt{-G}\mathcal{V}\mathcal{R}^4 \times\left(2\zeta(3)\mathcal{V}^{-3/2}E_{3/2}(\Omega,\bar{\Omega})+c_1+\frac{4\pi}{3}(\Lambda l_{11})^3\right), \tag{3.12}$$

which can be written in string theory coordinates and expanded for small string coupling constant in the form

$$S_{\mathcal{R}^4}=\frac{1}{3\times(4\pi)^7 l_s}\int d^9x\sqrt{-g^B}r_B\mathcal{R}^4 \times\left(2\zeta(3)e^{-2\phi^B}+\frac{2\pi^2}{3}+\frac{c_1+4\pi(\Lambda l_{11})^3/3}{r_B^2}+\cdots\right) =\frac{1}{3\times(4\pi)^7 l_s}\int d^9x\sqrt{-g^B}r_A\mathcal{R}^4 \times\left(2\zeta(3)e^{-2\phi^A}+\frac{2\pi^2}{3}\frac{1}{r_A^2}+c_1+\frac{4\pi}{3}(\Lambda l_{11})^3+\cdots\right), \tag{3.13}$$

in the type-IIA and type-IIB theories, respectively (ignoring the nonperturbative contributions). It follows that the only consistent value for c_1 which equates the type-IIA and -IIB expression is

$$c_1=\frac{2\pi^2}{3}-\frac{4\pi}{3}(\Lambda l_{11})^3. \tag{3.14}$$

This is the value which is also consistent with supersymmetry [10].

The momentum dependence of $I(S,T)$ in Eq. (3.4) is contained in the finite term $I'(S,T)$ in Eq. (3.5). We will separate the term with zero Kaluza-Klein momenta $I^0(S,T)$ ($l_1=l_2=0$), from the rest by writing

$$I'(S,T)=I^0(S,T)+\sum_{n=2}^{\infty}I_n(S,T). \tag{3.15}$$

The term $I^0(S,T)$, which contains the nonanalytic contribution to the amplitude [8,9] has the form in d dimensions

$$\begin{aligned} l_{11}^2\mathcal{V}I_d^0(S,T) &=2\pi^{d/2}\int_0^\infty d\sigma\ \ \sigma^{3-d/2}\int_{\mathcal{T}_{ST}}\prod_{r=1}^{3}d\omega_r(e^{-Q(S,T;\omega_r)\sigma}-1)\\ &=2\pi^{d/2}\Gamma(4-d/2)\\ &\times\int_{\mathcal{T}_{ST}}\prod_{r=1}^{3}d\omega_r Q(S,T;\omega_r)^{(d-8)/2}\\ &=2\pi^{d/2}\Gamma(4-d/2)(-\mathcal{G}_{ST})^{(d-8)/2}, \end{aligned} \tag{3.16}$$

where $\mathcal{G}_{ST}^n$ is defined by

$$\mathcal{G}_{ST}^n=\int_{\mathcal{T}_{ST}}\prod_{r=1}^{3}d\omega_r[-Q(S,T;\omega_r)]^n. \tag{3.17}$$

Similarly, $\mathcal{G}_{TU}^n$ and $\mathcal{G}_{US}^n$ will be defined by cyclically permuting S, T, and U in the function Q. Specializing to $d=9$ gives

$$\begin{aligned} l_{11}^2\mathcal{V}I^0(S,T)&\equiv-8\pi^5(-\mathcal{G}_{ST})^{1/2}\\ &=-8\pi^5\int_{\mathcal{T}_{ST}}\prod_{r=1}^{3}d\omega_r[Q(S,T;\omega_r)]^{1/2}. \end{aligned} \tag{3.18}$$

The terms in Eq. (3.15) with nonzero Kaluza-Klein charge I_n are homogeneous polynomials in S and T of degree n,

$$\begin{aligned} l_{11}^2\mathcal{V}I_n(S,T)&=2\pi^{9/2}\frac{\mathcal{G}_{ST}^n}{n!}\int_0^\infty\frac{d\sigma}{\sigma^{3/2-n}}\sum_{(l_1,l_2)\neq(0,0)}e^{-G^{IJ}l_Il_J\sigma}\\ &=4\pi^{9/2}\Gamma\left(n-\frac{1}{2}\right)\zeta(2n-1)l_{11}^{2n-1}\mathcal{V}^{n-1/2}\\ &\times\frac{\mathcal{G}_{ST}^n}{n!}E_{n-1/2}(\Omega,\overline{\Omega}). \end{aligned} \tag{3.19}$$

The Eisenstein series that enter this expression have the large-Ω_2 expansion

$$\begin{aligned} E_{n-1/2}(\Omega,\overline{\Omega})&=e^{-(2n-1)\phi^B/2}\\ &+\frac{\sqrt{\pi}\Gamma(n-1)\zeta(2n-2)}{\Gamma(n-\frac{1}{2})\zeta(2n-1)}e^{(3-2n)\phi^B/2}\\ &+\text{nonperturbative}. \end{aligned} \tag{3.20}$$

The first term will contribute to the tree-level amplitude when expressed in string coordinates and the second term is a n-loop contribution. All the other terms are nonperturbative D-instanton effects.

Using the expansion (3.20) and putting all the terms together, the complete expression for the $\mathcal{R}^4$ term in the one-loop effective action for eleven-dimensional supergravity in nine-dimensions is given by Eq. (2.5) with the function h defined by the amplitude A_4 in Eq. (3.1), where

$$\begin{aligned} &I(S,T)+I(T,U)+I(U,S)\\ &=I_0-\frac{4\pi^5}{l_{11}^2\mathcal{V}}(-\mathcal{W})^{1/2}\\ &+4\pi^{9/2}\sum_{n=2}^{\infty}l_{11}^{2n-3}\frac{\mathcal{W}^n\mathcal{V}^{n-3/2}}{n!}\\ &\times\left[\Gamma\left(n-\frac{1}{2}\right)\zeta(2n-1)\left(\frac{R_{10}}{R_{11}}\right)^{n-1/2}\right.\\ &\left.+\sqrt{\pi}\Gamma(n-1)\zeta(2n-2)\left(\frac{R_{10}}{R_{11}}\right)^{3/2-n}\right]\\ &+\text{nonperturbative} \end{aligned} \tag{3.21}$$

and

$$\mathcal{W}^n=\mathcal{G}_{ST}^n+\mathcal{G}_{TU}^n+\mathcal{G}_{US}^n. \tag{3.22}$$

There is no $n=1$ term after adding the contributions of $I(S,U)$ and $I(T,U)$ to $I(S,T)$ since the linear symmetric combination vanishes after using the mass shell condition.

The nonanalytic term (3.18) in nine-dimensional M-theory comes from the same massless thresholds that arise in either nine-dimensional type-II string theory [9,8] and have square root branch cuts of the form $(-S)^{1/2}$.

The two infinite series of terms on the right-hand side of (3.21) have very obvious origins from the dimensional reduction of the massless one-loop normal threshold of the eleven-dimensional loop. The first infinite series is a series of ascending powers of R_{10}. Although such terms appear to give singular behavior in the ten-dimensional IIA decompactification limit, $R_{10}\to\infty$, the series actually sums up to give the correct threshold behavior of the ten-dimensional theory, schematically of the form $S\ln(-S)$. More explicitly, the sum gives

$$
\begin{aligned}
&-\frac{4\pi^5}{l_{11}^2\mathcal{V}}(-\mathcal{W})^{1/2}+4\pi^{9/2}\sum_{n=2}^{\infty} l_{11}^{2n-3}\frac{\mathcal{W}^n\mathcal{V}^{n-3/2}}{n!}\\
&\quad\times\Gamma\left(n-\frac{1}{2}\right)\zeta(2n-1)\left(\frac{R_{10}}{R_{11}}\right)^{n-1/2}\\
&=-\frac{4\pi^5}{l_{11}^3\mathcal{V}}\sum_{r\in Z}\left(\frac{r^2}{R_{10}^2}-l_{11}^2\mathcal{W}\right)^{1/2}\\
&\quad-\frac{1}{l_{11}^3R_{11}R_{10}^2}4\pi^5\zeta(-1)\Gamma(-1/2). \qquad (3.23)
\end{aligned}
$$

The last term on the right-hand side cancels against the second term in the expansion of $E_{3/2}(\Omega,\overline{\Omega})$ in I_0 (3.7) [using Eq. (3.11)]. The net result is that the sum of the first infinite series of terms in Eq. (3.21) is given by

$$
\int_{T_{st}}\prod_{r=1}^{3} d\omega_r\left[-\frac{8\pi^2}{l_{11}^3\mathcal{V}}\sum_{r\in Z}\left(\frac{r^2}{R_{10}^2}+l_{11}^2Q(S,T;\omega_r)\right)^{1/2}\right]. \qquad (3.24)
$$

The sum over r can be evaluated in the large-R_{10} limit by approximating it by an integral by letting $r/R_{10}\to y$,

$$
\begin{aligned}
&\lim_{R_{10}\to\infty}-\frac{4\pi^5}{l_{11}^3\mathcal{V}}\sum_{r\in Z}\left(\frac{r^2}{R_{10}^2}-l_{11}^2\mathcal{W}\right)^{1/2}\\
&\quad=-\frac{8\pi^5}{l_{11}^3R_{11}}\int_0^{\infty}dy(y^2-l_{11}^2\mathcal{W})^{1/2}\\
&\quad=\frac{2\pi^5}{l_{11}R_{11}}\mathcal{W}[\ln(-\mathcal{W})-2], \qquad (3.25)
\end{aligned}
$$

where a constant has been absorbed into the implicit scale of the logarithm. This cancels out when the (S,T), (T,U), and (U,S) contributions are added. The result is that the series sums up to the expected massless logarithmic threshold in ten dimensions of the form

$$
\begin{aligned}
&\mathcal{R}^4(\mathcal{G}_{ST}\ln\mathcal{G}_{ST}+\mathcal{G}_{TU}\ln\mathcal{G}_{TU}+\mathcal{G}_{US}\ln\mathcal{G}_{US})\\
&\quad\sim\mathcal{R}^4(\mathcal{G}^s_{ST}\ln\mathcal{G}^s_{ST}+\mathcal{G}^s_{TU}\ln\mathcal{G}^s_{TU}+\mathcal{G}^s_{US}\ln\mathcal{G}^s_{US}), \qquad (3.26)
\end{aligned}
$$

where the expressions $\mathcal{G}^s_{ST}$, $\mathcal{G}^s_{TU}$, and $\mathcal{G}^s_{US}$ are defined in terms of the Mandelstam invariants of string theory, using the relation between the M-theory and string theory Mandelstam invariants, Eq. (2.10). Both Eqs. (3.18) and (3.26) have imaginary parts corresponding to the massless normal thresholds determined by unitarity. However, the real parts, which might have given rise to arbitrary constants, are here fixed to precise values. In addition to the threshold term (3.26), the tree-level part of I_0 also survives the limit $R_{10}\to\infty$ in the type-IIA amplitude, but the D-instanton terms are infinitely suppressed and disappear.

The second series of terms on the right-hand side of Eq. (3.21) is an ascending series of powers of R_{11}. In the ten-dimensional decompactification limit $R_{10}\to\infty$ this sums to a series of massive logarithmic thresholds of the ten-dimensional theory

$$
\begin{aligned}
&2\pi^5\sum_{n=1}^{\infty} l_{11}^{2n-3}\frac{\mathcal{W}^n\mathcal{V}^{n-3/2}}{n!}\Gamma(n-1)2\zeta(2n-2)\left(\frac{R_{10}}{R_{11}}\right)^{3/2-n}\\
&\quad=\frac{2\pi^5}{l_{11}^3R_{11}}\sum_{r\neq 0}\left(\frac{r^2}{R_{11}^2}-l_{11}^2\mathcal{W}\right)\left[\ln\left(1-\frac{R_{11}^2}{r^2}l_{11}^2\mathcal{W}\right)-2\right]. \qquad (3.27)
\end{aligned}
$$

This is a series of thresholds for the massive Kaluza-Klein states of M-theory on a circle of radius R_{11} which is to be added to the massless threshold that comes from Eq. (3.25). This sum can be evaluated in the decompactification limit $R_{11}\to\infty$ by approximating r/R_{11} by a continuous variable. Including the massless threshold term (3.26), the sum reduces to the eleven-dimensional threshold

$$
\begin{aligned}
l_{11}^2\mathcal{V}I&=2\pi^{9/2}\int_0^{\infty}d\sigma\sigma^{3-11/2}\int\prod_{r=1}^{3}d\omega_r e^{-\sigma Q(\omega_r)}\\
&=\frac{8\pi^5}{3}[(-\mathcal{G}_{ST})^{3/2}+(-\mathcal{G}_{TU})^{3/2}+(-\mathcal{G}_{US})^{3/2}]. \qquad (3.28)
\end{aligned}
$$

The tree-level type-IIA term vanishes in the $R_{11}\to\infty$ limit.

Rewriting the result of the one-loop calculation (3.21) in terms of the string theory parameters gives terms in the effective action of nine-dimensional type-IIA and -IIB string theory with derivatives acting on $\mathcal{R}^4$ that can be written in terms of the one-loop amplitude (3.1) as[6]

[6]These expressions correct certain coefficients in the corresponding formulas of Ref. [9].

$$A_4^{(1)}=(4\pi^8 l_{11}^{15} r_A^{-1})\hat{K}_{r_A}\Bigg[2\zeta(3)e^{-2\phi^A}+\frac{2\pi^2}{3r_A^2}+\frac{2\pi^2}{3}-8\pi^2 r_A l_s(-\mathcal{W}^s)^{1/2}+8\pi^{3/2}\sum_{n=2}^{\infty}\Bigg(\Gamma\Big(n-\frac{1}{2}\Big)\zeta(2n-1)\frac{1}{r_A^2}\frac{r_A^{2n}(l_s^2\mathcal{W}^s)^n}{n!}+\sqrt{\pi}\Gamma(n-1)\zeta(2n-2)e^{-2\phi^A}\frac{e^{2n\phi^A}(l_s^2\mathcal{W}^s)^n}{n!}\Bigg)\Bigg]+\text{nonperturbative} \tag{3.29}$$

or

$$A_4^{(1)}=(4\pi^8 l_{11}^{15} r_B)\hat{K}r_B\Bigg[2\zeta(3)e^{-2\phi^B}+\frac{2\pi^2}{3}+\frac{2\pi^2}{3}\frac{1}{r_B^2}-\frac{8\pi^2}{r_B^3}l_s(-\mathcal{W}^s)^{1/2}+8\pi^{3/2}\sum_{n=2}^{\infty}\Bigg(\Gamma\Big(n-\frac{1}{2}\Big)\zeta(2n-1)\frac{1}{n!}\frac{(l_s^2\mathcal{W}^s)^n}{r_B^{2n+2}}+\sqrt{\pi}\Gamma(n-1)\zeta(2n-2)\frac{e^{-2\phi^B}}{n!}\frac{(l_s^2\mathcal{W}^s)^n e^{2n\phi^B}}{r_B^{2n+2}}\Bigg)\Bigg]+\text{nonperturbative}, \tag{3.30}$$

where

$$\mathcal{W}^{s^n}=(\mathcal{G}_{ST}^s)^n+(\mathcal{G}_{TU}^s)^n+(\mathcal{G}_{US}^s)^n. \tag{3.31}$$

The overall factor of $4\pi^8 l_{11}^{15} r_A^{-1}=4\pi^8 l_{11}^{15} r_B$ that has been factored out in these expressions cancels with a factor in the measure in transforming the effective action from eleven-dimensional supergravity coordinates to string coordinates. This makes it easy to see the dependence of the effective interactions on the string-frame radius and the dilaton in in the remaining factors in Eqs. (3.29) and (3.30).

The infinite series of terms in the type-IIA theory are related by T duality to the series in the type-IIB theory. However, these terms appear asymmetrically between the two theories in Eqs. (3.29) and (3.30). In particular, all the terms in the series in the type-IIB action vanish in the limit $r_B \to \infty$ which is not true for the type-IIA series in the type-IIA decompactification limit. Since we saw in Sec. II C that the four-graviton amplitudes in the type-IIA and -IIB string theories are identical up to and including the contributions from two string loops there must be some more contributions that correct for this asymmetry. We will be concerned particularly with the $n=2$ terms on the right-hand side of the type-IIA action in Eq. (3.29),

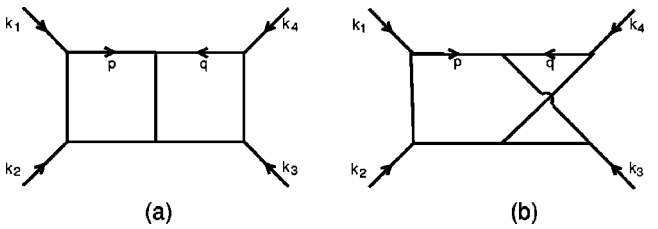

FIG. 3. The ''S-channel'' scalar field theory diagrams that contribute to the two-loop four-graviton amplitude of eleven-dimensional supergravity. (a) The (S,T) planar diagram $I^P(S,T)$; (b) the (S,T) nonplanar diagram $I^{NP}(S,T)$.

$$A_4^{(1)n=2}=4\pi^8 l_{11}^{19}e^{-(4/3)\phi^A}\times\pi^2(2\zeta(3)r_A^2+4\zeta(2)e^{2\phi^A})(\mathcal{W}^s)^2\hat{K}=\frac{4\pi^{10}}{6!}l_{11}^{19}e^{-(4/3)\phi^A}\Big(2\zeta(3)r_A^2+\frac{2\pi^2}{3}e^{2\phi^A}\Big)\times(s^2+t^2+u^2)\hat{K}, \tag{3.32}$$

where we have used

$$(\mathcal{G}_{ST}^s)^2=\int_{\mathcal{T}_{st}}\prod_{r=1}^{3}d\omega_r[-s\omega_1(\omega_3-\omega_2)-t(\omega_2-\omega_1)(1-\omega_3)]^2=\frac{1}{7!}(4s^2+4t^2+2st). \tag{3.33}$$

The expression (3.32) has a dependence on the dilaton that is characteristic of contributions in type-IIA string theory at one and two loops. Since the type-IIA and type-IIB string perturbation theories are identical up to two loops these type-IIA terms must be matched by identical terms in the type-IIB theory (with $r_A \to r_B$ and $\phi^A \to \phi^B$). We will see in the next section that these missing contributions to the type-IIB action arise from the compactification of two-loop terms in eleven-dimensional supergravity.

IV. THE TWO-LOOP SUPERGRAVITY AMPLITUDE

The evaluation of two-loop amplitudes in eleven-dimensional quantum supergravity would normally be a formidable task. However, it is known from the work of Ref. [7] that the two-loop four-graviton amplitude in maximally supersymmetric supergravity continues to have the feature that it can be written in terms of scalar field theory diagrams. The fact that the two-loop amplitude has such a simple expression was motivated in Ref. [7] in dimensions $\leqslant 10$ by use of the Kawai-Llewelyn-Tye (KLT) rules for constructing closed-string amplitudes out of open-string amplitudes [17]. This was shown to imply that the two-loop amplitude in the low-energy supergravity theory in d dimensions with maximum supersymmetry is given in terms of the two-loop amplitude of supersymmetric Yang-Mills theory with maximal supersymmetry. These rules were then independently derived by using unitarity in all channels. In eleven dimensions su-

pergravity is not the low-energy limit of a string theory so the strategy for determining the two-loop amplitude has to be a little different. In that case the expression may be determined by the requirement of unitarity and can also be checked by the requirement that it reduce to the lower-dimensional expressions upon trivial dimensional reduction.

The result is that the two-loop four-graviton amplitude $A_4^{(2)}(S,T,U)$, is given in terms of the sum of particular diagrams of φ^3 scalar field theory illustrated in Fig. 3. These are the planar diagram $I^P(S,T)$ and the nonplanar diagram $I^{NP}(S,T)$, together with the other diagrams obtained by permuting the external particles. The complete expression for the amplitude is (with same conventions as in Ref. [7])

$$A_4^{(2)} = i\frac{\kappa_{11}^6}{(2\pi)^{22}}\hat{K}\{S^2[I^P(S,T)+I^P(S,U)+I^{NP}(S,T) + I^{NP}(S,U)]+\text{perms.}\}, \tag{4.1}$$

where ''perms'' signifies the sum of terms with permutations of S, T, and U. This expression has an overall factor of $\mathcal{R}^4$ together with four powers of the momentum multiplying the loop integrals which means that these diagrams are much less divergent than they would naively appear. The loop integrals are given by

$$I^P(S,T) = \int d^{11}p d^{11}q \frac{1}{p^2(p+k_1)^2(p+k_1+k_2)^2(p+q)^2q^2(q+k_3+k_4)^2(q+k_4)^2} \tag{4.2}$$

and

$$I^{NP}(S,T) = \int d^{11}p d^{11}q \frac{1}{p^2(p+k_1)^2(p+q)^2(p+q+k_2)^2q^2(q+k_3+k_4)^2(q+k_4)^2} \tag{4.3}$$

which have ultraviolet divergences of order (momentum) [8] that will need to be regularized.

In addition to these two-loop diagrams there is a contribution to the amplitude from the one-loop diagram of Fig. 2(b), which is a triangle diagram in which there is one insertion of the linearized one-loop counterterm. Together with two-loop counterterm of Fig. 2(c), this will give an additional contribution $\delta A_4^{(2)}$ to the amplitude.

A. Evaluation of the two-loop amplitude on $\mathcal{T}^n$

We shall now consider the leading contribution to the derivative expansion arising from these two-loop diagrams when compactified on $\mathcal{T}^2$. As discussed earlier, this will contribute to the $D^4\mathcal{R}^4$ interaction. For convenience our considerations will be restricted to situations in which the polarization tensors and momenta of the gravitons are in directions transverse to torus and covariantize the final result. We will first be slightly more general and consider the case of compactification on an n-torus $\mathcal{T}^n$ with metric G_{IJ} and volume $\mathcal{V}_n$, in which case the planar diagram with external momenta k_r $r=1,\ldots,4$ is given by the expression

$$I^P(S,T) = \frac{1}{l_{11}^{2n}\mathcal{V}_n^2}\sum_{\{m_I,n_I\}}\int d^{11-n}p d^{11-n}q \int\prod_{r=1}^{7} d\sigma_r \times e^{-\left[G^{IJ}(\sigma m_I m_J+\lambda n_I n_J+\rho(m+n)_I(m+n)_J)+\sum_{r=1}^{7}K_r\sigma_r\right]}, \tag{4.4}$$

where $I,J=1,2$ label the directions in $\mathcal{T}^n$. The vector K_r is defined by

$$K_r = (p, p+k_1, p+k_3, q, q+k_3, q+k_4, p+q) \tag{4.5}$$

and

$$\sigma=\sigma_1+\sigma_2+\sigma_3, \quad \lambda=\sigma_4+\sigma_5+\sigma_6, \quad \rho=\sigma_7. \tag{4.6}$$

The nonplanar diagram is given by

$$I^{\mathrm{NP}}(S,T) = \frac{1}{l_{11}^{2n}\mathcal{V}_n^2}\sum_{\{m_I,n_I\}}\int d^{11-n}p d^{11-n}q \int\prod_{r=1}^{7} d\sigma_r e^{-[G^{IJ}(\sigma m_I m_J+\lambda n_I n_J+\rho(m+n)_I(m+n)_J)+\Sigma_{r=1}^{7}K_r'^2\sigma_r]}, \tag{4.7}$$

where

$$K_r' = (p, p+k_1, q, q+k_3, q+k_4, p+q+k_2) \tag{4.8}$$

and

$$\sigma = \sigma_1 + \sigma_2, \quad \lambda = \sigma_3 + \sigma_4 + \sigma_5, \quad \rho = \sigma_6 + \sigma_7. \tag{4.9}$$

The loop momentum integrals are performed in the standard manner by completing the squares in the exponent followed by Gaussian integration. We are envisioning introducing some sort of cutoff at large momenta by imposing a lower limit to the range of integration of the Schwinger parameters. The precise details will be clarified following suitable changes of variables below. Ignoring these for now, the resultant expressions for the planar and nonplanar loops are

$$\begin{aligned} I^P(S,T) = {} & \frac{\pi^{11-n}}{l_{11}^{2n}\mathcal{V}_n^2} \sum_{\{m_I,n_I\}} \int_0^\infty d\sigma d\lambda d\rho \, \frac{\sigma^2\lambda^2}{\Delta^{[(11-n)/2]}} \\ & \times e^{-G^{IJ}[\sigma m_I m_J + \lambda n_I n_J + \rho(m+n)_I(m+n)_J]} \int_0^1 dv_2 dw_2 \int_0^{v_2} dv_1 \int_0^{w_2} dw_1 \\ & \times e^{T(\sigma\lambda\rho/\Delta)(v_2-v_1)(w_2-w_1) + S[(\sigma\lambda\rho/\Delta)(v_1-w_1)(v_2-w_2) + \sigma v_1(1-v_2) + \lambda w_1(1-w_2)]} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} I^{\rm NP}(S,T) = {} & \frac{\pi^{11-n}}{l_{11}^{2n}\mathcal{V}_n^2} \sum_{\{m_I,n_I\}} \int_0^\infty d\sigma d\lambda d\rho \, \frac{2\sigma\lambda^2}{\Delta^{[(11-n)/2]}} e^{-G^{IJ}[\sigma m_I m_J + \lambda n_I n_J + \rho(m+n)_I(m+n)_J]} \\ & \times \int_0^1 du_1 dv_1 dw_2 \int_0^{w_2} dw_1 e^{T(\sigma\lambda\rho/\Delta)(w_2-w_1)(u_1-v_1) + S\{[(\sigma+\rho)\lambda^2/\Delta]w_1(1-w_2) + (\sigma\lambda\rho/\Delta)[w_1(1-u_1) + v_1(u_1-w_2)]\}} \end{aligned} \tag{4.11}$$

(where the variables u_1, v_1, v_2, w_1, and w_2 are rescalings of σ_i). These expressions can be expanded in powers of S, T, and U in order to determine their contributions to higher derivatives acting on $S^2\mathcal{R}^4$.

The leading term in the low energy expansion (of order $S^2\mathcal{R}^4$) is obtained by setting the external momenta to zero so that S, T, and U are set equal to zero in I^P and $I^{\rm NP}$. After summing these two zero-momentum contributions followed by a sum over all the diagrams required by Bose symmetrization the result is

$$\begin{aligned} I^P(0) + I^{\rm NP}(0) = {} & \frac{\pi^{11-n}}{3 l_{11}^{2n}\mathcal{V}_n^2} \sum_{\{m_I,n_I\}} \int_0^\infty d\sigma d\lambda d\rho - \frac{1}{\Delta^{[(7-n)/2]}} \\ & \times e^{-G^{IJ}[(\sigma m_I m_J + \lambda n_I n_J + \rho(m+n)_I(m+n)_J)]}, \end{aligned} \tag{4.12}$$

which is symmetric in the parameters σ, λ, and ρ. The integration in Eq. (4.12) is divergent for every value of m^I, n^I when $\Delta \sim 0$, which requires at least two of the parameters λ, ρ, σ to approach zero simultaneously. The sums contribute additional divergences, which makes this representation of the amplitude rather awkward to analyze.

As in the case of the one-loop amplitude it is convenient to analyze the divergences after performing a Poisson resummation over the Kaluza-Klein modes m_I, n_I, which transforms them into winding numbers $\hat{m}_I, \hat{n}_I$, and also to redefine the Schwinger parameters by

$$\hat{\sigma} = \frac{\sigma}{\Delta}, \quad \hat{\lambda} = \frac{\lambda}{\Delta}, \quad \hat{\rho} = \frac{\rho}{\Delta}, \tag{4.13}$$

where

$$\Delta = \sigma\lambda + \sigma\rho + \lambda\rho = \hat{\Delta}^{-1} = (\hat{\sigma}\hat{\lambda} + \hat{\sigma}\hat{\rho} + \hat{\lambda}\hat{\rho})^{-1}. \tag{4.14}$$

The amplitude (4.12) becomes

$$I^{P+\rm NP}(0) = \frac{\pi^7}{3} \sum_{\{m_I,n_I\}} \int_0^\infty d\hat{\sigma} d\hat{\lambda} d\hat{\rho} \hat{\Delta}^{1/2} e^{-\pi E\omega}, \tag{4.15}$$

where the exponent is defined by

$$E_w(\hat{\sigma},\hat{\lambda},\hat{\rho}) = G_{IJ}[\hat{\lambda}\hat{m}_I\hat{m}_J + \hat{\sigma}\hat{n}_I\hat{n}_J + \hat{\rho}(\hat{m}+\hat{n})_I(\hat{m}+\hat{n})_J], \tag{4.16}$$

and is a function of the winding numbers. The parameters $\hat{\phi}$, $\hat{\lambda}$ and $\hat{\rho}$ will be referred to as ''winding parameters.'' The classification of the divergences is simplified in the winding number basis. For example, the sector in which all the winding numbers vanish diverges at the end-point where all of the winding parameters reach their upper limits. This term is independent of the metric G_{IJ} and is the primitive

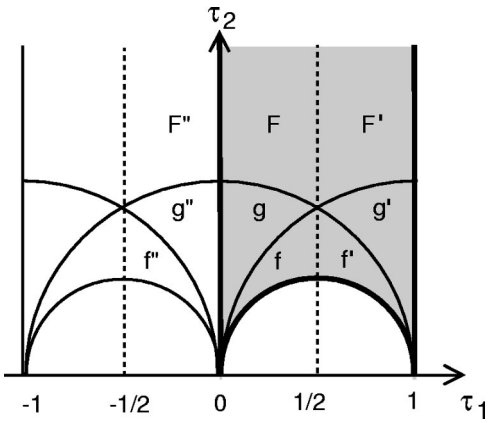

FIG. 4. The domain of integration over the parameters τ_1 and τ_2, bounded by the thick line, is the fundamental domain of $\Gamma_0(2)$.

two-loop divergence. There are many sectors that contribute to subleading divergences. The simplest examples are those sectors in which the winding numbers conjugate to a particular winding parameter vanish. In those cases the integral diverges at the endpoint where that parameter reaches its upper limit, which gives a subleading divergence. For example, the $\hat{\sigma}$ integral diverges in the $\hat{n}_I=0$ sector and behaves as Λ^3 if $\hat{\sigma}$ is cut off at the value Λ^2 (that was introduced in order to cut off the one-loop winding parameter). Sectors with less than n vanishing winding numbers give non-divergent contributions which are independent of any cutoff.

A more complete analysis of the divergences is greatly facilitated by the observation that the integrand possesses a secret SL(2,**Z**) symmetry that is not at all apparent in the $\hat{\lambda},\hat{\rho},\hat{\sigma}$ parametrization. This symmetry is made manifest by redefining the integration variables in Eq. (4.15) so that the parameters $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\phi}$, are replaced by the dimensionless volume V and complex structure $\tau=\tau_1+i\tau_2$ of a two-torus $\hat{\mathcal{T}}^2$ defined by

$$\tau_1=\frac{\hat{\rho}}{\hat{\rho}+\hat{\lambda}}, \quad \tau_2=\frac{\sqrt{\Delta}}{\hat{\rho}+\hat{\lambda}}, \quad V=l_{11}^2\sqrt{\hat{\Delta}}. \tag{4.17}$$

The Jacobian for the change of variables from $(\hat{\sigma},\hat{\lambda},\hat{\rho})$ to (V,τ) is

$$d\hat{\lambda}\,d\hat{\sigma}\,d\hat{\rho}=2l_{11}^{-6}dVV^2\frac{d^2\tau}{\tau_2^2}, \tag{4.18}$$

where $d^2\tau=d\tau_1 d\tau_2$. It is easy to see how the domain of integration of the Schwinger variables translates into the integration domain for V and τ. The volume V is integrated over $[0,\infty]$ and the domain of integration of τ is the fundamental domain of the $\Gamma_0(2)$ subgroup of SL(2,**Z**) (the shaded region in Fig. 4),

$$\mathcal{F}_{\Gamma_0(2)}=\left\{0\leqslant\tau_1\leqslant 1,\tau_2^2+\left(\tau_1-\frac{1}{2}\right)^2\geqslant\frac{1}{4}\right\}, \tag{4.19}$$

which consists of the sectors $F\oplus F'\oplus g\oplus g'\oplus f\oplus f'$. As is clear from the Fig. 4 this domain covers precisely three copies of $\mathcal{F}=F\oplus F''$, the fundamental domain of SL(2,**Z**). More concretely, in terms of the conventional generators of SL(2,**Z**):[7] region g is mapped into F'' by S; region g' is mapped into F by ST^{-1}; region f is mapped into F by TS; region f' is mapped into F'' by $T^{-1}ST^{-1}$; region F' is mapped into F'' by T^{-1}.

Substituting the change of variables (4.17) into the integral (4.15) gives

$$I^{P+\mathrm{NP}}(0)=\frac{2\pi^7}{l_{11}^8}\sum_{\{m_I,n_I\}}\int_0^\infty dVV^3\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^2} \times e^{-\pi(VG_{IJ}/l_{11}^2\tau_2)[(\hat{m}+\tau\hat{n})^I(\hat{m}+\bar{\tau}\hat{n})^J]}. \tag{4.20}$$

The integrand is precisely that which arises in one-loop diagrams in string theory compactified on $\mathcal{T}^n$ where the Lorentzian lattice is usually defined by

$$(\tau_2)^{n/2}\Gamma_{(n,n)}(G_{IJ}) =R_{11}^n\sum_{(\hat{m},\hat{n}_I)\in Z^2}e^{-\pi(VG_{IJ}/l_{11}^2\tau_2)[(\hat{m}+\tau\hat{n})^I(\hat{m}+\bar{\tau}\hat{n})^J]}. \tag{4.21}$$

In writing the integral (4.20) we have used the fact that the integrand is invariant under SL(2,**Z**) transformations to equate it to three times the integration over a single copy of $\mathcal{F}$. This invariance of the integrand can be seen by checking its transformations under T and S (with V being inert) which have a simple interpretation in terms of the original winding parameters. The T transformation is given by

$$\hat{\lambda}\to-\hat{\rho}, \quad \hat{\rho}\to 2\hat{\rho}+\hat{\lambda}, \quad \hat{\sigma}\to\hat{\sigma}+2\hat{\rho}, \tag{4.22}$$

while S is given by

$$\lambda\to\hat{\sigma}+2\hat{\rho}, \quad \hat{\rho}\to-\hat{\rho}, \quad \hat{\sigma}\to\hat{\lambda}+2\hat{\rho}. \tag{4.23}$$

The divergences of the loop amplitude are particularly easy to classify in terms of integration over τ and V (4.20). The leading and subleading divergences arise from two distinct kinds of boundaries of the integration domain.

(I) The leading divergence arises from the limit $V\to\infty$ with arbitrary fixed values of τ_1 and τ_2. We will regulate this by cutting off the upper limit at a value $V=V^c=a\Lambda^2 l_{11}^2$ (where a is an arbitrary constant) so that the amplitude is proportional to Λ^8. This is the two-loop primitive divergence which comes from the region in which the loop momenta are simultaneously of order Λ and corresponds to the region in which all three Schwinger parameters approach their lower endpoints.

(II) The three distinct kinds of subleading ultraviolet divergences arise from the region in which $\tau_2\to\infty$ with V

[7]Which are the translation T: $\tau\to\tau+1$ and the inversion S: $\tau\to-1/\tau$.

fixed, together with the SL(2,**Z**) images of this region obtained by the action of S and TS. These are the divergences which arise when one of the winding parameters approaches its upper limit, which is cut off at Λ. From Eq. (4.17) this translates into a cutoff on the upper τ_2 limit at $\tau_2^c = \Lambda^2 l_{11}^2 V^{-1}$, which means that the complex structure is integrated over the restricted fundamental domain,

$$\mathcal{F}_{\tau_2^c} = \{ -\tfrac{1}{2} \leqslant \tau_1 \leqslant \tfrac{1}{2}, \tau_2 \leqslant \tau_2^c, \tau_1^2 + \tau_2^2 \geqslant 1 \}. \qquad (4.24)$$

It is easy to see that this includes all the subdivergences, as follows. When τ_2 reaches its upper limit with fixed V, both $\hat{\rho}$ and $\hat{\lambda}$ approach zero while $\hat{\sigma}$ becomes infinite. This translates, via Eq. (4.13), into the region in which ρ and λ vanish, which corresponds to the ultraviolet subdivergence at which $q^2 \to \infty$. Similarly, the image under S of this limit is the boundary at $\tau_1 = 0$, $\tau_2 = 0$ which corresponds to the subdivergence at which $(p+q)^2 \to \infty$. The image under TS is the boundary at $\tau_1 = 1$, $\tau_2 = 0$ which corresponds to the subdivergence at which $p^2 \to \infty$.

We will now see how this description of the divergences is particularly well adapted to compactification on a toroidal target space since cases (I) and (II) arise from distinct classes of SL(2,**Z**) orbits in the mapping of $\hat{\mathcal{T}}$ into $\mathcal{T}^2$.

B. Compactification on $\mathcal{T}^2$

When the eleven-dimensional two-loop amplitude is compactified on a two-torus ($n=2$) of volume $\mathcal{V}$ and complex structure Ω the exponential factor (4.16) can be written as

$$E_w = \frac{\mathcal{V} V}{\Omega_2 \tau_2} |(1\,\Omega) A (\tau 1)|^2 - 2\mathcal{V} V \det(A), \qquad (4.25)$$

where we have used the usual formula for the metric on a two torus

$$G_{IJ} \hat{m}_I \hat{m}_J = l_{11}^2 \mathcal{V} \frac{|\hat{m}_1 + \hat{m}_2 \Omega|^2}{\Omega_2}, \qquad (4.26)$$

and defined a 2×2 matrix A with integer entries.

In this case the expression (4.20) becomes

$$\begin{aligned} I^{P+\mathrm{NP}}(0) &= \frac{2\pi^7}{l_{11}^8} \sum_{\hat{m}_I, \hat{n}_I} \int_0^{V^c} dV V^3 \int_{\mathcal{F}_{\tau_2^c}} \frac{d^2\tau}{\tau_2^2} \\ &\quad \times \exp\left(-\pi \frac{\mathcal{V}}{\Omega_2} \frac{V}{\tau_2} |(1\,\Omega) A(\tau 1)|^2 \right. \\ &\quad \left. + 2\pi \mathcal{V} V \det(A) \right) \\ &= \frac{2\pi^7}{l_{11}^8} \int_0^{V^c} dV V^3 \int_{\mathcal{F}_{\tau_2^c}} \frac{d^2\tau}{\tau_2^2} \tau_2 \Gamma_{(2,2)}(\mathcal{V}, \Omega; V, \tau). \end{aligned} \qquad (4.27)$$

This expression can be analyzed by splitting it into orbits for the right action of Sl(2,**Z**) on τ. There are three classes of orbits.

(I) Singular orbits are ones with $A=0$. By inspection, it is clear that these give the leading divergent contributions which are proportional to Λ^8 from the boundary $V = V^c = a l_{11}^2 \Lambda^2$.

(II) Degenerate orbits are those for which $\det A = 0$, A nonzero. In this case A can be transformed to the form $A = \left(\begin{smallmatrix} 0 & l \\ 0 & k \end{smallmatrix}\right)$ by a SL(2,**Z**) transformation that maps the fundamental domain to the strip. Again, by inspection these can be seen to give the subdivergences proportional to Λ^3.

(III) Nondegenerate orbits are ones with nonsingular A. An SL(2,**Z**) transformation that maps the fundamental domain to the upper half complex plane can be used to put A in the form $A = \pm \left(\begin{smallmatrix} m & j \\ 0 & n \end{smallmatrix}\right)$ where $0 \leqslant j \leqslant m-1$, $m > 0$, and $n \neq 0$ [18,19]. The nondegenerate orbits are the ones that contribute to the finite part of the two-loop integral after the divergent terms have been subtracted out. Since the string coupling constant appears in Ω both the degenerate and the nondegenerate cases will contribute to perturbative and nonperturbative string effects.

The singular term (I) comes from the zero winding number sector and does not depend on the shape of the torus. A term of the same form also arises by adding a local two-loop counterterm proportional to $S^2 \mathcal{R}^4$ to the effective action. We will argue later that consistency with string perturbation theory actually requires this renormalized coefficient of this interaction to vanish.

In order to extract the remaining contributions it is useful to first consider the action of the Laplace operator $\Delta_\Omega = 4\Omega_2^2 \partial_\Omega \overline{\partial}_\Omega$ on the expression (4.27) using the identity

$$\Delta_\Omega(\tau_2 \Gamma_{(2,2)}) = 4\tau_2^2 \partial_\tau \overline{\partial}_\tau (\tau_2 \Gamma_{(2,2)}) = \Delta_\tau (\tau_2 \Gamma_{(2,2)}). \qquad (4.28)$$

It follows that

$$\begin{aligned} \Delta_\Omega I(\mathcal{V}, \Omega) &= \frac{2\pi^7}{l_{11}^8} \int_0^{V^c} dV V^3 \int_{\mathcal{F}_{\tau_2^c}} d^2\tau (\partial_{\tau_1}^2 + \partial_{\tau_2}^2) \sum_{\{m^I, n^I\}} \\ &\quad \times \exp\left(-\pi \frac{\mathcal{V}}{\Omega_2} \frac{V}{\tau_2} |(1\,\Omega) A(\tau 1)|^2 \right. \\ &\quad \left. + 2\pi \mathcal{V} V \det(A) \right), \end{aligned} \qquad (4.29)$$

so that the τ integration at fixed V is simply the integral of a surface term which gets contributions from the boundary of the integration domain $\mathcal{F}_{\tau_2^c}$ at $\tau_2 = \tau_2^c \equiv l_{11}^2 \Lambda^2 / V$. The expression (4.29) reduces to

$$\begin{aligned} \Delta_\Omega I(\mathcal{V}, \Omega) &= \frac{2\pi^7}{l_{11}^8} \int_0^{V^c} dV V^3 \partial_{\tau_2} \sum_{A \in M(2,2,\mathbf{Z})} \\ &\quad \times \exp\left(-\pi \frac{\mathcal{V}}{\Omega_2} \frac{V}{\tau_2} |(1\,\Omega) A(\tau 1)|^2 \right. \\ &\quad \left. + 2\pi \mathcal{V} V \det(A) \right) \Big|_{r_2 = r_2^c} \end{aligned}$$

$$= -\frac{2\pi^8\mathcal{V}}{l_{11}^8}\int_0^{V^c} dV V^4 \times \sum_{A\in M(2,2,\mathbf{Z})} \partial_{\tau_2}\left(\frac{|(1\Omega)A(\tau 1)|^2}{\tau_2\Omega_2}\right) \times \exp\left(-\pi\frac{\mathcal{V}}{\Omega_2}\frac{V}{\tau_2^c}|(1\Omega)A(\tau^c 1)|^2 + 2\pi\mathcal{V}V\det(A)\right)\Bigg|_{\tau_2=\tau_2^c}. \quad (4.30)$$

In the limit $\Lambda^2\to\infty$ all the terms in the sum are exponentially suppressed apart from the degenerate orbits $A=\left(\begin{smallmatrix}0 & l\\ 0 & k\end{smallmatrix}\right)$. In this sector the exponent evaluated at the boundary $\tau_2=\tau_2^c=\Lambda^2 l_{11}^2/V$ reduces to

$$E_w=\frac{\mathcal{V}V}{\Omega_2\tau_2^c}|l+k\Omega|^2=\mathcal{V}\frac{|l+k\Omega|^2}{\Omega_2}\frac{V^2}{l_{11}^2\Lambda^2}, \quad (4.31)$$

so that Eq. (4.30) reduces to

$$\Delta_\Omega I(\mathcal{V},\Omega)=\frac{2\pi^8\mathcal{V}}{l_{11}^{12}\Lambda^4}\int_0^\infty dV V^6\sum_{(l,k)\neq(0,0)}\frac{|l+k\Omega|^2}{\Omega_2} \times e^{-\pi\mathcal{V}(|l+k\Omega|^2/\Omega_2)(v^2/l_{11}^2\Lambda^2)} = 2\pi^{9/2}\Gamma(7/2)\zeta(5)\Lambda^3 l_{11}^{-5}\mathcal{V}^{-5/2}E_{5/2}(\Omega), \quad (4.32)$$

where the upper limit is taken to infinity since the integral converges. Since $E_{5/2}(\Omega)$ satisfies the Laplace equation

$$\Delta_\Omega E_{5/2}(\Omega)=\frac{15}{4}E_{5/2}(\Omega), \quad (4.33)$$

we conclude that the two-loop integral has the form

$$I(\mathcal{V},\Omega)=a\Lambda^8+\pi^5\zeta(5)\Lambda^3 l_{11}^{-5}\mathcal{V}^{-5/2}E_{5/2}(\Omega)+I_{\rm fin}. \quad (4.34)$$

The first term, which has an undetermined value, is the leading regularized divergence and arises from the singular orbits. Its coefficient is modified by the addition of the two-loop $S^2\mathcal{R}^4$ counterterm with coefficient c_2. The second term in Eq. (4.34) is the contribution of the degenerate orbits. This has a Λ-dependent coefficient to which must be added the contribution that comes from Fig. 2(b) which includes the effect of the one-loop counterterm. As will be seen in Sec. IV C the combined coefficient is consistent with the equality of the type-IIA and -IIB string theory one-loop and two-loop amplitudes.

The term $I_{\rm fin}$ in Eq. (4.34) is independent of the cutoff and is the finite term that comes from the nondegenerate orbits and must satisfy $\Delta_\Omega I_{\rm fin}=0$. It can be evaluated explicitly using the ''unfolding trick.'' This allows one of the infinite sums in Eq. (4.27) to be used to rewrite the τ integral over $\mathcal{F}$ as an integral over the upper-half τ plane. This is similar to the analysis in Ref. [19], from which the result can be extracted in the form

$$I_{\rm fin}=\frac{4\pi^3}{l_{11}^8}\sum_{\substack{m>0,n\neq 0\\ 0\le j<m}}\int_0^\infty dV V^3\int_{C^+}\frac{d^2\tau}{\tau_2^2} \times e^{-(\mathcal{V}V/\Omega_2\tau_2)|m\tau+(j+n\Omega)|^2+2\mathcal{V}Vmn}. \quad (4.35)$$

In this case no cutoff is necessary and the result has a unique normalization. The τ_1 integration is Gaussian and can be carried out explicitly giving,

$$I_{\rm fin}=\frac{4\pi^{7/2}}{l_{11}^8}\sqrt{\frac{\Omega_2}{\mathcal{V}}}\sum_{\substack{0\le j<m\\ m>0,n\neq 0}}\frac{1}{m}\int_0^\infty dV V^3 \times\int_0^\infty\frac{d\tau_2}{\tau_2^2}\sqrt{\frac{\tau_2}{V}}e^{2\mathcal{V}Vmn}e^{-(\mathcal{V}/\Omega_2)(V/\tau_2)(m\tau_2+n\Omega_2)^2}. \quad (4.36)$$

Now setting $x=V/\tau_2$ and $y=V\tau_2$ we have

$$I_{\rm fin}=\frac{2\pi^{7/2}}{l_{11}^8}\sqrt{\frac{\Omega_2}{\mathcal{V}}}\sum_{\substack{0\le j<m\\ m>0,n\neq 0}}\frac{1}{m}\int_0^\infty dx x\int_0^\infty dy y^{1/2} \times e^{-(l_{11}^2\mathcal{V}/\Omega_2)(m^2y+n^2\Omega_2^2 x)} = 2\pi^4\zeta(3)\zeta(4)l_{11}^{-8}\mathcal{V}^{-4}. \quad (4.37)$$

C. Contribution from one-loop and two-loop counterterms

The sum of the contributions to the amplitude from Figs. 2(b) and 2(c) will be denoted $\delta A_4^{(2)}=\delta_1 A_4^{(2)}+\delta_2 A_4^{(2)}$. The term $\delta_1 A_4^{(2)}$ corresponds to Fig. 2(b) and is proportional to the one-loop counterterm so it has an overall factor of c_1, which has the value given by Eq. (3.14). The direct evaluation of this process would require a complicated sum over the different types of particles circulating in the loop. However, it is easy to check that the prescription of Ref. [7] for expressing the one-loop and two-loop supergravity diagrams in terms of scalar field theory Feynman rules generalizes to diagrams of this type, giving,

$$\delta_1 A_4^{(2)}=l_{11}^3\frac{\pi^3\kappa_{11}^6}{2(2\pi)^{22}}\hat{K}(S^2+T^2+U^2)\,\delta_1 I, \quad (4.38)$$

where the loop integral $\delta_1 I$ is given by using scalar field propagators and vertices in Fig. 2(b),

$$\delta_1 I=\int d^{11}q\,\frac{1}{q^2}\,\frac{1}{(q+k_1)^2}\,\frac{1}{(q+k_1+k_2)^2}. \quad (4.39)$$

The normalization in Eq. (4.38) can be obtained as a simple consequence of unitarity.

The evaluation of the integral (4.39) compactified on $\mathcal{T}^2$ follows closely the discussion in Sec. III of the box diagram. The only difference is that in this case there are only three internal propagators. The result is

$$\delta_1 I = \frac{\pi^{1/2}}{l_{11}^5}[(\Lambda l_{11})^5 + \mathcal{V}^{-5/2} 2\zeta(5)\Gamma(5/2)E_{5/2}(\Omega,\bar{\Omega})]. \tag{4.40}$$

The cutoff-dependent term comes from the zero winding sector, and upon inserting Eq. (4.40) in Eq. (4.38), contributes a term proportional to Λ^8 to the $\mathcal{V}$-independent part of the amplitude. Its coefficient will be absorbed into a redefinition of the coefficient a of the leading two-loop divergence in Eq. (4.34). The Ω-dependent part of Eq. (4.40) has the same form as the Λ^3 subdivergences of the two-loop amplitude in Eq. (4.34). After adding these two contributions and substituting the value of c_1 from Eq. (3.14) the net dependence on the cutoff cancels, leaving a specific finite contribution that will be discussed in the following section. The contribution to $\delta_2 A_4^{(2)}$ from the two-loop local counterterm is equal to

$$\delta_2 A_4^{(2)} = i c_2 \frac{\kappa_{11}^6}{(2\pi)^{22} l_{11}^8} \hat{K}(S^2+T^2+U^2), \tag{4.41}$$

where c_2 is a constant which, for the moment, is arbitrary.

D. Comparison of eleven-dimensional supergravity and type-II string theories

We now turn to the comparison of the results of the eleven-dimensional calculations to those of the type-II string theories. We will check that the normalization of the finite $S^2\mathcal{R}^4$ term (4.37) has precisely the value that is needed for the perturbative type-IIA and -IIB string theories to be equal at the order of one string loop. Furthermore, the value of the one-loop counterterm (3.14) will also be seen to lead to the equality of the type-IIA and -IIB string tree-level and two-loop terms. This strongly supports the impression that the two-loop contribution to $S^2\mathcal{R}^4$ does not get further contributions from higher-order Feynman diagrams.

In order to compare our two-loop supergravity results with string theory it is necessary to carefully specify our conventions. In either of the two string theories the four-graviton ampitude has the expansion at tree-level and one loop,

$$A_4^{\text{string}} = \kappa_{10}^2 \hat{K}\left[-e^{-2\phi} T + \frac{\kappa_{10}^2}{2^5\pi^6 l_s^8} I^{\text{1-loop}} + \cdots \right], \tag{4.42}$$

where the terms in the square brackets are dimensionless [recall that $2\kappa_{10}^2 = (2\pi)^7 l_s^8$] as in the analysis of Ref. [20]. The low-energy expansion of the tree-level and one-loop terms T and $I^{\text{1-loop}}$ are briefly described in the Appendix and in Ref. [20]. The one-loop and two-loop amplitudes in eleven-dimensional supergravity, together with the effects of the counterterms, are given by

$$\begin{aligned} A_4 + \delta A_4 = {} & \frac{\kappa_{11}^4}{(2\pi)^{11}} \hat{K}[I(S,T)+I(S,U)+I(T,U)] \\ & + i\frac{\kappa_{11}^6}{(2\pi)^{22}} \hat{K}[S^2(I^P(S,T)+I^{\text{NP}}(S,T)+I^P(S,U) \\ & + I^{\text{NP}}(S,U)) + \text{perms.}] + \delta A_4^{(1)} + \delta_1 A_4^{(2)} \\ & + \delta_2 A_4^{(2)}. \end{aligned} \tag{4.43}$$

The results of Sec. III show that the expansion up to order S^2 of the one-loop supergravity amplitude compactified on $\mathcal{T}^2$ is given by

$$\begin{aligned} A_4^{(1)}\delta A_4^{(1)} = {} & \frac{\kappa_{11}^4}{(2\pi)^{11} l_{11}^4} \hat{K}\left[\frac{2\zeta(3)}{R_{11}^3} + \frac{4\zeta(2)}{R_{11}R_{10}^2} + \frac{2\pi^2}{3} \right. \\ & + l_{11}^4 (S^2+T^2+U^2)\frac{\pi^2}{6!} \\ & \left. \times \left(4\zeta(2)R_{11} + 2\zeta(3)\frac{R_{10}^2}{R_{11}} \right) + \cdots \right], \end{aligned} \tag{4.44}$$

where the ellipsis indicates the infinite series of D-instanton contributions [4]. Converting into type-IIA string variables this becomes

$$\begin{aligned} A_4^{(1)} + \delta A_4^{(1)} = {} & (4\pi^8 l_{11}^{15} r_A^{-1}) r_A \hat{K}\left[2\zeta(3)e^{-2\phi^A} + \frac{4\zeta(2)}{r_A^2} \right. \\ & + \frac{2\pi^2}{3} + l_s^4(s^2+t^2+u^2)\frac{\pi^2}{6!}[4\zeta(2)e^{2\phi^A} \\ & \left. + 2\zeta(3)r_A^2] + \cdots \right]. \end{aligned} \tag{4.45}$$

The two-loop result can be written as

$$\begin{aligned} A_4^{(2)} + \delta A_4^{(2)} = {} & i\frac{\kappa_{11}^6}{(2\pi)^{22} l_{11}^8} \times \hat{K}(S^2+T^2+U^2)\left[a(\Lambda l_{11})^8 + c_2 \right. \\ & + \frac{\pi^6}{4}\left(\frac{2\zeta(5)}{R_{11}^5} + \frac{8}{3}\zeta(4)\frac{1}{R_{11}R_{10}^4} \right) \\ & \left. + \frac{2\pi^4\zeta(3)\zeta(4)}{(R_{10}R_{11})^4} + \cdots \right], \end{aligned} \tag{4.46}$$

where we have expanded the modular function $E_{5/2}(\Omega,\bar{\Omega})$ in powers of $\Omega_2^{-1} = R_{11}/R_{10}$. The constant a is meant to represent the sum of the primitive two-loop divergences that arise from the zero winding number sectors of Fig. 3 and Fig. 2(b). These combine with the coefficient of the two-loop counterterm c_2. After conversion into type-IIA string variables this becomes

$$A_4^{(2)}+\delta A_4^{(1)}=(4\pi^8 l_{11}^{15} r_A^{-1})\frac{i}{8\pi^6} r_A \hat{K} l_s^4 (s^2+t^2+u^2) \times\Bigg[(a(\Lambda l_{11})^8+c_2)e^{4\phi^A/3}+\frac{\pi^6}{4}\Bigg(2\zeta(5)e^{-2\phi^A}+\frac{8}{3}\zeta(4)\frac{e^{2\phi^A}}{r_A^4}\Bigg)+\frac{2\pi^4\zeta(3)\zeta(4)}{r_A^4}+\cdots\Bigg]. \tag{4.47}$$

Now we can use T-duality to replace the last term in Eq. (4.47), which has the dilaton dependence of a string one-loop term, by its type-IIB equivalent, which is proportional to r_B^2 and is also identified as a one-loop string contribution. Using the fact that the two string theories have identical four-graviton loop amplitudes (up to two loops) we must identify this term with the term in parentheses in Eq. (4.45) that is proportional to r_A^2 (which was deduced from a one-loop effect in eleven-dimensional supergravity). Gratifyingly the coefficients of these terms are indeed equal [using $\zeta(4)=\pi^4/90$] which appears to be a rather nontrivial check on our calculation. Similarly, we can check that the renormalized value for the subleading divergences respects this symmetry between the type-IIA and -IIB theories since the term proportional to $e^{2\phi^A}$ in Eq. (4.45) has the same coefficient as the term proportional to $e^{2\phi^A}$ in Eq. (4.47). Now we can check the consistency further by comparing the coefficient of the tree-level term proportional to $\zeta(5)e^{2\phi^A}s^2\mathcal{R}^4$ in Eq. (4.47) with the coefficient of the tree-level term proportional to $\zeta(3)e^{2\phi^A}\mathcal{R}^4$ in Eq. (4.45). These coefficients agree with the coefficients deduced from the expansion of the four-graviton tree amplitude reviewed in the Appendix.

We also need to consider the value of the leading divergent contribution to the $S^2\mathcal{R}^4$ interaction which arises from the loop amplitudes combined with the two-loop counterterm with coefficient c_2 and is independent of the parameters of the two-tours. Translating into type-IIA string theory coordinates this gives the term proportional to $R_{11}^2 s^2\mathcal{R}^4=e^{4\phi^A/3}s^2\mathcal{R}^4$ in Eq. (4.47) which is not proportional to an integer power of $e^{2\phi^A}$. Therefore, it cannot possibly arise in string perturbation theory, which means that its coefficient must vanish, i.e.,

$$c_2+a(\Lambda l_{11})^8=0. \tag{4.48}$$

A consequence of this is that the $D^4\mathcal{R}^4$ interaction vanishes in the decompactification limit, $R_{11}\to\infty$ so there is no $D^4\mathcal{R}^4$ interaction in the eleven-dimensional theory.

V. CONCLUSION

In earlier work the $\mathcal{R}^4$ interaction in the effective action for eleven-dimensional M-theory compactified on $\mathcal{T}^2$ was obtained by evaluating one-loop Feynman diagrams. Although the dependence on the complex structure was uniquely determined by this supergravity calculation, in order to pin down the value of the one-loop counterterm it was necessary to input the extra information that the four-graviton amplitudes in type-IIA and type-IIB superstring theories are equal at one string loop.

In this paper we have generalized these statements to obtain the exact scalar field dependence of the coefficient of the $S^2\mathcal{R}^4$ interaction based on consideration of two-loop Feynman diagrams for four-graviton scattering in eleven-dimensional supergravity. Using the value of the one-loop counterterm determined from the one-loop analysis, we have found that the renormalized value of the two-loop amplitude is

$$S_{D^4\mathcal{R}_4}=\frac{l_{11}^3}{48\times(4\pi)^7}\int d^9x\sqrt{-G}\mathcal{V}D^4\mathcal{R}^4\Bigg(\zeta(5)\mathcal{V}^{-5/2}E_{5/2}(\Omega,\bar{\Omega})+\frac{4}{\pi^2}\zeta(3)\zeta(4)\mathcal{V}^{-4}\Bigg) =\frac{l_s^3}{48\times(4\pi)^7}\int d^9x\sqrt{-g^B}r_B D^4\mathcal{R}^4\Bigg(\zeta(5)e^{(1/2)\phi^B}E_{5/2}(\Omega,\bar{\Omega})+\frac{4}{\pi^2}\zeta(3)\zeta(4)r_B^2\Bigg), \tag{5.1}$$

where the second equality expresses the amplitude in type-IIB parameters, recalling from Eq. (2.9) that $D^4\mathcal{R}^4$ is a symbolic way of representing the contraction of covariant derivatives and curvature tensors that gives rise to the kinematic factor $(S^2+T^2+U^2)\hat{K}$ in the four-graviton scattering amplitude.

The term in parentheses in Eq. (5.1) that is independent of Ω matches a corresponding term that arises in the type-IIA parametrization from the one-loop supergravity amplitude in Eq. (3.32). It should be easy to evaluate the string one-loop amplitude in nine dimensions and check the coefficient of this term. We saw in Sec. III that terms of this type, which appear to be singular in the decompactification limit $r_B\to\infty$, sum up to form the appropriate massless threshold singularity in ten dimensions. The dependence on the complex structure of the torus (the scalar field of the type-IIB theory) is contained entirely in the modular function $E_{5/2}(\Omega,\bar{\Omega})$ which survives the decompactification limit to the ten-dimensional type-IIB theory. This term has an expansion in the coupling e^{ϕ^B} that begins with a tree-level term followed by a two-loop term and then an infinite series of D-instanton contributions. We have seen that this is consistent with the little that is known from string perturbation theory—the tree-

level coefficient agrees with the string tree calculation reviewed in the appendix and the one string loop contribution to the $s^2\mathcal{R}^4$ interaction in ten dimensions is absent as it should be according to Ref. [20]. However, since no two-loop string amplitudes have yet been evaluated, the value we have obtained for the $s^2\mathcal{R}^4$ interaction at two string loops is not yet tested (although precise two-loop string calculations are feasible in principle [21]). The same is true for the infinite sequence of D-instanton contributions to this interaction.

Although we saw in the last section that the $S^2\mathcal{R}^4$ term cannot contribute to the eleven-dimensional theory in the decompactification limit $\mathcal{V}\to\infty$ the next term in the derivative expansion of the two-loop amplitude may. This is the term of the form $S^3\mathcal{R}^4$ which translates in type-IIA string parameters to a term of the form $e^{2\phi^A}s^3\mathcal{R}^4$ which would be a string two-loop effect. When $\mathcal{V}$ is finite there are other supergravity two-loop contributions to the prefactor multiplying $S^3\mathcal{R}^4$ which come from the expansion of the planar and nonplanar diagrams (4.10) and (4.11) to linear order in S, T, and U. The resulting expressions do not possess modular invariant integrands when expressed in terms of the integration variables V and τ and we have not made sense of the integrals. This suggests that extra contributions from higher-loop supergravity amplitudes are needed to give the full form of the prefactor. Since there are good dimensional arguments to expect this prefactor to be determined by supersymmetry it would be of interest to disentangle these contributions.

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APPENDIX: THE STRUCTURE OF THE TREE-LEVEL AND ONE-LOOP STRING CONTRIBUTIONS

The sum of the tree-level and one-loop contributions to the four-graviton amplitude in ten dimensions has the form [16,22,23]

$$A_4^{\rm string}=\kappa_{10}^2\hat{K}\Big[-e^{-2\phi}T(s,t,u)+\frac{\kappa_{10}^2}{2^5\pi^6 l_s^8}\int_{\mathcal{F}}\frac{d^2\Omega}{\Omega_2^2}F(\Omega,\bar{\Omega};s,t,u)\Big], \tag{A1}$$

where the functions T and F contain the dependence on the Mandelstam invariants of the tree-level and one-loop terms, respectively, $2\kappa_{10}^2=(2\pi)^7 l_s^8$ is defined as in Ref. [10] and $d^2\Omega=d\Omega_1 d\Omega_2$.

The function T contains the dynamical part of the tree amplitude for the elastic scattering of two gravitons in either type-II theory and is given

$$\begin{aligned}T&=\frac{64}{l_s^6 stu}\frac{\Gamma\left(1-\frac{l_s^2}{4}s\right)\Gamma\left(1-\frac{l_s^2}{4}t\right)\Gamma\left(1-\frac{l_s^2}{4}u\right)}{\Gamma\left(1+\frac{l_s^2}{4}s\right)\Gamma\left(1+\frac{l_s^2}{4}t\right)\Gamma\left(1+\frac{l_s^2}{4}u\right)}\\&=\frac{64}{l_s^6 stu}\exp\left(\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}\frac{l_s^{4n+2}}{4^{2n+1}}(s^{2n+1}+t^{2n+1}+u^{2n+1})\right).\end{aligned} \tag{A2}$$

Thus, the low-energy expansion of the amplitude begins with the terms

$$T=\frac{64}{l_s^6 stu}+2\zeta(3)+\frac{\zeta(5)}{16}l_s^4(s^2+t^2+u^2)+\frac{\zeta(3)^2}{96}l_s^6(s^3+t^3+u^3)+\frac{\zeta(7)}{512}l_s^8(s^2+t^2+u^2)^2+\cdots. \tag{A3}$$

This can be rewritten in terms of the coordinates of the eleven-dimensional theory by using the dictionary (2.4). When expressed in terms of the Mandelstam invariants in the M-theory metric the expression (A3) has the low-energy expansion

$$TR_{11}^{-3}=\frac{64}{l_{11}^6 STU}+\frac{2\zeta(3)}{R_{11}^3}+\frac{\zeta(5)}{16R_{11}^5}l_{11}^4(S^2+T^2+U^2)+\frac{\zeta(3)^2}{96R_{11}^6}l_{11}^6(S^3+T^3+U^3)+\frac{\zeta(7)}{512R_{11}^7}l_{11}^8(S^2+T^2+U^2)^2, \tag{A4}$$

The dynamical factor F in the loop amplitude is given in terms of the scalar Green function on the torus $\ln\chi_{ij}$,

$$F(\Omega,\bar{\Omega};s,t,u)=\int_{\mathcal{T}^2}\frac{d^2\Omega}{\Omega_2^2}\prod_{i=1}^{3}\frac{d^2\nu^{(i)}}{\Omega_2}(\chi_{12}\chi_{34})^{l_s^2 s}(\chi_{14}\chi_{23})^{l_s^2 t}(\chi_{13}\chi_{24})^{l_s^2 u}. \tag{A5}$$

The low-energy expansion of this expression is considered in Ref. [20] where the first few terms are shown to be

$$A_4^{\text{one-loop}} = \hat{K}\left(\frac{\pi}{6} + \frac{l_s^2}{16} I_{\text{nonan1}} + 0 \times l_s^4 (s^2+t^2+u^2) - \frac{2\pi}{288}\zeta(3) l_s^6 (s^3+t^3+u^3) + \frac{l_s^8}{256} I_{\text{nonan2}} + \cdots\right)$$
$$= \hat{K}\left(\frac{\pi}{6} + \frac{l_{11}^2}{16} I_{\text{nonan1}} + 0 \times l_{11}^2 (S^2+T^2+U^2) - \frac{2\pi}{288}\zeta(3) l_{11}^6 (S^3+T^3+U^3) + \frac{l_{11}^4}{256} I_{\text{nonan2}} + \cdots\right). \tag{A6}$$

The functions I_{nonan1} and I_{nonan2} are nonanalytic terms that contain logarithmic contributions to the two-particle normal thresholds that are defined by I' in Eq. (3.15) and are given more explicitly in Ref. [20].

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