PHYSICAL REVIEW D, VOLUME 61, 101503(R)

Gravitational waves from quasispherical black holes

Sean A. Hayward*
Center for Gravitational Physics and Geometry, 104 Davey Laboratory, The Pennsylvania State University, University Park, Pennsylvania 16802-6300

(Received 8 February 2000; published 21 April 2000)

A quasispherical approximation scheme, intended to apply to coalescing black holes, allows the waveforms of gravitational radiation to be computed by integrating ordinary differential equations.

PACS number(s): 04.30.−w, 04.20.Ha, 04.25.−g, 04.70.Bw

The coalescence of binary black holes is expected to be one of the main astrophysical sources for upcoming gravitational-wave detectors. The initial phase of inspiral and the final phase of ringdown are understood in terms of post-Newtonian and close-limit approximations respectively, but the coalescence is only qualitatively understood and generally thought to be tractable only by numerical methods [1–4]. Considerable problems have been encountered and currently there are no reliable predictions of waveforms.

This article presents an approximation scheme to compute the gravitational waveforms for space-times close to spherical symmetry. This is intended to apply to binary black holes once they have coalesced, i.e., when a marginal surface encloses both sources. The quasispherical approximation will be best when the angular momentum is small, but note that even the maximally rotating Kerr black hole is 70% spherically symmetric according to the ratio of the areal and equatorial radii of the horizon. Thus rough estimates may still be possible even for appreciable angular momentum.

The basic idea of a quasispherical approximation is to make a 2+2 decomposition of the space-time and linearize only those parts of the extrinsic curvature which vanish in spherical symmetry, cf. Bishop *et al.* [5]. Thus when the linearized fields vanish, spherical symmetry is recovered in full. This can be a highly dynamical situation; there will be no assumption of quasistationarity. Likewise, there will be no assumption of an exactly spherical background. Unlike previous work on null-temporal formulations [4–7], a dual-null formulation is adopted here, i.e., a decomposition of the space-time by two intersecting foliations of null hypersurfaces. This is adapted to the radiation problem in that the imposition of no ingoing radiation and the extraction of the outgoing radiation are immediate. It also allows a remarkable simplification from partial to ordinary differential equations.

A general Hamiltonian theory of dual-null dynamics [8] has been applied to Einstein gravity [9] and is summarized as follows. Denoting the space-time metric by g and labeling the null hypersurfaces by $x^\pm$, the normal 1-forms $n^\pm = -dx^\pm$ therefore satisfy

$$g^{-1}(n^\pm,n^\pm)=0. \tag{1}$$

The relative normalization of the null normals may be encoded in a function f defined by

$$e^f=-g^{-1}(n^+,n^-). \tag{2}$$

Then the induced metric on the transverse surfaces, the spatial surfaces of intersection, is found to be

$$h=g+2e^{-f}n^+\otimes n^- \tag{3}$$

where $\otimes$ denotes the symmetric tensor product. The dynamics is described by Lie transport along two commuting evolution vectors $u_\pm$:

$$[u_+,u_-]=0. \tag{4}$$

Specifically, the evolution derivatives, to be discretized in a numerical code, are

$$\Delta_\pm=\perp L_{u_\pm} \tag{5}$$

where $\perp$ indicates projection by h and L denotes the Lie derivative. There are two shift vectors

$$s_\pm=\perp u_\pm. \tag{6}$$

In a coordinate basis $(u_+,u_-;e_i)$ such that $u_\pm=\partial/\partial x^\pm$, where $e_i=\partial/\partial x^i$ is a basis for the transverse surfaces, the metric takes the form

$$g=h_{ij}(dx^i+s_+^i dx^+ + s_-^i dx^-)\otimes(dx^j+s_+^j dx^+ + s_-^j dx^-)-2e^{-f}dx^+\otimes dx^-. \tag{7}$$

Then $(h,f,s_\pm)$ are configuration fields and the independent momentum fields are found to be linear combinations of

$$\theta_\pm={*}L_\pm{*}1 \tag{8}$$

$$\sigma_\pm=\perp L_\pm h-\theta_\pm h \tag{9}$$

$$\nu_\pm=L_\pm f \tag{10}$$

$$\omega=\tfrac{1}{2}e^f h([l_-,l_+]) \tag{11}$$

where "*" is the Hodge operator of h and $L_\pm$ is shorthand for the Lie derivative along the null normal vectors

$$l_\pm=u_\pm-s_\pm=e^{-f}g^{-1}(n^\mp). \tag{12}$$

Then the functions $\theta_\pm$ are the expansions, the traceless bilinear forms $\sigma_\pm$ are the shears, the 1-form ω is the twist, measuring the lack of integrability of the normal space, and

*Email address: hayward@gravity.phys.psu.edu

the functions $\nu_\pm$ are the inaffinities, measuring the failure of the null normals to be affine. The fields $(\theta_\pm,\sigma_\pm,\nu_\pm,\omega)$ encode the extrinsic curvature of the dual-null foliation. These extrinsic fields are unique up to duality $\pm\mapsto\mp$ and diffeomorphisms which relabel the null hypersurfaces, i.e., $dx^\pm\mapsto e^{\lambda_\pm}dx^\pm$ for functions $\lambda_\pm(x^\pm)$.

It is also useful to decompose h into a conformal factor r and a conformal metric k by

$$h=r^2k \tag{13}$$

such that

$$\Delta_\pm \hat{*}1=0 \tag{14}$$

where "$\hat{*}$" is the Hodge operator of k, satisfying $*1=\hat{*}r^2$. Denoting the covariant derivative of h by D, the Ricci scalar of h is found to be

$$R=2r^{-2}(1-D^2\ln r) \tag{15}$$

by using the coordinate freedom on a given surface to fix k as the metric of a unit sphere.

The dual-null Hamilton equations and integrability conditions for vacuum Einstein gravity have been given previously [9] in a slightly different notation, so it will not be repeated here. They are linear combinations of the vacuum Einstein equation and a first integral of the contracted Bianchi identity. This is the vacuum Einstein system in first-order dual-null form. The vacuum case suffices for the application, outside the black holes.

In spherical symmetry, $(s_\pm,\sigma_\pm,\omega,D)$ vanish, while $(h,f,\theta_\pm,\nu_\pm,\Delta_\pm)$ are generally nonzero, e.g., [10,11]. The quasispherical approximation will therefore consist of linearizing in $(s_\pm,\sigma_\pm,\omega,D)$. In practice, one truncates the equations by setting to zero any second-order terms in $(s_\pm,\sigma_\pm,\omega,D)$. This greatly simplifies the equations, leaving the momentum definitions as

$$\Delta_\pm r=\tfrac{1}{2}r\theta_\pm \tag{16}$$

$$\Delta_\pm f=\nu_\pm \tag{17}$$

$$\Delta_\pm k=r^{-2}\sigma_\pm \tag{18}$$

$$\Delta_+ s_- -\Delta_- s_+=2e^{-f}h^{-1}(\omega) \tag{19}$$

and the remaining equations as

$$\Delta_\pm\theta_\pm=-\nu_\pm\theta_\pm-\tfrac{1}{2}\theta_\pm^2 \tag{20}$$

$$\Delta_\pm\theta_\mp=-\theta_+\theta_- -e^{-f}r^{-2} \tag{21}$$

$$\Delta_\pm\nu_\mp=-\tfrac{1}{2}\theta_+\theta_- -e^{-f}r^{-2} \tag{22}$$

$$\Delta_\pm\sigma_\mp=\tfrac{1}{2}(\theta_\pm\sigma_\mp-\theta_\mp\sigma_\pm) \tag{23}$$

$$\Delta_\pm\omega=-\theta_\pm\omega\pm\tfrac{1}{2}(D\nu_\pm-D\theta_\pm-\theta_\pm Df). \tag{24}$$

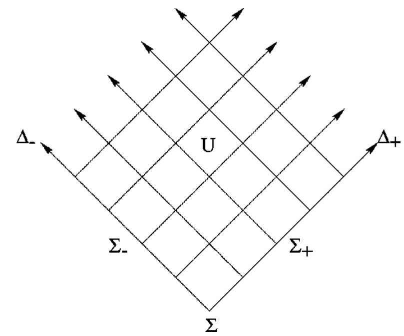

FIG. 1. The dual-null integration scheme. Initial data is prescribed on a spatial surface Σ and the null hypersurfaces Σ_+ and Σ_- generated from it.

This follows immediately from the full equations [9], the above expression for R, and the fact that

$$\Delta_\pm=\perp L_\pm \tag{25}$$

in this truncation. One may take quasispherical coordinates $x^i=(\vartheta,\varphi)$ on the transverse surfaces such that $\hat{*}1=\sin\vartheta d\vartheta\wedge d\varphi$, the standard area form of a unit sphere. Then r is the quasispherical radius.

The shear equations, composed into a second-order equation for k, become

$$\Box k=0 \tag{26}$$

where $\Box$ is the quasispherical wave operator:

$$\Box\phi=-2e^f(\Delta_{(+}\Delta_{-)}\phi+2r^{-1}\Delta_{(+}r\Delta_{-)}\phi). \tag{27}$$

Thus the conformal metric k satisfies the quasispherical wave equation. Then k may be interpreted as encoding the gravitational radiation. In particular, fixing u_+ to be the outgoing direction, the Bondi news at future null infinity $\mathfrak{I}^+$ is essentially $r^{-1}\sigma_-$ [12], as described explicitly below. Likewise, the no-ingoing-radiation condition is just $r^{-1}\sigma_+=0$ at past null infinity $\mathfrak{I}^-$. That k generally encodes the free gravitational data was suggested by d'Inverno and Stachel [13] and has been rediscovered by various authors, e.g., [7,9].

The dual-null initial-data formulation is based on a spatial surface Σ and the null hypersurfaces $\Sigma_\pm$ locally generated from Σ in the $u_\pm$ directions. The structure of the field equations shows that one may specify $(h,f,\theta_\pm,\omega)$ on Σ, (σ_+,ν_+) on Σ_+, (s_-,σ_-,ν_-) on Σ_-, and s_+ in U, a region to the future of $\Sigma_\pm$. In particular, the initial data is freely specifiable. There are no constraints as in the 3+1 formulation; these have been converted into evolution equations along $\Sigma_\pm$, which even in the general (nonquasispherical) case can be solved in closed form [14].

A numerical integration scheme runs as follows, as depicted in Fig. 1. First integrate the Δ_+ equations from Σ to obtain the full data on Σ_+. Then integrate the Δ_- equations one step along each ingoing null hypersurface, generating a new null hypersurface Σ'_+. Then repeat: integrate the Δ_+ equations along Σ'_+ to obtain the full data on Σ'_+, and so on. In practice, some interpolation between the two integrations is useful. There are many ways to perform the integrations in a different order, allowing flexibility which can be used, for instance, to avoid singularities. Any such scheme gives two estimates of $(r,k,f,\theta_\pm,\omega)$ at each point, since some of the equations play the role of integrability conditions. Thus one could ignore such equations to obtain a free rather than constrained integration scheme. This allows numerous internal checks on the accuracy of the numerical code, analogous to those of 3+1 integration schemes, e.g., Choptuik [15].

The equations for $\Delta_\pm(r,f,\theta_\pm,\nu_\pm)$, the quasispherical equations, decouple from the remaining equations. Thus there is a quasispherical background which may be found by integrating the quasispherical initial data. Since this background is independent of the linearized part, one may economize when computing different evolutions on the same background. It should be stressed that the quasispherical background is neither fixed in advance nor necessarily spherical, e.g., $Dr\neq 0$ in general.

To compute the outgoing radiation, one now needs only to integrate the equations for $(\Delta_\pm k,\Delta_\pm\sigma_\mp)$, i.e., the quasispherical wave equation for k. It is remarkable that this entire integration scheme involves only ordinary differential equations. The equations for $\Delta_\pm\omega$ are partial differential equations, containing transverse D derivatives, but the other equations decouple from the equations for $(s_\pm,\omega)$, which therefore need not be solved for the radiation problem. In short, most of the complexity of the system has been isolated and sidestepped.

Moreover, one may use a conformal transformation to obtain a scheme which is more accurate at large distances. Using the conformal factor

$$\Omega=r^{-1} \tag{28}$$

the rescaled expansions and shears

$$\vartheta_\pm=r\theta_\pm \tag{29}$$

$$\varsigma_\pm=r^{-1}\sigma_\pm \tag{30}$$

are finite and generally nonzero at $\Im^\mp$ for an asymptotically flat space-time. Rewriting the relevant equations yields the quasispherical equations

$$\Delta_\pm\Omega=-\tfrac{1}{2}\Omega^2\vartheta_\pm \tag{31}$$

$$\Delta_\pm f=\nu_\pm \tag{32}$$

$$\Delta_\pm\vartheta_\pm=-\nu_\pm\vartheta_\pm \tag{33}$$

$$\Delta_\pm\vartheta_\mp=-\Omega(\tfrac{1}{2}\vartheta_+\vartheta_-+e^{-f}) \tag{34}$$

$$\Delta_\pm\nu_\mp=-\Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_-+e^{-f}) \tag{35}$$

and the linearized equations

$$\Delta_\pm k=\Omega\varsigma_\pm \tag{36}$$

$$\Delta_\pm\varsigma_\mp=-\tfrac{1}{2}\Omega\vartheta_\mp\varsigma_\pm\,. \tag{37}$$

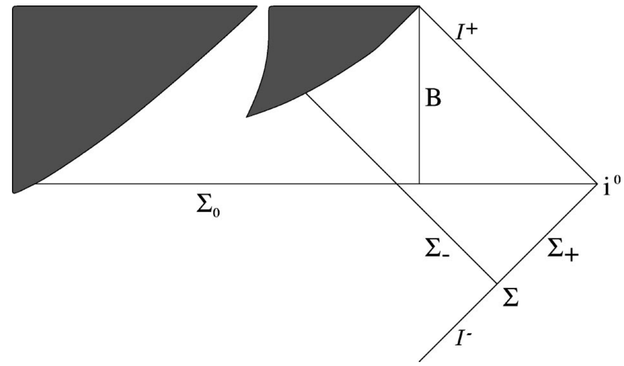

FIG. 2. Application to coalescing black holes. Shading indicates regions containing trapped surfaces, with the outermost trapped region being that of the coalesced black hole and the inner trapped region that of one of the original black holes. Initial data on the ingoing null hypersurface Σ_- may be extracted from a conventional 3+1 code based on an initial spatial hypersurface Σ_0 and outer boundary B.

One may take Σ_+ to be either part of $\Im^-$, as depicted in Fig. 2, or at sufficiently large distance for numerical purposes. Here, large distance means small Ω. For the quasispherical approximation to be valid at large distance, one may fix $(f,\vartheta_\pm,k)=(0,\pm\sqrt{2},\epsilon)$ on Σ, where $\epsilon=d\vartheta\otimes d\vartheta+\sin^2\vartheta d\varphi\otimes d\varphi$ is the standard metric of a unit sphere, and $\nu_+=0$ on Σ_+. The remaining coordinate data is given by ν_- on the ingoing null hypersurface Σ_-, which is left free so that one may adapt the foliation of Σ_- to the surfaces which are most spherical.

The no-ingoing-radiation condition is $\varsigma_+=0$ at $\Im^-$, leaving the gravitational initial data as ς_- on Σ_-. The outgoing radiation is found by computing ς_- at $\Im^+$, which is essentially the Bondi news. More precisely, the Bondi energy flux at $\Im^\mp$ would be [12]

$$\psi_\pm=-\frac{e^f\vartheta_\mp||\varsigma_\pm||^2}{64\pi} \tag{38}$$

where $||\sigma||^2=k^{ab}k^{cd}\sigma_{ac}\sigma_{bd}$ and such second-order terms are no longer being ignored. That is, the energy supply would be

$$\Delta_\pm E=\oint\hat{*}\psi_\pm \tag{39}$$

where E is the Bondi energy. In summary, the outgoing waveforms and their energy may be computed by integrating nine first-order ordinary differential equations and their duals, or a subset in the case of free evolution. For numerical purposes, this is a dramatic simplification. Numerical implementation of this scheme is in progress [16].

Before concluding, it should be noted that the domain of validity of the quasispherical approximation is not known in a precise sense. The guarantee is simply that spherically symmetric Einstein gravity is recovered in full when the linearized fields vanish. For the usual perturbative approximations, one may check successive orders of approximation to compare accuracy, but for the quasispherical approximation, the corresponding second-order approximation would be full Einstein gravity. If one wishes to know whether a given space-time is sufficiently spherical, the rough answer is that there should be a 2+2 decomposition such that the fields to be linearized are small compared to the remaining part. This depends on the choice of transverse surfaces, so that there will be some art to choosing the 2+2 foliation for optimal accuracy. For the Kerr black hole, the quasispherical null coordinates of Pretorius and Israel [17] may be useful. For a coalesced black hole, one might base the foliation on the marginal surfaces which locally define it, i.e., use the coordinate freedom in ν_- on Σ_- so that the foliation contains a marginal surface. There are some general laws of black-hole dynamics in terms of marginal surfaces and the trapping horizons they generate [18,19], including that outer trapping horizons are achronal and therefore cannot causally influence $\Im^+$. Thus Σ_- need not extend inside the trapped region in order for the domain U of integration to reach all of $\Im^+$.

To conclude, the intended scenario is an asymptotically flat space-time containing coalescing black holes, with an ingoing null hypersurface Σ_- chosen to intersect the coalesced black hole, i.e., the region of future trapped surfaces enclosing the original black holes. The initial data on Σ_- may be determined by extracting the relevant data from a conventional 3+1 numerical computation from an initial spatial hypersurface Σ_0, smoothed off to the past of Σ_0. The smoothing may be expected not to affect the results significantly if $\Sigma_- \cap \Sigma_0$ is sufficiently outside the black holes; there will be some spurious radiation at $\Im^+$ at early times, but not at the relevant late times, as this would involve backscatter of backscatter.

An advantage of this procedure is that it can avoid the outer-boundary problems which affect the conventional codes. Because the physically appropriate boundary conditions are not known, there is always some spurious gravitational radiation from the outer boundary which propagates inwards. In numerical simulations this can be seen to affect the coalesced black hole, for instance increasing its area, thereby rendering subsequent evolution unreliable. However, as depicted in Fig. 2, the outer boundary B cannot causally influence Σ_- if it intersects Σ_0 inside B. Thus the scheme requires only clean data from the 3+1 computation, uncontaminated by outer-boundary problems. A code implementing the scheme may be regarded as a black box which, taking input from any other code from which the required data on Σ_- can be extracted, computes approximate waveforms for the gravitational radiation. For this to work in practice, the outer boundary must be at sufficiently large distance that Σ_- does indeed intersect the coalesced black hole. This would be fine for the type of simulations mentioned above, where the black holes are close to coalescence, but currently impractical for a code which follows several orbital rotations before coalescence.

This suggests a quite general proposal to compute outgoing gravitational radiation from a 3+1 computation by a conformal dual-null code which extracts data on an ingoing null hypersurface intersecting the initial spatial hypersurface inside its outer boundary. One might expect this to be simpler than the usual matching on a temporal hypersurface [4–7] since the outer boundary is avoided and the problem is merely of extraction rather than dynamic matching. At present, there seems to be neither a general conformal dual-null code nor work on data extraction on a null hypersurface, though the null-temporal formulation can presumably be adapted [7]. These are tractable projects which would allow accurate computation of gravitational waveforms from coalescing black holes.

The author thanks Pablo Laguna, Luis Lehner, Keith Lockitch, Hisa-aki Shinkai, and Jeff Winicour for discussions, and Abhay Ashtekar and the Center for Gravitational Physics and Geometry for hospitality. This research was supported by the National Science Foundation under grant PHY-9800973.

[1] Binary Black Hole Grand Challenge Alliance, G.B. Cook *et al.*, Phys. Rev. Lett. **80**, 2512 (1998).

[2] P.R. Brady, J.D.E. Creighton, and K.S. Thorne, Phys. Rev. D **58**, 061501 (1998).

[3] J. Pullin, Prog. Theor. Phys. Suppl. **136**, 107 (1999).

[4] Binary Black Hole Grand Challenge Alliance, R. Gómez *et al.*, Phys. Rev. Lett. **80**, 3915 (1998).

[5] N.T. Bishop, R. Gómez, L. Lehner, and J. Winicour, Phys. Rev. D **54**, 6153 (1996).

[6] N.T. Bishop, R. Gómez, L. Lehner, M. Maharaj, and J. Winicour, Phys. Rev. D **56**, 6298 (1997).

[7] J. Winicour, Prog. Theor. Phys. Suppl. **136**, 57 (1999).

[8] S.A. Hayward, Ann. Inst. Henri Poincaré Phys. Theor. **59**, 399 (1993).

[9] S.A. Hayward, Class. Quantum Grav. **10**, 779 (1993).

[10] S.A. Hayward, Phys. Rev. D **53**, 1938 (1996).

[11] S.A. Hayward, Class. Quantum Grav. **15**, 3147 (1998).

[12] S.A. Hayward, Class. Quantum Grav. **11**, 3037 (1994).

[13] R.A. d'Inverno and J. Stachel, J. Math. Phys. **19**, 2447 (1978).

[14] S.A. Hayward, Class. Quantum Grav. **10**, 773 (1993).

[15] M.W. Choptuik, Phys. Rev. D **44**, 3124 (1991).

[16] H. Shinkai and S.A. Hayward (in preparation).

[17] F. Pretorius and W. Israel, Class. Quantum Grav. **15**, 2289 (1998).

[18] S.A. Hayward, Phys. Rev. D **49**, 6467 (1994).

[19] S. Mukohyama and S.A. Hayward, ''Quasi-local first law of black-hole dynamics,'' gr-qc/9905085.