# QED effective action at finite temperature: Two-loop dominance

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We calculate the two-loop effective action of QED for arbitrary constant electromagnetic fields at finite temperature $T$ in the limit of $T$ much smaller than the electron mass. It is shown that in this regime the two-loop contribution always exceeds the influence of the one-loop part due to the thermal excitation of the internal photon. As an application, we study light propagation and photon splitting in the presence of a magnetic background field at low temperature. We furthermore discover a thermally induced contribution to pair production in electric fields.

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## I. INTRODUCTION

In the low-energy sector of the theory, the effects of quantum electrodynamics can be summarized in an effective action, the Heisenberg-Euler action, which enlarges the classical theory of electrodynamics by non-linear self-interactions of the electromagnetic field. Technically speaking, this effective action arises from integrating out the massive (high-energy) degrees of freedom of electrons and positrons. This program has successfully been carried out to two-loop order [1–3].

The inclusion of finite-temperature effects at the one-loop level has also been considered in various papers [4–9], and the real-time [7] as well as the imaginary-time formalism [9] finally arrived at congruent results.

This paper is devoted to an investigation of the thermal QED effective action at the two-loop level. But contrary to the zero-temperature case, where the two-loop contribution only represents a 1%-correction to the one-loop effective action, we demonstrate that the thermal two-loop contribution is of a qualitatively different kind than the thermal one-loop part and exceeds the latter by far in the low-temperature domain.

The simple physical reason for this is the following: at one loop, one takes only the massive electrons and positrons as virtual loop particles into account [cf. Fig. 1(a)]. Because of the mass gap in the fermion spectrum, a heat bath at temperatures much below the electron mass $m$ can hardly excite higher fermion states. Hence, one expects thermal one-loop effects to be suppressed by the electron mass. In fact, in a low-temperature expansion of the thermal one-loop effective action [10], one finds that each term is accompanied by a factor of $\exp(-m/T)$, exhibiting an exponential damping for $T\to 0$.

On the other hand, the two-loop contribution to the thermal effective action involves a virtual photon within the fermion loop [cf. Fig. 1(b)]. Since the photon is massless, a heat bath of arbitrarily low temperature can easily excite higher photon states, implying a comparably strong influence of thermal effects on the effective action. In Sec. II, we are able to show that the dominant contribution to the thermal two-loop effective action in the low-temperature limit is proportional to $T^4/m^4$. This power-law behavior always wins out over the exponential damping of the one-loop case, leading to a *two-loop dominance* in the low-temperature domain. One might ask whether this inversion of the loop hierarchy signals the failure of perturbation theory for finite-temperature QED. But, of course, this is not the case, since the inclusion of a virtual photon does not ''amplify'' the two-loop graph and higher ones. Rather, calculating the one-loop graph should only be rated as an inconsistent truncation of the theory, since the one-loop approximation does not include all species of particles as virtual ones. Besides, effective field theory techniques indicate that the three-loop contribution is of the order of $T^8/m^8$ [11] for $T/m \ll 1$, thereby obeying the usual loop hierarchy.

The present paper is organized as follows: In Sec. II, we present the calculation of the two-loop effective QED action at finite temperature employing the imaginary-time formalism and concentrating on the low-temperature limit. The outcome will be valid for slowly varying external fields of arbitrary strength.

Section III is devoted to an investigation of light propagation at finite temperature. While, on the one hand, the well-known result for the velocity shift $\delta v \sim T^4/m^4$ is rediscovered [12–15], we are also able to determine further contributions to the velocity shift arising from a non-trivial interplay between temperature and an additional magnetic background field.

In Sec. IV, we study aspects of thermally induced photon splitting. Therein, we point out that the thermal two-loop contribution to the splitting process exceeds the zero-temperature and one-loop contributions in the low-

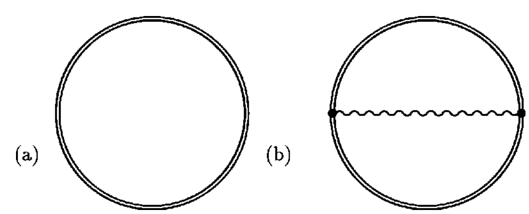


FIG. 1. Diagrammatic representation of the one-loop (a) and two-loop (b) contribution to the effective QED action. The fermionic double line represents the coupling to all orders to the external electromagnetic field.

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temperature and weak-field limit, but is negligible in comparison to other thermally induced scattering processes.

Sections III and IV are mainly concerned with the limit of a weak magnetic background field and low-frequency photons ($\omega \ll m$), and therefore represent only a first glance at these extensive subjects. In fact, the quantitative results for this energy regime describe only tiny effects; a relevance for astrophysical topics such as pulsar physics has not been identified up to now. However, the intention of the present work is a more categoric one, namely, to elucidate the mechanism for a violation of the usual loop hierarchy of perturbative thermal field theories involving virtual massless particles.

In Sec. V, we calculate the thermal contribution to Schwinger's famous pair-production formula [16] for constant electric background fields in the low-temperature limit. Here, a thermal one-loop contribution surprisingly does not exist [7,9], since the thermal one-loop effective action is purely real by construction. Hence, the findings of Sec. V prove the existence of thermally induced pair production — an effect which has been sought for 15 years [5,6,17–19]. In the low-temperature limit, we find that the situation of a strong electric field is dominated by the zero-temperature part (Schwinger formula), while the thermal contribution can become dominant for a weak electric field. Unfortunately, the experimentally more interesting high-temperature limit cannot be covered by our approach.

One last word of caution: the inclusion of electric background fields in finite-temperature QED is always plagued with the question of how violently this collides with assumptions on thermal equilibrium. In fact, electric fields and thermal equilibrium exclude each other, thus questioning the physical meaning of the results of Sec. V at least quantitatively. However, it is reasonable to assume the existence of an at least small window of parameters in the low-temperature and weak-field domain for which the thermal-equilibrium calculation represents a good approximation. Moreover, the knowledge of the effective Lagrangian including a full dependence on all possible field configurations is mandatory to derive equations of motion for the fields, even in the limit of vanishing electric fields.

## II. TWO-LOOP EFFECTIVE ACTION OF QED AT LOW TEMPERATURE

In the following, we will outline the calculation of the two-loop effective action, concentrating on the low-temperature limit where a *two-loop dominance* is expected. The calculation is necessarily very technical, wherefore some details are left for the Appendixes.[1]

But before we get down to business, it is useful to clarify our notation. From the field strength tensor $F^{\mu\nu}$ and its dual ${}^\star F_{\kappa\rho} = \frac{1}{2}\epsilon_{\kappa\rho\mu\nu}F^{\mu\nu}$, we can construct the following standard gauge and Lorentz invariants:

$$\mathcal{F} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) \equiv \frac{1}{2}(a^2 - b^2),$$

$$\mathcal{G} = \frac{1}{4}F^{\mu\nu\,\star}F_{\mu\nu} = -\mathbf{E}\cdot\mathbf{B} \equiv ab, \tag{1}$$

where, for reasons of convenience, we also introduced the *secular* invariants

$$a = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}}, \quad b = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}}, \tag{2}$$

and we assumed without loss of generality that a Lorentz system exists in which the electric and magnetic field are anti-parallel. In this particular frame, the secular invariants can be identified with the field strengths: $a = B \equiv |\mathbf{B}|$, $b = E \equiv |\mathbf{E}|$.

When the physical system involves another vector, say, a momentum 4-vector $k^\mu = (k^0, \mathbf{k})$, we can form another field invariant [metric: $g = (-,+,+,+)$]:

$$z_k := (k_\mu F^{\mu\alpha})(k_\nu F^\nu{}_\alpha) = |\mathbf{k}|^2 \mathbf{B}^2 \sin^2\theta_B + |\mathbf{k}|^2 \mathbf{E}^2 \sin^2\theta_E - k^2 E^2 + 2k^0\,\mathbf{E}\cdot(\mathbf{k}\times\mathbf{B}), \tag{3}$$

where $\theta_B$ ($\theta_E$) denotes the angle between the magnetic (electric) field and the 3-space vector $\mathbf{k}$.

In relativistic equilibrium thermodynamics, temperature can be associated with the invariant norm of a 4-vector $n^\mu$: $n^\mu n_\mu = -T^2$. On the other hand, $n^\mu$ is related to the 4-velocity vector $u^\mu$ of the heat bath by $n^\mu = T u^\mu$. E.g., in the heat-bath rest frame, $u^\mu$ takes the form: $u^\mu = (1,0,0,0)$. Hence, we can introduce one further invariant (beside the temperature itself):

$$\mathcal{E} = (u_\mu F^{\mu\alpha})(u_\nu F^\nu{}_\alpha). \tag{4}$$

E.g., in the heat-bath rest frame, $\mathcal{E}$ simply reduces to $\mathcal{E} = \mathbf{E}^2$. Since the effective Lagrangian is a Lorentz covariant and gauge-invariant quantity, it can only be a function of the complete set of invariants of the system under consideration. Hence, we expect a finite-temperature effective QED Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(\mathcal{E}, \mathcal{F}, \mathcal{G}, T). \tag{5}$$

Equipped with these conventions, we now turn to the calculation.

The two-loop contribution to the effective action/Lagrangian $\mathcal{L}^2$ is generally given by the diagram in Fig. 1(b). This translates into the following formula in coordinate space [2]:

$$\mathcal{L}^2 = \frac{e^2}{2}\int d^4x'\, \mathrm{tr}[\gamma^\mu G(x,x'|A)\,\gamma^\nu G(x',x|A)] \times D_{\mu\nu}(x-x'), \tag{6}$$

[1]The primarily phenomenologically interested reader may just take notice of the following conventions (1)–(5), then directly consult Eqs. (29)–(36), and skip the remainder of the section.

where $G(x,x'|A)$ represents the fermionic Green's function for the Dirac operator in the presence of an external electromagnetic field $A$. $D_{\mu\nu}$ denotes the photon propagator. Throughout the paper, we assume the background field to be constant or at least slowly varying compared to the scale of the Compton wavelength; therefore, the fermionic Green's function can be written as

$$G(x,x'|A)=\Phi(x,x')\int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p(x-x')}\, g(p), \tag{7}$$

where $g(p)$ denotes the Fourier transform of $G(x,x'|A)$ depending only on the field strength, and $\Phi(x,x')$ is the holonomy carrying the complete gauge dependence of the Green's function. Inserting Eq. (7) into Eq. (6) leads us to the object $\Phi(x,x')\Phi(x',x)\equiv\Phi(\bigcirc)$, where the right-hand side represents the holonomy evaluated for a closed path. For a simply connected manifold such as the Minkowski space, $\Phi(\bigcirc)=1$; hence, it does not contribute to the zero-temperature Lagrangian. For a non-simply connected manifold such as the finite-temperature coordinate space ($\mathbb{R}^3\times S^1$), $\Phi(\bigcirc)$ can pick up a winding number [9]. However, in the present case, we restrict our considerations to a situation with zero density, which implies the existence of a gauge in which $A_0=0$. Then, $\Phi(\bigcirc)=1$ and the influence of the holonomy can be discarded.

This leads us to the representation

$$\mathcal{L}^2=\frac{\mathrm{i}}{2}\int \frac{d^4k}{(2\pi)^4}\, D_{\mu\nu}(k)\, \Pi^{\mu\nu}(k) \tag{8}$$

for the two-loop Lagrangian, where $D_{\mu\nu}(k)$ denotes the photon propagator in momentum space, and we introduced the one-loop polarization tensor in an arbitrary constant external background field:

$$\Pi^{\mu\nu}(k)=-\mathrm{i}e^2\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}[\gamma^\mu g(p)\, \gamma^\nu g(p-k)]. \tag{9}$$

So we have finally arrived at the well-known fact that the two-loop effective action can be obtained from the polarization tensor in an external field by glueing the external lines together.

The transition to finite-temperature field theory can now be made within the imaginary-time formalism by replacing the momentum integration over the zeroth component in Eqs. (8) and (9) by a summation over bosonic and fermionic Matsubara frequencies, respectively. E.g., performing this procedure in Eq. (9) corresponds to thermalizing the fermions in the loop. Now we come to an important point: confining ourselves to the low-temperature domain where $T\ll m$, we know from the one-loop calculations [10,20] that thermal fermionic effects are suppressed by factors of $\mathrm{e}^{-m/T}$, indicating that the mass of the fermions suppresses thermal excitations. Hence, thermalizing the polarization tensor contributes at most terms of order $\mathrm{e}^{-m/T}$ to the two-loop Lagrangian for $T\ll m$; these are furthermore accompanied by an additional factor of the coupling constant $\alpha$ and can therefore be neglected compared to the one-loop terms. At low temperature, it is therefore sufficient to thermalize the internal photon only in order to obtain the leading $T$-dependence of $\mathcal{L}^2$.

Since, in Feynman gauge, the photon propagator reads

$$D_{\mu\nu}(k)=g_{\mu\nu}\frac{1}{k^2-\mathrm{i}\epsilon}, \quad k^2=-(k^0)^2+\mathbf{k}^2, \tag{10}$$

the introduction of bosonic Matsubara frequencies $(k^0)^2\to -\omega_n^2=-(2\pi Tn)^2$, $n\in\mathbb{Z}$, leads us to[2]

$$\mathcal{L}^{2+2T}=\frac{\mathrm{i}}{2}\,\mathrm{i}T\sum_{\omega_n}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2-\mathrm{i}\epsilon}\,\Pi^\mu{}_\mu(k). \tag{11}$$

From now on, we write $\mathcal{L}^2$ for the zero-temperature two-loop Lagrangian, $\mathcal{L}^{2T}$ for the purely thermal part, and $\mathcal{L}^{2+2T}$ for their sum. In Eq. (11), we need the trace of the polarization tensor in constant but otherwise arbitrary electromagnetic fields. In the literature, there are various equivalent representations for $\Pi_{\mu\nu}$. For the present purpose, it is useful to derive our own one which is based on a calculation of Urrutia [21]. Details are presented in Appendix A.

Inserting representation (A19) of the Appendix for $\Pi^\mu{}_\mu$ into Eq. (11), we obtain for the Lagrangian

$$\begin{aligned}\mathcal{L}^{2+2T}=&-\frac{T}{2}\frac{\alpha}{2\pi}\sum_{\omega_n}\int\frac{d^3k}{(2\pi)^3}\int_0^\infty\frac{ds}{s}\int_{-1}^1\frac{d\nu}{2}\\ &\times\frac{\mathrm{e}^{-\mathrm{i}s\phi_0}}{a^2+b^2}\,\frac{eas\, ebs}{\sin eas\,\sinh ebs}\\ &\times\Bigg[\frac{z_k}{k^2-\mathrm{i}\epsilon}(\tilde{N}_2-\tilde{N}_1)\\ &\quad+(2N_0(a^2+b^2)+b^2\tilde{N}_2+a^2\tilde{N}_1)\Bigg]\Bigg|_{(k^0)^2=-\omega_n^2},\end{aligned} \tag{12}$$

where the $\phi_0$, $N_0$, $\tilde{N}_i$ are functions of the integration variables $s$ and $\nu$ and of the invariants $a$ and $b$; only $\phi_0$ depends additionally on $z_k$ as defined in Eq. (3). Their explicit form can be looked up in Eqs. (A16), (A20) and (A22). In order to

[2]Of course, the present calculation does not necessarily have to be performed in the imaginary-time formalism. E.g., instead of Eq. (10), we could as well work with the real-time representation of the thermal photon propagator. We could even use the one-component formalism only, since we merely consider the photon to be thermalized. However, from our viewpoint, the calculations in the imaginary-time formalism appear a bit simpler since the momentum integrals will remain Gaussian. Of course, this might be just a matter of taste.

ensure convergence of the proper-time integrals, the causal prescription $m^2 \to m^2 - \mathrm{i}\epsilon$ for the mass term in $\phi_0$ is understood; this agrees with deforming the $s$-contour slightly below the real axis.

Now, the aim is to perform the $k$-momentum integration and/or summation; note that the $k$-dependence is contained in $\phi_0$, $z_k$ (and $k^2$, of course). Concentrating on this step, we encounter the integrals

$$I_1 = T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}s\phi_0}\bigg|_{(k^0)^2 = -\omega_n^2}, \tag{13}$$

$$I_2 = T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3}\, \frac{z_k}{k^2 - \mathrm{i}\epsilon}\, \mathrm{e}^{-\mathrm{i}s\phi_0}\bigg|_{(k^0)^2 = -\omega_n^2},$$

which allow us to write the Lagrangian (12) in terms of

$$\mathcal{L}^{2+2T} = -\frac{\alpha}{4\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{d\nu}{2}\, \frac{eas\, ebs}{(a^2+b^2)\sin eas \sinh ebs} \times [(\tilde{N}_2 - \tilde{N}_1)\, I_2 + (2N_0(a^2+b^2) + b^2\tilde{N}_2 + a^2\tilde{N}_1) I_1]. \tag{14}$$

Employing Eq. (A20) for $\phi_0$, we can put down the evaluation of $I_2$ to the one of $I_1$:

$$I_2 = T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3}\, \frac{z_k}{k^2 - \mathrm{i}\epsilon}\, \mathrm{e}^{-\mathrm{i}m^2 s} \mathrm{e}^{-A_z z_k}\, \mathrm{e}^{-A_k k^2}\bigg|_{(k^0)^2 = -\omega_n^2} = -\frac{\partial}{\partial A_z}\int_{A_k}^\infty dA_k'\, I_1, \tag{15}$$

where $A_z$ and $A_k$ again are functions of the integration variables $s$ and $\nu$ and of the invariants $a$ and $b$, and are defined in Eq. (A22). In view of Eq. (15), it is sufficient to consider the momentum integration-summation for $I_1$ only:

$$I_1 \overset{(A20)}{=} T\, \mathrm{e}^{-\mathrm{i}m^2 s} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3}\, \mathrm{e}^{-A_z z_k}\, \mathrm{e}^{-A_k k^2}\bigg|_{(k^0)^2 = -\omega_n^2}. \tag{16}$$

At this stage, the *finite-temperature coordinate frame* as introduced in [9] becomes extremely useful, since it enables us to perform the calculation in terms of the invariants. This coordinate system is adapted to the situation of electromagnetic fields at finite temperature in a way that the components of any tensor-valued function of the field strength can be expressed in terms of the invariants $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$. Again, details are presented in the appendix (Appendix B), from where we take the final formula for the exponent of Eq. (16) [cf. Eq. (B7)]:

$$A_z z_k + A_k k^2 = (A_k + (a^2 - b^2 + \mathcal{E})A_z)\left(k^2 - \frac{A_z\sqrt{d}}{A_z(2\mathcal{F}+\mathcal{E}) + A_k}\, k^0\right)^2 - \frac{(A_k + a^2A_z)(A_k - b^2A_z)}{A_k + (a^2 - b^2 + \mathcal{E})A_z}(k^0)^2 + \left(A_z\frac{a^2b^2}{\mathcal{E}} + A_k\right)\left(k^3 + \frac{A_z\dfrac{\sqrt{d}\mathcal{G}}{\mathcal{E}}}{A_z\dfrac{\mathcal{G}^2}{\mathcal{E}} + A_k}\, k^1\right)^2 + \frac{(A_k + a^2A_z)(A_k - b^2A_z)}{A_k\dfrac{a^2b^2}{\mathcal{E}} + A_k}(k^1)^2, \tag{17}$$

where $k^0, k^1, k^2, k^3$ represent the components of the rotated momentum vector $k^A = e^A{}_\mu k^\mu$, and $e^A{}_\mu$ denotes the vierbein which mediates between the given coordinate system and the finite-temperature coordinate frame [cf. Eq. (B1)]. Since the transformation into the new reference frame is only a rigid rotation in Minkowski space, no Jacobian arises for the measure of the momentum integral. Hence, only integrals of Gaussian type are present in Eq. (16), which can easily be performed to give

$$I_1 = T\, \frac{\mathrm{e}^{-\mathrm{i}m^2 s}}{(4\pi)^{3/2}}\, \frac{1}{\sqrt{p\, q_a\, q_b}} \sum_{\omega_n} \mathrm{e}^{-(q_a\, a_b/p)\omega_n^2}, \tag{18}$$

where it was convenient to introduce the short forms:

$$q_a := A_k + a^2 A_z, \quad q_b := A_k - b^2 A_z,$$
$$p := A_k + (a^2 - b^2 + \mathcal{E})A_z. \tag{19}$$

The sum in Eq. (18) can be rewritten with the aid of a Poisson resummation of the form

$$\sum_{n=-\infty}^{\infty} \exp(-\sigma(n-z)^2) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{\pi}{\sigma}}\, \exp\left(-\frac{\pi^2}{\sigma}\, n^2 - 2\pi \mathrm{i} z n\right). \tag{20}$$

With $z=0$ and $\sigma = (2\pi T)^2 (q_a q_b/p)$, we obtain for Eq. (18)

$$I_1 \equiv I_1^{T=0} + I_1^T = \frac{\mathrm{e}^{-\mathrm{i}m^2 s}}{16\pi^2}\frac{1}{q_a q_b} + \frac{\mathrm{e}^{-\mathrm{i}m^2 s}}{8\pi^2}\frac{1}{q_a q_b} \times \sum_{n=1}^{\infty} \exp\left(-\frac{p}{q_a q_b}\frac{n^2}{4T^2}\right), \tag{21}$$

where we separated the $(n=0)$-term from the remaining sum in order to find the $(T=0)$-contribution. The first term in Eq. (21) $[(n=0)$-term], namely, is independent of $T$ and $\mathcal{E}$, while the second term vanishes in the limit $T\to 0$ exponentially. It is straightforward to check explicitly that the first term of Eq. (21) indeed leads to the (unrenormalized) two-loop Lagrangian for arbitrary constant electromagnetic fields at zero temperature. E.g., for purely magnetic fields, the representation of Dittrich and Reuter [2] is rediscovered.

For our finite-temperature considerations, we will only keep the second term of Eq. (21), which we denote by $I_1^T$ in the following. Concerning the formula for $\mathcal{L}^{2T}$ in Eq. (14), $I_1^T$ is already in its final form (it will turn out later that this term is subdominant in the low-$T$ limit and only $I_2^T$ contains the important contributions). Hence, let us turn to the evaluation of $I_2^T$, i.e., the thermal part of Eq. (15); for this, we have to interpret $I_1^T$ as a function of $A_z$ and $A_k$ (remember: $q_a$, $q_b$ and $p$ are functions of $A_z$ and $A_k$):

$$I_2^T = -\frac{\partial}{\partial A_z}\int_{A_k}^{\infty} dA_k'\, I_1^T(A_k',A_z) = -\frac{\partial}{\partial A_z}\int_0^{\infty} ds'\, I_1^T(s'+A_k,A_z) =: -\frac{\mathrm{e}^{-\mathrm{i}m^2 s}}{8\pi^2}\sum_{n=1}^{\infty}\frac{\partial}{\partial A_z}J(A_z), \tag{22}$$

where we defined the auxiliary integral:

$$J(A_z) = \int_0^{\infty} ds' \frac{\exp\left(-\frac{s'+p}{(s'+q_a)(s'+q_b)}\frac{n^2}{4T^2}\right)}{(s'+q_a)(s'+q_b)}. \tag{23}$$

Upon a substitution of the integration variable,[3]

$$u := \frac{q_a q_b}{p}\frac{s'+p}{(s'+q_a)(s'+q_b)} \Rightarrow \frac{ds'}{(s'+q_a)(s'+q_b)} = -\frac{du}{\sqrt{\frac{q_a^2 a_b^2}{p^2} + \frac{2q_a q_b}{p}(2p-q_a-q_b)u + (q_a-q_b)^2 u^2}},$$

the auxiliary integral becomes

$$J(A_z) = \int_0^1 \frac{du\, \exp\left(-\frac{n^2}{4T^2}\frac{p}{q_a q_b}u\right)}{\sqrt{\frac{q_a^2 a_b^2}{p^2} + \frac{2q_a q_b}{p}(2p-q_a-q_b)u + (q_a-q_b)^2 u^2}}. \tag{25}$$

Now we come to an important point: since we only thermalized the photons, our effective Lagrangian $\mathcal{L}^{2T}$ is only valid for $T \ll m$ anyway. Nevertheless, our formulas also contain information about the high-temperature domain which we should discard, since it is incomplete. Regarding Eq. (25), the exponential function causes the integrand to be extremely small for small values of $T$, except where $u$ is also small. Hence, the auxiliary integral is mainly determined by the lower end of the integration interval.

Taking these considerations into account, we expand the square root for small values of $u$ and then extend the integration interval to infinity [in fact, maintaining 1 as the upper bound only creates terms of the order $\exp(-(2nm)/T)$, which are subdominant in the low-temperature limit]. The remaining $u$-integration can then easily be performed for each order in the $u$-expansion; up to $u^2$, we obtain

[3]Resolving for $s'=s'(u)$ leads to a quadratic equation from which the positive root has to be taken in order to take care of the integral boundaries.

$$J(A_z)=4\frac{T^2}{n^2}-16\frac{T^4}{n^4}(2p-q_a-q_b)-64\frac{T^6}{n^6}((q_a-q_b)^2-3(2p-q_a-q_b)^2)+\mathcal{O}(T^8/n^8). \tag{26}$$

Upon differentiation, the $T^2$-dependence drops out, and we get [cf. Eq. (19)]

$$\frac{\partial}{\partial A_z}J(A_z)=-2^5\frac{T^4}{n^4}(\mathcal{F}+\mathcal{E})-2^9\frac{T^6}{n^6}(\mathcal{F}^2+\mathcal{G}^2-3(\mathcal{F}+\mathcal{E})^2)A_z+\mathcal{O}(T^8/n^8). \tag{27}$$

In this equation, we indeed discover a power-law dependence on the temperature, which will directly translate into a power-law dependence of the two-loop effective action after insertion into Eqs. (22) and (14). Technically speaking, this arises from the fact that the omnipresent exponential factor $\exp(-(n^2/4T^2)(p/q_aq_b)u)$, which finally causes exponential damping for $T/m\to 0$, becomes equal to 1 after the $u$-integration at the lower bound at $u=0$.

At this stage, it is important to observe that the $u$-integration appears only in $I_2^T$ [via the $A_k'$-integration in Eq. (13)] and not in $I_1^T$. Therefore, $I_1^T$ will always contain exponential damping factors in the limit $T\to 0$. Even the remaining proper-time integrations do not provide for a mechanism similar to the $u$-integration, since for large $s$, the mass factor $\exp(-\mathrm{i}m^2 s)$ with the causal prescription $m^2\to m^2-\mathrm{i}\epsilon$ causes the integrand to vanish, and for small $s$, the combination $p/q_aq_b$ in the exponent becomes

$$\frac{p}{q_aq_b}=-\frac{4\mathrm{i}}{1-\nu^2}\frac{1}{s}+\mathcal{O}(s). \tag{28}$$

Obviously, inserting Eq. (28) into the exponent leads to an exponential fall off (bearing in mind that the $s$-contour will run slightly below the real axis). Similar conclusions can be drawn for the $\nu$-integration. To summarize these technical considerations, we conclude that only the term containing $I_2^T$ (thermal part of $I_2$) in Eq. (14) contributes dominantly to $\mathcal{L}^{2T}$ in the low-temperature limit.

Inserting the first and second term of $(\partial/\partial A_z)J(A_z)$ in Eq. (27) successively into Eq. (22) and then into Eq. (14), we obtain the dominant terms of order $T^4$ and $T^6$ of the two-loop effective QED Lagrangian at low temperature; particularly for the $T^4$-term, different useful representations can be given:

$$\mathcal{L}^{2T}|_{T^4}=-\frac{\alpha\pi}{90}\,T^4\,(\mathcal{F}+\mathcal{E})\int_0^\infty\frac{ds}{s}\int_{-1}^1\frac{d\nu}{2}\,\mathrm{e}^{-\mathrm{i}m^2s}\frac{eas\,ebs}{\sin eas\,\sinh ebs}\,\frac{(\tilde{N}_2-\tilde{N}_1)}{a^2+b^2}$$

$$=-\frac{\alpha\pi}{45}\,T^4\,(\mathcal{F}+\mathcal{E})\int_0^\infty\frac{ds}{s}\,\frac{1}{a^2+b^2}\,\mathrm{e}^{-\mathrm{i}m^2s}\left[ebs\coth ebs\,\frac{1-eas\cot eas}{\sin^2 eas}+eas\cot eas\,\frac{1-ebs\coth ebs}{\sinh^2 ebs}\right] \tag{29}$$

$$=\frac{\pi^2}{45}\,T^4\,(\mathcal{F}+\mathcal{E})\left(\frac{1}{a^2+b^2}(\partial_a^2+\partial_b^2)\right)\left[\frac{1}{8\pi^2}\int_0^\infty\frac{ds}{s^3}\,\mathrm{e}^{-\mathrm{i}m^2s}eas\cot eas\,ebs\coth ebs\right]. \tag{30}$$

The term proportional to $T^6$ reads

$$\mathcal{L}^{2T}|_{T^6}=-\frac{16\alpha\pi^3}{945}\,T^6\,(\mathcal{F}^2+\mathcal{G}^2-3(\mathcal{F}+\mathcal{E})^2)\int_0^\infty\frac{ds}{s}\int_{-1}^1\frac{d\nu}{2}\,\frac{\mathrm{e}^{-\mathrm{i}m^2s}}{a^2+b^2}\,\frac{eas\,ebs}{\sin eas\,\sinh ebs}(\tilde{N}_2-\tilde{N}_1)\,A_z, \tag{31}$$

where $\tilde{N}_i$ and $A_z$ are functions of the integration variables and the invariants $a$ and $b$ (not of $\mathcal{E}$), and are defined in Eqs. (A16) and (A22). The $\nu$-integration can be performed analytically, but the extensive result does not provide for new insights; hence we do not bother to write it down.

These equations represent the central result of the present work; therefore, a few of their properties should be stressed:

(1) While we worked explicitly in the low-temperature approximation $T\ll m$, we put no restrictions on the strength of the electromagnetic fields.

(2) The low-temperature Lagrangians contain arbitrary powers of the invariants $a$ and $b$ (equivalently $\mathcal{F}$ and $\mathcal{G}$), but the additional invariant at finite temperature $\mathcal{E}$ only appears linearly in the $T^4$-term and quadratic in the $T^6$-term. The small-$T$ expansion thus corresponds to a small-$\mathcal{E}$ expansion.

(3) The fact that only the integral $I_2^T$ with the prefactor $(\tilde{N}_2-\tilde{N}_1)$ contributes to the low-temperature Lagrangian in Eq. (14) implies that only the spatially transversal modes $\Pi_\parallel$ and $\Pi_\perp$ of the polarization tensor (A15) play a role in this thermalized virtual two-loop process. The time-like or longitudinal mode $\Pi_0$ (depending on the character of $k^\mu$) might become important at higher values of temperature.

(4) The fact that the invariant $\mathcal{E}$ always appears in the combination $\mathcal{F}+\mathcal{E}$ ensures a kind of dual invariance of the Lagrangian. Under the replacement $\mathbf{E}\to\mathbf{B}$ and $\mathbf{B}\to-\mathbf{E}$, the invariants change into $\mathcal{F}\to-\mathcal{F}$, $\mathcal{G}\to-\mathcal{G}$ and $\mathcal{E}\to\mathcal{E}+2\mathcal{F}$, so

that $\mathcal{F}+\mathcal{E}$ remains invariant.

(5) The $T^4$-term of $\mathcal{L}^{2T}$ as exhibited in Eq. (30) possesses the peculiarity of being derivable from the one-loop zero-temperature Lagrangian which we marked by square brackets in Eq. (30) after the derivative terms. This will be elucidated a bit further in the following section.

(6) The entire thermal contribution to the effective action is finite. This reflects the well-known fact that the counterterms which are necessary and sufficient in order to renormalize the zero-temperature effective action are also necessary and sufficient for the finite-temperature action. Even more conveniently, we were able to separate the zero-temperature from the thermal parts, implicitly assuming that the renormalization of the zero-temperature parts is performed without any reference to the finite-temperature system. As a consequence, we are dealing with the same renormalization point as at zero temperature which is naturally given by the zero-temperature electron mass. At finite-temperature, this does not have to be and indeed is not identical to the physical electron mass which undergoes further renormalization by finite-temperature effects. E.g., from a one-loop calculation of the mass operator one finds $m^2_{\rm phys} = m^2 + (2/3)\alpha\pi T^2$ for $T \ll m$ [22]. Therefore, the above-given thermal effective action must be viewed as ''off-shell'' renormalized. Nevertheless, since the physics is independent of the renormalization point, we can work with the zero-temperature as well as the physical electron mass.[4] The ''off-shell,'' i.e., zero-temperature renormalization is, of course, more transparent, since all temperature dependence is explicitly displayed, which would otherwise be partly hidden in the physical electron mass.

For the remainder of this section, we will discuss certain limiting cases of the two-loop low-temperature Lagrangian. First, let us concentrate on a weak-field expansion which corresponds to a small-$s$ expansion of the proper-time integral due to the exponential mass factor. Expanding the integrands for small values of $s$ (except the mass factor) and integrating over $\nu$ and $s$, leads us to the dominant terms in the weak-field limit:

$$\mathcal{L}^{2T}\big|_{T^4} = \frac{44\alpha^2\pi^2}{2025}\,\frac{T^4}{m^4}(\mathcal{F}+\mathcal{E}) - \frac{2^6\times 37\alpha^3\pi^3}{3^4\times 5^2\times 7}\,\frac{T^4}{m^4}\,\frac{\mathcal{F}(\mathcal{F}+\mathcal{E})}{m^4} + \mathcal{O}(3), \tag{32}$$

$$\mathcal{L}^{2T}\big|_{T^6} = \frac{2^{13}\alpha^3\pi^5}{3^6\times 5\times 7^2}\,\frac{T^6}{m^6}(2\mathcal{F}^2 + 6\mathcal{E}\mathcal{F} + 3\mathcal{E}^2 - \mathcal{G}^2)\frac{1}{m^4} + \mathcal{O}(3), \tag{33}$$

where $\mathcal{O}(3)$ signals that we omitted terms of third order in the field invariants (sixth order in the field strength). Note that no linear term in the field invariants to order $T^6$ exists. For the terms of quadratic order, the $T^6$-term is subdominant for $T/m \leqslant 0.05$, and amounts up to a 10%-correction to the $T^4$-term for $T/m \sim 0.1$. For even larger values of temperature, we expect the failure of the low-temperature approximation.

Finally, we consider $\mathcal{L}^{2T}|_{T^4}$ in the limit of purely magnetic background fields: $b \to 0$, $a \to B$, $\mathcal{F}+\mathcal{E} \to \frac{1}{2}B^2$. The $T^4$-term in Eq. (29) then reduces to

$$\mathcal{L}^{2T}(B)\big|_{T^4} = \frac{\alpha\pi}{90}\,T^4 \int_0^\infty \frac{dz}{z}\, \mathrm{e}^{-(m^2/eB)z} \times \left[ \frac{1 - z\coth z}{\sinh^2 z} + \frac{1}{3} z\coth z \right], \tag{34}$$

where we have performed the substitution $eas \to -\mathrm{i}z$ in concordance with the causal prescription $m^2 \to m^2 - \mathrm{i}\epsilon$. Incidentally, the limit of purely electric fields can simply be obtained by replacing $B \to -\mathrm{i}E$ and multiplying Eq. (34) by $(-1)$.

Introducing the critical field strength $B_{\rm cr} := m^2/e$, we can evaluate the integral in Eq. (34) analytically [15][5] and obtain:

$$\mathcal{L}^{2T}(B)\big|_{T^4} = \frac{\alpha\pi}{90} T^4 \Bigg[ \left( \frac{B_{\rm cr}^2}{2B^2} - \frac{1}{3} \right) \psi\left(1 + \frac{B_{\rm cr}}{2B}\right) - \frac{2B_{\rm cr}}{B}\ln\Gamma\left(\frac{B_{\rm cr}}{2B}\right) - \frac{3B_{\rm cr}^2}{4B^2} - \frac{B_{\rm cr}}{2B} + \frac{B_{\rm cr}}{B}\ln 2\pi + \frac{1}{6} + 4\zeta'\left(-1, \frac{B_{\rm cr}}{2B}\right) + \frac{B}{3B_{\rm cr}} \Bigg], \tag{35}$$

where $\psi(x)$ denotes the logarithmic derivative of the $\Gamma$-function, and $\zeta'(s,q)$ is the first derivative of the Hurwitz $\zeta$-function with respect to its first argument.

For strong magnetic fields, $B \gg B_{\rm cr}$, the last term in square brackets in Eq. (35) dominates the whole expression, and we find a linear increase of the effective Lagrangian:

$$\mathcal{L}^{2T}(B \gg B_{\rm cr})\big|_{T^4} = \frac{\alpha\pi}{270}\, T^4\, \frac{eB}{m^2}. \tag{36}$$

This contribution remains subdominant compared to the one arising from pure vacuum polarization $\sim B^2 \ln(eB/m^2)$, which is not astonishing, since the magnetization of (real) thermalized plasma particles is bounded: the spins can maximally be completely aligned. In contrast, the non-linearities of

[4] In the case of an ''on-shell'' renormalization, first, $m$ has to be replaced by $m_{\rm phys}$, and, secondly, we obtain an additional term $-(2/3)\alpha\pi T^2 (\partial\mathcal{L}^1/\partial m^2)$ from the mass renormalization at one-loop order.

[5] We take the opportunity to remark that there is a misprint in the corresponding integration result in [15]; the term $(+1/3)$ has to be replaced by $(+1/6)$ [cf. Eq. (35)].

vacuum polarization set no such upper bound. Quantitatively, the same result was found for the thermal one-loop contribution [23].

## III. LIGHT PROPAGATION

As a first application, we study the propagation of plane light waves at finite temperature and in a magnetic background. The subject of light propagation has recently gained renewed interest due to its accessibility to current experimental facilities [24].

In the limit of light of low-frequency $\omega \ll m$, the effective action for slowly varying fields has proved useful for obtaining velocity shifts, i.e., refractive indices of QED vacua which are modified by various external perturbations such as fields and temperature [25,15]. In this limit of low frequencies and smooth external perturbations, the terms involving derivatives of the fields in a derivative expansion of the effective action can be neglected, and the constant-field approximation is appropriate.

The case of light propagation at finite temperature has been investigated in [20] from a general viewpoint for a class of Lagrangians depending on the invariants $\mathcal{E},\mathcal{F},\mathcal{G},T$ in an arbitrary way. Therein, a light cone condition representing a sum-rule for the polarization modes of the propagating light has been derived; this has been exploited for a detailed investigation of light propagation at finite temperature to one-loop order by an insertion of the thermal one-loop effective Lagrangian of QED. It has been emphasized that these one-loop studies apply to a domain of intermediate values of temperature $\sim 0.1 \leqslant T/m \leqslant \sim 1$, where two-loop as well as plasma effects remain subdominant.

The famous results for the low-temperature velocity shift $\delta v \sim T^4/m^4$ [12–14] could not have been rediscovered by this first-principle investigation, because the thermal two-loop effective action was not at hand. In the present work, we intend to fill this last gap.

Let us first consider the situation of a thermalized QED vacuum without an additional background field. In the low-temperature domain, this vacuum is then characterized by the Lagrangian $\mathcal{L} = -\mathcal{F} + \mathcal{L}^{2T}$, where $-\mathcal{F}$ represents the classical Maxwell term. Following the lines of [20], the phase and group velocity $v$ of a propagating plane wave is then given by

$$v^2 = \frac{1}{1 + \dfrac{2\,\partial_{\mathcal{E}}\mathcal{L}}{(-\partial_{\mathcal{F}}\mathcal{L} + \partial_{\mathcal{E}}\mathcal{L})}}, \tag{37}$$

where $v = k^0/|\mathbf{k}|$ is constructed from the wave vector of the propagating light, and it is understood that the partial derivatives of $\mathcal{L}$ are evaluated in the zero-field limit. Inserting Eqs. (32) and (33) into Eq. (37), leads us to

$$v^2 = \frac{1}{1 + 2\dfrac{44}{2025}\alpha^2\pi^2\dfrac{T^4}{m^4}} \simeq 1 - 2\frac{44}{2025}\alpha^2\pi^2\frac{T^4}{m^4} + \mathcal{O}(T^8/m^8). \tag{38}$$

Note that there is no $T^6$-term, since $\mathcal{L}^{2T}|_{T^6}$ is at least quadratic in the field invariants. In Eq. (38), we rediscovered the well-known velocity shifts for light propagation in a thermal background as found in [12,14] via the two-loop polarization operator and in [13,15] via considering vacuum expectation values of field bilinears in a thermal background. The here-presented rederivation within the effective action approach from first principles thus can be viewed as an independent check of our calculations and of the light cone condition as derived in [20].

But we can go one step further and additionally take a weak external magnetic field into account; the light cone condition in this case reads [20]

$$0 = (\partial_{\mathcal{F}}\mathcal{L} - \partial_{\mathcal{E}}\mathcal{L} - \mathcal{F}\partial_{\mathcal{G}}^2\mathcal{L})k^2 + \frac{1}{2}(\partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2)\mathcal{L}\, z_k + 2\,\partial_{\mathcal{E}}\mathcal{L}\,(ku)^2, \tag{39}$$

where $u^\mu$ denotes the 4-velocity vector of the heat bath and $z_k$ is defined in Eq. (3). The Lagrangian describing a thermal QED vacuum with weak magnetic background fields at finite temperature is given by $\mathcal{L} = -\mathcal{F} + \mathcal{L}^1 + \mathcal{L}^{2T}$, where $\mathcal{L}^1$ denotes the one-loop effective Lagrangian at zero temperature. Up to the second order in the invariants, this famous Heisenberg-Euler Lagrangian $\mathcal{L}^1$ is given by

$$\mathcal{L}^1 = \frac{8}{45}\frac{\alpha^2}{m^4}\mathcal{F}^2 + \frac{14}{45}\frac{\alpha^2}{m^4}\mathcal{G}^2. \tag{40}$$

Inserting all the relevant contributions to $\mathcal{L}$ into the light cone condition Eq. (39), the light velocity to lowest order in the parameters $T$ and $B$ finally yields

$$v^2 = 1 - \frac{22}{45}\frac{\alpha^2}{m^4}B^2\sin^2\theta_B - 2\frac{44}{2025}\alpha^2\pi^2\frac{T^4}{m^4} + \frac{22}{45}\frac{\alpha^2}{m^4}\left(\frac{2^5\times 37}{3^2\times5\times7\times11}\alpha\pi^3\frac{T^4}{m^4}\right)B^2(1+\sin^2\theta_B), \tag{41}$$

where $\theta_B$ denotes the angle between the propagation direction and the magnetic field [cf. Eq. (3)]. The second and third term are the well-known velocity shifts for purely magnetic [26,27] and purely thermal vacua [cf. Eq. (38)], respectively. The last term describes a non-trivial interplay between these two vacuum modifications. The latter can best be elucidated in the various limits of the angle $\theta_B$; for orthogonal propagation to the magnetic field $\theta_B = \pi/2$, we get

$$v^2 = 1 - 2\frac{44}{2025}\alpha^2\pi^2\frac{T^4}{m^4} - \frac{22}{45}\frac{\alpha^2}{m^4}B^2\left(1 - (0.15\dots)\frac{T^4}{m^4}\right). \tag{42}$$

For parallel propagation to the magnetic field $\theta_B = 0$, we find

$$v^2 = 1 - 2\frac{44}{2025}\alpha^2\pi^2\frac{T^4}{m^4}\left(1 - (0.96\dots)\left(\frac{eB}{m^2}\right)^2\right). \tag{43}$$

Since $T/m$ and $eB/m^2$ are considered to be small in each case, the corrections to the pure effects in the mixed situation are comparably small. Note that the mixed thermal and magnetic corrections always diminish the values for the velocity shift of the pure magnetic or thermal situations. Let us finally remind the reader that the here-given velocities hold for low-frequency light ($\omega \ll m$) only, and represent averages over the two possible polarization modes. While for the purely thermal case the polarization modes cannot be distinguished, the situation involving an electromagnetic field generally leads to birefringence due to the existence of a preferred direction of the field lines.

Let us finally comment on the earlier works [13,15] related to the issue of light propagation in a thermal background. The philosophy therein was to calculate the velocity shifts in a purely (weak) electromagnetic background first, and then take thermal vacuum expectation values of the field bilinears. Expressing this in formulas, we first recall the expression for the propagation-direction-averaged light velocity in a weak electromagnetic background from [15]:

$$v^2 = 1 - \frac{2}{3}(\partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2)\mathcal{L}\ T^{00}, \tag{44}$$

where $T^{00} = \frac{1}{2}(E^2 + B^2)$ denotes the 00-component of the energy-momentum tensor, i.e., energy density of the electromagnetic field. In the weak-field limit, $(\partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2)\mathcal{L}$ is field independent: $2\frac{22}{45}(\alpha^2/m^4)$ [cf. Eq. (40)]; therefore, taking thermal vacuum expectation values of the field quantities in Eq. (44) is simply equivalent to replacing $T^{00}$ by $\langle T^{00}\rangle^T = (\pi^2/15)T^4$. This then leads to the correct result as given in Eq. (38).

From the viewpoint of the present work, the correctness of the approach of [13,15] arises from the special form of the low-temperature two-loop Lagrangian $\mathcal{L}^{2T}|_{T^4}$ as given in Eq. (30). Since

$$\frac{1}{a^2 + b^2}(\partial_a^2 + \partial_b^2) = \partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2, \tag{45}$$

Eq. (30) can also be written as

$$\partial_{\mathcal{E}}\mathcal{L}^{2T}|_{T^4} = \frac{2}{3}\langle T^{00}\rangle^T \frac{1}{2}(\partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2)\mathcal{L}^1. \tag{46}$$

Incidentally, Eq. (46) holds for arbitrary field strength, but, in this line of argument, it is required for weak fields only. Inserting Eq. (46) into the correct light cone condition at finite temperature, i.e., Eq. (37), we obtain to lowest order

$$v^2 \simeq 1 - 2\partial_{\mathcal{E}}\mathcal{L} = 1 - \frac{2}{3}(\partial_{\mathcal{F}}^2 + \partial_{\mathcal{G}}^2)\mathcal{L}\ \langle T^{00}\rangle^T, \tag{47}$$

which is equal to the heuristically deduced light cone condition for a thermal QED vacuum [13,15].

Note that the combined low-temperature–weak-field effects as given in Eqs. (41)–(43) could not have been found in [15], since the invariant structure is not completely taken into account in the heuristic approach. Whether the investigation of the intermediate-temperature domain to two-loop has been correctly modeled with the heuristic approach in [15], cannot be judged within the present work. Note, however, that the intermediate-temperature domain is controlled by one-loop effects, leading to a maximum velocity shift of $-\delta v_{\max}^2 = \alpha/3\pi$ [20]. As has been shown therein, the *two-loop dominance* is lost for $T/m \geqslant 0.058$.

## IV. PHOTON SPLITTING

Photon splitting in magnetic fields at zero temperature has been discussed comprehensively by Adler [26], stressing its relevance for the photon physics of compact astrophysical objects (see also [28]). For the description of the splitting process for low-frequency photons with $\omega \ll m$ at weak magnetic fields $eB/m^2 \ll 1$, the use of the one-loop effective Lagrangian for weak fields is sufficient for obtaining a good estimate of the absorption coefficient for photon splitting. To be precise, the lowest order contribution to the splitting process comes from the terms of third order in the invariants (sixth order in the field strength) of $\mathcal{L}^1$, i.e., the hexagon graph with one incoming, two outgoing photons and three couplings to the external magnetic field. Neglecting dispersion effects, the box graph vanishes because of $\mathcal{L}^1$ depending on $\mathcal{F}$ and $\mathcal{G}$ only, and because of the Lorentz kinematics of the photons.[6]

The question of thermally induced photon splitting has recently been investigated by Elmfors and Skagerstam [10] with the aid of the thermal one-loop effective QED Lagrangian; their studies were motivated by the fact that a vacuum may be a bad approximation for the surroundings of some astrophysical compact objects, while a thermalized environment at zero or finite density might be more appropriate. It turned out that, at temperatures and magnetic fields at the scale of the electron mass, the thermal contribution can exceed the zero-temperature one, but these effects then are superimposed by Compton scattering of the photons with the plasma. In realistic situations, the thermally induced process will thus be of subdominant importance.

In the following, we intend to complete these results about thermally induced photon splitting with the dominant low-temperature contributions stemming from the two-loop process. Hereby, we also concentrate on the splitting process $(\perp \rightarrow \|_1 + \|_2)$, where a photon, with its electric field vector orthogonal ($\perp$) to the plane spanned by the external magnetic field and the propagation direction, splits into two photons with their electric field vectors within ($\|$) that plane.[7] This is the only allowed process when dispersion effects are taken into account.

As pointed out in [10], the box-graph no longer vanishes at finite temperature, since the Lagrangian now involves an additional invariant. Hence, the lowest-order contribution to

[6]Taking dispersion effects into account, the box graph still is only an order $\alpha$ correction to the hexagon graph.

[7]Note that Adler's definition for the $\|,\perp$-mode rely on the direction of the magnetic field vector of the photon and thus are opposite to ours.

the photon-splitting matrix element is already produced by the terms of quadratic order in the invariants in Eqs. (32) and (33).

Without going into details, we recall that the splitting amplitude is obtained by attaching the external photon legs to the fermion loop, i.e., differentiating the effective action (which is represented by the loop) thrice with respect to the fields and then contracting the result with the field strengths of the involved photons. Hereby, one has to take into account that the effective Lagrangian now depends on three field invariants: $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$. The thermal amplitude arising from the box-graph finally yields

$$\mathcal{M}(\perp \to \|_1 + \|_2) = 2\,\omega\omega_1\omega_2\, B \sin\theta_B\, \partial_{\mathcal{E}\mathcal{F}}\mathcal{L}, \tag{48}$$

where $\omega, \omega_1, \omega_2$ denote the frequencies of the incoming and the two outgoing photons, respectively, and $\theta_B$ again represents the angle between the propagation direction and the magnetic field. From the splitting amplitude, we obtain the absorption coefficient $\kappa$ via the formula

$$\kappa = \frac{1}{32\pi\omega^2} \int_0^\omega d\omega_1 \int_0^\omega d\omega_2\; \delta(\omega - \omega_1 - \omega_2)\; \mathcal{M}^2. \tag{49}$$

Inserting Eq. (48) for the thermal splitting amplitude into Eq. (49) leads us to

$$\frac{\kappa}{m} = \frac{1}{2^6 \times 3 \times 5\,\pi^2} \left(\frac{eB}{m^2}\right)^2 \sin^2\theta_B \left(\frac{\omega}{m}\right)^5 (\partial_{\mathcal{E}\mathcal{F}}\mathcal{L})^2\, m^8. \tag{50}$$

Here, we encounter the typical $(\omega/m)^5$-dependence of the photon-splitting absorption coefficient for low-frequency photons. The appearance of the magnetic field to the second power is directly related to the fact that the box-graph exhibits only one coupling to the external field. In contrast, Adler's result for the absorption coefficient at zero temperature arising from the hexagon graph reads [26]

$$\frac{\kappa^{T=0}}{m} = \frac{13^2}{3^5 \times 5^3 \times 7^2} \frac{\alpha^3}{\pi^2} \left(\frac{eB}{m^2}\right)^6 \sin^6\theta_B \left(\frac{\omega}{m}\right)^5. \tag{51}$$

Here, the three couplings to the external magnetic field produce a $B^6$-dependence of the absorption coefficient. Therefore, any finite-temperature contribution will exceed the zero-temperature one for small enough magnetic fields; but, of course, the absorption coefficients may then become very tiny.

In order to obtain the one-loop and two-loop absorption coefficients for thermally induced photon splitting at low temperature, the derivatives of the corresponding Lagrangian are required in Eq. (50):

$$\partial_{\mathcal{E}\mathcal{F}}\mathcal{L}^{1T} = \left[\frac{8\alpha^2}{45}\left(\frac{m}{T}\right)^2 + \frac{4\pi\alpha^2}{45}\left(\frac{m}{T}\right)^3\right] \frac{e^{-m/T}}{m^4}, \tag{52}$$

$$\partial_{\mathcal{E}\mathcal{F}}\mathcal{L}^{2T} = \left[-\frac{2^6 \times 37\alpha^3\pi^3}{3^4 \times 5^2 \times 7}\left(\frac{T}{m}\right)^4 + \frac{2^{14}\alpha^3\pi^5}{3^5 \times 5 \times 7^2}\left(\frac{T}{m}\right)^6\right]\frac{1}{m^4}, \tag{53}$$

where we made use of the results of [10] for the low-temperature–weak-field approximation of the one-loop Lagrangian $\mathcal{L}^{1T}$, and employed Eqs. (32) and (33) for the two-loop one. Obviously, inserting the two-loop terms from Eq. (53) into Eq. (50) leads to a power-law dependence of the absorption coefficient $\sim T^8/m^8$, while the one-loop terms from Eq. (52) imply an exponential mass damping $\exp(-2m/T)$ for $T \to 0$.

As mentioned above, photons of frequencies below the pair-production threshold are not only exposed to splitting at finite temperature, but can also scatter directly with the plasma of electrons and positrons. Following [10], the absorption coefficient for the Compton process is given by

$$\frac{\kappa_C}{m} = \frac{\sigma_C}{m} \frac{2}{\pi^2} \int_0^\infty dp\, \frac{p^2}{e^{\omega_e/T} + 1}, \tag{54}$$

where $\omega_e$ denotes the fermion energy $\omega_e = \sqrt{p^2 + m^2}$, and the cross section $\sigma_C$ for unpolarized photons at $\omega/m \simeq 1$ is approximately given by

$$\sigma_C \simeq \frac{4\pi\alpha^2}{3m^2}. \tag{55}$$

Although $\omega/m \simeq 1$ formally represents the maximal limit of validity of our constant-field approximation for the effective action, we will continue to consider photons of that frequency in the following, since, on the one hand, this circumvents a suppression of the absorption coefficients by the common factor $(\omega/m)^5$, and on the other hand, it has been shown for the hexagon graph in [26] that the difference between $\omega/m = 1$- and $\omega/m \sim 0$-calculations is negligible for weak magnetic fields.

Finally, we have to consider another scattering process which arises from the presence of a heat bath: photon-photon scattering between the propagating photon and the black-body radiation of the thermal background. We estimate the absorption coefficient for this process by

$$\frac{\kappa_{\gamma\gamma}}{m} = \frac{\sigma_{\gamma\gamma} n_\gamma}{m}, \tag{56}$$

where $n_\gamma$ denotes the density of photons and is given by

$$n_\gamma = 2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\sqrt{p^2}/T} - 1} = \frac{2\zeta(3)}{\pi^2} T^3. \tag{57}$$

Here we encounter the Riemannian $\zeta$-function with $\zeta(3) \simeq 1.202$. The total polarization-averaged cross section for photon-photon scattering at low frequencies, as one obtains, e.g., from the Heisenberg-Euler Lagrangian [29], reads

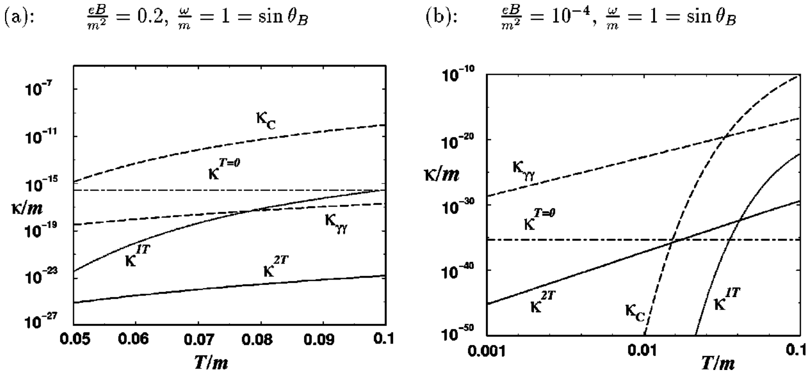


FIG. 2. Absorption coefficient $\kappa$ in units of the electron mass versus temperature $T$ in units of the electron mass. In (a), the various contributions are plotted for parameter values of a realistic astrophysical system. In (b), the parameters are chosen in such a way that the two-loop dominance over the one-loop and the Compton process is revealed; the photon-photon scattering contribution cannot be overtaken in the low-temperature limit.

$$\sigma_{\gamma\gamma}=\frac{973}{10125}\,\frac{\alpha^2}{\pi}\,\frac{\alpha^2}{m^2}\left(\frac{\omega_{\rm CM}}{m}\right)^6, \tag{58}$$

where $\omega_{\rm CM}$ denotes the frequency of both photons in the center-of-mass frame. In order to determine $\omega_{\rm CM}$, we first have to find the mean frequency at temperature $T$. Averaging over the thermal probability distribution for the photons, we find the mean value $\omega_T=[\pi^4/30\zeta(3)]T\simeq 2.701T$. According to relativistic kinematics, the average value for the CM-frequency $\omega_{\rm CM}$ is given by $\omega_{\rm CM}=\sqrt{\omega\omega_T/2}\simeq 1.16\sqrt{T\omega}$, where we averaged over the propagation direction of the thermal photons. Putting everything together, we obtain for the absorption coefficient for photon-photon scattering with the thermal background

$$\frac{\kappa_{\gamma\gamma}}{m}=\frac{7\times 139}{2^5\times 3^7\times 5^6}\,\frac{\pi^9}{\zeta(3)^2}\,\alpha^4\left(\frac{T}{m}\right)^6\left(\frac{\omega}{m}\right)^3$$
$$\simeq 5.21\times 10^{-11}\left(\frac{T}{m}\right)^6\left(\frac{\omega}{m}\right)^3. \tag{59}$$

Since the average frequency of the heat-bath photons is proportional to the temperature, this formula becomes invalid for $T\sim m$ and above, because we employed the low-frequency cross section in Eq. (56).

It is already clear from a qualitative viewpoint that there must be a domain where the two-loop splitting process at least exceeds the one-loop and the Compton contributions due to the power-law dependence on the temperature. But since $\kappa^{2T}\sim(T/m)^8$ and $\kappa_{\gamma\gamma}\sim(T/m)^6$, the two-loop contribution will eventually be surpassed by the photon-photon scattering for $T\to 0$.

However, quantitative results can only be revealed by numerical studies. In fact, as shown in Fig. 2(a), the two-loop contribution is completely irrelevant for parameter values which may be appropriate for a neutron star system and which are close to the upper bound of validity of our approximation: $eB/m^2=0.2$, $\omega/m=1$, $\sin\theta_B=1$, and $T/m=0.05\dots 0.1$. Even the one-loop contribution is small compared to the zero-temperature result; but all are negligible compared to the Compton process.

Concentrating on the relative strengths of the thermal splitting processes, the one-loop contribution loses its major role for $T/m\leqslant 0.041$, where its exponential decrease is surpassed by the two-loop power law.

In order to find a domain in which the two-loop splitting wins out over the zero-temperature process, we have to look at smaller values of the magnetic field strength; e.g., at values of temperature $T/m=0.025$, the two-loop process exceeds the zero-temperature one for $eB/m^2\leqslant 2.1\times 10^{-4}$. Since these are more moderate field strengths, the absorption coefficient naturally becomes very small: $\kappa/m\sim 10^{-34}\dots 10^{-33}$. Hence, in order to be able to measure the splitting rate, the extension of the magnetic field in which the photon propagates must be comparable to galactic scales.

Finally, we have plotted the Compton and photon-photon absorption coefficients, $\kappa_{\rm C}$ and $\kappa_{\gamma\gamma}$, and the two-loop coefficient $\kappa^{2T}$ for a weak magnetic field $eB/m^2=10^{-4}$ at $T/m=0.001\dots 0.1$ in Fig. 2(b). Obviously, the Compton process loses its dominant role for $T/m\leqslant 0.03$; below, the absorption coefficient is ruled by the photon-photon scattering as long as the temperature does not become so small that only the zero-temperature amplitude remains. As is also made visible in Fig. 2(b), the two-loop contribution does not exceed the photon-photon process, due to the weaker temperature dependence of the latter. Hence, we may summarize that the photon absorption coefficient in the low-temperature domain is either dominated by the zero-temperature contribution for strong magnetic fields or by the photon-photon scattering with the thermal background for weak fields. So the two-loop contribution always belongs to the top flight but is never ranked first.

In order to account for realistic astrophysical systems, it is compulsory to include a finite chemical potential. First estimates can be found in [10] to one-loop order, where signals have been found that a finite chemical potential of $\mu\simeq m$ may induce an increase of the thermal splitting amplitude at low temperatures. In order to settle this question properly, the present paper shows that an investigation of these systems should take the two-loop contributions into account. First progress in this direction has been achieved in [30] in which a two-loop calculation with an external magnetic field at finite density has been performed.

Let us conclude this section with the remark that in order to obtain the sum of the zero-temperature and the thermal contributions to the photon splitting absorption coefficient, the amplitudes must be summed up coherently, since the final states of the processes coincide, and the thermal vacuum with a constant background field does not provide for a mechanism of decoherence. While the zero-temperature amplitude as well as the thermal one-loop amplitude are strictly positive, the $T^4$-term in Eq. (53) contributes with a negative sign. Hence, an exceptional curve in the parameter space of $eB/m^2$ and $T/m$ exists where the thermal two-loop amplitude interferes with the thermal one-loop and zero-temperature amplitudes destructively so that photon splitting vanishes.

## V. PAIR PRODUCTION

Thermally induced pair production in electric fields has been searched for at the one-loop level for a long time [5,6,17–19] with extremely contrary results. In our opinion, the final concordant judgement in the real-time formalism [7], the functional Schrödinger approach [17], as well as the imaginary-time formalism [9] is that there is no imaginary part in the thermal contribution to the effective action to one loop, implying the absence of thermally induced pair production to this order of calculation. As already mentioned in the Introduction, drawing conclusions from an imaginary part of the thermal effective action to pair production is not as immediate and straightforward as at zero-temperature, since the presence of an electric pair-producing field and the thermal equilibrium assumption which is inherent to our approach contradict each other.

In the following, we simply assume that on the one hand, the time scale of pair production is much shorter than the time scale of the depletion of the electric field so that dynamical back-reactions can be neglected (this assumption is familiar from the zero-temperature Schwinger formula). On the other hand, we also assume that the state of the plasma is appropriately approximated by a thermal equilibrium although it is exposed to an electric field. Whether the assumption on thermal equilibrium is justified in concrete experimental situations such as, e.g., heavy ion collisions, is still under discussion.

Recently, pair production has been studied with the aid of a quantum kinetic equation (including non-homogeneous electric field configurations, back-reactions, and collisions), revealing the non-Markovian character of the creation process [31]. In these works, the Schwinger formula is rediscovered in the low-density limit for constant fields. We expect that our results hold in the same limit at finite temperature.

Let us now turn to the computation of the imaginary part of the two-loop thermal effective action for external electric fields. For this, we concentrate on the $T^4$-contribution as given in Eq. (29). For purely electric fields, $a\to 0$, $b\to E$, $\mathcal{E}+\mathcal{F}\to\frac{1}{2}E^2$, this reads

$$\mathcal{L}^{2T}(E)\big|_{T^4}=-\frac{\alpha\pi}{90}\,T^4\int_0^\infty\frac{dz}{z}\,\mathrm{e}^{-\mathrm{i}(m^2/eE)z}\times\left[\frac{1}{3}z\coth z+\frac{1-z\coth z}{\sinh^2 z}\right], \tag{60}$$

where we substituted $z=eEs$. For reasons of convenience, it is useful to abbreviate $\eta:=eE/m^2$, which denotes the dimensionless ratio between the electric field and the critical field strength $E_{\mathrm{cr}}:=m^2/e$. Integrating the $1/\sinh^2 z$-term by parts leads us to

$$\mathcal{L}^{2T}(E)\big|_{T^4}=-\frac{\alpha\pi}{90}T^4\lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon^2}+\frac{1}{2}+\frac{1}{4\eta^2}+\int_\epsilon^\infty dz\,\mathrm{e}^{-\mathrm{i}(z/\eta)}\left(\frac{1}{3}-\frac{\mathrm{i}}{\eta z}-\frac{1}{z^2}+\frac{1}{2\eta^2}\right)\coth z\right\}. \tag{61}$$

Here, it should be pointed out that the isolated pole in the first term of the curly brackets does not signal a divergence, but simply cancels the pole at the lower bound of the integral; the whole expression is still finite. Our aim is to evaluate the imaginary part of Eq. (61); for this, the behavior of the integral at the lower bound is of no interest. An imaginary part $\mathrm{Im}\,\mathcal{L}^{2T}(E)|_{T^4}$ arises from the poles of the $\coth z$-term on the imaginary axis at $z=\pm\mathrm{i}n\pi$, $n=1,2,\dots$ .

Decomposing the exponential function into $\cos+\mathrm{i}\sin$, it becomes obvious that the imaginary parts of the integrand are even functions in $z$, while the real parts are odd. Thus, extending the integration interval from $-\infty$ to $\infty$ exactly cancels the real parts and simply doubles the imaginary parts. We finally get

$$\mathrm{Im}\,\mathcal{L}^{2T}(E)\big|_{T^4}=-\frac{\alpha\pi}{90}\frac{T^4}{2\mathrm{i}}\int_{-\infty}^\infty dz\,\mathrm{e}^{-\mathrm{i}(z/\eta)}\times\left(\frac{1}{3}-\frac{\mathrm{i}}{\eta z}-\frac{1}{z^2}+\frac{1}{2\eta^2}\right)\coth z. \tag{62}$$

Now we can close the contour in the lower complex half plane, which is in agreement with the causal prescription $m^2\to m^2-\mathrm{i}\epsilon$. The value of the integral is then simply given by the sum of the residues of the $\coth z$-poles at $z=-\mathrm{i}\pi n$, $n=1,2,\dots$ . Hence, we arrive at

$$\mathrm{Im}\,\mathcal{L}^{2T}(E)\big|_{T^4}=\frac{\alpha\pi^2}{90}T^4\sum_{n=1}^\infty\mathrm{e}^{-n\pi/\eta}\times\left(\frac{1}{3}+\frac{1}{n\pi\eta}+\frac{1}{n^2\pi^2}+\frac{1}{2\eta^2}\right),$$

$$\eta=\frac{eE}{m^2}, \tag{63}$$

which represents our final result for the imaginary part of the thermal effective QED action at low temperature, and should be read side by side with Schwinger's one-loop result:

$$\mathrm{Im}\,\mathcal{L}^1(E)=\frac{m^4}{8\pi^3}\eta^2\sum_{n=1}^\infty\frac{\mathrm{e}^{-n\pi/\eta}}{n^2}. \tag{64}$$

The sum in Eqs. (63) and (64) can be carried out analytically; but here, it should be sufficient to consider the limiting cases of weak and strong electric fields.

In the weak-field limit, i.e., for small values of $\eta$, the sum over $n$ in Eq. (63) is dominated by the first term $n=1$. Furthermore, it is the last term which is the most important one in parentheses. These considerations then lead us to

$$\mathrm{Im}\,\mathcal{L}^{2T}(eE\ll m^2)\simeq \frac{\alpha\pi^2}{180}\,T^4\,\frac{\mathrm{e}^{-\pi/\eta}}{\eta^2}. \tag{65}$$

Combining this with the weak-field approximation of Eq. (64), we get roughly for the total imaginary part of the effective Lagrangian

$$\begin{aligned}\mathrm{Im}\,\mathcal{L}(eE\ll m^2)&=m^4\mathrm{e}^{-\pi/\eta}\left(\frac{\eta^2}{8\pi^3}+\frac{\alpha\pi^2}{180}\,\frac{1}{\eta^2}\,\frac{T^4}{m^4}\right)\\ &\simeq m^4\mathrm{e}^{-\pi/\eta}\left(4\times10^{-3}\,\eta^2+4\times10^{-4}\,\frac{T^4/m^4}{\eta^2}\right).\end{aligned} \tag{66}$$

E.g., for $T/m\simeq0.1$, where the present low-temperature approximation should still be appropriate, the thermal contribution can be neglected for $\eta\geqslant0.1$; both contributions become roughly equal for $\eta\simeq0.056$ (and $T/m=0.1$). For weaker fields and $T/m\simeq0.1$, the thermal contribution even becomes the dominant one.

In the opposite limit, where $\eta\gg1$, i.e., for strong electric fields beyond the critical field strength, the 1/3 in parentheses dominates the expression in Eq. (63), which then gives

$$\begin{aligned}\mathrm{Im}\,\mathcal{L}^{2T}(eE\gg m^2)&=\frac{\alpha\pi^2}{270}\,T^4\sum_{n=1}^{\infty}(\mathrm{e}^{-\pi/\eta})^n\\ &=\frac{\alpha\pi^2}{270}\,T^4\,\frac{\mathrm{e}^{-\pi/\eta}}{1-\mathrm{e}^{-\pi/\eta}}=\frac{\alpha\pi}{270}\,T^4\,\eta+\mathcal{O}(\eta^0).\end{aligned} \tag{67}$$

Together with the strong-field approximation of the Schwinger formula, this gives

$$\begin{aligned}\mathrm{Im}\,\mathcal{L}(eE\gg m^2)&=m^4\,\eta\left(\frac{\eta}{48\pi}+\frac{\alpha\pi}{270}\,\frac{T^4}{m^4}\right)\\ &\simeq m^4\,\eta\left(6.6\times10^{-3}\,\eta+8.5\times10^{-5}\,\frac{T^4}{m^4}\right).\end{aligned} \tag{68}$$

Since Eq. (68) is valid for $\eta\gg1$ and $T/m\ll1$, the low-temperature contribution to $\mathrm{Im}\,\mathcal{L}(E)$ can be neglected for strong electric fields. Similarly to the case of strong magnetic fields, we find that the non-linearities of pure (zero-$T$) vacuum polarization exceed the polarizability of the thermally induced real plasma by far in the strong field limit.

Nevertheless, in the limit of weak electric fields, thermal effects can increase the pair-production probability $P=1-\exp(-2\,\mathrm{Im}\,\mathcal{L}(E))$ significantly, as was shown in Eq. (66). Of course, for these values of $\eta$, the total imaginary part is very small due to the inverse power of $\eta$ in the exponential.

Since we did not consider thermalized fermions, our approach is not capable of describing high-temperature pair production, which would be desirable for forthcoming heavy-ion collision experiments. However, as can be read off from our results for light propagation and photon splitting, extrapolating the power-law behavior to higher temperature scales of $T\sim m$ or even $T/m\gg1$ overestimates a possible two-loop contribution by far, since, for these values of temperature, the one-loop contribution can be expected to be the dominant one. The latter increases at most logarithmically with $T$.

Therefore, it is reasonable to assume that the pair-production probability also increases at most logarithmically with $T$. In view of these considerations, a power-law growth as suggested in [6,18,19] does not appear plausible. Of course, in order to decide this question, the two-loop calculation has to be carried out for arbitrary values of temperature.

## VI. DISCUSSION

In the present work, we calculated the thermal two-loop contribution to the effective QED action for arbitrary constant electromagnetic fields in the low-temperature limit, $T/m\ll1$. Contrary to the usual loop hierarchy in a perturbation theory with small coupling, the thermal two-loop part is found to be dominating over the thermal one-loop part in the low-temperature limit, since the former exhibits a power-law behavior in $T/m$, while the latter is exponentially suppressed by factors of $\exp(-m/T)$. The physical reason behind this is that the one-loop approximation does not involve virtual photons, which, due to their being massless, can be more easily excited at low temperatures than massive fermions; thus, the one-loop approximation should be rated as an inconsistent truncation of finite-temperature QED for $T$ much below the electron mass.

The power-law dependence of the thermal effective action to two loop starting with $T^4/m^4$ implies a *two-loop dominance* in the low-energy domain of thermal QED, which holds roughly up to $T/m\simeq0.05$.

For the subject of light propagation at finite temperature, this two-loop dominance has been known for some time from studies of the polarization tensor [12,14]. Moreover, for the subject of QED in a Casimir vacuum like the parallel-plate configuration, the two-loop dominance is very natural and well known, since the fermions are not considered to be subject to the periodic boundary conditions anyway. This gives rise to a non-trivial check of our results, since Casimir and finite-temperature calculations highly resemble each other. Replacing, as usual, $T$ by $1/(2a)$ in Eq. (65) for the weak-field limit of the imaginary part of the effective Lagrangian, where $a$ denotes the separation of the Casimir plates, we obtain

$$\mathrm{Im}\,\mathcal{L}^{2a}(E)\big|_{a^{-4}} = \frac{\pi e^2}{2^8 \times 45}\frac{1}{a^4}\left(\frac{m^2}{eE}\right)^2 \mathrm{e}^{-\pi m^2/eE}, \qquad (69)$$

which agrees precisely with the findings of [32] for the Casimir corrections to the Schwinger formula.[8]

In order to illustrate the two-loop dominance, we studied light propagation and photon splitting in a weak magnetic background at low temperature. Since we are dealing with the two-loop level, the here-considered effects are naturally very tiny and a significant influence on, e.g., photon physics near astrophysical compact objects appears not very probable. One should rather take a closer look at photon physics on large galactic scales.

Furthermore, we calculated the imaginary part of the thermal two-loop effective action for electric background fields at low temperature. Under mild assumptions, this result can be related to a thermally induced production probability of electron-positron pairs. Especially in the weak-field limit, the thermal contribution has a significant influence on the production rate. Since no thermal one-loop imaginary part exists, any finite two-loop result automatically dominates at any temperature scale.

For the subjects of light propagation and photon splitting, the loop hierarchy is restored above $T/m \simeq 0.05$. Already at this comparably low value of temperature, the thermal excitation of the fermions begins to compete with that of the virtual photon. Hence, a calculation of the two-loop thermal Lagrangian at intermediate or high temperatures would appear as an imposition, were it not for the high-temperature pair-production probability which is beyond the range of the one-loop approximation and of great interest for, e.g., heavy-ion collisions.

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## APPENDIX A: ONE-LOOP POLARIZATION TENSOR

While the polarization tensor in an external magnetic field has been considered by many authors (a comprehensive study can, e.g., be found in [34]), a generalization to arbitrary constant electromagnetic fields in a straightforward manner is associated with a substantial increase in calculational difficulties. The problem was first solved by Batalin and Shabad [35]; their extensive result was later brought into a practical form by Artimovich [36]. In the following, we will briefly sketch a simpler derivation of the polarization tensor in arbitrary constant electromagnetic fields; our approach is based on the findings of Urrutia [21], who solved the problem for the special case of parallel electric and magnetic fields.

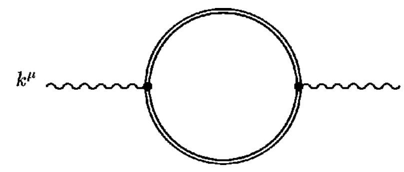


FIG. 3. Diagrammatic representation of the one-loop polarization tensor. The fermionic double line represents the coupling to all orders to the external electromagnetic field.

Assume that the $(-E)$-field and the $B$ field point along the 3-axis. 4-vectors like the external momentum (cf. Fig. 3) can then be decomposed into

$$k^\mu = k_\parallel^\mu + k_\perp^\mu, \quad k_\parallel^\mu = (k^0,0,0,k^3), \quad k_\perp^\mu = (0,k^1,k^2,0). \qquad (A1)$$

In the same manner, tensors can be decomposed, e.g., $g^{\mu\nu} = g_\parallel^{\mu\nu} + g_\perp^{\mu\nu}$. With respect to each subspace, we easily find the unique orthogonal vector to a given one:

$$\tilde{k}_\parallel^\mu = (k^3,0,0,k^0), \quad \tilde{k}_\perp^\mu = (0,k^2,-k^1,0). \qquad (A2)$$

Following Urrutia [21], the polarization tensor for the special field configuration can be written as

$$\Pi^{\mu\nu}(k|A) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^{1}\frac{d\nu}{2}\Bigg\{ \mathrm{e}^{-is\phi_0}\frac{zz'}{\sin z \sinh z'} \times [(g^{\mu\nu}k^2 - k^\mu k^\nu)N_0 + (g_\parallel^{\mu\nu}k_\parallel^2 - k_\parallel^\mu k_\parallel^\nu)N_1 + (g_\perp^{\mu\nu}k_\perp^2 - k_\perp^\mu k_\perp^\nu)N_2 - (\tilde{k}_\perp^\mu \tilde{k}_\parallel^\nu + \tilde{k}_\parallel^\mu \tilde{k}_\perp^\nu)N_3] + \text{c.t.}\Bigg\}. \qquad (A3)$$

The electric and magnetic field strengths $E,B$ are contained in the variables $z := eBs$ and $z' := eEs$. The exponent $\phi_0$ is given by[9]

$$\phi_0 := m^2 + \frac{k_\parallel^2}{2}\frac{\cosh z' - \cosh \nu z'}{z' \sinh z'} + \frac{k_\perp^2}{2}\frac{\cos \nu z - \cos z}{z \sin z}. \qquad (A4)$$

The functions $N_i$ read[10]

$$N_0 = \cosh \nu z' \cos \nu z - \sinh \nu z' \sin \nu z \cot z \coth z',$$

$$N_1 = 2\cos z \frac{\cosh z' - \cosh \nu z'}{\sinh^2 z'} - N_0 \quad =: \tilde{N}_1 - N_0,$$

[8]Actually, Eq. (69) agrees with the findings of [32] except for a global sign; however, as was pointed out by one of the authors in a footnote of [33], the expression in [32] is wrong by a minus sign, which saves the day.

[9]This formula has been misprinted in Ref. [21].

[10]$N_3$ differs from Urrutia's findings by a minus sign, since he considers *parallel* $E$- and $B$-fields.

$$N_2 = 2\cosh z' \frac{\cos \nu z - \cos z}{\sin^2 z} - N_0 \qquad =: \tilde{N}_2 - N_0, \tag{A5}$$

$$N_3 = -\frac{1-\cos z \cos \nu z}{\sin z}\,\frac{\cosh \nu z' \cosh z' - 1}{\sinh z'} + \sin \nu z \sinh \nu z',$$

where we have incidentally defined the functions $\tilde{N}_{1,2}$ for later use. The determination of the contact term corresponds to a charge and field strength renormalization and yields

$$\text{c.t.} = -\,\mathrm{e}^{-\mathrm{i}m^2 s}(1-\nu^2)(g^{\mu\nu}k^2 - k^\mu k^\nu). \tag{A6}$$

Now, one can show [37] that the Lorentz invariant form of the polarization tensor for arbitrary constant electromagnetic fields can be completely reconstructed from the special form given above for anti-parallel electric and magnetic fields. This is achieved by, first, a one-to-one mapping between Urrutia's scalar variables $(k_\parallel^2, k_\perp^2, E, B)$ and a set of invariants which reduce to Urrutia's variables in the special system:

$$a \to B, \qquad b \to -E,$$
$$z_k \to -E^2 k_\parallel^2 + B^2 k_\perp^2, \quad k^2 \to k_\parallel^2 + k_\perp^2. \tag{A7}$$

The inverse map is obtained by a simple calculation; the non-trivial relations are

$$k_\parallel^2 \to \frac{a^2k^2 - z_k}{a^2+b^2}, \quad k_\perp^2 \to \frac{b^2k^2 + z_k}{a^2+b^2}. \tag{A8}$$

Secondly, the reconstruction requires a one-to-one mapping between Urrutia's tensor structures in Eq. (A3) and Lorentz covariant tensors which reduce to Urrutia's in the special system. For this, we need to introduce the following definitions. First, we employ a set of four linearly independent 4-vectors:

$$k^\mu, \quad Fk^\mu \equiv F^{\mu\alpha}k_\alpha, \quad F^2k^\mu \equiv F^{\mu\alpha}F_{\alpha\beta}k^\beta,$$
$${}^\star Fk^\mu \equiv {}^\star F^{\mu\alpha}k_\alpha. \tag{A9}$$

From these, we construct the 4-vectors:

$$v_\parallel^\mu := \frac{1}{a^2+b^2}(a\,{}^\star Fk^\mu - b\,Fk^\mu) \;\to\; \tilde{k}_\parallel^\mu,$$
$$v_\perp^\mu := \frac{1}{a^2+b^2}(b\,{}^\star Fk^\mu + a\,Fk^\mu) \;\to\; \tilde{k}_\perp^\mu, \tag{A10}$$

where the subscripts $\parallel$ and $\perp$ are to remind us of the meaning of $v_\parallel$ and $v_\perp$ in the special Lorentz system (longitudinal and transversal part of $\tilde{k}$). Incidentally, they obey the relations [cf. Eq. (A8)]

$$v_\parallel^2 = v_\parallel^\mu v_{\parallel\mu} = -\frac{a^2k^2 - z_k}{a^2+b^2},$$
$$v_\perp^2 = v_\perp^\mu v_{\perp\mu} = \frac{b^2k^2 + z_k}{a^2+b^2}, \quad v_\parallel^\mu v_{\perp\mu} = 0. \tag{A11}$$

Finally introducing the projectors

$$P_0^{\mu\nu} := \frac{1}{k^2\left[2\mathcal{F}\frac{z_k}{k^2} + \mathcal{G}^2 - \left(\frac{z_k}{k^2}\right)^2\right]} \times \left(F^2k^\mu + \frac{z_k}{k^2}k^\mu\right)\left(F^2k^\nu + \frac{z_k}{k^2}k^\nu\right),$$

$$P_\parallel^{\mu\nu} := \frac{v_\parallel^\mu v_\parallel^\nu}{v_\parallel^2}, \quad P_\perp^{\mu\nu} := \frac{v_\perp^\mu v_\perp^\nu}{v_\perp^2}, \tag{A12}$$

which satisfy the usual projector identities, $P_{0,\parallel,\perp}^2 = P_{0,\parallel,\perp}$, $P_{0,\parallel,\perp}{}^\mu{}_\mu = 1$, we can establish the one-to-one mapping:

$$-v_\parallel^\mu v_\parallel^\nu \to (g_\parallel^{\mu\nu}k_\parallel^2 - k_\parallel^\mu k_\parallel^\nu),$$
$$v_\perp^\mu v_\perp^\nu \to (g_\perp^{\mu\nu}k_\perp^2 - k_\perp^\mu k_\perp^\nu),$$
$$Q^{\mu\nu} := v_\perp^\mu v_\parallel^\nu + v_\parallel^\mu v_\perp^\nu \to (\tilde{k}_\perp^\mu \tilde{k}_\parallel^\nu + \tilde{k}_\parallel^\mu \tilde{k}_\perp^\nu), \tag{A13}$$
$$k^2[P_0^{\mu\nu} + P_\parallel^{\mu\nu} + P_\perp^{\mu\nu}] \to (g^{\mu\nu}k^2 - k^\mu k^\nu).$$

In the third line, we have defined the object $Q^{\mu\nu}$, which is neither a projector nor orthogonal to the $P_{\parallel,\perp}^{\mu\nu}$'s but orthogonal to $P_0^{\mu\nu}$.

We are finally in a position to transform the polarization tensor for the parallel field configuration into its generalized form for arbitrary constant electromagnetic fields:

$$\Pi^{\mu\nu}(k|A) = \Pi_0\,P_0^{\mu\nu} + \Pi_\parallel\,P_\parallel^{\mu\nu} + \Pi_\perp\,P_\perp^{\mu\nu} + \Theta\,Q^{\mu\nu}, \tag{A14}$$

where $\Pi_{0,\parallel,\perp}$ and $\Theta$ are functions of the invariants and read

$$\left\{\begin{matrix}\Pi_0\\ \Pi_\parallel\\ \Pi_\perp\\ \Theta\end{matrix}\right\} = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^{1}\frac{d\nu}{2}\left[\mathrm{e}^{-\mathrm{i}s\phi_0}\frac{eas\,ebs}{\sin eas \sinh ebs} \times \left\{\begin{matrix}k^2N_0\\ N_0v_\perp^2 - \tilde{N}_1 v_\parallel^2\\ \tilde{N}_2 v_\perp^2 - N_0 v_\parallel^2\\ -N_3\end{matrix}\right\} + \text{c.t.}\right]. \tag{A15}$$

Substituting the invariants into Eqs. (A4) and (A5), the functions $N_i$ and $\phi_0$ yield

$$\phi_0 = m^2 - \frac{v_\parallel^2}{2}\frac{\cosh ebs - \cosh \nu ebs}{ebs \sinh ebs} + \frac{v_\perp^2}{2}\frac{\cos \nu eas - \cos eas}{eas \sin eas},$$

$$N_0 = \cosh \nu ebs \cos \nu eas - \sinh \nu ebs \sin \nu eas \cot eas \coth ebs,$$

$$\tilde{N}_1 = 2\cos eas \frac{\cosh ebs - \cosh \nu ebs}{\sinh^2 ebs},$$

$$\tilde{N}_2 = 2\cosh ebs \frac{\cos \nu eas - \cos eas}{\sin^2 eas},$$

$$N_3 = -\frac{1-\cos eas \cos \nu eas}{\sin eas} \times \frac{\cosh \nu ebs \cosh ebs - 1}{\sinh ebs} + \sin \nu eas \sinh \nu ebs. \tag{A16}$$

The scalars $v_{\parallel,\perp}^2$ are given by certain combinations of the invariants and can be found in Eq. (A10). The contact term given in Eq. (A6) contributes equally to the $\Pi_i$'s,

$$\text{c.t.} = -\mathrm{e}^{-\mathrm{i}m^2 s} k^2 (1-\nu^2), \tag{A17}$$

but does not modify the function $\Theta$, which is already finite.

Note that Eq. (A14) almost appears in a diagonalized form except for the term $\Theta\, Q^{\mu\nu}$. While $P_0^{\mu\nu}$ indeed projects onto an eigenspace of $\Pi^{\mu\nu}$ with eigenvalue $\Pi_0$, this is generally not the case for the projectors $P_{\parallel,\perp}^{\mu\nu}$, due to $\Theta\, Q^{\mu\nu}$. Although a further diagonalization is straightforward, we will not bother to write it down, since we only need the trace of $\Pi^{\mu\nu}$, which is simply given by

$$\Pi^\mu{}_\mu = \Pi_0 + \Pi_\parallel + \Pi_\perp, \quad Q^\mu{}_\mu = 0. \tag{A18}$$

In the actual two-loop calculation, the contact terms can be omitted for two reasons: first, it does not contribute to the thermal part, since the latter is finite; secondly, for the zero-temperature Lagrangian, a renormalization procedure is required anyway and, in particular, the mass renormalization would not be covered by an inclusion of the contact terms.

Inserting Eq. (A15) into Eq. (A18) brings us to the explicit representation of the trace:

$$\Pi^\mu{}_\mu = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{d\nu}{2}\,\frac{\mathrm{e}^{-\mathrm{i}s\phi_0}}{a^2+b^2}\,\frac{eas\, ebs}{\sin eas \sinh ebs} \times [z_k(\tilde{N}_2 - \tilde{N}_1) + k^2(2N_0(a^2+b^2) + b^2\tilde{N}_2 + a^2\tilde{N}_1)]. \tag{A19}$$

This is the desired expression which is required in Eq. (11). For reasons of convenience, it is useful to rewrite the function $\phi_0$ in terms of the variables $k^2$ and $z_k$. For this, we insert Eq. (A10) into the first line of Eq. (A16); a reorganization yields

$$\mathrm{e}^{-\mathrm{i}s\phi_0} = \mathrm{e}^{-\mathrm{i}m^2 s}\,\mathrm{e}^{-A_z z_k}\,\mathrm{e}^{-A_k k^2}, \tag{A20}$$

where we implicitly defined

$$A_z := \frac{\mathrm{i}s}{2}\,\frac{1}{a^2+b^2}\left(\frac{\cos \nu eas - \cos eas}{eas \sin eas} + \frac{\cosh \nu ebs - \cosh ebs}{ebs \sinh ebs}\right), \tag{A21}$$

$$A_k := \frac{\mathrm{i}s}{2}\frac{1}{a^2+b^2}\left(b^2\frac{\cos \nu eas - \cos eas}{eas \sin eas} - a^2\frac{\cosh \nu ebs - \cosh ebs}{ebs \sinh ebs}\right). \tag{A22}$$

This provides us with the required necessities for the two-loop calculation in Sec. II.

## APPENDIX B: FINITE-TEMPERATURE COORDINATE FRAME

In order to make the paper self-contained, we briefly review the construction of the finite-temperature coordinate frame as introduced in [9], and then apply it to the present problem.

First, we define the *vierbein* $e^{A\mu}$ which mediates between the given system labeled by $\mu,\nu,\ldots = 0,1,2,3$ and the desired system labeled by the (Lorentz) indices $A,B,\ldots = 0,1,2,3$ by

$$e_0{}^\mu := u^\mu,$$

$$e_1{}^\mu := \frac{u_\alpha F^{\alpha\mu}}{\sqrt{\mathcal{E}}},$$

$$e_2{}^\mu := \frac{1}{\sqrt{d}}(u^\alpha F_{\alpha\beta}F^{\beta\mu} - \mathcal{E}\, e_0{}^\mu),$$

$$e_3{}^\mu := \epsilon^{\alpha\beta\gamma\mu}\, e_{0\alpha}\, e_{1\beta}\, e_{2\gamma}, \tag{B1}$$

where the quantity $d$ abbreviates the combination of invariants:

$$d := 2\mathcal{F}\mathcal{E} - \mathcal{G}^2 + \mathcal{E}^2. \tag{B2}$$

The vierbein satisfies the identity

$$e_{A\mu}\, e_B{}^\mu = g_{AB} \equiv \mathrm{diag}(-1,1,1,1), \tag{B3}$$

where $g_{AB} \sim g^{AB}$ denotes the metric which raises and lowers capital indices. By a direct computation, we can transform the field strength tensors and the heat-bath vector:

$$n^A := g^{AB} e_B{}^{\mu} n_{\mu} = (T,0,0,0),$$

$$F_{AB} := e_{A\mu} F^{\mu\nu} e_{B\nu} = \begin{pmatrix} 0 & \sqrt{\mathcal{E}} & 0 & 0 \\ -\sqrt{\mathcal{E}} & 0 & \sqrt{d/\mathcal{E}} & 0 \\ 0 & -\sqrt{d/\mathcal{E}} & 0 & -\mathcal{G}/\sqrt{\mathcal{E}} \\ 0 & 0 & \mathcal{G}/\sqrt{\mathcal{E}} & 0 \end{pmatrix}. \tag{B4}$$

Obviously, the new system corresponds to the heat-bath rest frame with the spatial axes oriented along the electromagnetic field in some sense. The components of the field strength tensor are now given by combinations of the invariants.

In order to determine the form of $z_k = k_\mu F^{\mu\alpha} k_\nu F^{\nu}{}_{\alpha} \equiv -k^A F_{AC} F^C{}_B k^B$, we need the square of the field strength tensor:

$$F^2_{AB} \equiv F_{AC} F^C{}_B = \begin{pmatrix} -\mathcal{E} & 0 & \sqrt{d} & 0 \\ 0 & \mathcal{E} - \dfrac{d}{\mathcal{E}} & 0 & -\dfrac{\sqrt{d}\mathcal{G}}{\mathcal{E}} \\ \sqrt{d} & 0 & -\dfrac{\mathcal{G}^2 + d}{\mathcal{E}} & 0 \\ 0 & -\dfrac{\sqrt{d}\mathcal{G}}{\mathcal{E}} & 0 & -\dfrac{\mathcal{G}^2}{\mathcal{E}} \end{pmatrix}. \tag{B5}$$

This allows us to write $z_k$ in the form

$$z_k = \mathcal{E}(k^0)^2 - 2\sqrt{d}\, k^0 k^2 + (2\mathcal{F} + \mathcal{E})(k^2)^2 + \left(\frac{d}{\mathcal{E}} - \mathcal{E}\right)(k^1)^2 + 2\frac{\sqrt{d}\mathcal{G}}{\mathcal{E}} k^1 k^3 + \frac{\mathcal{G}^2}{\mathcal{E}}(k^3)^2, \tag{B6}$$

where $k^0, k^1, k^2, k^3$ represent the components of the rotated momentum vector $k^A = e^A{}_\mu k^\mu$.

Now we can finally determine the desired form for the exponent in Eq. (16) in terms of finite-temperature coordinates:

$$A_z z_k + A_k k^2 = (A_k + (a^2 - b^2 + \mathcal{E})A_z)\left(k^2 - \frac{A_z\sqrt{d}}{A_z(2\mathcal{F}+\mathcal{E}) + A_k} k^0\right)^2 - \frac{(A_k + a^2 A_z)(A_k - b^2 A_z)}{A_k + (a^2 - b^2 + \mathcal{E})A_z}(k^0)^2$$
$$+ \left(A_z \frac{a^2 b^2}{\mathcal{E}} + A_k\right)\left(k^3 + \frac{A_z \dfrac{\sqrt{d}\mathcal{G}}{\mathcal{E}}}{A_z \dfrac{\mathcal{G}^2}{\mathcal{E}} + A_k} k^1\right) + \frac{(A_k + a^2 A_z)(A_k - b^2 A_z)}{A_k \dfrac{a^2 b^2}{\mathcal{E}} + A_k}(k^1)^2, \tag{B7}$$

where again $k^0, k^1, k^2, k^3$ represent the components of $k^A$.


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