(Meta)stable closed vortices in 3+1 dimensional gauge theories with an extended Higgs sector

Antti J. Niemi*
Department of Theoretical Physics, Uppsala University, P.O. Box 803, S-75108, Uppsala, Sweden and Helsinki Institute of Physics, P.O. Box 9, FIN-00014 University of Helsinki, Finland

Kaupo Palo†
Evotec BioSystems AG, Instituudi tee 11, Harku EE76902, Estonia

Sami Virtanen‡
Material Physics Laboratory, Helsinki University of Technology, Box 2200, FIN-02015 HUT, Finland

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In gauge theories with an extended Higgs sector the classical equations of motion can have solutions that describe stable, closed finite energy vortices. Such vortices separate two disjointed Higgs vacua, with one of the vacua embedded in the other in a manner that forms a topologically nontrivial knot. The knottedness stabilizes the vortex against shrinkage in 3+1 dimensional space time. But in a world with extra large dimensions we expect the configuration to decay by unknotting. As an example, we consider the semilocal $\theta_W \to \pi/2$ limit of the Weinberg-Salam model. We present numerical evidence for the existence of a stable closed vortex, twisted into a toroidal configuration around a circular Higgs vacuum at its core.

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I. INTRODUCTION

In unified field theory models the existence of several degenerate Higgs vacua is generic [1]. This degeneracy has numerous consequences, and leads to the appearance of domain walls [2] and vortices [3]. The role and properties of such configurations have been widely discussed. They are expected to be highly relevant in a variety of problems in high energy physics and early Universe cosmology.

The energy of a line vortex scales with its length and a stable vortex is expected to have an infinite energy. Any isolated, finite length vortex such as a vortex loop in the standard Abelian Higgs model with a single complex scalar field is expected to rapidly collapse. But here we propose that (meta)stable, or at least very long-lived, finite length closed vortices can actually be present in unified field theory models with an *extended* Higgs sector. Even though the energy of these vortices does scale in proportion to their length, they can still be prevented from shrinking by their topological nontriviality. This becomes possible when the vortex is deformed so that it forms a knot, separating two Higgs vacua which are tangled around each other in a nontrivial manner. The topological nontriviality of a knotted structure then provides a repulsive force which stabilizes the vortex and prevents it from collapsing.

An actual dynamical stability of a knotted finite energy closed vortex depends on the dynamical details of the underlying field theory model, and must be verified separately by inspecting the classical equations of motion. But we shall argue that with a properly extended Higgs sector dynamical stability can occur. The vortex becomes classically protected from a collapse by a finite energy barrier, very much in analogy with the mechanism that has been introduced in the case of one dimensional models in Ref. [4]. Indeed, following Ref. [4] we propose that such a finite energy barrier persists even at low coupling, where semiclassical methods become reliable. This means that quantum mechanically the vortex can have a very long lifetime, even though it may eventually decay by a tunneling process [4].

The sample field theory model that we consider in the following is related to the Weinberg-Salam model. It emerges when we take the limit where Weinberg angle $\theta_W \to \pi/2$. This suggests that our configurations may have relevance to standard model physics, at energy scales which may be reachable in the future accelerators such as CERN Large Hadron Collider (LHC). As a consequence, a detailed inspection of knotted vortices in realistic (supersymmetric) extensions of the standard model could become rewarding: Since a nontrivial knot is topologically stable exactly in three space dimensions, the existence of stable knotted solitons may provide a direct test of the dimensionality of space time at the TeV scale, presently under an active debate [5,6]. If extra dimensions can indeed be seen by the standard model at TeV scales, it should be difficult to preserve the stability of any TeV (or higher) scale knotted configuration. Such a configuration should rapidly decay, by unknotting itself via the extra dimensions. Instead of knotted vortices, in an extension of the standard model with TeV scale extra dimensions it becomes relevant to inspect the stability of knotted configurations with co-dimensionality two in the extended world.

II. MODEL

The topological nontriviality of a knot is characterized by its self-linkage, and the ensuing linking number coincides with the Hopf invariant [7]. This is an integral invariant that

*Email address: Niemi@teorfys.uu.se
†Email address: Palo@ebi.ee
‡Email address: Sami@focus.hut.fi

relates self-linking to the $\pi_3(S^3)\sim\pi_3(S^2)$ homotopy classes. In the case of static and localized configurations such as a knot, we may compactify $R^3\rightarrow S^3$. This suggests that in the case of a Yang-Mills-Higgs model, a knotted vortex might be present when the Higgs vacuum sector allows for a nontrivial $S^3\rightarrow S^3$ or $S^3\rightarrow S^2$ mapping. This is possible when the symmetry group of the Higgs vacuum sector contains a $SU(2)$ subgroup. A familiar example is the Weinberg-Salam model, and we shall exemplify our ideas by considering its simplified version, the so-called semilocal, i.e., $\theta_W=\pi/2$ limit of the Weinberg-Salam model [3]. This is an Abelian gauge theory with a two component complex Higgs scalar and renormalizable Lagrangian

$$L=-\frac{1}{4}F_{\mu\nu}^2+\overline{D_\mu}Z^\dagger D_\mu Z-\lambda(Z^\dagger Z-1)^2, \tag{1}$$

where $D_\mu=\partial_\mu-iA_\mu$ is the $U(1)$ covariant derivative and

$$Z=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=|\Phi|\cdot\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} \tag{2}$$

the Higgs field, and we normalize $\Sigma\phi_i^2=1$. The Higgs vacuum manifold of Eq. (1) is then characterized by $|\Phi|=1$ which determines the topology of S^3. In this model Higgs mechanism occurs, the gauge field A_μ and one component of the complex φ_a combine into a massive vector field. In addition we have a massive scalar and two Goldstone bosons.

The model (1) is known to support straight, infinite-length, infinite-energy line vortices as stable classical solutions to its equations of motion, and the properties of these vortices have been widely discussed [3]. In the following we argue that the model (1) also supports static, finite length, and energy knotted vortices as stable solutions. The vortex now appears as a domain wall that separates two different Higgs vacua. Modulo global $SU(2)$ rotations we can select these two Higgs vacua so that in R^3 they coincide with the preimages of the two S^1, which are determined by setting either $|\varphi_2|(\mathbf{x})=1$ or $|\varphi_1|(\mathbf{x})=1$ [with $|\Phi|(\mathbf{x})=1$]. In the sequel we shall denote these two vacua as S^1_{out} and S^1_{in}, respectively. For a potentially stable closed vortex to occur, these two vacua should be linked with a nontrivial linking number which is computed by the Hopf invariant Q_H [8]:

$$Q_H=\frac{1}{12\pi^2}\int \epsilon_{ijkl}\phi_i d\phi_j\wedge d\phi_k\wedge d\phi_l\,. \tag{3}$$

We note that this coincides with the $\pi_3(S^3)$ winding number of the field ϕ_a.

We argue that for a nontrivial Q_H the vortex which separates the two Higgs vacua can be classically (meta)stable, at least for some range of values for the coupling constant λ. Indeed, the classical equation of motion obtained by varying Eq. (1) with respect to A_i is (we consider static configurations in the $A_0=0$ gauge)

$$A_i=-\frac{1}{2}\frac{1}{Z^\dagger Z}(\partial_i Z^\dagger Z-Z^\dagger\partial_i Z)+\frac{1}{4}\frac{1}{Z^\dagger Z}\partial_j F_{ij}\,. \tag{4}$$

By sending $\lambda\rightarrow\infty$ we force $Z^\dagger Z=|\Phi|\equiv 1$, and if we define a three component unit vector by $\hat{\mathbf{n}}=Z^\dagger\hat{\sigma}Z$ and use Eq. (4) to eliminate A_i from Eq. (1), we find that Eq. (1) reduces to

$$L\rightarrow|\partial_\mu\hat{\mathbf{n}}|^2+\frac{1}{4e}(\hat{\mathbf{n}}\cdot d\hat{\mathbf{n}}\times d\hat{\mathbf{n}})^2+(\text{higher derivatives}). \tag{5}$$

The first two terms that we have presented in Eq. (5) reproduce the model studied in Refs. [8] and [9]. There, it has been established that Eq. (5) admits stable knotlike solitons with a nontrivial Hopf invariant. Those results then suggest, that at least for large values of λ [10] the model (1) should either support stable knotted configurations with nontrivial Q_H, or then it should describe metastable knotted configurations with a lifetime that approaches infinity as $\lambda\rightarrow\infty$.

III. STABLE TOROIDAL VORTICES

The previous arguments are suggestive but not sufficient to conclude that for finite, even for weak λ the model (1) could support stable closed vortices. The equations of motion are highly nonlinear and for finite coupling the higher order terms in Eq. (5) cannot be ignored. Consequently, an explicit construction with a finite λ becomes imperative. For this we remind that Eq. (1) supports stable, infinitely long line vortices [3,10]. If we construct a finite energy vortex in Eq. (1) by cutting a piece of the line vortex studied in Ref. [3] which we then deform into the shape of a finite radius torus by joining the ends, the resulting configuration becomes unstable against shrinkage. However, if we form such a toroidal vortex ring by first twisting the line vortex once around its core before joining its ends, the nontriviality of the twist might produce a repulsive interaction that stabilizes the configuration against shrinkage [8]. In order to form an appropriate configuration that allows for the introduction of a nontrivial twist, we recall that the present model reduces to the standard Abelian Higgs model in the limit where we truncate one of the two complex fields φ_a. In the standard Abelian Higgs model the number of degrees of freedom is insufficient for describing a nontrivial twist around the core of a vortex, and a toroidal configuration is unstable. But in the present case the Higgs sector is extended with the broken phase containing two additional Goldstone bosons. This ensures that the number of degrees of freedom is now sufficient for describing a nontrivial twist around the core of a vortex. For this, we form a toroidal configuration in such a manner that outside and inside of a toroidal surface we have a different asymptotic large distance Higgs vacuum of the standard line vortex in the conventional Abelian Higgs model. This means that outside of the toroidal surface we select, e.g., the Higgs vacuum S^1_{out} which is characterized by $|\Phi|=|\varphi_2|=1$, and inside of the torus we select the Higgs vacuum S^1_{in} with $|\Phi|=|\varphi_1|=1$. These two Higgs vacua become then separated by a toroidal domain wall configuration that interpolates between the two vacua. We identify this domain wall as our closed vortex, wrapped around the toroidal surface.

The two Higgs vacua S^1_{in} and S^1_{out} are now linked in a nontrivial manner. If we assume that $|\Phi| \neq 0$ everywhere, the Hopf invariant (3) is everywhere well defined and it computes the linking number of these two Higgs vacua. A nonvanishing Hopf invariant provides topological stability for the toroidal vortex that interpolates these Higgs vacua, the vortex can unwind only if the norm $|\Phi|$ in Eq. (2) vanishes for some $\mathbf{x} \in R^3$ so that the Hopf invariant becomes ill-defined. The Higgs potential provides an energy barrier that prevents $|\Phi|$ from vanishing. For finite λ this energy barrier is finite. It increases with an increasing λ and becomes infinitely high in the limit where $\lambda \to \infty$. For finite values of λ the vortex can then be stable, provided the equations of motion indeed restrict the Higgs field so that it is everywhere constrained to $|\Phi(\mathbf{x})| > 0$. In this manner classical (meta)stability becomes a dynamical issue, it can be resolved only by actually solving the equations of motion.

Following Ref. [4] we note that since the energy barrier is finite, quantum mechanically the vortex may still decay. It may unwind itself by barrier penetration, when the Higgs fields fluctuates so that for some region of space $|\Phi(\mathbf{x})|$ vanishes. However, if the classical solution has $|\Phi(\mathbf{x})| \approx 1$ as we expect, e.g., for large λ, the vortex has an exponentially long quantum mechanical lifetime. Indeed, since the present situation is quite similar to the one dimensional case studied in Ref. [4], we expect that also in the present case a classical solution remains stable even for relatively small values of λ.

IV. A NUMERICAL SOLUTION

The equations of motion for Eq. (1) are highly nonlinear, to the extent that an analytical investigation of an actual vortex configuration appears entirely hopeless. Consequently, we resort to numerical methods. We have performed an extensive search for a (locally stable) vortex configuration in a toroidal shape. For this, we specify a torus on the (x,y) plane, oriented so that the toroidal symmetry axis coincides with the z axis. With (ρ,z,ψ) cylindrical coordinates, we use rotation invariance to argue that the most general $SO(2) \times SO(2)$ invariant ansatz for the Higgs field can be represented in the functional form

$$Z = \Phi(\rho,z) \cdot \begin{pmatrix} \cos k\psi \cdot \sin \frac{1}{2}\theta(\rho,z) \\ \sin k\psi \cdot \sin \frac{1}{2}\theta(\rho,z) \\ \cos \vartheta(\rho,z) \cdot \cos \frac{1}{2}\theta(\rho,z) \\ \sin \vartheta(\rho,z) \cdot \cos \frac{1}{2}\theta(\rho,z) \end{pmatrix}, \qquad (6)$$

where $0 \leq \vartheta < 2\pi n$ and $0 \leq \theta < \pi$ and k,n are integers. For the gauge field A_i the rotation invariance restricts the components into functions of (ρ,z) only. With this ansatz the static equations of motion for Eq. (1) reduce into a set of coupled equations for six unknown functions, which we then study numerically. We use the following boundary conditions: Both at $\rho = 0$ and for ρ large $\theta(\rho)$ vanishes and $\Phi(\rho) = 1$, corresponding to our Higgs vacuum configuration S^1_{out}. For some point at a distance $\rho = \rho_c$ we have $\theta(\rho_c,0) = \pi$ with $\Phi(\rho_c,0) = 1$, corresponding to our Higgs vacuum configuration S^1_{in}. These boundary conditions determine a toroidal vortex wrapped around the circle S^1_{in}, it appears as a domain wall that separates the Higgs vacuum S^1_{in} from the asymptotic Higgs vacuum S^1_{out}. The Hopf invariant of this configuration (when $\Phi \neq 0$ everywhere) is $Q_H = k \cdot n$ so that the vortex is topologically stable: It wraps the torus k times, and twists n times around its center. We expect that this twist produces a repulsive force that balances the vortex against shrinkage, resulting into a (meta)stable configuration.

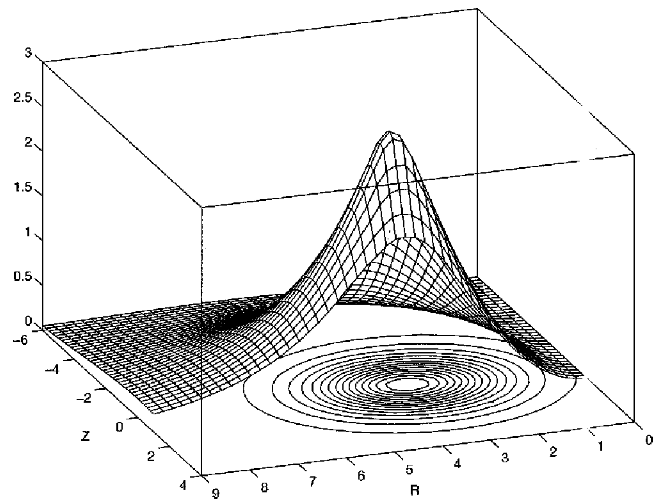

FIG. 1. A typical profile (7) for a stable configuration with $n = 1$ and $k = 2$ and $\lambda = \mathcal{O}(10)$. At the center of the torus, here at a distance $R \approx 3.8$ from the z axis, we have $\Theta = \pi$ corresponding to the Higgs vacuum $|\varphi_1| = 1$ and $\varphi_2 = 0$. Outside of the torus we approach the Higgs vacuum with $|\varphi_2| = 1$ and $\varphi_1 = 0$. The closed vortex is a toroidal domain wall that interpolates between these two Higgs vacua. Also in the figure we have the contours for energy density, which peaks near $\Theta = \pi$ but at a somewhat smaller distance R from the z axis.

We have employed our ansatz to perform very extensive numerical simulations to search for a toroidal vortex configuration. In our simulations we have used the techniques described in Ref. [8], by extending the Hamiltonian equation into a parabolic flow equation with respect to an auxiliary time variable and then following the flow towards a fixed point of the Hamiltonian. We have found definite convergence towards torus-shaped configurations, that appear stable under our parabolic flow. This indicates that the Hessian is positive definite when evaluated at the fixed point, which is a manifestation that our final configuration is indeed classically stable. As an example, in Fig. 1 we describe a solution for $k = 2$ and $n = 1$. In this figure we plot the profile for the angular variable

$$\Theta(\rho,z) = 2 \arctan\left(\frac{|\varphi_1|}{|\varphi_2|}\right). \qquad (7)$$

It varies between $\Theta_{in}=\pi$ at $\rho_c\approx 3.8$ and $\Theta_{out}=0$ outside of the configuration, so that we indeed have a domain wall configuration that separates the two Higgs vacua S^1_{in} and S^1_{out} in the expected manner. In this figure we have also plotted the contours for the energy density, which we find peaks near $\rho_c\approx 3.3$ for this particular configuration. We note that when we approach the z axis, the energy density vanishes. The qualitative behavior of the energy density is consistent with the expected behavior of the Higgs field near the two vacua S^1_{in} and S^1_{out}. When we approach the Higgs vacuum S^1_{in}, we expect that $|\Phi|\approx 1+\mathcal{O}(|\rho-\rho_c|^n)$ so that with $n=1$ the derivative terms in Eq. (1) yield a finite contribution to the energy density even at $\rho=\rho_c$, and when we approach the z axis we expect $|\Phi|\approx 1+\mathcal{O}(\rho^k)$ so that with $k=2$ the energy density vanishes on the z axis.

The equations of motion, even with the simplified ansatz (6), are highly complex and practical simulations become very time consuming. A typical run consumes over a hundred hours of CPU time, when we use a finite element method implemented with the PDE2D program [11] and a single EV56 processor in a Digital AlphaServer 8400. We have inspected the solutions to the classical equations of motion for various values of the coupling λ, and found evidence of convergence even for moderately small values of $\lambda \sim \mathcal{O}(1)$. However, for such small values of λ the simulations become increasingly involved, and numerical convergence becomes increasingly sensitive to the choice of an initial condition. This is expected: When we use the analogy with Ref. [4], we conclude that for convergence towards a stable vortex we need to locate an initial configuration which cannot ''slip off the tip of the Mexican hat,'' using the analogy with the discussion in Ref. [4]. When λ decreases the choice of a suitable initial configuration becomes then increasingly more difficult, and the ensuing simulation similarly increasingly more time consuming. Unfortunately, at the moment we do not have access to sufficiently effective computers and numerical methods to locate a lower bound λ_c for the existence of (locally) stable solutions in a reliable manner. Indeed, it appears that a detailed numerical investigation of the properties of our vortices must be postponed to the future [12], when more powerful computers become available. However, we expect that these simulations will eventually reveal a qualitative behavior which is very similar to that found in the lower dimensional analogue [4]. Among the conclusions in Ref. [4] are that the configurations should be generic in gauge models with an extended Higgs sector. Moreover, the solutions can be stable even in the weak coupling limit where semiclassical methods become reliable. If these properties indeed persists in 3+1 dimensions, we can expect that our vortices are generic in models with an extended Higgs sector, even in the low coupling limit where semiclassical methods become reliable. If so, they should have relevance in a number of physical scenarios, including applications to CERN Large Hadron Collider (LHC) physics and early universe cosmology.

V. CONCLUSIONS

In conclusion, we have argued that closed finite energy knotted vortices may be generic in gauge theories with an extended Higgs sector. These vortices appear as domain walls that separate two different Higgs vacua. They are topologically nontrivial if the vacua are tangled into knots with a nontrivial Hopf invariant. The actual dynamical stability of a closed vortex depends on the details of the model. For this the Higgs potential should provide a sufficiently high, ''tip of the Mexican-hat,'' type energy barrier so that in combination with the derivative terms it prevents the vortex from unwinding. Studies of an analogous behavior in one dimension suggests that closed vortices could be generic in theories with an extended Higgs sector, and their stability could persist even at low coupling where semiclassical methods become reliable. If present, these configurations should have relevance to LHC physics and early Universe cosmology. In particular, since knots are topologically stable only when embedded in three space dimensions, the very existence of (meta)stable knotted vortices could develop into a viable method for experimentally testing the dimensionality of the world, as seen by the standard model at TeV scales.

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