# Casimir energy of massive MIT fermions in an Aharonov-Bohm background

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We study the effect of a background flux string on the vacuum energy of massive Dirac fermions in 2+1 dimensions confined to a finite spatial region by MIT boundary conditions. We treat two admissible self-adjoint extensions of the Hamiltonian and compare the results. In particular, for one of these extensions, the Casimir energy turns out to be discontinuous at integer values of the flux.

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## I. INTRODUCTION

In a by now classic paper, Aharonov and Bohm [1] pointed out that even a locally trivial vector potential can give rise to observable effects in a nontrivial topology. Since then, the relevance of Aharonov-Bohm scenarios both in particle physics and condensed matter has been recognized.

More recently, much attention has been paid to the inclusion of spin, mainly in connection with the interaction of cosmic strings with matter [2–6]. In this context, the need to consider self-adjoint extensions of the radial Dirac Hamiltonian was realized [7,8].

Thereafter, the vacuum properties of Dirac fields in the background of a singular [9–14] as well as extended [15,16] magnetic flux string have been extensively studied. The scalar field and the electromagnetic field have also been considered, some pertinent references being Refs. [17–20].

As is well known, the presence of background fields modifies the energy spectrum giving rise to a nontrivial vacuum, or Casimir, energy [21,22]. Furthermore, the Casimir energy is altered by the presence of boundaries, and the consequent imposition of boundary conditions on the quantum fields. For Dirac fields, many examples of both situations have been studied in the literature (see, for instance, Refs. [16,23–25]).

In particular, the combined effect of a classical magnetic fluxon and MIT boundary conditions on the vacuum energy of a massless Dirac field in 2+1 dimensions was treated in Ref. [26]. There, just one of the possible self-adjoint extensions of the radial Hamiltonian was considered.

In this paper, we will consider the more realistic case of massive fermions. In this context, it is important to mention that, despite general belief, in the presence of curved boundaries the effect of a mass is not exponentially small [27] as it is for parallel plates [22]. On the contrary, in some situations it might even lead to a sign change in the Casimir force and is by no means negligible [27]. In the present context, we will see that properties such as existence of a minimum or continuity of the Casimir energy as a function of the flux depend on the mass underlining its crucial importance.

Moreover, we will analyze two possible self-adjoint extensions, both known to be compatible with the presence of a Dirac delta magnetic field at the origin [28]. Treating the origin as an excluded point, different self-adjoint extensions are manifestations of different physics within the vortex (considered as a black box) [8]. It is well understood that the parameter characterizing the self-adjoint extension together with the flux determine the Hamiltonian outside the vortex [8]. As a result, the Casimir energies obtained for different self-adjoint extensions are quite different emphasizing again that they describe *nontrivial* physics in the core, see also Ref. [8].

The organization of the paper is as follows. In Sec. II we summarize the generalities of the model. In Sec. III, we present a discussion of self-adjoint extensions distinguished by the behavior of the wave function at the origin. We determine the energy eigenfunctions corresponding to two different cases of these extensions. The first one is a minimal divergence extension. As shown in Ref. [23,24], it arises when imposing Atiyah-Patodi-Singer boundary conditions [29–34] at a finite radius, which is then shrunk to zero (an idea first suggested in [35]). The second one follows from the zero radius limit of a cylindrical flux shell [2,5,6]. In Sec. IV, the implicit equations for the energy spectrum are found in both cases, once the theory is confined to a circle of radius $R$ and MIT conditions (see, for instance, Ref. [36], and references therein) imposed at the exterior boundary. The expression of the Casimir energy for both types of behavior at the origin is given in the framework of $\zeta$-function regularization [37–40]. In Sec. V, we evaluate the vacuum energies, following the methods developed in [26,41–43,27]. Finally, Sec. VI contains a discussion of the results.

## II. SETTING OF THE PROBLEM

We study the Dirac equation for a massive particle in $(2+1)$-dimensional Minkowski space:

 

$$(i\partial\!\!\!/ - A\!\!\!/ - m)\Psi = 0 \tag{1}$$

in the presence of a flux string located at the origin, i.e.,

$$\vec{H} = \vec{\nabla} \wedge \vec{A} = \frac{\kappa}{r}\,\delta(r)\check{e}_z, \tag{2}$$

where $\kappa = \Phi/2\pi$ is the reduced flux.

We assume the flux string to be radially symmetric; so, a gauge can be chosen such that the (covariant) vector potential is given by

$$A_\theta(r) = -\frac{\kappa}{r}, \quad \text{for} \quad r > 0. \tag{3}$$

We will consider the chiral representation for the Dirac matrices

$$\gamma^0 = \rho_3 \otimes \sigma_3, \quad \gamma^1 = i\rho_3 \otimes \sigma_2, \quad \gamma^2 = -i\rho_3 \otimes \sigma_1, \tag{4}$$

which, together with

$$\gamma^3 = i\rho_2 \otimes \sigma_0, \tag{5}$$

give a closed Clifford algebra.

Then, the eigenvalue equation for the Dirac Hamiltonian takes the form

$$\begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \Psi_E = E\Psi_E, \tag{6}$$

where the two-by-two blocks are given by

$$H_\pm = \begin{pmatrix} \mp m & L^\dagger \\ L & \pm m \end{pmatrix}, \tag{7}$$

and we have introduced $L = -ie^{i\theta}(-\partial_r + B)$, $L^\dagger = ie^{-i\theta}(\partial_r + B)$, $B = -(i/r)\partial_\theta - \kappa/r$. (Notice that these two ''polarizations'' correspond to the two inequivalent two by two irreducible representations of the gamma matrices [6].)

The general solution to Eq. (6) can be written as a combination of

$$\Psi_E^{(I)} = \begin{pmatrix} \psi_E^+ \\ 0 \end{pmatrix}, \quad \Psi_E^{(II)} = \begin{pmatrix} 0 \\ \psi_E^- \end{pmatrix}, \tag{8}$$

with

$$H_\pm \psi_E^\pm = E\psi_E^\pm. \tag{9}$$

After separating variables, and for noninteger $\kappa = l + a$ (where $l$ is the integer part of the reduced flux, and $a$ its fractionary part), the eigenfunctions in Eq. (6) turn out to be

$$\Psi_E(r,\theta) = \begin{pmatrix} \sum_{n=-\infty}^{\infty} f_n^+(r)e^{in\theta} \\ \sum_{n=-\infty}^{\infty} g_n^+(r)e^{i(n+1)\theta} \\ \sum_{n=-\infty}^{\infty} f_n^-(r)e^{in\theta} \\ \sum_{n=-\infty}^{\infty} g_n^-(r)e^{i(n+1)\theta} \end{pmatrix} = \begin{pmatrix} \sum_{n=-\infty}^{\infty} (A_n^+ J_{n-\kappa}(kr) + B_n^+ J_{\kappa-n}(kr))e^{in\theta} \\ \sum_{n=-\infty}^{\infty} -i\frac{k}{E-m}(A_n^+ J_{n+1-\kappa}(kr) - B_n^+ J_{\kappa-n-1}(kr))e^{i(n+1)\theta} \\ \sum_{n=-\infty}^{\infty} (A_n^- J_{n-\kappa}(kr) + B_n^- J_{\kappa-n}(kr))e^{in\theta} \\ \sum_{n=-\infty}^{\infty} -i\frac{k}{E+m}(A_n^- J_{n+1-\kappa}(kr) - B_n^- J_{\kappa-n-1}(kr))e^{i(n+1)\theta} \end{pmatrix}, \tag{10}$$

where $k = +\sqrt{(E^2 - m^2)}$. (Of course, for integer $\kappa$, a linear combination of Bessel and Neumann functions must be taken.)

### III. BEHAVIOR AT THE ORIGIN

As is well known [2,7,8], the radial Dirac Hamiltonian in the background of an Aharonov-Bohm gauge field requires a self-adjoint extension for the critical subspace $n = l$. In fact, imposing regularity of all components of the Dirac field at the origin is too strong a requirement, except for integer flux. Rather, one has to apply the theory of Von Neumann deficiency indices [44], which leads to a one-parameter family of allowed boundary conditions [8], characterized by

$$i\lim_{r\to 0}(mr)^{1-a} g_l^\pm(r) \sin\left(\frac{\pi}{4} + \frac{\Theta^\pm}{2}\right) = \lim_{r\to 0}(mr)^a f_l^\pm(r) \cos\left(\frac{\pi}{4} + \frac{\Theta^\pm}{2}\right). \tag{11}$$

Here, $\Theta^\pm$ parameterize the admissible self-adjoint extensions of $H_\pm$ respectively. Which of these extensions to choose depends on the physical situation under study.

Throughout this paper we will, for noninteger $\kappa$, consider two different behaviors at the origin. The first one, from now on called behavior I, is characterized by

$$\Theta^{\pm}=\begin{cases} -\dfrac{\pi}{2} & \text{for} \quad a\geqslant\dfrac{1}{2}, \\ \dfrac{\pi}{2} & \text{for} \quad a<\dfrac{1}{2}. \end{cases} \tag{12}$$

As shown in Refs. [23,24], this is the extension arising when boundary conditions of the Atiyah-Patodi-Singer (APS) type [29–34] are imposed at a finite radius, which is then taken to zero. The second self-adjoint extension we will consider, from now on called behavior II, corresponds to

$$\Theta^{\pm}=\begin{cases} -\dfrac{\pi}{2} & \text{for} \quad \kappa>0, \\ \dfrac{\pi}{2} & \text{for} \quad \kappa<0. \end{cases} \tag{13}$$

As shown in Ref. [5], this extension arises when a finite radius flux tube is considered, thus asking for continuity of the components of the spinor, and then shrinking the radius to zero.

Outside the critical subspace, the eigenfunctions in Eq. (10) are determined by the requirement of square integrability at the origin, and are thus identical for the behaviors I and II. They are given by

$$\Psi_E^{n\leqslant l-1}(r,\theta)=\begin{pmatrix} B_n^+ J_{l+a-n}(kr)e^{in\theta} \\ i\dfrac{k}{E-m}B_n^+ J_{l+a-n-1}(kr)e^{i(n+1)\theta} \\ B_n^- J_{l+a-n}(kr)e^{in\theta} \\ i\dfrac{k}{E+m}B_n^- J_{l+a-n-1}(kr)e^{i(n+1)\theta} \end{pmatrix} \tag{14}$$

and

$$\Psi_E^{n\geqslant l+1}(r,\theta)=\begin{pmatrix} A_n^+ J_{n-l-a}(kr)e^{in\theta} \\ -i\dfrac{k}{E-m}A_n^+ J_{n+1-l-a}(kr)e^{i(n+1)\theta} \\ A_n^- J_{n-l-a}(kr)e^{in\theta} \\ -i\dfrac{k}{E+m}A_n^- J_{n+1-l-a}(kr)e^{i(n+1)\theta} \end{pmatrix}. \tag{15}$$

In the critical subspace $(n=l)$, the eigenfunction for behavior I is given by

$$\Psi_E^{l}(r,\theta)=\begin{pmatrix} B_l^+ J_{a}(kr)e^{il\theta} \\ i\dfrac{k}{E-m}B_l^+ J_{a-1}(kr)e^{i(l+1)\theta} \\ B_l^- J_{a}(kr)e^{il\theta} \\ i\dfrac{k}{E+m}B_l^- J_{a-1}(kr)e^{i(l+1)\theta} \end{pmatrix} \quad \text{for} \quad a\geqslant\frac{1}{2}, \tag{16}$$

and

$$\Psi_E^{l}(r,\theta)=\begin{pmatrix} A_l^+ J_{-a}(kr)e^{il\theta} \\ -i\dfrac{k}{E-m}A_l^+ J_{1-a}(kr)e^{i(l+1)\theta} \\ A_l^- J_{-a}(kr)e^{il\theta} \\ -i\dfrac{k}{E+m}A_l^- J_{1-a}(kr)e^{i(l+1)\theta} \end{pmatrix} \quad \text{for} \quad a<\frac{1}{2}. \tag{17}$$

It is easy to see that this extension satisfies the condition of minimal irregularity (the radial functions diverge as $r\to 0$ at most as $r^{-p}$, with $p\leqslant\frac{1}{2}$). Moreover, it is compatible with periodicity in $\kappa$, a natural requirement when the origin is an excluded point.

When behavior II is imposed at the origin, the eigenfunctions in the critical subspace are given by Eq. (16) for $\kappa>0$, and by Eq. (17) for $\kappa<0$. It is worth pointing out that for integer $\kappa=l$, both APS boundary conditions and the finite radius flux tube lead, when taking the singular limit, to the requirement of regularity of all components at the origin. In this case

$$\Psi_E(r,\theta)=\sum_{n=-\infty}^{\infty}\begin{pmatrix} A_n^+ J_{n-\kappa}(kr)e^{in\theta} \\ -i\dfrac{k}{E-m}A_n^+ J_{n+1-\kappa}(kr)e^{i(n+1)\theta} \\ A_n^- J_{n-\kappa}(kr)e^{in\theta} \\ -i\dfrac{k}{E+m}A_n^- J_{n+1-\kappa}(kr)e^{i(n+1)\theta} \end{pmatrix}. \tag{18}$$

## IV. THE THEORY IN A BOUNDED REGION: ENERGY SPECTRUM AND CASIMIR ENERGY

From now on, we will confine the Dirac fields to a bounded region, by introducing a boundary at $r=R$, and imposing MIT bag boundary conditions. The Casimir energy is formally given by

$$E_C=-\frac{1}{2}\left(\sum_{E>m}E_\rho-\sum_{E<-m}E_\rho\right), \tag{19}$$

where $\rho$ represents all indices appearing in the eigenvalue equation that arises after local MIT conditions are imposed. In doing so, one must consider a boundary operator $B$ which, with the representation of the Dirac matrices given in Eq. (4), is also block diagonal, and can be written as

$$B=1-i\not{n}=1+i(\gamma^1 n^1+\gamma^2 n^2)=\begin{pmatrix} B_+ & 0_{2\times 2} \\ 0_{2\times 2} & B_- \end{pmatrix}, \tag{20}$$

where $n$ is the exterior normal and

$$B_{\pm}=\begin{pmatrix} 1 & \pm ie^{-i\theta} \\ \mp ie^{i\theta} & 1 \end{pmatrix}. \tag{21}$$

Consider, in the first place, behavior I (II) for $a \geqslant 1/2$ ($\kappa > 0$). Then, the eigenvalue equations for the upper (+) polarization are

$$J_{n+a}(kR) = \frac{k}{E-m} J_{n-1+a}(kR) \quad \text{for} \quad n=1,\ldots,\infty, \tag{22}$$

$$J_{n-a}(kR) = \frac{-k}{E-m} J_{n+1-a}(kR) \quad \text{for} \quad n=1,\ldots,\infty, \tag{23}$$

coming from noncritical subspaces, and

$$J_a(kR) = \frac{k}{E-m} J_{a-1}(kR), \tag{24}$$

from the critical one.

The eigenvalue equations corresponding to the lower (−) polarization are

$$J_{n+a}(kR) = \frac{-k}{E+m} J_{n-1+a}(kR) \quad \text{for} \quad n=1,\ldots,\infty, \tag{25}$$

$$J_{n-a}(kR) = \frac{k}{E+m} J_{n+1-a}(kR) \quad \text{for} \quad n=1,\ldots,\infty, \tag{26}$$

from noncritical subspaces, and

$$J_a(kR) = \frac{-k}{E+m} J_{a-1}(kR), \tag{27}$$

from the critical one.

For $a < 1/2$ ($\kappa < 0$), the contributions from noncritical subspaces are the same, while those due to $n=l$ are

$$J_{-a}(kR) = \frac{-k}{E-m} J_{1-a}(kR), \tag{28}$$

for the upper polarization, and

$$J_{-a}(kR) = \frac{k}{E+m} J_{1-a}(kR), \tag{29}$$

for the lower one.

It is easy to verify that positive energies coming from one polarization correspond to negative energies coming from the other. Thus, both polarizations give identical contributions to the Casimir energy in Eq. (19).

Then, the formal expression for the Casimir energy (19) is

$$E_C = -\frac{1}{2} 2 \sum_k (k^2+m^2)^{1/2}, \tag{30}$$

where $k$ denotes the solutions of

$$J_{n+a}^2(kR) - J_{n-1+a}^2(kR) - \frac{2m}{k} J_{n+a}(kR) J_{n-1+a}(kR) = 0 \quad \text{for} \quad n=0,\ldots,\infty, \tag{31a}$$

$$J_{n-a}^2(kR) - J_{n+1-a}^2(kR) + \frac{2m}{k} J_{n-a}(kR) J_{n+1-a}(kR) = 0 \quad \text{for} \quad n=1,\ldots,\infty, \tag{31b}$$

when $a \geqslant 1/2$ ($\kappa > 0$) while, for $a < 1/2$ ($\kappa < 0$), the first equation in Eq. (31) holds for $n=1,\ldots,\infty$ and the second one applies for $n=0,\ldots,\infty$.

Of course, a regularization method must be introduced in order to give sense to the divergent sum in Eq. (30). In the framework of the $\zeta$ regularization [37,38] (for several applications see Refs. [39,40]),

$$\begin{aligned} E_C &= -\frac{1}{2} M \lim_{s\to -1/2} M^{2s} \zeta(s) \\ &= -\frac{1}{2} M \lim_{s\to -1/2} 2 \sum_k \left( \frac{k^2+m^2}{M^2} \right)^{-s}, \end{aligned} \tag{32}$$

where the parameter $M$ is introduced for dimensional reasons.

Here, it is useful to define the so-called partial zeta function

$$\zeta_\mu(s) = 2 \sum_{l=1}^{\infty} (k_{\mu,l}^2 + m^2)^{-s}, \tag{33}$$

where $k_{\mu,l}$ are the roots of

$$J_\mu^2(kR) - J_{\mu-1}^2(kR) - \frac{2m}{k} J_\mu(kR) J_{\mu-1}(kR) = 0. \tag{34}$$

So, after introducing $\nu = n + \frac{1}{2}$ and $\alpha = a - \frac{1}{2}$, the Casimir energy for the behavior I at the origin can be written, for any $a$, as

$$E_C^I = -\frac{1}{2} M \lim_{s\to -1/2} M^{2s} \times \left\{ \sum_{\nu=1/2,3/2,\ldots} [\zeta_{\nu+\alpha}(s) + \zeta_{\nu-\alpha}(s)] - \zeta_{1/2-|\alpha|}(s) \right\}, \tag{35}$$

while for the behavior II at the origin, it is given by

$$E_C^{II} = -\frac{1}{2} M \lim_{s\to -1/2} M^{2s} \times \left\{ \sum_{\nu=1/2,3/2,\ldots} [\zeta_{\nu+\alpha}(s) + \zeta_{\nu-\alpha}(s)] - \zeta_{1/2-\mathrm{sgn}(\kappa)\alpha}(s) \right\}. \tag{36}$$

From Eq. (35) it is clear that, as mentioned before, the Casimir energy for a behavior of type I at the origin is independent of the integer part $l$ of the reduced flux. Moreover, it is invariant under $\alpha \to -\alpha$ ($a \to 1-a$). Thus, it is enough to study it for $0 < \alpha < \frac{1}{2}$, where the absolute value in the last term can be ignored, and to use $E_C^I(\alpha) = E_C^I(-\alpha)$.

Similarly, from Eq. (36), the Casimir energy for behavior II is seen to be invariant under $\kappa \to -\kappa$. Thus we only study the case $\kappa > 0$, where the last term is again $\zeta_{1/2-\alpha}(s)$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$ is considered.

## V. EVALUATION OF THE CASIMIR ENERGY

The Casimir energies in Eqs. (35) and (36) contain two contributions: the term inside the square brackets, which is summed over $\nu$, and the last term, which is a partial zeta function. In both cases, it is useful to introduce, as in Refs. [26,41,42], an integral representation for the partial zeta function

$$\zeta_\mu(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \int_z^\infty dx [x^2 - z^2]^{-s} \frac{d}{dx} \ln \times [x^{-2(\mu-1)} F_\mu(x)], \tag{37}$$

where

$$F_\mu(x) = I_\mu^2(x) + I_{\mu-1}^2(x) + \frac{2z}{x} I_\mu(x) I_{\mu-1}(x), \tag{38}$$

which have to be summed according to Eqs. (35) and (36). Here, the dimensionless variable $z = mR$ has been introduced.

In order to identify the divergences and evaluate the finite parts of the terms in Eqs. (35) and (36) an analytical continuation of the zeta function $\zeta(s)$ to $s = -\frac{1}{2}$ has to be constructed. A method of doing this has been developed in Ref. [43] and for details of the procedure we refer to this reference. For the part of the zeta functions involving the angular momentum sum the method consists of adding and subtracting several orders of the uniform Debye expansion of Eq. (38) so as to make the sum as well as the integral in Eqs. (35), (36), and (37) well defined in an increasing strip of the complex $s$-plane. For the partial zeta function, subtracting and adding the asymptotic terms for large arguments of the Bessel functions will be sufficient.

Let us first study the terms summed over $\nu$. By making use of the recurrence relations for Bessel functions, it is immediate to obtain

$$T(\mu,x,z) = \frac{d}{dx} \ln[x^{-2(\mu-1)} F_\mu(x)]$$
$$= \frac{2}{\mu}\left(\frac{\mu}{x}\right) \frac{1 + z - \left(\frac{\mu}{x}\right)^2 z + 2 d_\mu(x) + \frac{1}{\mu^2}\left(\frac{\mu}{x}\right)^2 z d_\mu^2(x)}{1 + \left(\frac{\mu}{x}\right)^2 + \frac{2}{\mu}\left(\frac{\mu}{x}\right)^2 z + \frac{2}{\mu}\left(\frac{\mu}{x}\right)^2 d_\mu(x) + \frac{2}{\mu^2}\left(\frac{\mu}{x}\right)^2 z d_\mu(x) + 1/\mu^2 (\mu/x)^2 d_\mu^2(x)}, \tag{39}$$

where $d_\mu(x) = x(d/dx)\ln I_\mu(x)$. This expression can be developed in powers of $1/\mu$, through the Debye expansion of Bessel functions, after taking $(\mu/x) = \tau/\sqrt{1-\tau^2}$, with $\tau$ the variable of the recursive polynomials $u_k(\tau)$ [45].

If $D^{(N)}(\mu,x,\tau,z)$ is such an expansion up to the order $1/\mu^N$, the partial zeta function can be written as

$$\zeta_\mu(s) = \zeta_\mu^a(s) + \zeta_\mu^d(s), \tag{40}$$

where

$$\zeta_\mu^a(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \int_z^\infty dx [x^2 - z^2]^{-s} \times [T(\mu,x,z) - D^{(N)}(\mu,x,\tau,z)] \tag{41}$$

is the analytic part of the partial zeta function for $s = -\frac{1}{2}$, while

$$\zeta_\mu^d(s) = 2\frac{\sin \pi s}{\pi} R^{2s} \int_z^\infty dx [x^2 - z^2]^{-s} D^{(N)}(\mu,x,\tau,z) \tag{42}$$

is the asymptotic contribution.

In order to make the integral in Eq. (41), and the subsequent sum over $\nu$, absolutely convergent at $s = -\frac{1}{2}$, it is necessary to take $N \geqslant 2$ [43]. We will choose $N = 4$ to improve the convergence of the sum of the analytic term (41), thus decreasing the computational time needed to get accurate numerical results.

Now, the term in square brackets in Eqs. (35) and (36) involves the combination $\zeta_{\nu+\alpha}(s) + \zeta_{\nu-\alpha}(s)$. In order to use previous results of Ref. [25] we further expand in powers of $1/\nu$. We introduce the corresponding combination of asymptotic expansions

$$\Delta^{(N)}(\nu,x,t,z) = D^{(N)}\left(\nu\left(1+\frac{\alpha}{\nu}\right), x, t\left(1+\frac{\alpha}{\nu}\right) \times \left[1 + \frac{2\alpha t^2}{\nu}\left(1 + \frac{\alpha}{2\nu}\right)\right]^{-1/2}, z\right) + D^{(N)}\left(\nu\left(1-\frac{\alpha}{\nu}\right), x, t\left(1-\frac{\alpha}{\nu}\right) \times \left[1 - \frac{2\alpha t^2}{\nu}\left(1 - \frac{\alpha}{2\nu}\right)\right]^{-1/2}, z\right), \tag{43}$$

expanded up to the order $1/\nu^N$. In the above expression $t=\nu/\sqrt{\nu^2+x^2}$.

The asymptotic expansion can be written as

$$\Delta^{(N)}(\nu,x,t,z)=\Delta_{-1}+\Delta_0+\sum_{i=1}^{N}\Delta_i, \tag{44}$$

where

$$\Delta_{-1}=\frac{4x}{\nu}\frac{t}{1+t},\quad \Delta_0=\frac{2x}{\nu^2}\frac{t^2}{1+t},$$

$$\Delta_i=\frac{1}{\nu^i}\frac{d}{dx}\sum_{j=0}^{2i}b_{(i,j)}t^{i+j}, \tag{45}$$

and the coefficients $b_{(i,j)}$ are listed in Appendix A.

Then, the term inside the square brackets in Eqs. (35) and (36) can be written as

$$\lim_{s\to-1/2}M^{2s}\sum_{n=0}^{\infty}[\zeta_{\nu+\alpha}(s)+\zeta_{\nu-\alpha}(s)]$$
$$=Z\left(-\frac{1}{2}\right)+\lim_{s\to-1/2}M^{2s}\left(A_{-1}(s)+A_0(s)+\sum_{i=1}^{N-1}A_i(s)\right), \tag{46}$$

where

$$Z\left(-\frac{1}{2}\right)=-\frac{2}{\pi MR}\sum_{n=0}^{\infty}\int_z^{\infty}dx[x^2-(z)^2]^{1/2}\{T(\nu+\alpha,x,z)+T(\nu-\alpha,x,z)-\Delta^{(N)}(\nu,x,t,z)\}, \tag{47}$$

$$A_{-1}(s)=2\frac{\sin\pi s}{\pi}R^{2s}4\sum_{n=0}^{\infty}\nu\int_{\frac{z}{\nu}}^{\infty}dx[(\nu x)^2-z^2]^{-s}\times\frac{\sqrt{1+x^2}-1}{x}, \tag{48}$$

$$A_0(s)=2\frac{\sin\pi s}{\pi}R^{2s}2\sum_{n=0}^{\infty}\int_{\frac{z}{\nu}}^{\infty}dx[(\nu x)^2-z^2]^{-s}\times\frac{1}{\sqrt{1+x^2}}\frac{\sqrt{1+x^2}-1}{x}, \tag{49}$$

$$A_i(s)=2\frac{\sin\pi s}{\pi}R^{2s}\sum_{j=0}^{2i}b_{(i,j)}\sum_{n=0}^{\infty}\nu^{-i}\int_{\frac{z}{\nu}}^{\infty}dx[(\nu x)^2-z^2]^{-s}\times\frac{d}{dx}\left(\frac{1}{\sqrt{1+x^2}}\right)^{i+j}. \tag{50}$$

Equations (48), (49), and (50) can be expressed in a systematic way by introducing the functions [25]

$$f(s;a,b;x)=\sum_{n=0}^{\infty}\nu^a\left[1+\left(\frac{\nu}{x}\right)^2\right]^{-s-b}, \tag{51}$$

studied further in Appendix B, which allow one to write the asymptotic parts as

$$A_{-1}(s)=\frac{2R^{2s}}{\sqrt{\pi}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}z^{-2s+1}\int_0^1\frac{dy}{\sqrt{y}}f\left(s;0,-\frac{1}{2};z\sqrt{y}\right), \tag{52}$$

$$A_0(s)=\frac{2R^{2s}}{\sqrt{\pi}}\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)}z^{-2s-1}\int_0^1\frac{dy}{y^{3/2}}f\left(s;1,\frac{1}{2};z\sqrt{y}\right), \tag{53}$$

$$A_i(s)=\sum_{j=0}^{2i}b_{(i,j)}\mathcal{A}_{(i,j)}(s), \tag{54}$$

where

$$\mathcal{A}_{(i,j)}(s)=-2R^{2s}z^{-(i+j)}\frac{\Gamma\left(s+\frac{i+j}{2}\right)}{\Gamma\left(\frac{i+j}{2}\right)\Gamma(s)}z^{-2s}f\left(s;j,\frac{i+j}{2};z\right). \tag{55}$$

The complete expressions for these asymptotic parts around $s=-\frac{1}{2}$ are derived in Appendix C. Here, we list their residues, which will be relevant to the discussion of the renormalization in the next section:

$$\mathrm{Res}|_{s=-1/2}A_{-1}=0, \tag{56a}$$

$$\mathrm{Res}|_{s=-1/2}A_0=\frac{1}{R}\left[\frac{z^2}{\pi}+\frac{1}{12\pi}\right], \tag{56b}$$

$$\mathrm{Res}|_{s=-1/2}A_1=0, \tag{56c}$$

$$\mathrm{Res}|_{s=-1/2}A_2=\frac{1}{R}\left[\frac{1}{64}-\frac{1}{12\pi}-\frac{\alpha^2}{\pi}-\frac{z}{4}+\frac{z}{\pi}-\frac{z^2}{2}\right]. \tag{56d}$$

Next, we study the partial-zeta contribution

$$e_c=-\frac{1}{2}M\lim_{s\to-1/2}M^{2s}[-\zeta_{1/2-\alpha}(s)] \tag{57}$$

to the Casimir energy in Eqs. (35) and (36), and for reasons already given we omit the absolute value in the index.

In order to isolate the singularities, it is enough to consider the three leading terms in the asymptotic expansion of Bessel functions for large arguments, which will be denoted by $L(\frac{1}{2}-\alpha,x,z)$; thus, the partial zeta function can be written as

$$\zeta_{1/2-\alpha}(s)=\zeta^a_{1/2-\alpha}(s)+\zeta^d_{1/2-\alpha}(s), \tag{58}$$

where

$$\zeta^{a}_{1/2-\alpha}(s)=2\frac{\sin\pi s}{\pi}R^{2s}\int_z^\infty dx[x^2-z^2]^{-s}\times\left[T\left(\frac{1}{2}-\alpha,x,z\right)-L\left(\frac{1}{2}-\alpha,x,z\right)\right], \quad (59)$$

$$\zeta^{d}_{1/2-\alpha}(s)=2\frac{\sin\pi s}{\pi}R^{2s}\int_z^\infty dx[x^2-z^2]^{-s}L\left(\frac{1}{2}-\alpha,x,z\right). \quad (60)$$

Now, the subtracted terms can be written as

$$L\left(\frac{1}{2}-\alpha,x,z\right)=\delta_0+\delta_1+\delta_2, \quad (61)$$

where

$$\delta_0=2, \quad \delta_1=\frac{1}{x}2\alpha, \quad \delta_2=\frac{1}{x^2}(\alpha^2-z). \quad (62)$$

Then

$$\lim_{s\to-1/2}M^{2s}\zeta_{1/2-\alpha}(s)=z\left(-\frac{1}{2}\right)+\lim_{s\to-1/2}M^{2s}\times[a_0(s)+a_1(s)+a_2(s)], \quad (63)$$

with

$$z\left(-\frac{1}{2}\right)=-\frac{2}{\pi MR}\int_z^\infty dx\,[x^2-z^2]^{1/2}\left\{T\left(\frac{1}{2}-\alpha,x,z\right)-L\left(\frac{1}{2}-\alpha,x,z\right)\right\}, \quad (64)$$

which will be evaluated numerically, and

$$a_0(s)=\frac{1}{R}\left\{\frac{z^2}{\pi}\frac{1}{\left(s+\frac{1}{2}\right)}-\frac{z^2}{\pi}\left[1+2\log\left(\frac{z}{2R}\right)\right]+O\left(s+\frac{1}{2}\right)\right\}, \quad (65)$$

$$a_1(s)=\frac{1}{R}\left[2\alpha z+O\left(s+\frac{1}{2}\right)\right], \quad (66)$$

$$a_2(s)=\frac{1}{R}(\alpha^2-z)\left\{-\frac{1}{\pi}\frac{1}{\left(s+\frac{1}{2}\right)}+\frac{1}{\pi}\times\left[2+2\log\left(\frac{z}{2R}\right)\right]+O\left(s+\frac{1}{2}\right)\right\}. \quad (67)$$

So, the residues at $s=-\frac{1}{2}$ are given by

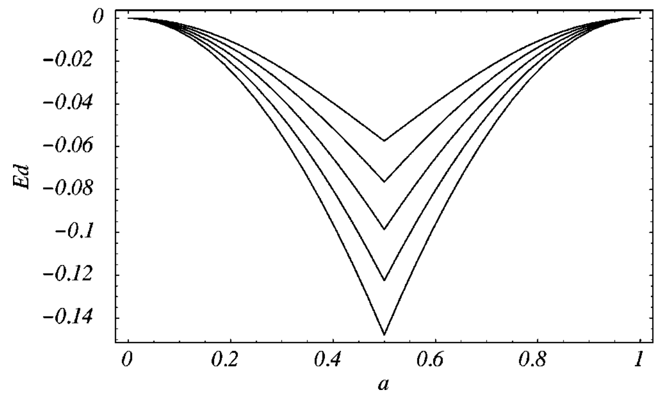


FIG. 1. Difference $E_d$ of Casimir energies. Behavior I at the origin. Top to bottom: $z=\frac{1}{128},\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2}$.

$$\mathrm{Res}|_{s=-1/2}a_0=\frac{1}{R}\left[\frac{z^2}{\pi}\right], \quad (68a)$$

$$\mathrm{Res}|_{s=-1/2}a_1=0, \quad (68b)$$

$$\mathrm{Res}|_{s=-1/2}a_2=\frac{1}{R}\left[-\frac{\alpha^2}{\pi}+\frac{z}{\pi}\right]. \quad (68c)$$

## VI. DISCUSSION OF THE RESULTS

Clearly the Casimir energy is divergent and using Eqs. (56) and Eqs. (68) in Eqs. (35) and (36), the total residue is given by

$$\mathrm{Res}|_{s=-1/2}E_C=-\frac{1}{2R}\left\{\frac{1}{64}-\frac{z}{4}-\frac{z^2}{2}\right\}, \quad (69)$$

which is independent of the flux. Thus, the difference between Casimir energies with arbitrary and with integer flux is finite and contains the relevant information about the effect of the flux.

In Fig. 1, we plot the dimensionless difference $E_d=R[E_C(a)-E_C(0)]$ for a behavior of type I at the origin, as a function of $a$ (the fractionary part of the reduced flux), for different values of $z$. Since the finite part of the Casimir energy is continuous in $a$, the difference goes to zero both at $a=0$ and $a=1$. It shows a minimum at $a=\frac{1}{2}$ as well as a jump in the derivative. This jump can be traced back to $a_1(s)$, Eq. (66), which effectively contains the absolute value $|\alpha|$ [see comment below Eq. (36)]. It is interesting to note, that around $a=0$, the vacuum energy decreases when the flux grows and that this effect is enhanced with increasing mass.

For a type II behavior at the origin, the same difference is plotted in Fig. 2. With decreasing mass our curves tend to the $m=0$ result of Ref. [26] and already $z=\frac{1}{128}$ shows quite good agreement with the corresponding figure in that reference (except for a factor of 2, due to the fact that only one polarization was considered in Ref. [26]).

Whereas for small values of the mass the energy exhibits a minimum at $a\neq0$, for larger values of $m$ this minimum is

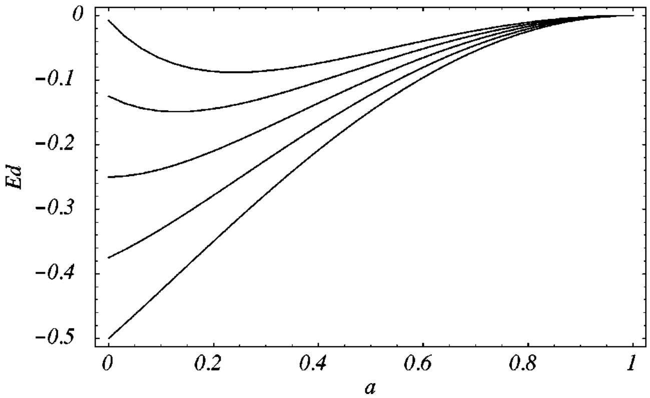


FIG. 2. Difference $E_d$ of Casimir energies. Behavior II at the origin. Top to bottom: $z=\frac{1}{128},\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2}$.

shifted towards $a=0$. Furthermore, for $m\neq 0$ a nonzero value is seen to arise for $a\to 0^+$. This is due to the discontinuous behavior of the finite part of the vacuum energy, more precisely of the contribution of the critical subspace, at integer values of the flux.

The origin of this discontinuity can be traced back to the appearance, for $a\to 0^+$, of a root of the combination of Bessel functions involved in the partial zeta $\zeta_a$. Such a root is absent when $a=0$. For $a\to 0^+$, this root goes to zero and, thus, gives rise to a gap, which equals $m$. The quantity $J_a^2(kR)-J_{a-1}^2(kR)-(2m/k)J_a(kR)J_{a-1}(kR)$ is shown in Fig. 3 as a function of $kR$, for various values of $a$ in order to clarify this discontinuous behavior. For $m=0$ the discontinuous behavior turns into a jump in the derivative at integer values of $a$ [26].

In summary, we have seen that the presence of the mass as well as the choice of the self-adjoint extension of the Hamiltonian have a considerable influence on the dependence of the Casimir energy on the flux. We have analyzed and discussed in detail the behavior of the Casimir energy as a function of the parameters, namely, flux and mass.

We have clearly shown that the Casimir energy depends strongly on the self-adjoint extension chosen. Although this might seem surprising, one has to remember that different values of $\Theta$ describe different physics. For example, Eq. (12) arises by imposing APS boundary conditions at a finite radius which is sent to zero afterwards. Equation (13) results when a finite radius flux tube is considered with no physics (e.g., potential) inside the flux and the finite radius taken to zero. Given the fact that different extensions arise from different physical setups, the dependence of the Casimir energy on $\Theta$ seems very reasonable. This dependence has already been observed when considering the scattering cross section [2,8].

As a consequence of the mass, in the case of the self-adjoint extension (13), a discontinuity of the energy is found. It would be interesting to see how this effect comes about, starting with the finite radius flux [5] and shrinking the radius to zero. Does the effect persist or is it only a result of the singular vortex? The same question arises for the jump in the derivative of the Casimir energy which appears for all masses $m$ in case I and for $m=0$ in case II. We think that in all cases the idealized situation of a singular flux is the origin of this behavior. This conjecture is supported by the fact that no such features were observed in the nonsingular cases [16,20], where, however, no boundary was present. Furthermore, one should try to understand better the physical meaning of the different self-adjoint extensions by considering how the Casimir energy depends on the parameter $\Theta$ of the one-parameter family of self-adjoint extensions. Finally, more realistic $(3+1)$-dimensional calculations should be envisaged.

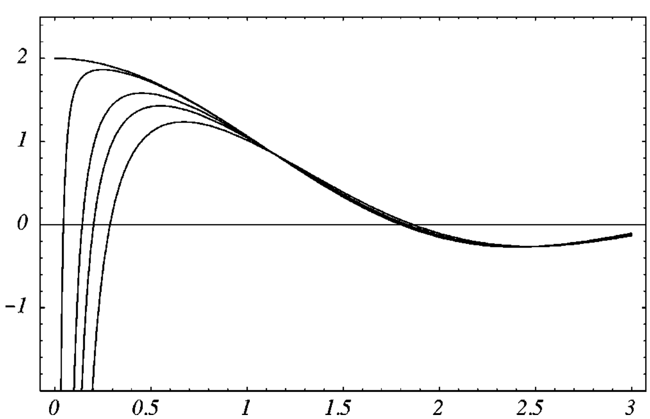


FIG. 3. $J_a^2(kR)-J_{a-1}^2(kR)-(2m/k)J_a(kR)J_{a-1}(kR)$. Top to bottom: $a=0$, 0.001, 0.01, 0.02.

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## APPENDIX A: COEFFICIENTS

In this appendix we list the coefficients $b_{(i,j)}$ defined by Eq. (45)

$$b_{(1,0)}=-\frac{1}{2}-2\alpha^2+2z,$$

$$b_{(1,1)}=0,$$

$$b_{(1,2)}=\frac{1}{6},$$

$$b_{(2,0)}=-(z^2),$$

$$b_{(2,1)}=-\frac{1}{4}-\alpha^2+z,$$

$$b_{(2,2)}=\frac{1}{4}-z,$$

$$b_{(2,3)}=\frac{1}{4},$$

$$b_{(2,4)}=-\frac{1}{4},$$

$$b_{(3,0)}=\frac{5}{96}+5\frac{\alpha^2}{12}+\frac{\alpha^4}{6}-\left(\frac{1}{4}+\alpha^2\right)z+\frac{2z^3}{3},$$

$$b_{(3,1)}=-\frac{1}{4}+z-z^2,$$

$$b_{(3,2)}=\frac{9}{160}-2\alpha^2-\frac{\alpha^4}{2}-\left(\frac{1}{2}-3\alpha^2\right)z+z^2,$$

$$b_{(3,3)}=1-2z,$$

$$b_{(3,4)}=-\frac{23}{32}+\frac{5\alpha^2}{4}+\frac{7z}{4},$$

$$b_{(3,5)}=-\frac{3}{4},$$

$$b_{(3,6)}=\frac{179}{288},$$

$$b_{(4,0)}=\frac{1}{16}+\frac{\alpha^2}{4}-\left(\frac{1}{4}+\alpha^2\right)z+\left(\frac{1}{4}+\alpha^2\right)z^2-\frac{z^4}{2},$$

$$b_{(4,1)}=-\frac{17}{64}+\frac{15\alpha^2}{8}+\frac{3\alpha^4}{4}-\left(-\frac{7}{8}+\frac{9\alpha^2}{2}\right)z-z^2+z^3,$$

$$b_{(4,2)}=-\frac{1}{4}-4\alpha^2-\left(-\frac{1}{2}-10\alpha^2\right)z+\left(\frac{1}{4}-4\alpha^2\right)z^2-z^3,$$

$$b_{(4,3)}=\frac{165}{64}-\frac{25\alpha^2}{4}-\frac{5\alpha^4}{4}-\left(6-\frac{15\alpha^2}{2}\right)z+\frac{5z^2}{2},$$

$$b_{(4,4)}=-\frac{37}{32}+\frac{39\alpha^2}{4}-(-4+12\alpha^2)z-2z^2,$$

$$x_{(4,5)}=-\frac{327}{64}+\frac{35\alpha^2}{8}+\frac{49z}{8},$$

$$b_{(4,6)}=\frac{57}{16}-6\alpha^2-\frac{21z}{4},$$

$$b_{(4,7)}=\frac{179}{64},$$

$$b_{(4,8)}=-\frac{71}{32}.$$

## APPENDIX B: FUNCTIONS $f(s;a,b;x)$

Here we are going to provide all analytical properties of the functions $f(s;a,b;x)$ defined in Eq. (51). As in Ref. [25], we will make use of

$$\sum_{n=0}^{\infty} h(\nu)=\int_0^{\infty} d\nu h(\nu)-i\int_0^{\infty} d\nu \frac{h(i\nu+\epsilon)-h(-i\nu+\epsilon)}{1+e^{2\pi\nu}} \tag{B1}$$

in the limit $\epsilon\rightarrow 0$.

When applied to

$$h(\nu)=\nu^a\left[1+\left(\frac{\nu}{x}\right)^2\right]^{-t},$$

the previous equation gives

$$f(t;a,0;x)=\sum_{n=0}^{\infty}\nu^a\left[1+\left(\frac{\nu}{x}\right)^2\right]^{-t}=x^{a+1}\left\{\frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(t-\frac{a+1}{2}\right)}{\Gamma(t)}+2\sin\left(\frac{\pi a}{2}\right)\int_0^1 du\frac{u^a}{1+e^{2\pi ux}}(1-u^2)^{-t}\right.$$
$$\left.+2\sin\left(\frac{\pi a}{2}-\pi t\right)\int_1^{\infty} du\frac{u^a}{1+e^{2\pi ux}}(u^2-1)^{-t}\right\}. \tag{B2}$$

Now, we are interested in $f(s;a,b;x)$, for arbitrary $b$. From the definition, it is clear that $f(s;a,b;x)=f(s+b,a,0;x)$. However, as $b$ grows, the integrals in Eq. (B2) eventually diverge at $u=1$. In order to avoid such divergences, we will perform an adequate number of integrations by parts, thus obtaining

$$f(t;a,0;x)=x^{a+1}\Bigg\{\frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(t-\frac{a+1}{2}\right)}{\Gamma(t)}-\left(\frac{e^{i\pi a}+1}{2}\right)\frac{(-1)^{a/2}}{2^{(a-2)/2}}\sin(\pi t)\frac{\Gamma\left(t-\frac{a}{2}\right)}{\Gamma(t)}\int_1^\infty udug\left(\frac{a}{2},a-1;u,x\right)(u^2-1)^{-t+a/2}$$
$$-\left(\frac{e^{i\pi a}-1}{2}\right)\frac{1}{2^{(a-3)/2}}\frac{\Gamma\left(t-\frac{a-1}{2}\right)}{\Gamma(t)}\Bigg[\int_0^1 udug\left(\frac{a-1}{2},a-1;u,x\right)(1-u^2)^{-t+(a-1)/2}$$
$$+(-1)^{(a-1)/2}\cos(\pi t)\int_1^\infty udug\left(\frac{a-1}{2},a-1;u,x\right)(u^2-1)^{-t+(a-1)/2}\Bigg]\Bigg\}, \tag{B3}$$

where

$$g(a,b;u,x)=\left(\frac{1}{u}\frac{d}{du}\right)^a\frac{u^b}{1+e^{2\pi ux}}. \tag{B4}$$

However, the number of integrations by parts is bounded by the requirement that the integrated terms are well behaved at $u=0$. In what follows, we will thus keep the number of integrations admissible, by making use of the following recurrence relationship:

$$f(s;a,b;x)=f(s;a,b-1;x)-\frac{1}{x^2}f(s;a+2,b;x). \tag{B5}$$

In this way, all the required functions can be reduced to four different cases

$$f(s;2n,n;x),\quad f\left(s;2n,n+\frac{1}{2};x\right),\quad n=0,1,2,3,4,5,6,7,$$
$$f(s;2n+1,n;x),\quad f\left(s;2n+1,n+\frac{1}{2};x\right),$$
$$n=0,1,2,3,4,5,6.$$

Finally, after expanding in powers of $s+\frac{1}{2}$, we get

$$f(2n,n;x)=x^{2n+1}\Bigg(-\frac{1}{2}\frac{1}{s+\frac{1}{2}}\left(n-\frac{1}{2}\right)-\frac{1}{2}\left\{\left(n+\frac{1}{2}\right)+\left(n-\frac{1}{2}\right)\left[\psi(1)-\psi\left(n+\frac{1}{2}\right)\right]\right\}$$
$$-\frac{\pi}{2^{n-2}}\frac{\left(n-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}\times\int_1^\infty udug(n,2n-1;u,x)(u^2-1)^{\frac{1}{2}}+O\left(s+\frac{1}{2}\right)\Bigg), \tag{B6}$$

$$f\left(2n,n+\frac{1}{2};x\right)=x^{2n+1}\Bigg\{-n\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}-n\frac{1}{2^{n-1}}\frac{\pi}{\Gamma(n+1)}\int_1^\infty udug(n,2n-1;u,x)+O\left(s+\frac{1}{2}\right)\Bigg\}, \tag{B7}$$

$$f(2n+1,n;x) = x^{2(n+1)}\left\{ \frac{2}{3}\frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\left(n-\frac{1}{2}\right) - \frac{1}{2^{n-2}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\left(n-\frac{1}{2}\right) \times \int_0^1 u\,du\,g(n,2n;u,x)(1-u^2)^{1/2} + O\left(s+\frac{1}{2}\right)\right\}, \tag{B8}$$

$$f\left(2n+1,n+\frac{1}{2};x\right) = x^{2(n+1)}\left( -\frac{1}{2s+\frac{1}{2}}n\left[1-\frac{1}{2^{n-2}}\frac{1}{\Gamma(n+1)} \times \int_0^\infty udug(n,2n;u,x)\right] - \frac{1}{2}\{1+n+n[\psi(1) - \psi(n+1)]\} + \frac{1}{2^{n-1}}\frac{1}{\Gamma(n+1)}\left\{\{1+n[\psi(1) - \psi(n+1)]\}\int_0^\infty udug(n,2n;u,x) - n\int_0^\infty udug(n,2n;u,x)\ln|u^2-1|\right\} + O\left(s+\frac{1}{2}\right)\right), \tag{B9}$$

where $\psi(x)$ is the Euler psi function.

These expressions generate all the $f$-functions necessary for the evaluation of the required $A_i(s)$, for $i=1,2,\ldots$ . Notice that, for $j=0$ and $i=1$, the prefactor $\Gamma[s+(i+j)/2]$ in Eq. (55) has a pole at $s=-\frac{1}{2}$. Thus, the order $s+\frac{1}{2}$ in the expansion of $f(s;0,\frac{1}{2};x)$ must be retained,

$$f\left(s;0,\frac{1}{2};x\right) = -\pi x\left(s+\frac{1}{2}\right)\left[1+2\int_1^\infty udug(0,-1;u,x)\right] + O\left[\left(s+\frac{1}{2}\right)^2\right]. \tag{B10}$$

## APPENDIX C: EVALUATION OF $A_{-1}$ AND $A_0$

Finally, in this appendix we are going to describe some details of the calculation of the $A_i(s)$, Eqs. (52)–(55), at $s=-\frac{1}{2}$. We will, in the first place, obtain an expression for $A_{-1}$ in Eq. (52)

$$A_{-1}(s) = \frac{2R^{2s}}{\sqrt{\pi}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}z^{-2s+1}\int_0^1\frac{dy}{\sqrt{y}}f\left(s;0,-\frac{1}{2};z\sqrt{y}\right). \tag{C1}$$

By using equation (B2) in the previous appendix, this can be put in the form

$$A_{-1}(s) = \frac{2R^{2s}}{\sqrt{\pi}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}z^{-2s+1}\int_0^1\frac{dy}{\sqrt{y}} \times\left\{\frac{1}{2}\sqrt{y}z\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s-1)}{\Gamma\left(s-\frac{1}{2}\right)} + 2\sqrt{y}z\sin \times\left[\pi\left(\frac{1}{2}-s\right)\right]\int_1^\infty du(u^2-1)^{1/2-s}\frac{1}{1+e^{2\pi u\sqrt{y}z}}\right\}. \tag{C2}$$

After interchanging the integrals, one gets

$$A_{-1}(s) = \frac{R^{2s}z^{-2s+2}}{s-1} + \frac{4R^{2s}z^{-2s+1}}{\sqrt{\pi}\Gamma(s)\Gamma\left(\frac{3}{2}-s\right)}\int_1^\infty du\frac{(u^2-1)^{1/2-s}}{u} \times\ln(1+e^{-2\pi uz}) - \frac{2R^{2s}z^{-2s}}{\sqrt{\pi}\Gamma(s)\Gamma\left(\frac{3}{2}-s\right)}\left\{\frac{\pi^2\Gamma(s)\Gamma\left(\frac{3}{2}-s\right)}{24\Gamma\left(\frac{3}{2}\right)} + \int_1^\infty du\frac{(u^2-1)^{1/2-s}}{u^2}\mathrm{Li}_2(-e^{-2\pi zu})\right\}, \tag{C3}$$

where $\mathrm{Li}_j(x)=\sum_{n=1}^\infty(x^n/n^j)$.

Finally, expanding around $s=-\frac{1}{2}$, one has

$$A_{-1}(s)=\frac{2z^3}{R}\left[-\frac{1}{3}-\frac{1}{12z^2}+\frac{1}{2\pi^2z^2}\int_1^\infty du\,\frac{u^2-1}{u^2}\times\mathrm{Li}_2(-e^{-2\pi uz})-\frac{1}{\pi z}\int_1^\infty du\,\frac{u^2-1}{u}\times\log(1+e^{-2\pi uz})\right]+O\left(s+\frac{1}{2}\right), \tag{C4}$$

which is a useful representation for numerical calculations.

Let us now go to the evaluation of $A_0$ in Eq. (53),

$$A_0(s)=\frac{2R^{2s}}{\sqrt{\pi}}\,\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)}\,z^{-2s-1}\int_0^1\frac{dy}{y^{3/2}}f\left(s;1,\frac{1}{2};z\sqrt{y}\right). \tag{C5}$$

As before, using Eq. (B2), and interchanging the integration order, one gets

$$A_0(s)=\frac{2R^{2s}z^{-2s-1}}{\sqrt{\pi}}\,\frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s)}\times\left\{\frac{z^2}{2\left(s-\frac{1}{2}\right)}\int_0^1\frac{dy}{y^{1/2}}+2z^2\int_0^1du\,u(1-u^2)^{-(s+1/2)}\times\int_0^1dy\frac{1}{y^{1/2}(1+e^{2\pi uz\sqrt{y}})}\right.$$
$$\left.+2\cos\left[\pi\left(s+\frac{1}{2}\right)\right]z^2\int_1^\infty du\,u(u^2-1)^{-(s+1/2)}\times\int_0^1dy\frac{1}{y^{1/2}(1+e^{2\pi uz\sqrt{y}})}\right\}. \tag{C6}$$

Now, after analytically extending, and developing around $s=-\frac{1}{2}$, one has

$$A_0(s)=\frac{z^2}{\pi R}\,\frac{1}{\left(s+\frac{1}{2}\right)}\left(1+\frac{1}{12z^2}\right)-\frac{z^2}{\pi R}\left[1+\frac{1}{6z^2}+2\left(1+\frac{1}{12z^2}\right)\log\left(\frac{z}{2R}\right)+\frac{2}{\pi z}\int_0^\infty du\times\log|1-u^2|\log(1+e^{-2\pi uz})\right]+O\left(s+\frac{1}{2}\right), \tag{C7}$$

where a simple pole appears at $s=-\frac{1}{2}$.

Clearly, in both cases the finite parts must be evaluated numerically. We will not go into the detailed calculation of the $A_i$ for $i>0$, since it is a direct consequence of the properties of $f(s;a,b;x)$ described in the previous appendix.


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