Wegner-Houghton equation in low dimensions

J. M. Carmona*
Dipartimento di Fisica, Università di Pisa, Via Buonarroti, 2, Ed. B, 56127 Pisa, Italy

J. Polonyi†
Laboratoire de Physique Théorique, Université Louis Pasteur, 3, rue de l'Université, 67084 Strasbourg Cedex, France and Department of Atomic Physics, L. Eötvös University, Budapest, Hungary

A. Tarancón‡
Departamento de Física Teórica, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain
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We consider scalar field theories in dimensions lower than four in the context of the Wegner-Houghton renormalization group equations (WHRG). The renormalized trajectory makes a non-perturbative interpolation between the ultraviolet and the infrared scaling regimes. Strong indication is found that in two dimensions and below the models with polynomial interaction are always non-perturbative in the infrared scaling regime. Finally we check that these results do not depend on the regularization and we develop a lattice version of the WHRG in two dimensions.

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I. INTRODUCTION

Asymptotically free models gained importance both in high energy and in condensed matter physics. This is obvious for the latter, where the low dimensional phenomena invoke effective theories below the upper critical dimension, $d=4$, which are super-renormalizable and their Hamiltonians contain asymptotically free coupling constants only. In the former case the asymptotically free models can provide structureless high energy physics where the cutoff can be pushed away at will. What kind of non-perturbative mechanisms do we encounter as we follow the renormalization group flow of an asymptotically free model? To distinguish different finite energy mechanisms we recall that unless the model possesses unbroken scale invariance there are always two scaling regimes, an ultraviolet and an infrared one separated by a crossover at the characteristic scale of the model. If there is a dimensional coupling constant, say a mass parameter m, in the Lagrangian then the UV and IR scaling regimes are separated by a crossover at $p=p_{cr}\approx m$ because m^2/p^2 or p^2/m^2 is treated as a small quantity in the UV or the IR side, respectively. The IR scaling laws are trivial because the radiative corrections to the evolution are suppressed by p^2/m^2 and what is left is the scale dependence governed by the canonical dimensions of the coupling constants. If there are no dimensional coupling constants in the Lagrangian, i.e. the model possesses classical scale invariance, then a crossover can be identified where one of the coupling constants reach the value 1, $g(p_{cr})=1$. The expression for the dimensionless running coupling constant $g(p)$ must contain a dimensional parameter, usually called the Λ-parameter and $p_{cr}\approx\Lambda_{\phi^4}$.[1]

Non-perturbative effects may originate in the IR scaling regimes. The classically scale invariant models at the lower critical dimension, such as the two-dimensional sigma and the Gross-Neveu model do not support long range order. Thus the IR scaling laws must contain a dynamically generated mass scale, an effect generated by the infrared or collinear divergences.

The asymptotically free coupling constants may generate non-perturbative effects in the UV scaling regime, as well, due to their growth. The question is whether the scale parameter Λ_{ϕ^4} or the mass m is reached first as we lower the cutoff. The system remains perturbative for $m>\Lambda_{\phi^4}$ because the IR scaling laws cut off the growth of the asymptotically free coupling constants before they would reach a dangerously large value. The dynamics of the modes $p<\Lambda_{\phi^4}$ becomes non-perturbative when $\Lambda_{\phi^4}>m$. The four-dimensional Yang-Mills models develop linearly rising potential between static external charges. This is believed to happen due to the large value of the asymptotically free coupling constant at the ultraviolet side of the crossover, indicated by the infrared Landau pole in perturbative QCD. The quark masses are supposed to play no important role in the vacuum structure and the chiral limit $m\to 0$ is assumed to be safe and convergent, though non-perturbative.

The infrared singularities are easier to isolate than the

*Email address: carmona@mailbox.difi.unipi.it
†Email address: polonyi@fresnel.u-strasbg.fr
‡Email address: tarancon@sol.unizar.es

[1]It is worthwhile noting that one can introduce the Λ-parameter even in the presence of any parameter with positive mass dimension in the Lagrangian so long as the IR dynamics remains stable when this latter is removed. In fact, let us write this parameter as m^l where m defined in this manner is the mass scale of the model. The running coupling constant depends on the ratio m/p, $g(p)=f(m/p)$ where the limit $m\to 0$ is convergent. In other words, the evolution of the classically scale invariant system is recovered as $p\to\infty$.

non-perturbative effects arising at the IR end of the UV scaling regime. It has already been achieved for non-asymptotically free models [1] and superrenormalizable theories [2]. The more complex non-perturbative features should arise at the ultraviolet side of the crossover where no asymptotic analysis is available.

The goal of the present paper is the study of the scaling laws and their consequences at the IR end of the UV scaling regime. According to our best knowledge this source of the non-perturbative effects in asymptotically free theories has not been studied before in a systematic manner. The problem is rather involved and demanding, we restrict ourselves to outline only the way the more detailed analysis should be done. In order to defuse the infrared problem we turn to the scalar model with polynomial interaction in the phase where no spontaneous symmetry breaking occurs. The term ϕ^n in the Lagrangian of the *d*-dimensional scalar model is asymptotically free for $n<2d/(d-2)$, so we have to trace the evolution of a large number of coupling constants at low dimensions. We present numerical evidences that the polynomial interactions at the lower critical dimension or below where infinitely many operators are relevant in the UV scaling regime always become strong, non-perturbative.

The study of the asymptotically free three-dimensional ϕ^6 model can bring some light into the IR Landau pole problem of four-dimensional gauge theories by considering the role of the non-renormalizable couplings in the IR extrapolation of the UV scaling regime. We will show that they affect considerably the evolution of the relevant couplings. This points to the possibility that the non-renormalizable couplings could suppress the Landau pole. The full solution of the theory where all possible coupling constants are followed by the renormalization group equation should necessarily yield a non-singular renormalization group flow so long as the theory possesses local interactions. This suggests that the introduction of the hadronic composite operators in QCD could defuse its Landau pole problem [3].

Another issue addressed in this paper is the manner in which the influence of the non-renormalizable operators on the dynamics is suppressed and the universal physics is reached as the cutoff is lowered. The usual argument based on the linearization of the blocking relation is not obviously applicable due to the presence of infinitely many relevant operators. This question is discussed in the framework of lattice regulation by tailoring the Wegner-Houghton equation for lattice regulated models. As a result, the approach to the low energy physics can be compared for the momentum space cutoff regularization in the continuum and for the lattice regularization, and the universality, the regulator independence, can be established well beyond the linearized approximation of the blocking relations.

The organization of this paper is the following. The infinitesimal renormalization group step, the Wegner-Houghton equation, is described in Sec. II. The running coupling constants are introduced and their evolution equations are given in Sec. III for scalar models. The asymptotic UV evolution is discussed in Sec. IV for $d=2$, 3, and 4. Section V is devoted to the demonstration of the difficulties in finding a perturbative model in $d=2$. The Wegner-Houghton equation is derived in the lattice regularization and its solution is presented in Sec. VI. Section VII is for the conclusions.

II. THE WEGNER-HOUGHTON EQUATION

There are different ways the mixing of a large number of operators can be traced down. The Wegner-Houghton equation [4], which we use in the local potential approximation in this work, is the simplest implementation of the Kadanoff-Wilson blocking [5] in the momentum space and produces the cutoff dependence of the bare action along the renormalized trajectory. Other methods work with the effective action where the infrared cutoff dependence is sought [6]. Different schemes should agree in the infrared limit where few long wavelength modes are left only in the system. We shall make two approximations in computing the blocked action, the truncation of the gradient expansion at the leading order, the local potential approximation [7], and the truncation of the Taylor expansion of the local potential in the field variable [8]. The higher order terms of the gradient expansion are non-renormalizable according to the power counting. We believe that these coupling constants which are irrelevant in the ultraviolet scaling regime do not modify our qualitative conclusion.

The Wegner-Houghton (WH) equation [4] describes the evolution of the effective action as the cutoff is lowered. As mentioned in the Introduction we shall consider a scalar model with an intrinsic mass scale which allows to clearly distinguish the UV and IR scaling regimes. We derive the WH equation for a scalar field theory by using a sharp momentum space cutoff [3,9]: we will call this regularization procedure the *continuum regularization.* In Sec. VI we consider an alternative, the *lattice regularization.*

Denote the bare action by $S_k[\phi]$, where k is the UV cutoff. Then, according to the usual Wilson-Kadanoff procedure,

$$e^{-(1/\hbar)S_{k'}[\phi]}=\int \mathcal{D}\phi' \, e^{-(1/\hbar)S_k[\phi+\phi']} \tag{1}$$

where $k'<k$ in the Euclidean space-time. The Fourier transform of the fields $\phi(x)$ and $\phi'(x)$ is non-vanishing only for $p<k'$ and $k'<p<k$, respectively. The right hand side is evaluated by means of the loop expansion, so that Eq. (1) gives

$$S_{k'}[\phi]=S_k[\phi+\phi_0']+\frac{\hbar}{2}\operatorname{tr}\log\delta^2 S+\mathcal{O}(\hbar^2), \tag{2}$$

where

$$\delta^2 S(x,y)=\frac{\delta^2 S_k[\phi+\phi_0']}{\delta\phi'(x)\,\delta\phi'(y)}, \tag{3}$$

and the saddle point, ϕ_0', is defined by the extremum condition

$$\frac{\delta S_k[\phi+\phi_0']}{\delta\phi'}=0, \tag{4}$$

in which the infrared background field, $\phi(x)$, is held fixed. It can be proved that the saddle point is trivial, $\phi_0'=0$, as long as the matrix $\delta^2 S(x,y)$ is invertible and the IR background field is homogeneous, $\phi(x)=\Phi$.

Now, each successive loop integral in the n-loop contributions which are not explicitly written in Eq. (2) brings a suppression factor

$$\frac{k^d-k'^d}{k'^d}=\mathcal{O}\left(\frac{k-k'}{k'}\right) \tag{5}$$

due to the integration volume in the momentum space. Thus $\delta k/k=(k-k')/k$ appears as a new small parameter which suppresses the higher loop contributions in the blocking relation and the ''exact'' functional differential equation obtained in the limit $\delta k\to 0$ includes the one-loop contribution only. But we should bear in mind that the loop expansion had to be used at the initial stage of the derivation so the resulting ''exact'' equation might be unreliable in the strong coupling situation. All we know is that the loop corrections to the evolution equation obtained in the one loop level are vanishing.

We will use the gradient expansion for the action,

$$S[\phi]=\sum_{n=0}^{\infty}\int \mathrm{d}^d x\, U_n(\phi(x),\partial^{2n}), \tag{6}$$

where U_n is an homogeneous function of order $2n$ in the derivative. In the leading order of this expansion, the so-called local potential approximation, we have

$$S[\phi]=\int \mathrm{d}^d x\left[\frac{Z(\phi)}{2}(\partial_\mu\phi)^2+U(\phi)\right], \tag{7}$$

and furthermore the simplification $Z(\phi)=1$ will be used to derive a simple differential equation for the potential U. This local potential will be then the only function characterizing the action. If we use now a homogeneous infrared background field, $\phi(x)=\Phi$, we obtain from Eqs. (2) and (7) an equation for the local potential $U_k(\phi=\Phi)$:

$$U_{k-\delta k}(\Phi)=U_k(\Phi)+\frac{1}{2}\mathrm{tr}\log[\Box+U_k''(\Phi)]+\mathcal{O}(\delta k^2), \tag{8}$$

where we have introduced the notation

$$U_k''(\Phi)=\frac{\partial^2 U_k(\Phi)}{\partial\Phi^2},\quad \Box=-\partial_\mu\partial^\mu \tag{9}$$

and the trace is taken in the subspace of the eliminated modes. We can explicitly write the trace in momentum space and get

$$U_{k-\delta k}(\Phi)=U_k(\Phi)+\frac{1}{2}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\log[p^2+U_k''(\Phi)] +\mathcal{O}(\delta k^2), \tag{10}$$

where the integration extends over the shell $k-\delta k<p<k$. In the limit $\delta k\to 0$ one then finds the differential equation

$$k\frac{\partial}{\partial k}U_k(\Phi)=-\frac{\Omega_d k^d}{2(2\pi)^d}\log[k^2+U_k''(\Phi)], \tag{11}$$

where Ω_d denotes the d-dimensional solid angle

$$\Omega_d=\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}. \tag{12}$$

The Wegner-Houghton equation (11) represents the one-loop resummed mixing of the coupling constants of the potential $U_k=\Sigma_n(g_n/n!)\Phi^n$. In fact, an expansion of the logarithm in the second derivative of the potential gives

$$k\frac{\partial}{\partial k}U_k(\Phi)=-\frac{\Omega_d k^d}{2(2\pi)^d}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{-U_k''(\Phi)}{k^2+U_k''(\Phi)}\right)^n, \tag{13}$$

up to a field independent constant. This is the usual one loop resummation of the effective potential [10] except that the loop momentum is now restricted to the subspace of the modes to be eliminated. Actually, the fact that the right-hand side (RHS) includes the running potential $U_k(\Phi)$ rather than the bare one, $U_\Lambda(\Phi)$, indicates that the contributions of the successive eliminations of the degrees of freedom are piled up during the integration of the differential equation and the solution of the renormalization group equation resums the perturbation series. The solution of the differential equation interpolates between the bare and the effective potential as k is lowered from the original cutoff Λ to zero.

Finally, let us note that the derivation of Eq. (11) shows that the restoring force for the fluctuations into the equilibrium is proportional to the argument of the logarithm function. Thus a nontrivial saddle point should be used when

$$k^2+U_k''(\Phi)\leqslant 0. \tag{14}$$

III. EVOLUTION OF THE COUPLING CONSTANTS

Our effective action is

$$S_k=\int \mathrm{d}^d x\left[\frac{1}{2}(\partial_\mu\phi(x))^2+U_k(\phi(x))\right], \tag{15}$$

and the initial condition for the evolution equation is given at $k=\Lambda$. The potential $U_k(\Phi)$ is assumed to be polynomial so it is expanded as

$$U_k(\Phi)=\sum_{n=0}^{N}\frac{1}{n!}g_n(\Phi_0)(\Phi-\Phi_0)^n. \tag{16}$$

We study the model in the symmetric phase, where the saddle point is trivial, $\Phi_0=0$. The polynomial structure of the potential is consistent because we avoid the singularity of Eq. (11) at $k_{\rm cr}^2=-U''_{k_{\rm cr}}(\Phi)$ [recall Eq. (14)] which occurs in a region around $\Phi=0$ in the symmetry broken phase [9].

Taking the n-th derivative of Eq. (16) at $\Phi=0$ we obtain the coupling constant $g_n(k)$, and define the corresponding beta function by

$$g_n(k)=\frac{\partial^n}{\partial\Phi^n}U_k(\Phi)_{|\Phi=0},\qquad \beta_n=k\frac{\mathrm{d}}{\mathrm{d}k}g_n(k)=\frac{\partial^n}{\partial\Phi^n}k\frac{\partial}{\partial k}U_k(\Phi). \tag{17}$$

By taking the successive derivatives of Eq. (11), we obtain

$$\beta_n=-\frac{\Omega_d k^d}{2(2\pi)^d}\mathcal{P}_n(G_2,\ldots,G_{n+2}), \tag{18}$$

where

$$G_n=\frac{g_n}{k^2+g_2} \tag{19}$$

and

$$\mathcal{P}_n=\frac{\partial^n}{\partial\Phi^n}\log[k^2+U''_k(\Phi)] \tag{20}$$

is a polynom of order n in the variables G_j, $j=2,\ldots,n+2$,

$$\begin{aligned}
\mathcal{P}_1&=G_3,\\
\mathcal{P}_2&=G_4-G_3^2,\\
\mathcal{P}_3&=G_5-3G_3G_4+2G_3^3,\\
\mathcal{P}_4&=G_6-4G_5G_3-3G_4^2+12G_3^2G_4-6G_3^4,\\
\mathcal{P}_5&=G_7-5G_6G_3-10G_5G_4+20G_5G_3^2\\
&\quad+30G_4^2G_3-60G_4G_3^3+24G_3^5,\\
\mathcal{P}_6&=G_8-6G_7G_3-15G_6G_4-10G_5^2+30G_6G_3^2\\
&\quad+120G_5G_4G_3+30G_4^3-120G_5G_3^3-270G_4^2G_3^2\\
&\quad+360G_4G_3^4-120G_3^6.
\end{aligned} \tag{21}$$

The coupling constants defined through Eq. (16) are dimensional parameters [the field variable ϕ has dimension $(d-2)/2$]. However, the corresponding dimensionless parameters have more physical sense. We obtain them in the following way:

$$\tilde{g}_n(k)=k^{-n(1-d/2)-d}g_n(k)=(\Lambda\tilde{k})^{-n(1-d/2)-d}g_n(k), \tag{22}$$

where now $\tilde{k}$ runs from 1 to 0. Their beta functions are

$$\tilde{\beta}_n=-\left[n\left(1-\frac{d}{2}\right)+d\right]\tilde{g}_n+k^{-n(1-d/2)-d}\beta_n, \tag{23}$$

where the first and the second term stands for the tree-level and the loop corrections, respectively. One can see that the super-renormalizable coupling constants follow asymptotically free scaling law at the tree level.

IV. UV SCALING LAWS AND THEIR EXTENSIONS

One can distinguish an ultraviolet and an infrared scaling regime, for $k^2\gg|m^2(k)|$ and for $k^2\ll|m^2(k)|$, respectively. In the UV regime the scale dependence comes dominantly from the k^2 term of the propagator, see the denominator of Eq. (19); the k-dependence is generated by the phase factor k^d in the IR regime where k^2 could be neglected in the inverse propagator. We will begin at the UV scale with the usual ϕ^4 potential ($g_2\equiv m^2$)

$$V_\Lambda(\phi)=\frac{1}{2}m^2\phi^2+\frac{1}{4!}g_4\phi^4+\frac{1}{6!}g_6\phi^6, \tag{24}$$

and see how the different couplings are generated when we move towards the IR regime.

One ignores the g_2 term in the denominator of Eq. (19) in the asymptotic UV regime and finds

$$\begin{aligned}
\frac{\mathrm{d}g_2}{\mathrm{d}k}&=-\frac{\Omega_d}{2(2\pi)^d}k^{d-3}g_4,\\
\frac{\mathrm{d}g_4}{\mathrm{d}k}&=\frac{\Omega_d}{2(2\pi)^d}k^{d-3}\left(\frac{3}{k^2}g_4^2-g_6\right),\\
\frac{\mathrm{d}g_6}{\mathrm{d}k}&=-\frac{\Omega_d}{2(2\pi)^d}3g_4k^{d-5}\left(\frac{10}{k^2}g_4^2-5g_6\right),
\end{aligned} \tag{25}$$

where in the last equation we omitted the contribution of g_8. Consider the usual strategy in which the coupling constants g_n are neglected for $n>4$ and the resulting equation is easy to integrate,

$$\frac{1}{g_4(k)}=\frac{1}{g_4(\Lambda)}+\frac{3\Omega_d\left(1-\left(\frac{k}{\Lambda}\right)^{4-d}\right)}{(4-d)2(2\pi)^dk^{4-d}}. \tag{26}$$

This expression agrees with the result of the minimal subtraction (MS), a scheme which proved to be specially convenient in the ultraviolet scaling regime. It is based on the analytical continuation of the loop integrals in the ultraviolet domain so the resulting beta functions are mass independent, i.e. the terms $\mathcal{O}(g_2/k^2)$ are neglected. When extrapolating to the infrared regime we find erroneously the mass independent result $g_4\sim k^{4-d}$ ($g_4\sim\log k$ in $d=4$), g_4 tends to zero as

$k\to 0$. This can be understood by inspecting Eq. (25) where we find large positive values in the infrared for $g_4\neq 0$ (this conclusion remains valid for finite g_6, as well). When the mass term is retained the beta function assumes the correct behavior and becomes $\mathcal{O}(k^d)$ in the infrared. Note that the term with g_6 acts in the opposite manner than g_4, cf. the different signs in the right-hand side of Eq. (25), so that it can change the evolution considerably.

It is instructive to look into the evolution of the dimensionless coupling constant:

$$\frac{1}{\tilde{g}_4(k)}=\frac{\left(\frac{k}{\Lambda}\right)^{4-d}}{\tilde{g}_4(\Lambda)}+\frac{3\Omega_d\left(1-\left(\frac{k}{\Lambda}\right)^{4-d}\right)}{(4-d)2(2\pi)^d}. \tag{27}$$

For $4-d>0$ the one-loop $\omega=4-d$ universal critical exponent is reached for k values sufficiently below the cutoff where $k/\Lambda\approx 0$. The latter condition is needed to get rid of the non-universal cutoff effects. The scaling changes qualitatively as $d\to 4$ because the non-universal $k\approx\Lambda$ behavior is spread over the whole k range due to the smallness of $4-d$. This is what happens in the expansion

$$1-\left(\frac{k}{\Lambda}\right)^{4-d}\to(d-4)\ln\frac{k}{\Lambda}, \tag{28}$$

employed in the dimensional regularization scheme. This generalizes to any dimension: the marginal coupling constant follows the scaling law extended from the non-universal cutoff regime.

The evolution for g_2 is of the form

$$\frac{\mathrm{d}g_2}{\mathrm{d}k}=-\frac{\Omega_d}{2(2\pi)^d}g_4k^{d-3}. \tag{29}$$

It predicts

$$\frac{\mathrm{d}g_2}{\mathrm{d}k}\sim k, \quad g_2\sim k^2+\text{const} \tag{30}$$

in lack of any dimensional constant.

In the IR scaling regime we neglect the k^2 term in the denominator of Eq. (19) and using Eqs. (18) and (21), we get, for g_4 (assuming again $g_6=0$),

$$\frac{\mathrm{d}g_4}{\mathrm{d}k}=\frac{3\Omega_d}{2(2\pi)^d}k^{d-1}\frac{g_4^2}{g_2^2}. \tag{31}$$

This evolution is much slower comparing with Eq. (26). In fact, for $d>1$ we have now a suppression factor k^{d-1} which makes the coupling to stabilize at the attractive IR fixed point. In the same way, we obtain, for g_2,

$$\frac{\mathrm{d}g_2}{\mathrm{d}k}=-\frac{\Omega_d}{2(2\pi)^d}k^{d-1}\frac{g_4}{g_2}, \tag{32}$$

with a variation which is slower than that predicted by the UV scaling and a suppression factor for $k\to 0$.

We see that the extraction of the scaling in the limit $k\to 0$ from the UV scaling laws, which is the commonly accepted practice in perturbation theory is incorrect when the MS scheme is used. One has to come back to the complete scaling laws in order to describe correctly the IR scaling.

Four-dimensional asymptotically free gauge models present an infrared Landau pole in perturbation theory. But this behavior results from the extrapolation from the UV scaling laws. As we have just remarked, the IR limit of the UV regime is not correct in general, and the mass term can change considerably the actual behavior in the IR. Moreover, at the IR side of the UV regime, there are nonlinear effects that make important the contribution of the irrelevant (nonrenormalizable) couplings [see Fig. 3(b), commented on in the next section], therefore even the IR limit of the UV scaling can be influenced by these couplings. These ideas have been considered qualitatively in the previous paragraphs, after inspection of Eq. (25). Let us now examine them more quantitatively.

We take as an example of asymptotically free model the scalar theory in three dimensions. We know from the epsilon-expansion result [11] that below four dimensions, the $\lambda\phi^4$ theory does not present an infrared Landau pole, but a fixed point located at

$$\tilde{\lambda}^*=\frac{16\pi^2}{3}\epsilon, \tag{33}$$

at order $\epsilon=4-d$. This is the Wilson-Fisher fixed point. For finite ϵ, for example in two and three dimensions, we can get this one-loop result from our beta functions of the dimensionless coupling constants obtained from Eqs. (23) and (18) in the asymptotic UV regime (that is, ignoring g_2), which for the $\tilde{g}_4$ coupling give

$$\tilde{\beta}_4=-(4-d)\tilde{g}_4+3\tilde{g}_4^2\frac{\Omega_d}{2(2\pi)^d}, \tag{34}$$

giving the IR fixed point

$$\tilde{g}_4^*=\frac{(4-d)2(2\pi)^d}{3\Omega_d}. \tag{35}$$

Restricting ourselves to the $d=3$ case, we find

$$\tilde{g}_4^*=\frac{4\pi^2}{3}\simeq 13.15947. \tag{36}$$

However, in three dimensions g_6 is a marginal coupling, and it can be generated by the RG flow, modifying the position of the fixed point (36). If we include $\tilde{g}_6$ in our analysis, we get the beta functions

$$\tilde{\beta}_4=-\tilde{g}_4+\frac{3}{4\pi^2}\tilde{g}_4^2-\frac{1}{4\pi^2}\tilde{g}_6,$$

$$\tilde{\beta}_6=\frac{-3}{4\pi^2}\tilde{g}_4(10\tilde{g}_4^2-5\tilde{g}_6). \tag{37}$$

It is immediate to see that the zeros of these beta functions are at the point

$$\tilde{g}_4^* = 4\pi^2 \simeq 39.4784,$$
$$\tilde{g}_6^* = 32\pi^4 \simeq 3117.091. \tag{38}$$

But now let us consider the inclusion of a non-renormalizable coupling, g_8. The beta functions for the (g_4, g_6, g_8) model are

$$\tilde{\beta}_4 = -\tilde{g}_4 + \frac{3}{4\pi^2}\tilde{g}_4^2 - \frac{1}{4\pi^2}\tilde{g}_6,$$

$$\tilde{\beta}_6 = \frac{-3}{4\pi^2}\tilde{g}_4 (10\tilde{g}_4^2 - 5\tilde{g}_6) - \frac{1}{4\pi^2}\tilde{g}_8, \tag{39}$$

$$\tilde{\beta}_8 = \tilde{g}_8 - \frac{7}{4\pi^2}(-90\tilde{g}_4^4 + 60\tilde{g}_4^2\tilde{g}_6 - 5\tilde{g}_6^2 - 4\tilde{g}_4\tilde{g}_8).$$

From them, one obtains the fixed point

$$\tilde{g}_4^* = \frac{4\pi^2}{210}(195 + \sqrt{29625}) \simeq 69.01,$$

$$\tilde{g}_6^* = -4\pi^2\tilde{g}_4^* + 3\tilde{g}_4^{*2} \simeq 11563, \tag{40}$$

$$\tilde{g}_8^* = 15\tilde{g}_4^{*3} - 60\pi\tilde{g}_4^{*2} \simeq 2.11\times 10^6.$$

These values are also obtained in the numerical integration of Eqs. (39), independently of the initial values for the different couplings (if they are different from zero, which corresponds to the Gaussian fixed point).

To assess the importance of this result, the difference between the physics around the fixed points (38) and (40), recall that the modification of the irrelevant operator set at the cutoff influences the overall scale of the model. Thus one has to consider dimensionless quantities in comparing the two coupling constant regions. The most obvious candidate, the dimensionless ratio between the mass and the four point vertex, g_2/g_4^2, is trivially vanishing in our approximation. But g_6 is dimensionless and its variation at the fixed points indicates that no adjustment of the overall scale could bring the physics of these two fixed points together.

As we have expected, the non-renormalizable coupling g_8 modifies the position of the fixed point without changing the blocking procedure, turning to a situation of strong coupling dynamics in the IR. This is because the linearity which one assumes to ignore the irrelevant coupling constants is no longer valid in the strong coupling regime. As g_4 and g_6 approach their large IR fixed point the linearization fails and new scaling laws are found which in turn generate new relevant operators [9], overlapping with ϕ^8. Thus the strong coupling dynamics may induce a new (and artificial) IR scaling regime even if the UV scaling laws are extrapolated down to low energies.

Nothing unusual happens for infinitesimal ϵ when one stays in the vicinity of the Gaussian fixed point. In that case, linearity applies all along the renormalization group (RG) flow from the Gaussian fixed point to the Wilson-Fisher fixed point given by Eq. (33). However, as we have seen, its location is changed for finite ϵ by nonlinear effects produced in the flow from one fixed point to the other. In the vicinity of this infrared fixed point (which, we stress again, is artificial in the sense that it neglects the influence of the mass term) we of course have again the classification of relevant and irrelevant terms, which, in the case of three and two dimensions, is different from the one obtained from power counting. For example, this IR fixed point has in two dimensions just the mass and the fourth order coupling as relevant parameters, while higher order couplings are irrelevant.

It is well known that the poles of the fixed point action at complex values of the field variable make the Taylor expansion in the field unreliable [8]. We do not see any reason to reject a blocked action only because the potential is diverging beyond a given field strength. This kind of internal space singularity might only indicate a maximal particle density in the system. We should stay only sufficiently far from this limiting value of the field variable when the evolution equation is truncated. We interpret the difference of the two fixed points as an indication of the breakdown of the simple universality which is based on the linearized flow equation around the UV fixed point.

So far we considered the extrapolation of the UV scaling laws to the IR regime. Does the conclusion concerning the importance of the non-renormalizable coupling constant g_8 remains valid when the true evolution equation, with $g_2 \neq 0$, is considered? The mass slows down the evolution but this may happen ''too late'' and the strong coupling effects can be found on the true renormalization group trajectory for small enough renormalized mass, close enough to the critical point. When the mass is large then crossover freezes the evolution of the coupling constants ''earlier'' and the linearization remains valid. To demonstrate this case recall that the IR limit of the UV regime means a fixed point for the dimensionless couplings. For a super-renormalizable coupling constant such as g_4, which has positive dimension, this would mean that the dimensional coupling goes to zero when $k \to 0$. However, we know that for a relevant coupling, the dimensionless quantity diverges when $k \to 0$, so that the dimensional coupling will take a finite value at the IR. This reasoning can be explicitly checked in Fig. 1, in which we consider the $d=3$ scalar theory with just one coupling, g_4. The white points follow the evolution of the UV regime and its extrapolation to $k=0$. We observe that indeed the dimensionless coupling reaches the fixed point given by Eq. (36), while the dimensional coupling goes to zero. However, if one considers the complete beta function, i.e. retaining g_2 (black points), the behavior is the same in the UV regime, but then the true trajectory separates from the IR limit of this regime and enters into the actual IR regime, which implies a divergent dimensionless coupling at $k=0$, and a certain finite value of the dimensional coupling, as explained above. One can however see numerically that this finite value is stable and almost does not change when one introduces more and

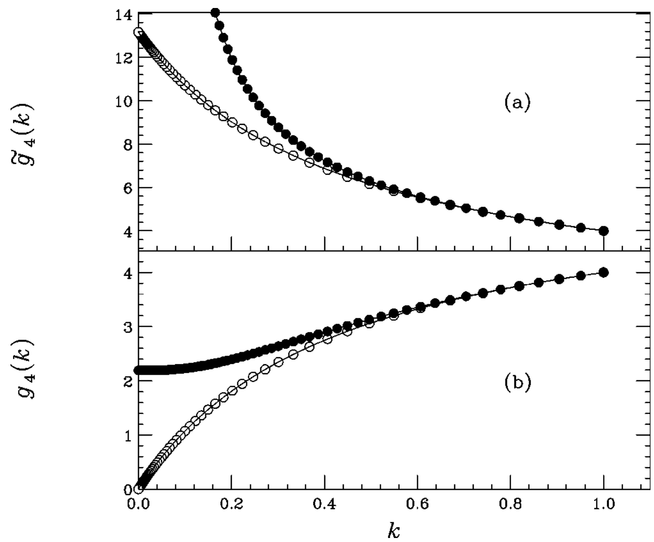

FIG. 1. Numerical RG evolution for the (a) dimensionless $\tilde{g}_4$ coupling and (b) dimensional g_4 coupling in the $d=3$ $g_4\phi^4$ scalar theory. Black points result from the integration of the complete beta function with $g_2=0.052$, while white points show just the UV regime (see text). In (a), the IR limit of the UV regime is given by the fixed point (36).

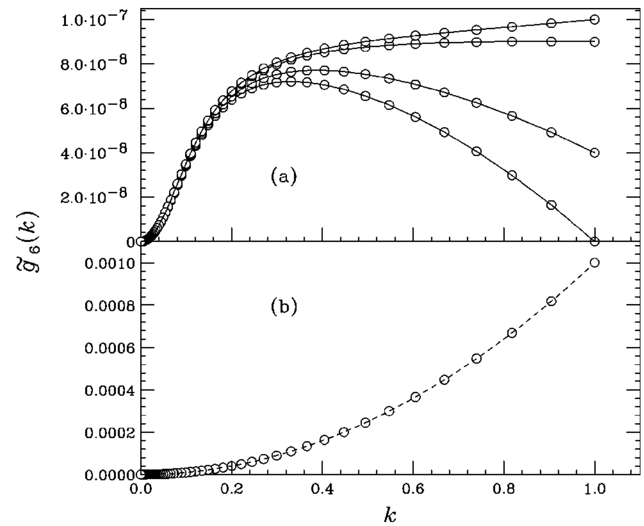

FIG. 2. Renormalization group flow for the dimensionless coupling constant $\tilde{g}_6(k)$ in $d=4$. The initial conditions are $\tilde{g}_2(1)=\tilde{g}_4(1)=0.01$, together with the values of $\tilde{g}_6(1)=0.0, 4\times10^{-8}, 9\times10^{-8}, 10^{-7}$ for the different lines in plot (a), and $\tilde{g}_6(1)=10^{-3}$ for plot (b). In this last plot the points of the numerical renormalization group flow are joined by the linearized scaling $\tilde{g}_6\approx k^2$, shown by a dashed line.

more non-renormalizable couplings in the RG evolution. This is what one expects when the crossover captures the coupling constants and slows down their evolution in the regime of linearizability where the non-renormalizable couplings are unimportant.

In the same way, it might well be that the IR Landau pole observed in four-dimensional gauge theories is just an artifact of a wrong IR limit or the truncation of the renormalized action. First, the nature of the singularity can change when one adds non-renormalizable couplings, and then the IR Landau pole would be the reflection of the insufficient functional form of the blocked action, and second, the true IR trajectory can be quite different from the IR limit of the UV scaling.

V. ASYMPTOTIC FREEDOM AND THE PERTURBATION EXPANSION

We examine in this section the scaling laws in dimensions $d=2$, 3 and 4 from the point of view of the applicability of the perturbation expansion.

As we have seen in the previous section, in the RG evolution of our model there are two asymptotic scaling regimes, $k\to\infty$ and $k\to0$. The latter one is trivial as mentioned above, because the beta functions (18) are suppressed by the factor k^d and the evolution of the dimensional coupling constants slows down as $k\to0$. The asymptotic UV scaling is however more involved. The super-renormalizable (relevant) and the renormalizable (marginal) coupling constants, g_n with $n<2d/(d-2)$ and $n=2d/(d-2)$ according to the power counting, respectively, follow their autonomous evolution, the universal renormalized trajectory[2]. The non-renormalizable (irrelevant) coupling constants ''forget quickly'' their initial value and take values which are generated by the universal flow. This general trend is demonstrated by the renormalization group flow shown for $d=4$ and 3 in Figs. 2 and 3, respectively. In those figures the dimensionless coupling constants are displayed as the functions of the cutoff which is measured in the units of the initial cutoff value, $k\to k\Lambda$.

In the case of theories with non-Gaussian asymptotically free couplings (the case of our model for $d<4$), an excessive growth of these couplings in the UV regime may produce a non-perturbative situation in the infrared. We want to study this by comparing the values of the couplings at $k=0$. To do so, we will adopt the convention that a model with positive renormalized mass square is non-perturbative in the vertex $g_n\phi^n$ if the radiative correction $\mathcal{O}(g_n)$ to the self energy is stronger than the mass term, i.e.,

$$\frac{\tilde{g}_n(k)}{((n-2)/2)!\,2^{(n-2)/2}\tilde{m}^2(k)}\gg1 \tag{41}$$

where $\tilde{m}^2(k)=\tilde{g}_2(k)$. In case this inequality were satisfied in the infrared ($k=0$), this would mean a non-perturbative situation and the invalidity of renormalized perturbation expansion.

[2]Ignoring the triviality in $d=4$ where the tree level marginal coupling constant g_4 is actually irrelevant due to the radiative corrections.

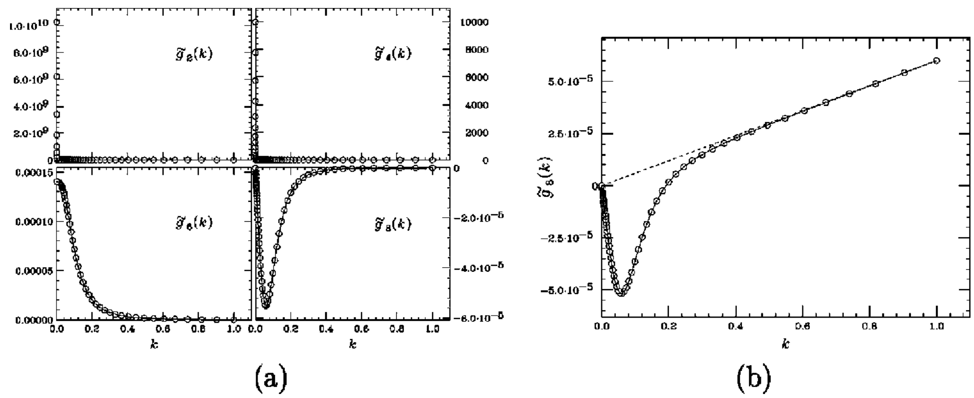

FIG. 3. (a) Renormalization group flow for the dimensionless coupling constants $\tilde{g}_n(k)$, $n=2$, 4, 6 and 8 in $d=3$. The initial conditions are $\tilde{g}_2(1)=\tilde{g}_4(1)=0.01, \tilde{g}_n(1)=0$ for $n\geq 6$. (b) Evolution of $\tilde{g}_8(k)$ with the same initial conditions that before except for $\tilde{g}_8(1)=6\times 10^{-5}$, together with the linearized scaling law $\tilde{g}_8(k)\approx k$, indicated by the dashed line.

We remark that we are asking here about the validity of the perturbative condition at the true, $k=0$, infrared fixed point, where the RG flow ends. We are not considering for example the situation at the Wilson-Fisher IR fixed point in $d<4$, which, as was explained in Sec. IV, is the IR limit of the UV scaling behavior (a fixed point which describes the behavior of the system at the end of the UV regime), and not the real IR fixed point if we ask for the behavior of the system at energy scales much lower than the mass $m(k)$ (for example, in dimension two, we will see that at the crossover, or the end of the UV regime, the high order couplings start to take large values, while they are irrelevant couplings for the Wilson-Fisher IR fixed point; this is because the true RG flow separates from the extrapolation of the UV flow, as was explicitly shown in Fig. 1 in the case of $d=3$).

We now turn to a detailed analysis of the situation at dimensions four, three and two. Since the neglected higher order vertices may influence the evolution while we lower the cutoff, we have to address the problem of the system of coupled equations numerically.

The evolution of a non-renormalizable coupling constant, $\tilde{g}_6$ in four dimensions, is shown in Fig. 2. The irrelevance is expressed by the independence of $\tilde{g}_6(k)$ on the initial value $\tilde{g}_6(1)$ for $k\ll 1$.[3] The value given in the leading order of the perturbation expansion,[4] $\mathcal{O}(\tilde{g}_4^3)$, is reached at $k\approx 0.3$ for not too large values of $\tilde{g}_6(1)$. Since the model is in the weak coupling regime the evolution is rather slow after arriving at this universal value if the cutoff is high enough to provide a long scaling regime. In our case the scale window $0.3<k<1$ was insufficient and the plateau is reduced into a peak at $k\approx 0.3$ before the crossover. But the bringing of $\tilde{g}_6(1)$ close to the universal value creates a plateau even with this limited range of the scales as it can be seen for $\tilde{g}_6(1)=9\cdot 10^{-8}$. Such a scale independence is the ultimate goal of the improved action program [12]. As the initial value increases beyond the plateau level the coupling constant decreases in a monotonous manner as the cutoff is lowered. In order to check the critical exponent coming from the linearized blocking we need a $\tilde{g}_6(k)$ larger than the universal value since the latter originates in the nonlinear level, $\mathcal{O}(\tilde{g}_4^3)$. The evolution follows the linear relation $\tilde{g}_6\approx k^2$ for $k<1$ indicated by the dashed line in the last plot of Fig. 2. The infrared scaling regime is $0<k<0.1$ where $\tilde{g}_6(k)$ tends to zero with k along the universal trajectory. The theory remains perturbative in $d=4$ since it has no relevant non-Gaussian coupling constant.

The three-dimensional renormalization group flow is depicted in Fig. 3. The asymptotic infrared scaling laws are rather simple, the super-renormalizable coupling constants diverge, the renormalizable one $n=6$ converges and the non-renormalizable ones tend to zero in the infrared, $k\to 0$. The ultraviolet scaling law, above $k\approx 0.1$ indicates the weak, radiative correction generated relevance of $\tilde{g}_6$ for the given initial conditions, $\tilde{g}_n(1)$. The insensitivity on the initial condition $\tilde{g}_8(1)$ for $k<0.3$ seen in the last plot supports the irrelevance of $\tilde{g}_8$. In fact, the evolution of $\tilde{g}_8(k)$ follows the linearized scaling law as long as its value is far form the universal $\mathcal{O}(\tilde{g}_4^4)$ value. It is worthwhile noting that the non-renormalizable coupling constants $\tilde{g}_n(k)$ always develop a peak of sign $(-1)^{1+n/2}$ around the crossover, $k\approx 0.1$. The appearance of the peak can be understood as the result of the increase of $|\tilde{g}_n(k)|$ from zero as the cutoff is lowered in the ultraviolet scaling regime and the decrease in the infrared side of the crossover.

Non-perturbative phenomena may arise at the low energy edge of the UV scaling regime due to the increase of the asymptotically free coupling constants, g_4 and g_6 if the scaling regime is long enough and the initial value of the coupling constants $\tilde{g}_4(1)$ and $\tilde{g}_6(1)$ are large enough. There is however no problem in finding a perturbative, asymptotically

[3]The different initial value for the non-renormalizable coupling constants may induce a different overall scale factor. This effect is very weak in our case due to the smallness of the renormalizable coupling constants.

[4]Which is not applicable for the strongly coupled case (38) or (40).

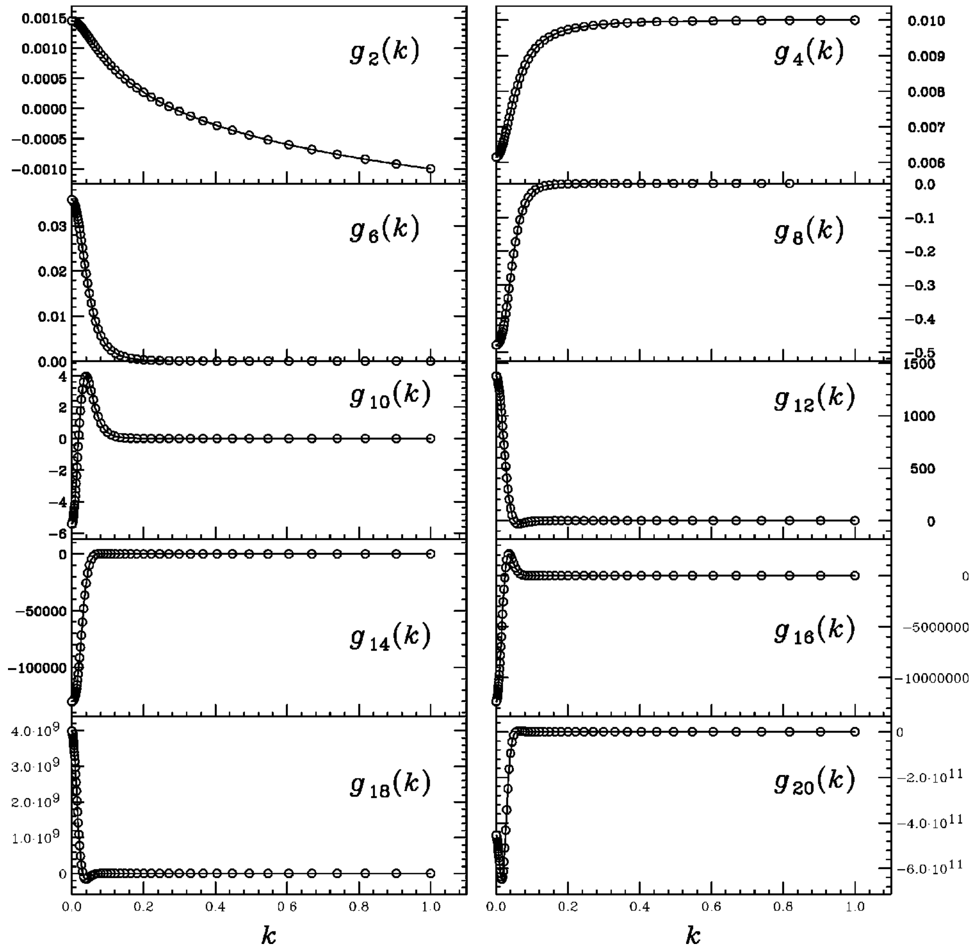

FIG. 4. Renormalization group flow for the dimensional coupling constants in $d=2$. The initial conditions are $g_2(1)=g_4(1)=0.01$, $g_n(1)=0$ for $n\geqslant 6$.

free theory in the infrared. The parameter with the highest energy dimension in the Lagrangian is $g_2=m^2$ for $d>2$. Thus $\tilde{g}_2$ is the largest among the dimensionless coupling constants according to Eq. (23), and it dominates the action and renders the theory perturbative in the IR limit. The comparison of this conclusion with Eq. (40) reveals the necessity of treating the IR scaling laws properly in establishing the validity of the renormalized perturbation expansion.

As d approaches 2 more and more coupling constants become super-renormalizable. The fastest increasing dimensionless non-Gaussian coupling constant during the decrease of the cutoff is $\tilde{g}_4$. The critical exponent, the measure of the speed of the increase, becomes degenerate for infinitely many coupling constants when $d=2$. The specialty of the lower critical dimension is the existence of infinitely many super-renormalizable coupling constants, $\tilde{g}_n$, with equal critical dimension. This degeneracy of the dimensions evolves the non-Gaussian pieces of the action with the same rate as the mass term on the tree level and the theories are not obviously perturbative any more. In other words, it remains for the radiative corrections, the last term in the right hand side of Eq. (23) to determine if the theory runs into weak- or strong-coupling regime at the infrared.

The renormalization group equations were integrated out numerically in two dimensions with the initial conditions $(g_2^i=-0.001, g_4^i=0.01, g_n^i=0, n>4)$ to find the evolution of the coupling constants. The result, depicted in Fig. 4, shows a marked increase at the low energy end of the UV scaling regime. This increase originates from the asymptotically free evolution. The relevant behavior of a coupling constant is defined on the linearized level of the blocking, i.e. in the leading order of the perturbation expansion. This one-loop result can be obtained by replacing the running coupling constants in the beta functions by their initial values at the cutoff. The local potential obtained in the one-loop approximation is

$$\begin{aligned} U_k(\phi) &= V_\Lambda(\phi)+\frac{1}{2}\int_{k<p<\Lambda}\frac{\mathrm{d}^2p}{(2\pi)^2}\log[p^2+V''_\Lambda(\phi)] \\ &= \frac{1}{8\pi}\{[\Lambda^2+V''_\Lambda(\phi)]\log[\Lambda^2+V''_\Lambda(\phi)] \\ &\quad -[k^2+V''_\Lambda(\phi)]\log[k^2+V''_\Lambda(\phi)]-\Lambda^2+k^2\}, \end{aligned} \tag{42}$$

with

$$V''_\Lambda(\phi)=m^2(\Lambda)+\frac{g_4(\Lambda)}{2}\phi^2. \tag{43}$$

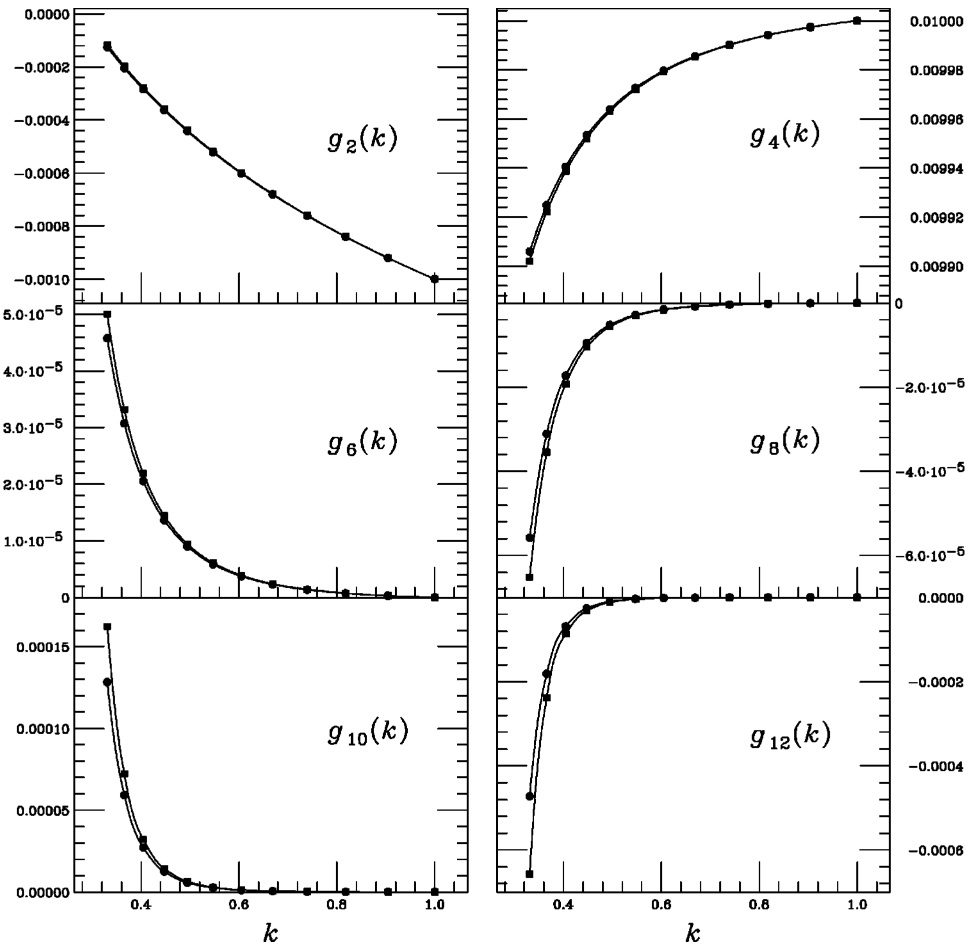

FIG. 5. Evolution of the dimensional coupling constants obtained in the one-loop approximation (squares) and numerically (circles) for $d=2$.

The comparison of the numerical solution with the one-loop evolution is shown in Fig. 5. The one-loop formula (42) cannot be extended down to $k=0$, because there will be a value of k such that $k^2+m^2(\Lambda)=0$, since $m^2(\Lambda)$ is negative. But we only want to compare the results in the perturbative regime. So we have stopped the evolution in Fig. 5 at $k\sim 0.3$. One can also see an increase in the one-loop solution at small k which accumulates and drives the system non-perturbative at lower values of k.

The numerical results of Fig. 4 show that the initial conditions ($g_2^i=-0.001, g_4^i=0.01, g_n^i=0, n>4$) correspond to a non-perturbative system. Can we find initial conditions which yield perturbative dynamics? In order to answer this question the left-hand side of the inequality (41) is plotted against the initial value for g_2 on Fig. 6, for the different couplings up to $N=20$, at a value of $g_4^i=0.001$. The result does not change qualitatively for different values of bare g_4. It supports the general trend of having the systems more perturbative when the Gaussian part of the action is increased. The higher order coupling constants tend to grow faster but it seems that g_n can be brought into the perturbative regime for sufficiently large initial mass square. At $g_2^i=0.01$, for example, all the couplings are perturbative, and this perturbative character is more pronounced for the high couplings. However, the separation between the values of the LHS ratio of Eq. (41) at $g_2^i=0.01$ is smaller as we go to higher couplings, and one can ask whether it has got a limiting value or the trend can be reversed for a sufficiently high order coupling. Indeed, Fig. 7 reveals that this happens for the $n=24$ coupling ($N=26$), which suggests that one will have a non-perturbative situation also at this value of bare g_2 going to a sufficiently high order coupling constant. The situation is the same, even stronger, below two dimensions: the existence of an infinite number of relevant couplings makes that one cannot assure perturbativity by looking to a finite number of couplings, no matter what the initial conditions

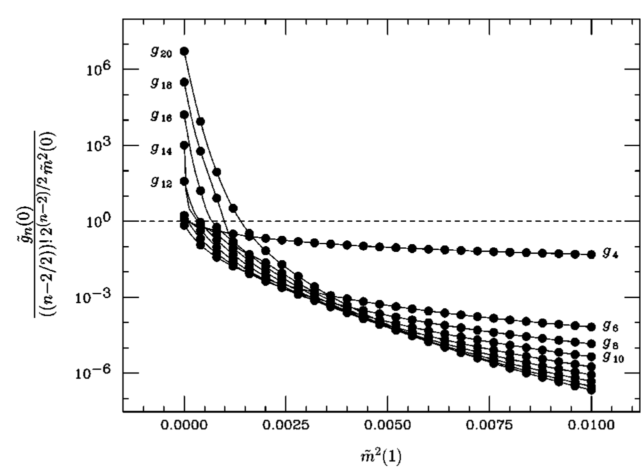

FIG. 6. The left-hand side of Eq. (41) as a function of the bare mass square for $n=4,\ldots,20$ in $d=2$.

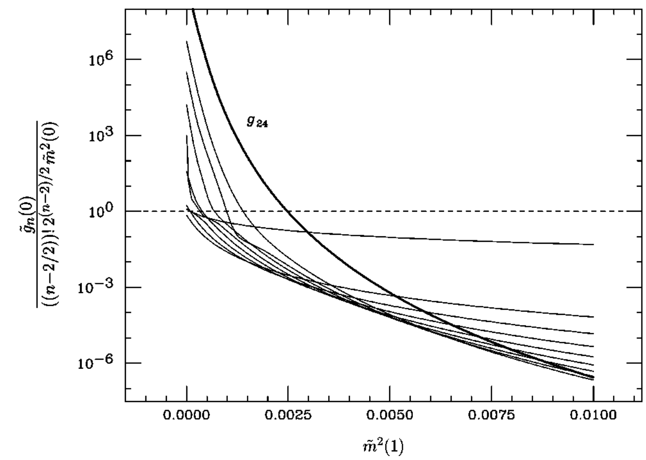

FIG. 7. The same as Fig. 6 but now including the $n=24$ coupling.

are. We take this and similar other failures in finding a perturbative theory observed at different initial conditions as a strong numerical indication of the non-perturbative nature of *any* two and lower dimensional scalar field theory with polynomial interaction.

VI. LATTICE WEGNER-HOUGHTON EQUATION

The previous sections dealt with the renormalization group flow at finite scales. We address now a different, asymptotic problem, the manner the sensitivity on the initial values of the irrelevant coupling constants is suppressed during the renormalization. This question is usually rendered trivial by the universality argument. But there are two reasons to suspect that such a reasoning which is based on the linearization of the blocking relation might be oversimplified; both are related to an infinite set of operators.

The reason motivating a more careful check of the universality, mentioned in the Introduction, is that the models at or below the lower critical dimension contain infinitely many relevant operators. It is not obvious whether the sum over the interaction vertices is always convergent enough to make the linearization of the blocking relation a reliable approximation.

Another potential problem shows up if one changes infinitely many irrelevant terms in the action by choosing another regulator. Let us compare the momentum space cutoff in the continuum with the lattice regularization. The propagator is a monotonic function of the momentum in the continuum. This is not the case on the lattice. In fact, the fermion doubling problem on the lattice [13] results from the periodicity of the propagator in the first Brillouin zone, the appearance of 2^d-1 new maxima in the propagators in the UV, non-universal regime. The existence of a maximum of the propagator in the UV regime contradicts an assumption of the studies of the continuum models, namely that the propagator decreases monotonically as $p\to\infty$, and renders the perturbation expansion non-universal for lattice fermionic models [14]. There is no species doubling for bosons but their propagator remains periodic on the lattice and we find 2^d-1 lattice extrema in the UV regime. The existence of these extrema is an evidence of the slowing down in the decrease of the propagator as the momentum approaches the boundary of the first Brillouin zone. This in turn indicates the weaker suppression of the high energy modes compared to the continuum regularization. Does this mean that the UV scaling laws are different in the continuum than on the lattice? We shall find an affirmative answer to this question but this result does not contradict the universality.

Some words of caution are in order at this point. One would object the interpretation of the modification of the cutoff as the introduction of new irrelevant coupling constants by recalling that the theory ceases to be renormalizable in the presence of the non-renormalizable (irrelevant) couplings. The resolution of the apparent paradox is based on the difference between the ways the renormalization group is used in statistical and high energy physics. We are interested in the dynamics close to the cutoff in statistical physics and this is respected by the employment of the blocking which keeps the *complete* dynamics unchanged below the actual cutoff. The price of this precision is the appearance of the infinitely many irrelevant coupling constants in the action. We seek the dynamics at finite, fixed scales in high energy physics. Since the cutoff is sent to the infinity this boils down the problem of keeping the physics cutoff independent *far from the cutoff* only. The obvious gain of such an ease of the conditions is the freedom from the adjustment of the non-renormalizable parameters. Thus one can remove the cutoff when the non-renormalizable parameters are present in the action without any problem[5] as long as the renormalization conditions are imposed far from the cutoff.

The lattice regularization of the scalar model can be described by using the momentum space as the introduction of the non-renormalizable higher order derivative terms:

$$(\partial_\mu\phi)^2\to\frac{4}{a^2}\left(\sin\frac{a\partial_\mu}{2i}\phi\right)^2=\left(\sum_{l=0}^{\infty}\frac{1}{(2l+1)!}\left(\frac{a}{2}\right)^{2l}\partial^{2l+1}\phi\right)^2. \tag{44}$$

The cutoff dependence of the non-renormalizable coupling constants follows a tree-level relation arising from the Taylor expansion of the sine function. This is sufficient to establish convergent physics at finite scales when $a\to 0$ [14], a claim to be verified in this section numerically by means of the implementation of the Wegner-Houghton scheme on the lattice. But this convergence cannot rule out a modification of the scaling laws in the asymptotical UV regime. In fact, we shall find a new scaling regime between the region where the usual universal UV scaling is observed and the UV fixed point. The only effect the different adjustments of the non-renormalized coupling constants may leave on the finite scale physics can be comprised in an overall scale factor.

It is rather straightforward to repeat the steps leading to Eq. (11) on the lattice. It is shown in the Appendix that the only change required is the modification of the ‘‘solid

[5]Ignoring again the possibility of the triviality, the appearance of an UV Landau pole.

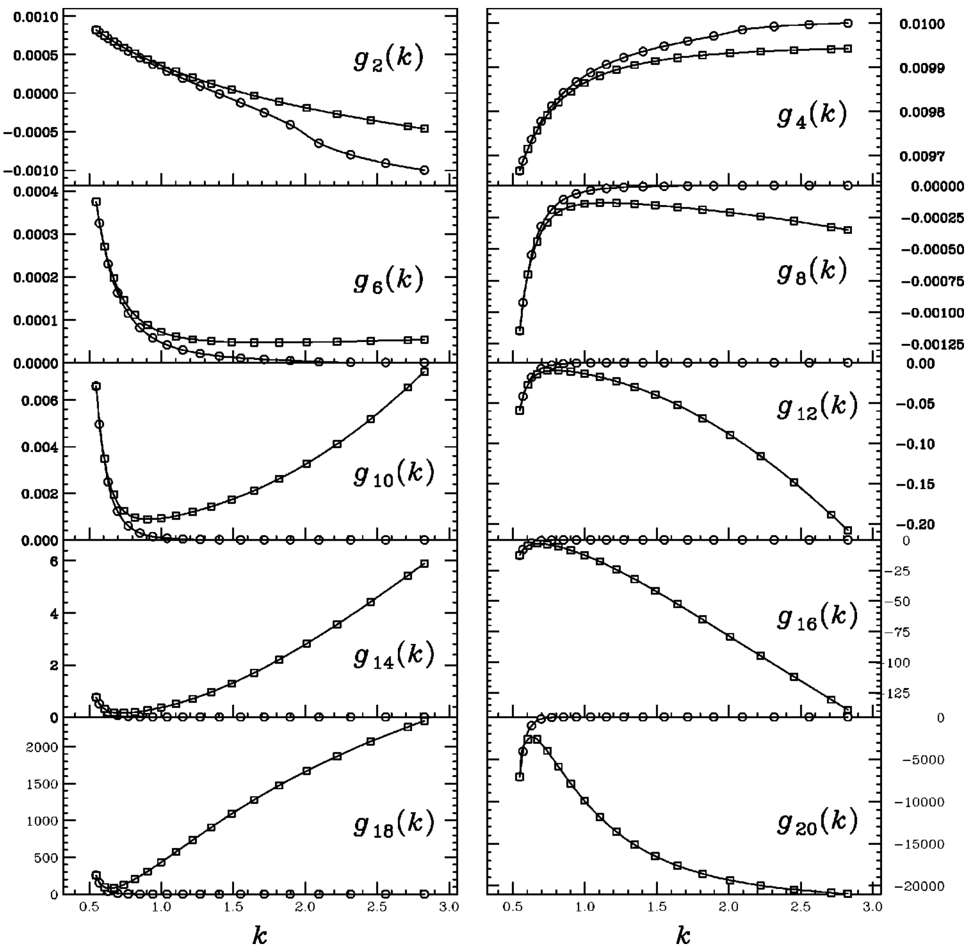

FIG. 8. Matching of lattice and continuum regularizations; we start with lattice field theory at the UV, go down to the IR (circles) and then we come back to the UV with the continuum regularization (squares); see text.

angle'' factor, $\Omega(k)$: the lattice evolution equation (A25) is obtained from Eq. (11) by replacing Eq. (12) by Eq. (A22). One recovers the continuum solid angle for $d=2,\Omega_2=2\pi$, in Eq. (A22) as $k\rightarrow 0$, thus the WH equations agree in the IR limit. In fact, one sees numerically that the behavior of the evolution of the different coupling constants in the lattice RG is qualitatively the same as in the continuum case. But the question we are interested in is the relation between the regularizations in the UV, where the coupling constants are introduced, when the physics is the same at finite scales. It is shown in the Appendix that there is a natural relation between the cutoffs, $\Lambda^2=8/a^2$, which matches the finite scale physics. We shall follow the renormalization group flow in terms of the coupling constants whose dimension is removed by the initial value of the cutoff,

$$g_n \rightarrow g_n/\Lambda^2, \tag{45}$$

in order to avoid the singularities at $k=0$.

Let us consider $\lambda\phi^4$ lattice theory which can be studied either numerically or analytically and whose properties can be matched to those of the continuum theory by the adjustment of g_2 and g_4. But the situation is more involved in two dimensions. The reason is again that there are infinitely many renormalizable coupling constants and one cannot match the finite scale physics by adjusting g_2 and g_4. This is demonstrated in Fig. 8 where the lattice model with the initial conditions $m^2(8)=g_2(8)=-0.001$, $g_4(8)=0.01$, and $g_n(8)=0$, $n>4$ [where we have already made the rescaling Eq. (45)] was evolved in the infrared direction. As the system reached $k=k_{\rm end}$ the continuum WH equation was used to increase the cutoff. The result is a ''perfect matching'' of the models in the UV which gives the same low energy physics in the IR. As we can see in Fig. 8, the lattice $\lambda\phi^4$ model in two dimensions is *not* the continuum $\lambda\phi^4$ theory. It contains contributions of infinitely many renormalizable other coupling constants. Of course, numerically we had to truncate the equations at a certain coupling [here, at $\mathcal{O}(\Phi^{22})$] but we checked that these ''truncation effects'' hardly influence the values of the low order coupling constants.

We have taken for the parameter $k^2_{\rm end}$ the value $k^2=0.3$ in Fig. 8, while the crossover is at $k^2_{\rm cr}\sim 0.01$. We had to use $k^2_{\rm end}>k^2_{\rm cr}$, because the high order couplings have very large values at the crossover which requires very fine discretization in the numerical resolution of our differential equations to ensure that the way back to the UV is done accurately. However, $k^2_{\rm end}$ should also be sufficiently small that the flow be universal there, in other words to make sure that the irrelevant lattice contributions are suppressed for $k^2<k^2_{\rm cr}$.

The choice $k^2_{\rm end}>0$ introduces an uncertainty in the matching. To assess it we repeated the ''go-return'' evolution described above and checked the discretization errors for $k^2_{\rm end}=0.8$, 0.5 and 0.3. After then we took the appropriate

TABLE I. Relative differences between the continuum and lattice coupling constants at $k=0$, Eq. (46), after having matched the couplings at the UV using the parameter $k_{\rm end}^2$.

	$k_{\rm end}^2=0.8$	$k_{\rm end}^2=0.5$	$k_{\rm end}^2=0.3$
Δg_2	0.0166	0.0096	0.0052
Δg_4	0.0028	0.0011	0.0002
Δg_6	0.0188	0.0116	0.0069
Δg_8	0.0199	0.0135	0.0092
Δg_{10}	0.1793	0.0992	0.0494
Δg_{12}	0.0680	0.0424	0.0259
Δg_{14}	0.0522	0.0340	0.0221
Δg_{16}	0.9300	0.5558	0.3226
Δg_{18}	0.2116	0.1261	0.0731
Δg_{20}	0.0584	0.0389	0.0259

bare parameters at the UV end points in both regularizations and followed the evolutions down to $k=0$. The relative difference,

$$\Delta g_n \equiv \frac{|g_n^{\rm cont}(0)-g_n^{\rm latt}(0)|}{|g_n^{\rm latt}(0)|} \tag{46}$$

is shown in Table I for the different coupling constants at $k=0$. The smallness of the deviation assures that the IR behavior has practically been obtained with $k_{\rm end}^2=0.3$, and one can trust the conclusions, the approach of the flow to a universal curve, extracted from Fig. 8.

The first two plots in Fig. 8 show that the mass and the quartic coupling constant run parallel in the SUV region of the lattice regularization (see Appendix) and in the continuum. There is no convergence between the two regularizations in this unusual scaling regime, anticipated above. The approach to the universal curve starts for $k^2<4$, below the SUV regime only. The fact that the renormalization group flow converges to the universal one in the 2^{-d}-th part of the Brillouin zone only sets an unexpected high lower limit on the lattice size when the continuum limit is sought in numerical simulations. The higher order vertices seem to converge to the universal curve from the very beginning but the difference between the two regularizations is surprisingly large. The universal trajectory of the ϕ^4 model is reached later by the higher order vertices. This effect appears to be a counterpart of the non-perturbative features seen in Figs. 6,7 and introduces a large uncertainty in identifying two-dimensional models in different regularizations.

VII. CONCLUSIONS

The renormalization group flow of scalar models with polynomial interaction is considered in the first part of this paper by solving the Wegner-Houghton equation numerically in the local potential approximation for $d=2$, 3 and 4.

The numerical results showed in this paper suggest that the length of the UV scaling regime which is needed to generate non-perturbative dynamics in the infrared shrinks to zero as the number of the asymptotically free coupling constants tends to infinity. In other words, the Λ-parameter tends to the cutoff as the lower critical dimension is approached, $d\to 2$. Such a behavior limits considerably the values of the coupling constants for a perturbative system in dimension 3 and renders *all* two- and lower dimensional field theories with polynomial couplings non-perturbative. This makes the understanding of the noncritical low dimensional condensed matter systems more involved. The one-dimensional models belong to first quantized quantum mechanics and our result is a manifestation of the failure of the convergence of the perturbation expansion for an anharmonic oscillator.

Such a conclusion does not invalidate the well known results for two-dimensional systems, such as the applicability of the Bethe ansatz, bosonization and the availability of certain exact information for models with conformal invariance. Instead, it makes the asymptotic state structure and the relation between the the dressed particles and the states created by the application of the field operator from the vacuum more involved.

We found an interesting analogy between the infrared Landau pole of the confining four-dimensional Yang-Mills theories and the low dimensional scalar models which opens the possibility of an unexpected, nontrivial structure in the asymptotic states in the low dimensional scalar models. Viewed with interest in particle physics our conclusion suggests that one can avoid the IR Landau pole by following the evolution of the non-renormalizable operators.

How to find the non-renormalizable operators whose presence stabilizes the theories at low energies? It is well known that massive Lagrangians generate trivial infrared scaling laws, i.e. the Gaussian mass term is the only relevant operator in the infrared scaling regime. This is because the fluctuations are exponentially suppressed beyond the correlation length so the evolution of the coupling constants slows down at the infrared side of the crossover. The theories with dimensional transmutation, i.e. dynamically generated scale parameter or infrared instability only can support non-perturbative dynamics in the IR scaling regime. Thus the operators sought should be relevant in the IR regime, their growth being fed by IR or collinear divergences. There are few known cases only where the low modes are controlled by non-renormalizable operators. These include the four fermion contact term in solids inducing the BCS transition [15], the higher order derivative terms in the action which generate inhomogeneous vacuua [16], the common element being the onset of a Bose-Einstein condensation [9].

In the second part of the paper the infinitesimal renormalization group scheme is generalized for lattice regularization. The matching of the continuum and the lattice regularizations is carried out numerically and the approach of the universal renormalization group flow is demonstrated for the two-dimensional ϕ^4 lattice model. This result suggests that the naive argument for the universality, which is based on the linearization of the blocking relations remains valid in the presence of infinitely many relevant operators. Other potential troublemakers, the infinitely many higher order derivatives contained in the lattice kinetic energy do generate a new, ''super UV'' scaling regime but universality is restored at the IR end of the usual UV scaling regime. Another use of the lattice regulated version of the Wegner-Haughton equation is the estimate of the finite size effects in a nonperturbative manner. This provides a useful check of the thermodynamic limit of the numerical results obtained in general on small lattices.

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APPENDIX

The details of the derivation of the Wegner-Houghton equation in the lattice regularization are given in this Appendix. Let us consider the scalar field theory regularized on a lattice of lattice spacing $a=1$. We want to derive a WH equation similar to Eq. (11). We integrate over spherical shells in the momentum space for the continuum regularization, because the propagator has spherical symmetry. This is no longer the case on the lattice, where we have

$$\Box=\sum_{\mu=1}^{d}\hat{p}_\mu^2, \quad \hat{p}_\mu=2\sin\frac{p_\mu}{2}. \tag{A1}$$

Let us see the surfaces of equal value of the lattice propagator in two dimensions by performing the following change of variables:

$$(p_x,p_y)\rightarrow(p,\theta)$$

$$4\sin^2\frac{p_x}{2}+4\sin^2\frac{p_y}{2}=p^2 \tag{A2}$$

$$\tan\theta=\frac{\sin(p_y/2)}{\sin(p_x/2)}.$$

We can see in Fig. 9 the form of the curves of constant propagator for several values of p^2. p can be identified as the ''momentum scale'' that runs from the cutoff at $p^2=\Lambda^2=8$ to the IR $p^2=0$. It is also clear in that figure that the value $p^2=4$ separates two regimes, still in the ultraviolet region, that we could call super-UV (SUV), for $8>p^2>4$, and normal-UV regimes. For $p^2\sim0$, the lines are spheres, and our change of variables (A2) reduces to the usual relation between Cartesian and polar coordinates.

The absolute value of the Jacobian of the transformation (A2) is found to be

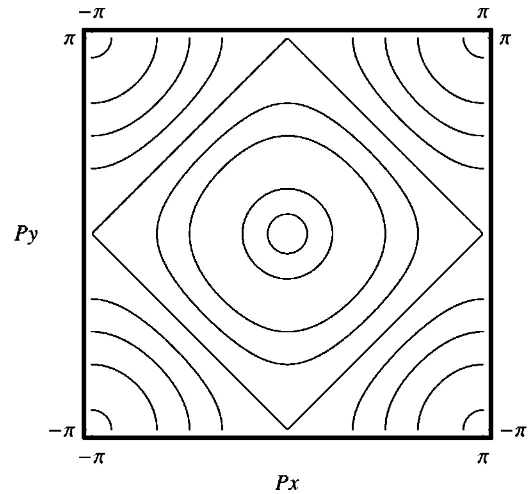

FIG. 9. Lines of equal value of the lattice propagator. To the inside, the lines drawn correspond to $p^2=7.9,7,6,5,4,3,2,0.5,0.1$.

$$J=\frac{J_{\rm p}}{\sqrt{\left(1-\frac{p^2}{4}\cos^2\theta\right)\left(1-\frac{p^2}{4}\sin^2\theta\right)}}, \tag{A3}$$

where $J_{\rm p}$ is the usual Jacobian for the polar change of variables, $J_{\rm p}=p$. The transformation (A2) can be easily generalized to three and four dimensions; however, we can only treat analytically the integral that appears in the derivation of the WH equation in the $d=2$ case.

To derive the equivalent of the Wegner-Houghton equation (11) in the bidimensional lattice regularization we will start from Eq. (8), and calculate the trace by integrating in momentum space over a shell $k-\delta k<p<k$, where p is the parameter we have introduced in Eq. (A2),

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}\log[\Box+U_k'']&=\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\log\left[4\sin^2\frac{p_x}{2}+4\sin^2\frac{p_y}{2}+U_k''\right]\\&=\frac{1}{2(2\pi)^2}\int d\theta\int_{k-\delta k}^{k}dp\,J\log[p^2+U_k'']\\&\approx\frac{\delta k}{2(2\pi)^2}k\log[k^2+U_k'']\,\Omega(k),\end{aligned} \tag{A4}$$

with

$$\Omega(k)=\int d\theta\frac{8}{\sqrt{64-16k^2+k^4\sin^2 2\theta}}. \tag{A5}$$

We have to distinguish two different regimes in making the integration: (i) $\underline{k^2<4}$. In this region the range of values for θ is $(0,2\pi)$,

$$\Omega(k)=\int_0^{2\pi}\mathrm{d}\theta\frac{8}{\sqrt{64-16k^2+k^4\sin^2 2\theta}}$$
$$=\sum_{i=1}^{4}\Omega_i(k), \tag{A6}$$

where we have split the interval $(0,2\pi)$ into four intervals $(0,\pi/2)$, $(\pi/2,\pi)$, etc. Let us consider $\Omega_1(k)$. With the change of variable $x=\tan\theta$ and the notation

$$\overline{k^2}=4-k^2, \tag{A7}$$

this integral can be brought into the form

$$\Omega_1(\overline{k})=\frac{1}{\overline{k}}\int_0^\infty\frac{2\,\mathrm{d}x}{\sqrt{(x^2+b^2)(x^2+b^{-2})}}, \tag{A8}$$

where $b^2=4/\overline{k^2}$. The integral in Eq. (A8) is related to the elliptic integral of the first kind [17] $F(\phi,t)$:

$$\Omega_1(k)=F\left[\frac{\pi}{2},\frac{k}{4}\sqrt{8-k^2}\right],\quad F(\varphi,t)=\int_0^\varphi\frac{\mathrm{d}\alpha}{\sqrt{1-t^2\sin^2\alpha}}. \tag{A9}$$

The result is the same for the other integrals Ω_i, $i=2,3,4$. But $F(\pi/2,t)$ is a complete elliptic integral, which can be expressed in terms of the hypergeometric function [17,18]

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!} \tag{A10}$$

as

$$\Omega(k)=2\pi F\left(\frac{1}{2},\frac{1}{2};1;\frac{(8-k^2)k^2}{16}\right). \tag{A11}$$

(ii) $4<k^2<8$. This is the SUV region. We split again the integral into four integrations in the corresponding quadrants. By using the same change of variables as above, we have to calculate

$$\Omega_i(\overline{k})=\int\frac{4\,\mathrm{d}x}{\sqrt{x^2(\overline{k^4}+16)-4\overline{k^2}(1+x^4)}}, \tag{A12}$$

where

$$\overline{k^2}=k^2-4. \tag{A13}$$

Special care is needed at the limits of integration (recall Fig. 9). It can be seen that the four integrals can be put together in the form

$$\Omega(\overline{k})=4\int_{\overline{k}/2}^{2/\overline{k}}\frac{4\,\mathrm{d}x}{\sqrt{x^2(\overline{k^4}+16)-4\overline{k^2}(1+x^4)}}$$
$$=\frac{8}{\overline{k}}\int_{\overline{k}/2}^{2/\overline{k}}\frac{\mathrm{d}x}{\sqrt{(x-\overline{k}/2)(2/\overline{k}-x)(x+\overline{k}/2)(x+2/\overline{k})}}. \tag{A14}$$

This integral is related again [17] to an elliptic integral and an hypergeometric function:

$$\Omega(k)=\frac{32}{k^2}F\left[\frac{\pi}{2},\frac{8}{k^2}-1\right]=\frac{16\pi}{k^2}F\left(\frac{1}{2},\frac{1}{2};1,\frac{(8-k^2)^2}{k^4}\right). \tag{A15}$$

We would like to have a common expression for $\Omega(k)$ for both cases (i) and (ii). From Eqs. (A11) and (A15), we find in fact that the expressions differ in a factor 2 for $k^2=4$. The reason is that actually our integral is divergent at this point. From Eq. (A5) we see that the divergent integral is

$$\Omega(k^2=4)=\int_0^{2\pi}\mathrm{d}\theta\frac{2}{|\sin 2\theta|} \tag{A16}$$

[in fact, the hypergeometric function (A10) converges in general only in the unit circle $|z|<1$ [17]]. We will see, however, that this divergence is integrable during the RG evolution from $k^2=8$ to $k^2=0$, and therefore it has no physical significance. In order to have a consistent, single expression for the cases $8<k^2<4$ and $4<k^2<0$, we will make use of the following property of the hypergeometric functions [17]:

$$F\left(\frac{1}{2},\frac{1}{2};1,z^2\right)=\frac{1}{1+z}F\left(\frac{1}{2},\frac{1}{2};1,\frac{4z}{(1+z^2)^2}\right),\quad 0\leqslant z<1. \tag{A17}$$

Let us consider the expression (A11) which is valid for $k^2<4$. Using the property (A17) one finds

$$\Omega(k)=2\pi F\left(\frac{1}{2},\frac{1}{2};1,\frac{4z}{(1+z^2)^2}\right)$$
$$=2\pi(1+z)F\left(\frac{1}{2},\frac{1}{2};1;z^2\right), \tag{A18}$$

where we have set

$$\frac{4z}{(1+z^2)^2}=\frac{(8-k^2)k^2}{16}. \tag{A19}$$

This equation has two solutions for z as a function of k:

$$z=\frac{k^2}{8-k^2},\quad\frac{8-k^2}{k^2}, \tag{A20}$$

but only the first one is admissible for Eq. (A18), because it gives $z<1$ for $k^2<4$, while the second solution gives a value greater than 1 in this region. Now, if we define $\tilde{k}=8-k^2$, then we have $z=(8-\tilde{k}^2)/\tilde{k}^2$, and Eq. (A18) becomes

$$\Omega(\tilde{k})=2\pi\frac{8}{\tilde{k}^2}F\left(\frac{1}{2},\frac{1}{2};1;z^2\right). \tag{A21}$$

Comparing this last expression with the result (A15) for the case $4<k^2<8$, we can finally write

$$\Omega(k)=\frac{16\pi}{\overline{k^2}}F\left(\frac{1}{2},\frac{1}{2};1;z^2\right),\ \text{with}\ z=\frac{8-\overline{k^2}}{\overline{k^2}}$$

$$\text{and}\quad \begin{cases}\overline{k^2}=k^2 & \text{if}\ k^2=8\ldots4,\\ \overline{k^2}=8-k^2 & \text{if}\ k^2=4\ldots0.\end{cases} \tag{A22}$$

The hypergeometric function $F(\frac{1}{2},\frac{1}{2};1;z^2)$ can be computed directly from its definition (A10). One can obtain a high precision in the evaluation of the series with a reasonable number of terms (say, around 50) when z is not very close to 1, say, for $0.7>z>0$. For $1>z>0.7$ we have used the following alternative formula [18]:

$$\begin{aligned}F(\alpha,\beta;\alpha+\beta;z)=&\frac{\Gamma(\alpha+\beta)}{(\Gamma(\alpha)\Gamma(\beta))^2}\\&\times\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{(n!)^2}\\&\times[2\psi(n+1)-\psi(\alpha+n)-\psi(\beta+n)\\&-\log(1-z)](1-z)^n,\\&(|\arg(1-z)|<\pi,|1-z|<1),\end{aligned} \tag{A23}$$

where [18]

$$\psi(z)=\frac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z}, \tag{A24}$$

which gives a better convergence for the function $F(\frac{1}{2},\frac{1}{2};1;z^2)$ near $z=1$ because it is a series in the variable $(1-z^2)$.

In conclusion, our generalization of the WH equation (11) for a lattice regularization in two dimensions is

$$k\frac{\partial}{\partial k}U_k(\Phi)=-\frac{\Omega(k)k^2}{2(2\pi)^2}\log[k^2+U_k''(\Phi)], \tag{A25}$$

where k is the parameter p of Eq. (A2), and $\Omega(k)$ is given by Eq. (A22).

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