BF models, duality, and bosonization on higher genus surfaces

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The generating functional of two dimensional BF field theories coupled to fermionic fields and conserved currents is computed in the general case when the base manifold is a genus g compact Riemann surface. The Lagrangian density $L=dB\wedge A$ is written in terms of a globally defined 1-form A and a multivalued scalar field B. Consistency conditions on the periods of dB have to be imposed. It is shown that there is a nontrivial dependence of the generating functional on the topological restrictions imposed to B. In particular if the periods of the B field are constrained to take values $4\pi n$, with n any integer, then the partition function is independent of the chosen spin structure and may be written as a sum over all the spin structures associated with the fermions even when one started with a fixed spin structure. These results are then applied to the functional bosonization of fermionic fields on higher genus surfaces. A bosonized form of the partition function which takes care of the chosen spin structure is obtained.

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I. INTRODUCTION

In this paper we compute the generating functional of BF [1] topological systems coupled to fermions on a two dimensional compact manifold of arbitrary genus. We then apply the result to discuss the bosonization [3] of the fermionic fields. Our computation which is closely related to the work presented in Refs. [4,5] has two different motivations. First, it has been observed [6] that the allowed world hypersurfaces described by classical sources (p-branes) coupled to a BF theory are subject to restrictions of a topological nature. One is then led to ask the question of how this effect is translated to the quantum theory. The second aspect which motivates this work concerns the relation between topological models and duality transformations. For a large class of systems, duality transformations have been devised along the lines of the T-duality transformation in the sigma models [7]. The method consists essentially in a two step elimination of one field in terms of its dual variable. First one introduces an auxiliary gauge field constrained at the beginning to have zero curvature. This allows us to decouple the original variable from the currents and then one may perform the remaining integration in the quadratic approximation. At this point the connection with the BF [1] theory appears, since to impose the zero curvature condition into the functional integral one may introduce the partition function of a BF topological model. When the duality transformation is applied to the generating functional of free fermionic fields on genus zero manifolds [8], it leads to the bosonized [3] representation of the theory. In the operatorial approach bosonization in two [3] and three dimensions [9,10] has been related to the construction of dual soliton operators [11] in bosonic theories and this gives an additional meaning to the duality transformation discussed in the papers of Ref. [8]. At the intermediate step after introducing the gauge field one is dealing with a BF theory coupled to fermions which is the subject of this paper. This point of view has been also extended to higher dimensions [12,13].

Over topologically trivial manifolds, the procedure described above allows in some cases to determine exact equivalences between fields theories. When the base manifold has genus g one has to take care of the global definition of the geometrical objects appearing in the formulation [14,15]. The global aspects introduced by the auxiliary fields in the path integral have been explicitly tested for example in the purely bosonic self-dual vectorial model in $3-D$. This model is known to be locally equivalent to the topologically massive model [16] and in fact can be viewed as a gauge fixed version of it [17]. Nevertheless it has been shown that in topologically nontrivial manifolds this equivalence has to be reinterpreted [15,18,19] since the partition function of the topologically massive model has an additional factor of topological origin. When matter fields are included, the coupling with the topological field theory may be related to self-interaction terms for the fermions [20].

In the specific case of the fermionic models there are other reasons to explore the consequences of defining the system on higher genus surfaces. Even in genus zero surfaces, when coupled to gauge fields with nontrivial topological properties, the fermions show dynamical effects. The most notorious of these is the nonvanishing of fermion condensates [21–23] due to the contributions of the instantons associated in four dimensions to the resolution of the U(1) problem [24]. The vanishing of such condensates in the topological trivial case is enforced by gauge symmetry. The nonvanishing result in the most general situation may be traced in the functional approach to an explicit contribution of the zero modes of the fermionic fields [21]. In higher genus surfaces one expects a more rich structure in the gauge field sector, but also further complications are introduced in the duality transformation when gauge fields of nonvanishing topological index have to be considered. For this reason in this article we exclude this possibility.

This paper is organized as follows. In Sec. II we review

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some useful concepts and notation and discuss the result [4,5] of the computation of the fermionic determinant in genus g manifolds. In Sec. III we compute the generating functional of a particular BF topological theory coupled to fermions. This model which as we will see later appears naturally in the bosonization of fermions in higher genus surfaces is described in terms of a 1-form A globally defined and a multivalued B field. Since dB should remain univalued one has to impose restrictions over the periods of dB. When these periods are chosen to be integral multiples of 4π the partition function is shown to be a sum over all spin structures even if one starts with a fixed spin structure. This is an interesting result which in particular implies that the partition function is independent of the spin structure originally chosen. In Sec. IV we discuss the bosonization [3] of fermionic fields on higher genus Riemann surfaces. Here again, bosonization may be understood as a duality transformation [8] between the fermionic current and the Hodge dual of the field intensity tensor of a vector field. A careful treatment of the global aspects in the formulation leads naturally to a bosonized effective action in terms of a multivalued 0-form.

II. THE FERMIONIC DETERMINANT ON HIGHER GENUS COMPACT RIEMANN SURFACES

We consider BF models coupled to fermionic fields over higher genus Riemann surfaces. In order to compute its partition function one needs the explicit formula for the fermionic determinant in the case of zero curvature gauge potentials A on trivial U(1) line bundles. This determinant was computed in Refs. [4,5]. To express the result, let us introduce some notation concerning the properties of the manifold and the fields. We take a_i and b_j to be a basis of homology of closed curves over Σ, a compact Riemann surface of genus g. The set of curves a_i and b_j will be denoted by $\mathcal{C}^I$. If one deform continuously the fermionic field along the curves of the basis, after returning to the original point the fermionic field may change sign or not. A spin structure over Σ is determined by taking one of these possibilities for each of the curves of the basis. The gauge potentials A on a trivial U(1) bundle are characterized by the vanishing of the Chern class

$$\int_\Sigma F(A)=\int_\Sigma dA=0. \tag{1}$$

The index of the corresponding Dirac operator is then zero and consequently there are no zero modes in the fermionic sector.

The potential may be decomposed into its exact, co-exact and harmonic parts:

$$A=ds+{}^*dp+A_h. \tag{2}$$

The harmonic part of the field is expressed in terms of a base of real harmonic forms α_i and β_i, $i,j=1,\dots,g$ as follows:

$$A_h=2\pi\sum_i^g\ (u_i\alpha_i-v_i\beta_i). \tag{3}$$

The real harmonic basis α_i and β_j is constructed from two normalized holomorphic basis ω_j and $\hat{\omega}_j$, $j=1,\dots,g$

$$\int_{a_i}\omega_j=\delta_{ij},\qquad \int_{a_i}\hat{\omega}_j=\hat{\Omega}_{ij},$$

$$\int_{b_i}\omega_j=\Omega_{ij},\qquad \int_{b_i}\hat{\omega}_j=\delta_{ij}, \tag{4}$$

where Ω is the period matrix. In terms of ω_j and $\hat{\omega}_j$, α_i and β_j are given by

$$\beta_j=\frac{1}{2i}(\omega_k-\overline{\omega_k})[\mathrm{Im}\,\Omega]^{-1}_{kj},$$

$$\alpha_i=\frac{1}{2i}(\hat{\omega}_k-\overline{\hat{\omega}_k})[\mathrm{Im}\,\hat{\Omega}]^{-1}_{ki}. \tag{5}$$

The imaginary part of the period matrix $\mathrm{Im}\,\Omega$ is always an invertible matrix. Let us now consider a fermionic field defined over a genus g compact Riemann surface with a definite but arbitrary spin structure. The spin structure is fixed by specifying two g dimensional vectors ϵ_i and κ_j with components 0 or $\frac{1}{2}$ so that the periodicities of the fermions about the cycles a_i and b_j are respectively $\exp(2\pi i\epsilon_i)$ and $\exp(-2\pi i\kappa_j)$. The partition function which defines the fermionic determinant is

$$\begin{aligned}\hat{Z}_f[A,\epsilon,\kappa]&=\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(\int_\Sigma d^2x\sqrt{g}\,\overline{\psi}(-i\not{D}+\not{A})\psi\right)\\&=\det[-i\not{D}+\not{A}],\end{aligned} \tag{6}$$

where we take $\not{D}$ to be the covariant derivative for the fermions.

The fermionic determinant in this situation may be obtained from the results in [4,5] and is given by

$$\begin{aligned}\hat{Z}_f[A,\epsilon,\kappa]=\exp\Bigg(&-\frac{1}{2\pi}\int_\Sigma F(A)\frac{1}{\Delta_0}{}^*F(A)\\&\times\left[\frac{\det\mathrm{Im}\,\Omega\ \mathrm{Vol}(\Sigma)}{\det'\Delta_0}\right]^{1/2}\Bigg)\left|\theta\begin{bmatrix}u+\epsilon\\v+\kappa\end{bmatrix}(0|\Omega)\right|^2.\end{aligned} \tag{7}$$

Here Δ_0 is the Laplacian operator acting on 0-forms and $\mathrm{Vol}(\Sigma)$ is the area of the Riemann surface. The third factor is a θ function given by

$$\theta\begin{bmatrix}u\\v\end{bmatrix}(0|\Omega)=\sum_{n\in\mathcal{Z}^g}\exp[i\pi(n+u)\Omega(n+u)+i2\pi(n+u)v]. \tag{8}$$

We note that in Eq. (7) the first factor depends only on the coexact component of the gauge field. This contribution corresponds to the result for genus zero surfaces [25] which is usually written in the form

$$\frac{\det(-i\partial\!\!\!/+A\!\!\!/)}{\det(-i\partial\!\!\!/)}=\exp\left[-\frac{1}{2\pi}\int d^2x\sqrt{g}A_\mu\left(\delta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu\right]. \tag{9}$$

The other two factors in Eq. (7) give the Dirac determinant for a purely harmonic potential A_h of the form (3). The result (7) has been used to investigate the Schwinger model in higher genus surfaces [26].

Finally by summing over all spin structures, we may also define

$$\hat{Z}_f[A]=\sum_{\epsilon,\kappa}\hat{Z}_f[A,\epsilon,\kappa], \tag{10}$$

which will play a role in what follows.

III. TWO DIMENSIONAL *BF* THEORIES COUPLED TO FERMIONS

In this section we compute the generating functional for a particular BF system coupled to fermions over a genus g Riemann surface. The action functional of a BF theory is written in terms of a connection A and a field B which may be interpreted as a Lagrange multiplier which enforces the A field to have zero curvature [1]. In its usual form it is given by

$$S_{BF}=\int_\Sigma dA\wedge B. \tag{11}$$

Here the 0-form B and dA are defined globally on the manifold Σ. The connection A may be allowed to have transitions over Σ. The computation of the partition function of this system was discussed in [2]. The off-shell BRST charge was computed in [27]. We will consider a modification of this system which appears naturally in the context of bosonization. We consider the action

$$S_{BF}^{\text{mod}}=\int_\Sigma dB\wedge A, \tag{12}$$

which may be different from the action above, for trivial bundles, only over higher genus surfaces. The one forms A and dB have to be globally defined but B may be multivalued. Due to the nontrivial topological structure of the manifold, one may distinguish three cases in the definition of the generating functional. One may consider the following conditions on the periods of dB:

$$\oint_{C^I}dB=0, \tag{13}$$

$$\oint_{C^I}dB=2\pi m^I, \tag{14}$$

$$\oint_{C^I}dB=4\pi m^I. \tag{15}$$

The first case is the usual BF model. In the second and third cases we consider the summation on all the values of m^I in the functional integral (16). Each of the choices defines a different model.

The generating functional of these systems coupled to conserved currents j and J is given in all the cases by

$$Z[j,J,\epsilon,\kappa]=\sum_{m^I}\int\mathcal{D}C\mathcal{D}A\mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_{\text{eff}}}, \tag{16}$$

$$S_{\text{eff}}=\int d^2x\sqrt{g}[\bar{\psi}(i\not{D}-A\!\!\!/-j\!\!\!/)\psi+L_g] + \int_\Sigma\left(\frac{i}{2\pi}dB\wedge A-{}^*J\wedge A\right) \tag{17}$$

where L_g includes the gauge fixing term and the contributions of the auxiliary fields (ghosts fields and Lagrange multipliers) and $\mathcal{D}C$ stands for the integration measure in those fields. The sum in m^I is included to stress the fact that we are summing over the B field configurations which satisfy either Eqs. (13), (14), or (15). The spin connection is fixed and identified by the g dimensional vectors ϵ_i and κ_j.

The functional integration on the Lagrange multiplier B of course provides the factor $\delta(F(A))$ in the measure of the generating functional but as we will see presently, the additional summation over the periods gives rise to a factor which constrain also the periods of A. Let us see how this works. Suppose for example that we compute the generating functional (16) summing over the B field configurations which satisfies Eq. (15). Given two different configurations of B, say B_1 and B_2, satisfying this condition we have

$$B_2-B_1=b, \tag{18}$$

where b is univalued over Σ. In general we may then write

$$B=B_{m^I}+b, \tag{19}$$

with B_{m^I} a specific configuration satisfying Eq. (15) with a set of values m^I. The functional integration on the multivalued B field has been expressed as an integration on the univalued function b and a sum over all possible choices m^I. Consider now the BF action in the sector defined by one of such choices. We have

$$\int_\Sigma dB\wedge A=\int_\Sigma d(B_{m^I}A)+\int_\Sigma(-B_{m^I})\wedge dA+\int_\Sigma(-b)\wedge dA. \tag{20}$$

The generating functional becomes

$$Z[j,J,\epsilon,\kappa]=\sum_{m^I}\int\mathcal{D}A\mathcal{D}b\mathcal{D}C\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_{\text{eff}}} \tag{21}$$

$$\mathcal{S}_{\rm eff}=\int_{\Sigma}d^2x\sqrt{g}[\bar{\psi}(i\not{D}-\not{j}-\not{A})\psi+L_g]+\frac{i}{2\pi}\int_{\Sigma}[d(B_{m^I}A)-(B_{m^I}+b)\wedge dA]-\int_{\Sigma}{}^*J\wedge A.$$

At this point, one recovers the factor $\delta(dA)$ in the measure of the generating functional by performing the functional integral in b and in the ghost fields introduced to guarantee the Becchi-Rouet-Stora-Tyutin (BRST) invariance of the effective action. In particular this makes the second term in Eq. (20) to vanish and to disappear also from Eq. (21).

To evaluate the remaining functional integral, consider a triangulation of Σ in terms of elementary domains U_i, $i\in[1,\ldots,N]$. Since Σ is compact the triangulation exists and the covering is provided by a finite number of elementary domains. Let A^i and $B^i_{m^I}$ be the restrictions of the fields to the domain U_i. Then, in the functional space projected by $\delta(dA)$, we have

$$\left[\int_{\Sigma}dB\wedge A\right]_{|dA=0}=\int_{\Sigma}d(B_{m^I}A)=\sum_{i=1}^{N}\int_{U_i}d(B^i_{m^I}A^i)=\sum_{U_i\cap U_j\neq\emptyset}\int_{U_i\cap U_j}(B^i_{m^I}-B^j_{m^I})A=\sum_{U_i\cap U_j\neq\emptyset}4\pi m^{(ij)}\int_{U_i\cap U_j}A \qquad (22)$$

where $m^{(ij)}$ are integers. Using again that the connection is flat we finally get

$$e^{i/2\pi\int_{\Sigma}dB\wedge A}=e^{\,i\Sigma_I\,2m^I\oint_{C^I}A}. \qquad (23)$$

Here one recognizes the coefficients of the Fourier expansion of a delta function with period π. Upon summing over all m^I the total contribution is

$$\delta(F(A))\sum_I\delta\left(\oint_{\mathcal{C}^I}A-\pi n^I\right). \qquad (24)$$

When the B field in Eq. (16) is taken to satisfy Eq. (13) only the factor $\delta(F(A))$ appears. We will not discuss this case furthermore. When the B field is taken to satisfy Eq. (14), the second delta function has period 2π and the factor turns out to be

$$\delta(F(A))\sum_I\delta\left(\oint_{\mathcal{C}^I}A-2\pi n^I\right). \qquad (25)$$

Let us see now how the conditions (24) or (25) enter in the complete evaluation of Eq. (16). Using the decomposition (2) for the A field, the factor $\delta(dA)$ in the measure of Eq. (16) allows the integration of the coexact part of A and we are left with the task of determining which are the configurations of A_h that contribute. It is now straightforward to show that the delta functions in Eq. (24) [or respectively Eq. (25)] constrain the values of the coefficients in the expansion (3) of A_h to be half-integers (or integers). To continue we use this fact and perform the functional integration in the fermions. Defining u^0 and v^0 to be the coefficients in the expansion of the harmonic part of j,

$$j_h=2\pi\sum_i^g(u_i^0\alpha_i-v_i^0\beta_i), \qquad (26)$$

we obtain

$$Z[j,J,\epsilon,\kappa]=\sum_{u,v}\left[\frac{\det \mathrm{Im}\,\Omega\ \mathrm{Vol}(\Sigma)}{\det'\Delta_0}\right]^{1/2}\left|\theta\begin{bmatrix}u+u^0+\epsilon\\ v+v^0+\kappa\end{bmatrix}(0|\Omega)\right|^2\exp\left[\int_{\Sigma}\left({}^*J\wedge A_h-\frac{1}{2\pi}dj\frac{1}{\Delta_0}{}^*dj\right)\right]. \qquad (27)$$

The sum in Eq. (27) is over the allowed values of u and v which as we already said are all the integers or all the half-integers depending which case we are considering. From here on we have to distinguish between the two cases.

Let us take first the case when the B field satisfies Eq. (14). Then in Eq. (27) we have a sum over the g-tuples with integral entries which we label by m and l. The factor with the theta function in Eq. (27) takes the form

$$\left|\theta\begin{bmatrix}u^0+m+\epsilon\\ v^0+l+\kappa\end{bmatrix}(0|\Omega)\right|^2. \qquad (28)$$

It is straightforward to see from Eq. (8) that this becomes independent of l due to the square norm that we are taking. Moreover Eq. (28) also is independent of m, since in Eq. (8) one may redefine $n+m=n'$ and one still will have summation in all n'. We can then factorize the contribution of the harmonic part of the field to the partition function in the form

$$Z_{2n}[j,J,\epsilon,\kappa]=\hat{Z}_f[j,\epsilon,\kappa]\sum_{m,l}\exp\left(\int_{\Sigma}{}^*J\wedge A_h\right) \qquad (29)$$

with $\hat{Z}_f[j,\epsilon,\kappa]$ given by Eq. (7);

$$\hat{Z}_f[j,\epsilon,\kappa]=\left[\frac{\det \mathrm{Im}\,\Omega\ \mathrm{Vol}(\Sigma)}{\det'\Delta_0}\right]^{1/2}\left|\theta\begin{bmatrix}u^0+\epsilon\\ v^0+\kappa\end{bmatrix}(0|\Omega)\right|^2\times\exp\left(-\frac{1}{2\pi}\int_{\Sigma}dj\frac{1}{\Delta_0}{}^*dj\right). \qquad (30)$$

Note that the external current J only couples to the harmonic part of the vector field. When J is zero we obtain

$$Z_{2n}[j,0,\epsilon,\kappa]=\mathcal{N}\hat{Z}_f[j,\epsilon,\kappa] \tag{31}$$

with $\mathcal{N}$ a constant which measures the volume of the harmonic space. This factor is expected from the original expression (16) since in that case the volume of the zero modes factorizes from the functional integral.

Consider now the situation when Eq. (15) holds. We have instead of Eq. (28) the expression

$$\left|\theta\begin{bmatrix} u^0+\dfrac{m}{2}+\epsilon \\ v^0+\dfrac{l}{2}+\kappa \end{bmatrix}(0|\Omega)\right|^2 \tag{32}$$

where $m/2$ and $l/2$ are the half-integer periods of A. We consider the following decomposition,

$$\frac{m}{2}=m'+\eta,$$

$$\frac{l}{2}=l'+\mu, \tag{33}$$

where m' and l' are integer numbers while η and μ are g-tuples with components 0 or $\frac{1}{2}$. Summation in all m and l is equivalent to summation in all (η,μ) and all (m',l'). The summation in the integers may be handled as before. Then the summation in the half-integers (η,μ) may be reinterpreted as a sum over all spin structures (weighted by a factor which depends on J). When J is zero we have

$$Z_{4n}[j,0,\epsilon,\kappa]=\mathcal{N}\sum_{\epsilon',\kappa'}\hat{Z}[j,\epsilon',\kappa']=\mathcal{N}Z_f[j]. \tag{34}$$

The factor $\mathcal{N}$ here gives the same measure of the space of harmonic 1-forms with integral periods as in Eq. (31). We started with a fixed spin structure, however the final result corresponds to the partition function of spinor fields with summation n in all spin structures. In particular it shows that $Z_{4n}[j,0,\epsilon,\kappa]$ is independent of the spin structure (i.e., of ϵ and κ).

IV. BOSONIZATION IN HIGHER GENUS SURFACES

As an application we use the results of the previous section to discuss the bosonization of fermions over higher genus compact Riemann surfaces. Equations (29) and (34) already establish the relation between the partition function of the fermions and the partition function of the *BF* model. In this section we obtain this result using the constructive approach of [8].

Let us begin with a quick review of the situation in the topologically trivial case. Consider the generating functional of a fermion field coupled to a conserved current j:

$$Z_f[j]=\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\int d^2x\sqrt{g}\,\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/)\psi\right). \tag{35}$$

We suppose here that the current j has a topological index zero. In two dimensions, on a genus zero surface, this fermion determinant is explicitly known [25] and given by Eq. (9). The duality-bosonization transformation allows us to express this result in terms of a bosonic field. To construct this transformation one begins observing that the system has a global U(1) gauge invariance. Then [8,7] one makes a change of variables with the functional form of a local gauge transformation and identify the spurious contributions which appear in the action as coupling terms with a gauge field of zero curvature. The adequate change of variables in this case is

$$\psi(x)\to e^{i\Lambda(x)}\psi(x) \tag{36}$$

where $\Lambda(x)$ is an arbitrary parameter with local dependence on x. The fermionic generating functional turns out to be

$$Z_j[j]=\mathcal{K}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\int d^2x\sqrt{g}\,\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/+\partial\!\!\!/\Lambda)\psi\right), \tag{37}$$

where $\mathcal{K}$ is the Jacobian of the transformation (which in this case is a nonrelevant constant). This can be reinterpreted as the partition function of a model consisting of a flat connection A_μ coupled to the fermions in the particular gauge where

$$A_\mu=\partial_\mu\Lambda. \tag{38}$$

The zero curvature condition on A_μ implies, of course, that the connection is locally a pure gauge. Since the vanishing of $*F(A)=\epsilon^{\mu\nu}F_{\mu\nu}(A)$ implies that of $F_{\mu\nu}(A)$, one introduces the 1-form connection restricted by the condition

$$*F(A)=\epsilon^{\mu\nu}F_{\mu\nu}(A)=0. \tag{39}$$

After imposing this constraint in the functional integral one gets

$$Z_j[j]=\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\frac{\delta^*(F(A))}{\mathrm{Vol}(\mathcal{G}_A)} \times\exp\left(\int d^2x\sqrt{g}\,\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/+A\!\!\!/)\psi\right), \tag{40}$$

where $\mathcal{G}_A$ is the gauge group of A. Now one introduces a Lagrange multiplier B to raise the $\delta(F)$ to the exponential but has to take into account that since there are infinitely many solutions of the equation $*F(A)=0$, the functional $\delta(*F)$ has to be defined with some care. It is properly defined [15] in terms of the generating functional of a *BF* topological field theory [27]. Using the BRST invariance as a guide to guarantee that the functional integral remains well defined, we obtain

$$Z_f[j]=\int \mathcal{D}A\mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}C\exp\left[\int d^2x\sqrt{g}\left(\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/+A\!\!\!/)\psi-\frac{i}{2\pi}\epsilon^{\mu\nu}\partial_\mu BA_\nu+L_g\right)\right], \tag{41}$$

where again $\mathcal{D}C$ stands for the measure of the ghosts and auxiliary fields and L_g for the contributions of those fields plus the gauge fixing term to the Lagrangian. The appearance of the BF effective action should be expected since the factor which comes from the exterior derivative in $\delta(*F(A))$ may be expressed as a function of the Ray-Singer torsion an hence related to the BF effective action [2]. In two dimensions the Ray-Singer torsion turns out to be equal to one.

To complete the bosonization of the generating functional one makes a shift $A+j\rightarrow A$. The fermionic field remains coupled only to the new A field. Then one uses the result (9) for the fermionic determinant, chooses an adequate gauge fixing condition which allow to make the quadratic functional integral in A and ends up with

$$Z_f[j]=Z[0]\mathcal{N}\int \mathcal{D}B\exp\left[-\int d^2x\sqrt{g}\left(\frac{1}{4}\partial_\mu B\partial_\mu B-\frac{i}{2\pi}\epsilon_{\mu\nu}\partial_\mu Bj_\nu\right)\right], \tag{42}$$

where $\mathcal{N}$ is the factor which appears after the quadratic integral on A has been performed. This is the bosonized effective action. The external current j appears in this expression coupled to the topological current of the Lagrange multiplier B.

Let us now turn to the general case on an arbitrary genus g, compact Riemann surface. On the light of Eq. (34) we start with

$$\hat{Z}_f[j]=\sum_{\epsilon_i,\kappa_j}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\int d^2x\sqrt{g}\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/)\psi\right). \tag{43}$$

Instead of using Eq. (34) directly let us argue how one can adapt the discussion presented for the genus zero surfaces and recover the BF partition function in a constructive way. Let us introduce the change of variables (36). In order to have a uniform change of variables in the functional integral, $\Lambda(x)$ must satisfy

$$\oint_{\mathcal{C}^I} d\Lambda=\pi n^I \tag{44}$$

where n^I are integers. If all the n^I are even the change of variables does not change the spin structure that we have defined over Σ. Otherwise we change from one to another spin structure but since we are summing over all of them this is not a problem here. We get again,

$$\hat{Z}_f[j]=\sum_{\epsilon_i,\kappa_j}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(\int_\Sigma d^2x\sqrt{g}\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/+\partial\!\!\!/\Lambda)\psi\right). \tag{45}$$

In this case we also wish to rewrite this in terms of a globally defined flat connection A. For two dimensional surfaces this means that A should be a flat connection over a trivial U(1) line bundle. To achieve consistency with Eq. (44) we have to impose that

$$G(A)=\oint_{\mathcal{C}^I} A=\pi n^I. \tag{46}$$

This is exactly the condition forced by Eq. (24) and in fact its appearance at this point provided the original motivation to the discussion presented in the previous section. Things now follow smoothly. First, in order to introduce A satisfying Eq. (46) in the functional integral one extends the functional integral to the space of connections and introduces factors $\delta(F(A))$ and $\delta(G(A))$ in the measure. We get

$$\hat{Z}_f[j]=\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\frac{\delta(F(A))\delta(G(A))}{\mathrm{Vol}(\mathcal{G}_A)}\times\exp\left(\int_\Sigma d^2x\sqrt{g}\bar{\psi}(-i\partial\!\!\!/+j\!\!\!/+A\!\!\!/)\psi\right), \tag{47}$$

where $\mathcal{G}_A$ is the group of allowed gauge transformations of A, that is of those gauge transformations with an uniform gauge function.

Now we want to raise the δ functions to the exponential. From our results of the previous section, the right way to do that is to take a multivalued Lagrange multiplier B over Σ satisfying

$$\oint_{\mathcal{C}^I} dB=4\pi m^I \tag{48}$$

and to integrate over the functional space of B with all possible m^I. In order to have a well-defined functional integral, the measure has to be defined in terms of precisely the BF topological field theory we considered previously. We then recover Eq. (34):

$$\hat{Z}_f[j]=Z_{4n}[j,0,\epsilon,\kappa]=\sum_{m^I}\int \mathcal{D}C\mathcal{D}A\mathcal{D}B\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S_{\mathrm{eff}}}, \tag{49}$$

$$S_{\mathrm{eff}}=\int d^2x\sqrt{g}[\bar{\psi}(iD\!\!\!\!/-A\!\!\!/-j\!\!\!/)\psi+L_g]+\frac{i}{2\pi}\int_\Sigma(dB\wedge A). \tag{50}$$

Here as we discussed earlier the result does not depend on the spin structure (ϵ,κ). To obtain the bosonized representation of Eq. (43) we now choose the gauge fixing and ghost terms in Eq. (49) and perform the fermionic integral. We can work more generally with $J\neq 0$ and use Eq. (16). Making first a shift

$$\tilde{A}=A+j \tag{51}$$

in Eq. (16), taking the gauge condition

$$*d*\tilde{A}=0 \tag{52}$$

and performing the fermionic integral we have

$$Z_{4n}[j,J,\epsilon,\kappa]=[\det{}'\Delta_0]^{1/2}[\det \mathrm{Im}\,\Omega\ \mathrm{Vol}(\Sigma)]^{1/2} \times\left|\theta\begin{bmatrix}u+\epsilon\\ v+\kappa\end{bmatrix}(0|\Omega)\right|^2\sum_{m^l}\int \mathcal{D}B\mathcal{D}\tilde{A}e^{-S[\tilde{A},B]}, \tag{53}$$

$$S[\tilde{A},B]=\frac{1}{2\pi}\int_\Sigma\left(d\tilde{A}\frac{1}{\Delta_0}*d\tilde{A}+idB\wedge(\tilde{A}-j) +\frac{1}{2}d*\tilde{A}\wedge*d*\tilde{A}-2\pi*J\wedge(\tilde{A}-j)\right) \tag{54}$$

where a factor $[\det{}'\Delta_0]$ arises from the integration on the ghost and antighost fields. The arguments u and v in the theta function are the coefficients in the expansion of $\tilde{A}_h$ and are not restricted until now. To write out our final expression we introduce the decomposition (2) for $\tilde{A}$,

$$\tilde{A}=d\tilde{s}+*d\tilde{p}+\tilde{A}_h$$

and observe that (i) integration in $\tilde{s}$ contributes with a factor $(\det{}'\Delta_0)^{-1}$. (ii) Integration in $\tilde{p}$ and the Jacobian of the transformation contribute a factor $(\det{}'\Delta_0)^{1/2}$ and a term in the action of the form

$$S(B,J)=-\frac{1}{2\pi}\int_\Sigma(dB+2\pi i*J)_{\mathrm{exact}}\wedge*(dB+2\pi i*J)_{\mathrm{exact}}$$

since only the exact part of $(dB+2\pi i*j)$ couples with $\tilde{p}$. (iii) One is left with the integration in $\tilde{A}_h$. Using the decomposition (19), for the the field B one may show again that the summation over the periods of B leads to the half integral periodicity conditions in $\tilde{A}_h$. The integral in $\tilde{A}_h$ is then a summation over the half-integral periods. We finally obtain

$$Z_{4n}[j,J,\epsilon,\kappa]=\sum_{l,m}\int\mathcal{D}be^{-S[\tilde{A}_h,b]}[\det \mathrm{Im}\,\Omega\ \mathrm{Vol}(\Sigma)]^{1/2} \times|\theta\begin{bmatrix}\frac{l}{2}\\ \frac{m}{2}\end{bmatrix}(0|\Omega)|^2 \tag{55}$$

where the contribution of the spin structure is included in the argument of the θ function and we define

$$S[\tilde{A}_h,b]=\frac{1}{2\pi}\int_\Sigma(db+2\pi i*J_{\mathrm{exact}})\wedge*(db+2\pi i*J_{\mathrm{exact}}) -i(db+2\pi*J_{\mathrm{exact}})\wedge j+\int_\Sigma*J\wedge\tilde{A}_h, \tag{56}$$

with A_h given by Eq. (3) restricted to half-integers periods. When J is zero this gives the bosonized expression for the fermionic partition function in higher genus surfaces. A similar expression for the partition function over a single spin structure may be obtained straightforwardly, following the same lines, starting from Eq. (29).

V. CONCLUSION

The results presented in this paper show that when one investigates the properties of a system of fermions on a Riemann manifold of arbitrary genus, the information about the spin structure of the manifold may be expressed in terms of the topological properties of the fields of a *BF* model. Moreover the nontrivial topological properties of the *BF* fields are shown to be included in the path integral which defines the generating functional of the coupled system. The bosonized version of the generating functional which, in this approach is obtained after integrating out the fermionic fields, is expressed in terms of these *BF* fields. It may be view as a dual model in a way that generalize in a nontrivial way the result for genus zero surfaces.

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