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Einstein constraints on asymptotically Euclidean manifolds

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We consider the Einstein constraints on asymptotically Euclidean manifolds M of dimension $n \geqslant 3$ with sources of both scaled and unscaled types. We extend to asymptotically Euclidean manifolds the constructive method of proof of existence. We also treat discontinuous scaled sources. In the last section we obtain new results in the case of non-constant mean curvature.

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I. INTRODUCTION

The geometric initial data for the $(n+1)$ dimensional Einstein equations are a properly Riemannian metric $\overline{g}$ and a symmetric 2-tensor K on an n-dimensional smooth manifold M. These data must satisfy the constraints, which are the Gauss-Codazzi equations linking the metric $\overline{g}$ induced on M by the spacetime metric g with the extrinsic curvature K of M as a submanifold imbedded in the spacetime (V,g) and the value on M of the Ricci tensor of g.

As equations on M, these constraints read

$$R(\overline{g}) - K\cdot K + (\mathrm{trK})^2 = 2\rho \quad \text{Hamiltonian constraint,} \tag{1}$$

$$\overline{\nabla}\cdot K - \overline{\nabla}\mathrm{trK} = j \quad \text{momentum constraint.} \tag{2}$$

$R(\overline{g})$ is the scalar curvature and the center dot denotes a product defined by the metric $\overline{g}$. The quantity ρ is a scalar and j a vector on M determined by the stress energy tensor of the sources. In coordinates adapted to the problem, where the equation of M in V is $x^0 = 0$, one has

$$j_i = \overline{N} T^0_i, \quad \rho = \overline{N}^2 T^{00} \tag{3}$$

with $\rho \geqslant 0$ if the sources satisfy the weak energy condition and if $\rho \geqslant \overline{g}(j,j)^{1/2}$ the sources satisfy the dominant energy condition. The space scalar $\overline{N}$ is the spacetime lapse function.

A classical method of solving the constraints, initiated by Lichnerowicz when $n=3$, is the conformal method (cf. [1] and references therein anterior to 1980, [2]). In these papers solutions were obtained under the condition that the initial submanifold will have constant mean extrinsic curvature, i.e., trK = const. Recently the results have been extended to the non-constant mean curvature case with some hypotheses on the smallness of its variations. The case of a compact manifold M is treated in [3] and [4], the first by using the Leray-Schauder theory, the second through a constructive method. Results for asymptotically Euclidean M are given in [5], using again the Leray-Schauder theory. All the quoted papers treat the case of scaled and continuous sources on a three-dimensional manifold M.

We will in this article consider the case where the manifold M has an arbitrary dimension $n \geqslant 3$ and the sources are the sum of scaled and unscaled ones. We will extend to asymptotically Euclidean manifolds the constructive method. We will extend the existence proof to discontinuous scaled sources.

In the last section we obtain results in the non-constant trK case. In the asymptotically Euclidean case, non-constant trK denotes non-maximal submanifolds. A simple smallness assumption on the variations of trK is sufficient to insure existence of solutions for metrics in the positive Yamabe-Brill-Cantor class when there are no unscaled sources. In the other cases the study is more delicate, as pointed out by O'Murchadha, and we obtain some results, in particular for unscaled sources.

We do not claim to have constructed solutions with scaled sources in the negative Yamabe class on non-maximal manifolds. The problem of the existence of solutions with large variations of trK also remains open.

We will use the conformal thin sandwich formulation developed recently by one of us [6] to express the momentum constraint. It gives a better understanding of the splitting between given and unknown initial data.

II. CONFORMAL METHOD IN ITS THIN SANDWICH FORMULATION

One turns the Hamiltonian constraint into an elliptic equation for a scalar function φ by considering the metric $\overline{g}$ as given up to a conformal factor. A convenient choice is to set, when $n>2$,

$$\overline{g} \equiv \gamma\varphi^{2p}, \quad \text{i.e.,} \quad \overline{g}_{ij} = \varphi^{2p}\gamma_{ij}, \quad \text{with} \quad p = \frac{2}{n-2}. \tag{4}$$

Then the following identity holds:

$$R(\bar{g}) \equiv \varphi^{-(n+2)/(n-2)}\left(\varphi R(\gamma) - \frac{4(n-1)}{n-2}\triangle_\gamma \varphi\right). \quad (5)$$

The Hamiltonian constraint becomes a semi-linear elliptic equation for φ with a non-linearity of a fairly simple type when γ and K are known—namely

$$\triangle_\gamma \varphi - k_n R(\gamma)\varphi + k_n(K\cdot K - \tau^2 + 2\rho)\varphi^{(n+2)/(n-2)} = 0 \quad (6)$$

with

$$\tau \equiv \mathrm{trK}, \quad k_n = \frac{n-2}{4(n-1)}. \quad (7)$$

We now explain the conformal form of the momentum constraint as recently deduced by one of us [6] from thin sandwich considerations. It can be construed to include previous methods as special cases, but no tensor splitting is needed. The initial metric $\bar{g}$ being known up to a conformal factor, it is natural to consider that the time derivative of this metric (the other ingredient of the initial data in a thin sandwich formulation) is known only for its conformal equivalence class. We have above

$$\bar{g}_{ij} = \varphi^{4/(n-2)}\gamma_{ij}. \quad (8)$$

If $\bar{g}_{ij}$ and γ_{ij} depend on t, their time derivatives are linked by

$$\bar{u}_{ij} = \varphi^{4/(n-2)}u_{ij}, \quad \bar{u}^{ij} = \varphi^{-4/(n-2)}u^{ij} \quad (9)$$

with

$$\partial_t \bar{g}_{ij} - \frac{1}{n}\bar{g}_{ij}\bar{g}^{hk}\partial_t \bar{g}_{hk} \equiv \bar{u}_{ij} \quad (10)$$

and an analogous expression for u_{ij} constructed with γ_{ij}.

We will consider the traceless symmetric two-tensor u_{ij} as given on the manifold (M,γ). Recall the identity

$$K_{ij} \equiv (2\bar{N})^{-1}\{-\partial_t \bar{g}_{ij} + \bar{\nabla}_i \bar{\beta}_j + \bar{\nabla}_j \bar{\beta}_i\}, \quad (11)$$

where $\bar{\beta}$ and $\bar{N}$ will be respectively the shift and the lapse in the imbedding spacetime. The shift vector $\bar{\beta}^i$ is not to be weighted; it is not a dynamical variable. The other non-dynamical variable is not the lapse $\bar{N}$ but a scalar density α of weight -1 such that $\bar{N} = \alpha \det(\bar{g})^{1/2}$ (cf. [8]). We therefore consider as given in this context a function N with the spacetime lapse $\bar{N}$ linked to it by the relation:

$$\bar{N} = \varphi^{2n/(n-2)}N. \quad (12)$$

We denote by $\bar{\nabla}$ and ∇ the covariant derivatives in the metrics $\bar{g}$ and γ respectively. We denote by $\mathcal{L}$ the conformal Killing operator

$$(\overline{\mathcal{L}\beta})^{ij} \equiv \bar{\nabla}^i \bar{\beta}^j + \bar{\nabla}^j \bar{\beta}^i - \frac{2}{n}\bar{g}^{ij}\bar{\nabla}_h \beta^h. \quad (13)$$

We have

$$(\overline{\mathcal{L}\beta})^{ij} \equiv \varphi^{-4/(n-2)}(\mathcal{L}\beta)^{ij}, \quad \bar{\beta}^i \equiv \beta^i \quad (14)$$

and

$$K^{ij} = \frac{1}{n}\bar{g}^{ij}\tau + \varphi^{-2(n+2)/(n-2)}A^{ij} \quad (15)$$

with

$$A^{ij} \equiv (2N)^{-1}\{-u^{ij} + (\mathcal{L}\beta)^{ij}\}. \quad (16)$$

One finds by straightforward calculation that the momentum constraint now reads as an equation on (M,γ) with unknown β (and φ if $D\tau \not\equiv 0$):

$$\nabla_j\{(2N^{-1})(\mathcal{L}\beta)^{ij}\} = \nabla_j\{(2N^{-1})u^{ij}\} + \frac{n-1}{n}\varphi^{2n/n-2}\nabla^i \tau + \varphi^{2(n+2)/(n-2)}j \quad (17)$$

where N, τ, and u are given.

The Hamiltonian constraint now reads

$$\triangle_\gamma \varphi - k_n R(\gamma)\varphi + k_n \varphi^{(-3n+2)/(n-2)}A\cdot A - \frac{n-2}{4n}\varphi^{(n+2)/(n-2)}\tau^2 = -2k_n \rho \varphi^{(n+2)/(n-2)}. \quad (18)$$

The scaling of the quantities ρ and j appearing in (17) and (18) depends upon the nature of the source fields. For generic fluid sources, with no independent field equations of their own, we may (a) leave the fields unscaled, (b) scale them in a way that is convenient for analysis of the constraints, or (c) combine the two approaches. In this last case, we set

$$j \equiv J + \varphi^{-2(n+2)/(n-2)}v, \quad \frac{n-2}{2(n-1)}\rho = c + q\varphi^{-2(n+1)/(n-2)}. \quad (19)$$

Here J and c are unscaled, while v and q are scaled.

When the source fields do have their own field equations, the scaling is to an extent dictated by these source field equations. Here, we discuss two examples; see Isenberg and Nester, as referenced in [1], for further discussion of the scaling of sources.

Examples of source field scalings

(1) $n=3$, *the source is an electromagnetic or Yang-Mills field F*. The electric and magnetic fields relative to a spacetime observer at rest with respect to the initial manifold M (i.e., with 4-velocity orthogonal to this manifold) are

$$\bar{E}^i \equiv \bar{N}^{-1}F^i_0 = \varphi^{-6}N^{-1}F^i_0 \equiv \varphi^{-6}E^i \quad (20)$$

$$\bar{H}^i = \frac{1}{2}\bar{\eta}^{ijk}F_{jk} = \frac{1}{2}\varphi^{-6}\eta^{ijk}F_{jk} \equiv \varphi^{-6}H^i \quad (21)$$

with η and $\bar{\eta}$ respectively the volume forms of γ and $\bar{g}$.

Note that if $(\overline{E^i},\overline{H^i})$ satisfy the Maxwell constraints $\overline{\nabla}_i\overline{E^i}=0$ and $\overline{\nabla}_i\overline{H^i}=0$ in the metric $\overline{g}$, the fields (E^i,H^i) satisfy these constraints in the metric γ. We consider that it is these last fields which are known on M.

The energy density is

$$\rho=\frac{1}{2}\overline{g}_{ij}(\overline{E^i}\overline{E^j}+\overline{H^i}\overline{H^j})\equiv\varphi^{-8}q \tag{22}$$

with q, considered as known on M, given by

$$q\equiv\frac{1}{2}\gamma_{ij}(E^iE^j+H^iH^j). \tag{23}$$

The momentum density is

$$j^i=\overline{N}T^{i0}=\overline{N}F^{0j}F^i_j=-\overline{E^j}\overline{g^{ik}}\eta_{kjl}H^l=\varphi^{-10}v^i \tag{24}$$

with v^i the quantity considered given as

$$v^i=-\gamma^{ik}\eta_{kjl}E^jH^l. \tag{25}$$

The sources are scaled as defined [compare Eq. (19)] and the constraints decouple if $D\tau=0$. Note that if $q\geqslant(\gamma_{ij}v^iv^j)^{1/2}$, then $\rho\geqslant(\overline{g_{ij}}j^ij^j)^{1/2}$.

(2) *General n, the source is a Klein-Gordon field.* The energy density on M of a Klein-Gordon field ψ with respect to an observer at rest is

$$\rho=\frac{1}{2}(\overline{N}^{-2}|\partial_0\psi|^2+\overline{g^{ij}}\partial_i\psi\partial_j\psi+m\psi^2), \tag{26}$$

i.e.,

$$\rho=\frac{1}{2}\{\varphi^{-4n/(n-2)}N^{-2}|\partial_0\psi|^2+\varphi^{-4/(n-2)}\gamma^{ij}\partial_i\psi\partial_j\psi+m\psi^2\}. \tag{27}$$

If we consider as known on M the initial data $\psi|_M$ and $\partial_0\psi|_M$ together with γ and N, then neither of the terms in ρ scales as indicated in Eq. (19). The term $N^{-2}|\partial_0\varphi|^2$ adds in the Hamiltonian constraint to $A\cdot A$, the term $m^2\psi$ is unscaled and gives a contribution to c, the middle term gives a new, positive contribution to the φ term which adds to $-R(\gamma)$. The momentum density is

$$j^i=-\overline{N}^{-1}\overline{g^{ij}}\partial_j\psi\partial_0\psi=-\varphi^{-2(n+2)/(n-2)}\gamma^{ij}\partial_j\psi\partial_0\psi. \tag{28}$$

We see that the momentum scales as in Eq. (19). The constraints decouple if $D\tau=0$.

The methods we give below to study the constraints with scaled or unscaled sources can be applied to more general scalings, such as this example.

Summary. The given initial data on a manifold M are on the one hand (geometric initial data) a set (γ,u,τ,N), with γ a properly Riemannian metric, u a traceless symmetric 2-tensor, τ and N scalar functions, and on the other hand (source data) a set (J,v,c,q), two vectors and two scalars. The initial data to be determined by the constraints make a pair (φ,β) with φ a scalar function and β a vector on M. In the conformal thin-sandwich formalism the constraints reduce to Eqs. (17) and (18) which read, taking Eq. (19) into account,

$$\nabla_j\{(2N^{-1})(\mathcal{L}\beta)^{ij}\}=h^i(\cdot,\varphi) \tag{29}$$

with

$$h^i(\cdot,\varphi)\equiv\nabla_j\{(2N^{-1})u^{ij}\}+\frac{n-1}{n}\varphi^{2n/(n-2)}\nabla^i\tau+\varphi^{2(n+2)/(n-2)}J^i+v^i, \tag{30}$$

and

$$\triangle_\gamma\varphi=f(\cdot,\varphi), \tag{31}$$

where

$$f(\cdot,\varphi)\equiv r\varphi-a\varphi^{(-3n+2)/(n-2)}+d\varphi^{(n+2)/(n-2)}-q\varphi^{-n/(n-2)}, \tag{32}$$

with r, a, and d defined as functions of the geometric data as in Eq. (45).

When τ is constant on M and the sources have no unscaled momentum (i.e., $J=0$) these constraints decouple in the following sense: the momentum constraint (29) is a linear equation for β, independent of φ, and the Hamiltonian constraint (31) is a non-linear equation for φ when β is known.

When the constraints are solved the spacetime metric on M reads

$$ds^2=-\overline{N}^2dt^2+\overline{g}_{ij}(dx^i+\beta^idt)(dx^j+\beta^jdt), \tag{33}$$

with $\overline{g}$ and $\overline{N}$ given by the formulas (8) and (12). The extrinsic curvature of M is determined by Eqs. (15) and (16), the derivative $\partial_t\overline{g}_{ij}$ on M by Eq. (11). The derivatives $\partial_t\overline{N}$ and $\partial_t\beta$ remain arbitrary.

We now express in our setting the conformal invariance of the conformal constraints.

Lemma. The constraint equations (17) and (18) are conformally invariant in the following sense: If (β,φ) is a solution of the constraints with data $(\gamma,u,\tau,N;J,v,c,q)$ then $(\beta,\tilde{\varphi})$ is a solution of the constraints with data $[\tilde{\gamma}=(\tilde{\varphi}\varphi^{-1})^{4/(n-2)}\gamma,\ \tilde{u}=(\tilde{\varphi}\varphi^{-1})^{4/(n-2)}u,\ \tau,\ \tilde{N}=(\tilde{\varphi}\varphi^{-1})^{2n/(n-2)}N;\ \tilde{J}=J,\ \tilde{v}=(\tilde{\varphi}\varphi^{-1})^{-2(n+2)/(n-2)}v,\ \tilde{c}=c,\ \tilde{q}=(\tilde{\varphi}\varphi^{-1})^{-2(n+1)/(n-2)}q]$.

Proof. If (β,φ) together with the considered given data is a solution of the conformal constraints, the corresponding Einstein initial data set $(\overline{g},K)$ is a solution of the Einstein constraints with sources j, ρ given by Eq. (19). The Einstein initial data set and sources constructed with the approximate quantities are identical with $(\overline{g},K)$ and (j,ρ). Since the Einstein constraints are satisfied, the conformal constraints written with the approximate quantities are also satisfied. □

Remark. In the case $n=2$, equations analogous to the ones obtained here for the conformal factor φ and the vector β are obtained by setting (cf. [9]):

$$\bar{g}=e^{2\varphi}\gamma, \tag{34}$$

and in the thin sandwich point of view,

$$\bar{N}=e^{2\varphi}N \tag{35}$$

which gives

$$K^{ij}=e^{4\varphi}A^{ij}+\frac{1}{2}\bar{g}^{ij}\tau. \tag{36}$$

However we will not consider $n=2$ because it poses special problems in what could correspond to an asymptotically Euclidean case.

III. ASYMPTOTICALLY EUCLIDEAN MANIFOLDS AND WEIGHTED SOBOLEV SPACES

The *Euclidean space* $\mathbb{E}^n$ is the manifold $\mathbb{R}^n$ endowed with the Euclidean metric which reads in canonical coordinates $\Sigma(dx^i)^2$. A C^∞, n-dimensional Riemannian manifold (M,e) is called *Euclidean at infinity* if there exists a compact subset S of M such that $M-S$ is the disjoint union of a finite number of open sets U_i, and each (U_i,e) is isometric to the exterior of a ball in $\mathbb{E}^n$. Each open set $U_i\subset M$ is sometimes called an *end* of M. If M is diffeomorphic to $\mathbb{R}^n$, it has only one end; and we can then take for e the Euclidean metric.

A Riemannian manifold (M,γ) is called *asymptotically Euclidean* if there exists a Riemannian manifold (M,e) Euclidean at infinity, and γ tends to e at infinity in each end. Consider one end U and the canonical coordinates x^i in the space $\mathbb{E}^n$ which contains the exterior of the ball to which U is diffeomorphic. Set $r\equiv\{\Sigma(x^i)^2\}^{1/2}$. In the coordinates x^i the metric e has components $e_{ij}=\delta_{ij}$. The metric γ tends to e at infinity if in these coordinates $\gamma_{ij}-\delta_{ij}$ tends to zero. A possible way of making this statement mathematically precise is to use weighted Sobolev spaces. (One can also use in these elliptic constraint problems weighted Hölder spaces, but they are not well adapted to the related evolution problems.)

A *weighted Sobolev space* $W^p_{s,\delta}$, $1\leqslant p<\infty$, $s\in\mathbb{N}_+$, $\delta\in\mathbb{R}$, of tensors of some given type on the manifold (M,e) Euclidean at infinity is the closure of C_0^∞ tensors of the given type (C^∞ tensors with compact support in M) in the norm

$$\|u\|_{W^p_{s,\delta}}=\left\{\sum_{0\leqslant m\leqslant s}\int_V|\partial^m u|^p(1+d^2)^{1/2\,p(\delta+m)}d\mu\right\}^{1/p}, \tag{37}$$

where ∂, $|\ |$ and $d\mu$ denote the covariant derivative, norm and volume element in the metric e, and d is the distance in the metric e from a point of M to a fixed point. If (M,e) is a Euclidean space one can choose $d=r$, the Euclidean distance to the origin. We recall the multiplication and imbedding properties (cf. [10,11])

$$W^p_{s_1,\delta_1}\times W^p_{s_2,\delta_2}\subset W^p_{s,\delta}\ \text{ if }\ s\leqslant s_1,s_2, s<s_1+s_2-\frac{n}{p},$$

$$\delta<\delta_1+\delta_2+\frac{n}{p},$$

$$W^p_{s,\delta}\subset C^m_\beta\ \text{ if }\ m<s-\frac{n}{p},\ \ \beta<\delta+\frac{n}{p},$$

$$\|u\|_{C^m_\beta}\equiv\sum_{0\leqslant l\leqslant m}\sup_M(|\partial^l u|(1+d^2)^{1/2(\beta+l)}). \tag{38}$$

The imbedding of the space $W^p_{s,\delta}$ into $W^p_{s',\delta'}$, $s\geqslant s'$, $\delta\geqslant\delta'$ is compact if $s>s'$, $\delta>\delta'$. We have on the other hand

$$(1+d^2)^{-\beta/2}\in W^p_{s,\delta}\ \text{ if }\ \ \beta>\delta+\frac{n}{p},\ \ s\geqslant 0. \tag{39}$$

Let (M,e) be a manifold Euclidean at infinity. Then the Riemannian manifold (M,γ) is said to be "$W^p_{\sigma,\rho}$ asymptotically Euclidean" if $\gamma-e\in W^p_{\sigma,\rho}$. When we speak of "asymptotically Euclidean manifolds" without further specification, we suppose that $\gamma-e\in W^p_{\sigma,\rho}$ with $\sigma>n/p+1$, $\rho>-n/p$. These hypotheses imply that γ is C^1 and $\gamma-e$ tends to zero at infinity.

IV. MOMENTUM CONSTRAINT

In the thin sandwich conformal formulation the momentum constraint reads

$$\nabla_j\{(2N^{-1})(\mathcal{L}\beta)^{ij}\}=h(\cdot,\varphi) \tag{40}$$

with

$$h^i(\cdot,\varphi)\equiv\nabla_j\{(2N^{-1})u^{ij}\}+\frac{n-1}{n}\varphi^{2n/(n-2)}\nabla^i\tau$$
$$+\varphi^{2(n+2)/(n-2)}J^i+v^i, \tag{41}$$

where N and τ are given functions on M and u a given symmetric traceless tensor field. The sources J and v are also considered as known. We suppose momentarily that φ is also a known function; in fact, it disappears from the equation if $\nabla\tau\equiv 0$ and $J\equiv 0$.

The momentum constraint is a linear elliptic system for the unknown β on the manifold (M,γ). (The symbol of the principal operator is an isomorphism.)

Theorem. Let (M,γ) be a $W^p_{\sigma,\rho}$ asymptotically Euclidean manifold with $\sigma>n/p+2,\rho>-n/p$. Let u, $\tau\in W^p_{s+1,\delta+1}$ be given, $(1-N^{-1})$ and $(1-\varphi)\in W^p_{s+2,\delta}, N>0, \varphi>0$, and $J,v\in W^p_{s,\delta+2}$. The momentum constraint has one and only one solution $\beta\in W^p_{s+2,\delta}$ if $s>n/p-2$ and $0\leqslant s\leqslant\sigma-2$, $-n/p<\delta<n-2-n/p$.

Proof. The operator on the left-hand side of Eq. (40) is injective on $W^p_{s+2,\delta}$ because a solution $\beta\in W^p_{s,\delta}$, $\delta>-n/p$ of the equation

$$\nabla_j\{(2N)^{-1}(\mathcal{L}\beta)^{ij}\}=0 \tag{42}$$

is necessarily a conformal Killing field. Indeed if $\beta \in C_0^\infty$ the equation implies by integration on M that

$$\int_M \beta_i \nabla_j \{(2N)^{-1}(\mathcal{L}\beta)^{ij}\}\mu_\gamma = \int_M (2N)^{-1}\mathcal{L}\beta . \mathcal{L}\beta \mu_\gamma = 0. \tag{43}$$

The same is true if $\beta \in W^p_{s+2,\delta'}$ with $\delta' > -n/p + n/2 - 2$ (respectively $\delta' \geqslant -2$ if $p=2$). There is such a δ' if $\beta \in W^p_{s+2,\delta}$ satisfies the homogeneous second order equation (cf. a similar proof for the Laplace operator in Appendix B). It is known that there are no conformal Killing vector fields tending to zero at infinity on an asymptotically Euclidean manifold (cf. [1] where a proof requiring only low regularity is cited).

Because the elliptic operator on β is injective, the isomorphism theorem applies to give the existence and uniqueness of β.

V. HAMILTONIAN CONSTRAINT

In the conformal method the Hamiltonian constraint reads as a non-linear elliptic equation for the conformal factor φ. We write it

$$\triangle_\gamma \varphi = f(\cdot,\varphi)$$

$$f(\cdot,\varphi) \equiv r\varphi - a\varphi^{(-3n+2)/(n-2)} + d\varphi^{(n+2)/(n-2)} - q\varphi^{-n/(n-2)}, \tag{44}$$

with A given by Eq. (16), and

$$r \equiv k_n R(\gamma), \quad a \equiv k_n A\cdot A, \quad k_n \equiv (n-2)/4(n-1)$$

$$d \equiv b - c, \quad b \equiv (n-2)/(4n)\tau^2. \tag{45}$$

By their definitions we have

$$a \geqslant 0, \quad b \geqslant 0, \quad c \geqslant 0, \quad q \geqslant 0. \tag{46}$$

The functions q and c, scaled and unscaled sources, are considered as given on M. We will suppose that τ (hence b) is also known on M. The function a is known when the momentum constraint has been solved: this can be done independently of φ if τ is constant and the unscaled sources have zero momentum.

The constructive method of sub and super solutions used by one of us [2] to solve non-linear elliptic equations on a compact manifold can be extended to asymptotically Euclidean manifolds.

The following theorem is a particular case of the theorem proven in the Appendix B.

Theorem. Let (M,γ) be a (p,σ,ρ) asymptotically Euclidean manifold with $\sigma > n/p+1$, $\rho > -n/p$. Suppose r, a, q, $d \in W^p_{s,\delta+2}, \sigma - 1 \geqslant s > n/p, -n/p < \delta$. Suppose the equation $\triangle_\gamma \varphi = f(x,\varphi)$ admits a subsolution $\varphi_- > 0$ and a uniformly bounded supersolution φ_+, functions in C^2 such that

$$\triangle_\gamma \varphi_- \geqslant f(\cdot,\varphi_-), \quad \triangle_\gamma \varphi_+ \leqslant f(\cdot,\varphi_+) \tag{47}$$

and

$$\lim_\infty \varphi_- \leqslant 1, \quad \lim_\infty \varphi_+ \geqslant 1$$

$$\varphi_- \leqslant \varphi_+ \quad \text{on } M. \tag{48}$$

Suppose that $D\varphi_-$, $D\varphi_+ \in W^p_{s'-1,\delta'+1}$, $s' \geqslant s$, $\delta' > -n/p$.

Then the equation admits a solution φ such that

$$\varphi_- \leqslant \varphi \leqslant \varphi_+, \quad 1-\varphi \in W^p_{s+2,\delta} \tag{49}$$

if

$$-\frac{n}{p} < \delta < n-2-\frac{n}{p}. \tag{50}$$

Remark. When $r \equiv k_n R(\gamma)$ we have $r \in W^p_{\sigma-2,\delta+2}$ if $\sigma > n/p+2$, $\delta > -n/p$.

We will use this theorem directly in Sec. XI, with constant sub and super solutions. We will give and use in Secs. VI and X intermediate simple steps to obtain non-constant sub and supersolutions.

VI. BRILL-CANTOR THEOREM

The constraints in their conformal formulation are invariant under conformal rescaling (cf. Sec. II).

In the case of a compact manifold M a convenient first step before studying the solution of the Lichnerowicz equation is to use the Yamabe theorem which says that each manifold (M,γ) is conformal to a manifold with constant scalar curvature which can be chosen to be 1, -1 or zero. The positive, negative and zero Yamabe classes correspond to the signs of these constants and are conformal invariants. There is no known analogous theorem for asymptotically Euclidean manifolds. (In any case the curvatures could not be non-zero constants.) However an interesting theorem has been proved by Brill and Cantor, with the following definition.

Definition. The asymptotically Euclidean manifold (M,γ) is in the positive Yamabe class if for every function f on M with $f \in C_0^\infty$, $f \not\equiv 0$, it is true that

$$\int_M \{|Df|^2 + r(\gamma)f^2\}\mu_\gamma > 0. \tag{51}$$

The positive Yamabe class is a conformal invariant due to the identity

$$\triangle_\gamma f - r(\gamma)f \equiv \varphi^{(n+2)/(n-2)}\{\triangle_{\gamma'} f' - r(\gamma')f'\} \tag{52}$$

$$\gamma' = \varphi^{4/(n-2)}\gamma, \quad f' = f\varphi^{-1}, \tag{53}$$

which gives after integration by parts with $f \in C_0^\infty$, because $\mu_{\gamma'} = \varphi^{2n/n-2}\mu_\gamma$,

$$\int_M \{|Df|^2 + r(\gamma)f^2\}\mu_\gamma = \int_M \{|Df'|^2 + r(\gamma')f'^2\}\mu_{\gamma'}. \tag{54}$$

We will say, following O'Murchada, that the asymptotically Euclidean manifold (M,γ) is in the negative Yamabe class if it is not in the positive one [7]. However, analogy with the case of a compact manifold can be misleading, as shown in the following theorem.

Theorem. ([12]). The asymptotically Euclidean manifold (M,γ) is conformal to a manifold with zero scalar curvature, that is, the equation $\triangle_{\gamma}\varphi - r(\gamma)\varphi = 0$ has a solution $\varphi > 0$, if and only if (M,γ) is in the *positive Yamabe class*.

The physical metric $\bar{g}$ that solves the constraints together with the symmetric two-tensor K has a non-negative scalar curvature $R(\bar{g})$ if the sources have positive energy and the initial manifold has constant mean extrinsic curvature (necessarily zero in the asymptotically Euclidean case). Thus, $R(\bar{g}) \geq 0$, with $R(\bar{g}) \not\equiv 0$ except in vacuum for an instant of time symmetry, i.e., $K \equiv 0$. Therefore, the physical metric $\bar{g}$ on an initial maximal submanifold is in the positive Yamabe class and all metrics γ used as substrata to obtain it must be in that class.

We will prove a more general theorem. We will also make fewer restrictions than Brill-Cantor on the weighted spaces.

Theorem. On a (p,σ,ρ) asymptotically Euclidean manifold the equation

$$\triangle_{\gamma}\varphi - \alpha\varphi = \mathrm{v}, \tag{55}$$

where α, $\mathrm{v} \in W^p_{s,\delta+2}$, $\mathrm{v} \leq 0$, has a solution $\varphi > 0$, $\varphi - 1 \in W^p_{s+2,\delta}$, $s \geq 0$, $\delta > -n/p$ only if for all $f \in C^\infty_0$, $f \not\equiv 0$, the following inequality holds

$$\int_M \{|Df|^2 + \alpha f^2\}\mu_\gamma > 0. \tag{56}$$

Under the same hypothesis the solution φ exists with $\varphi - 1 \in W^p_{s+2,\delta}$, and $\varphi > 0$ if one supposes moreover $s > n/p$, $\delta > n/2 - n/p - 1$ if $p \neq 2$ (respectively $\delta \geq -1$ if $p = 2$), and that either $\mathrm{v} < 0$ or $\mathrm{v} \equiv 0$ on M, or $\alpha = r(\gamma)$ with, in this last case, $\sigma \geq 2$.

The theorem of Brill and Cantor corresponds to the case $\mathrm{v} \equiv 0$ and $\alpha = r(\gamma)$. They make the additional hypothesis $p > n$.

Proof. (1) ("only if") Suppose φ exists and solves the equation satisfying the hypothesis of the theorem. Then we will show that for any $f \not\equiv 0$, $f \in C^\infty_0$,

$$\int_M \{|Df|^2 + \alpha f^2\}\mu_\gamma > 0. \tag{57}$$

Indeed, let $f \in C^\infty_0$, $f \not\equiv 0$. The function $\theta = f\varphi^{-1}$ has compact support, belongs to $W^p_{2,\delta'}$ for any δ' and is such that $D\theta \not\equiv 0$ since θ, having compact support, cannot be a constant without being identically zero. We have by elementary calculus:

$$|Df|^2 = |D\theta|^2\varphi^2 + \varphi D\varphi \cdot D(\theta^2) + \theta^2|D\varphi|^2. \tag{58}$$

The following integration by parts holds for the considered functions:

$$\int_M \varphi D\varphi \cdot D(\theta^2)\mu_\gamma = \int_M -\theta^2 D\cdot(\varphi D\varphi)\mu_\gamma; \tag{59}$$

therefore,

$$\int_M \varphi D\varphi \cdot D(\theta^2)\mu_\gamma = \int_M -\theta^2(\varphi\triangle_\gamma\varphi + |D\varphi|^2)\mu_\gamma \tag{60}$$

and

$$\int_M |Df|^2\mu_\gamma > \int_M -\theta^2\varphi\triangle_\gamma\varphi\mu_\gamma. \tag{61}$$

Hence when φ satisfies the given equation and $\theta\varphi = f$:

$$\int_M \{|Df|^2 + \alpha f^2\}\mu_\gamma > \int_M -\mathrm{v}\theta^2\varphi\mu_\gamma \geq 0 \tag{62}$$

if $\varphi > 0$ and $\mathrm{v} \leq 0$.

(2) ("if":existence) Setting $\varphi = 1 + u$ the equation reads:

$$\triangle_\gamma u - \alpha u = \mathrm{v} + \alpha. \tag{63}$$

The operator $\triangle_\gamma - \alpha$ is injective on $W^p_{2,\delta}$ (cf. Appendix A).

The general theorem on linear elliptic equations on an asymptotically Euclidean manifold shows that our equation has one solution $u \in W^p_{s+2,\delta}$, $s \geq 0$, $-n/p < \delta < n - 2 - n/p$. The problem is to prove that $\varphi = 1 + u$ is positive. We will use the maximum principle, supposing the solution to be C^2, i.e., $s > n/p$. Since α is not necessarily positive we cannot apply directly the maximum principle. One proceeds as in the Brill-Cantor proof. One considers the family of equations, which all satisfy the criterion for the existence of a solution φ_λ with $\varphi_\lambda - 1 \in W^p_{s+2,\delta}$,

$$\triangle_\gamma\varphi - \lambda\alpha\varphi = \lambda\mathrm{v}, \quad \lambda \in [0,1]. \tag{64}$$

The solutions φ_λ depend continuously on λ and we have $\varphi_0 = 1$. If the function $\varphi_1 \equiv \varphi$ takes negative values there is one of these functions φ_{λ_0} which takes positive or zero values. The points where φ_{λ_0} vanishes are minima of this function. It is incompatible with the equation satisfied by φ_{λ_0} if v is negative at that point. Therefore we have $\varphi_\lambda > 0$ for $\lambda \in [0,1]$ if $\mathrm{v} < 0$.

To prove that $\varphi_{\lambda_0} > 0$, and hence $\varphi_\lambda > 0$ for $\lambda \in [0,1]$, when $\mathrm{v} \equiv 0$, we use, as Brill-Cantor, a theorem of Alexandrov: if there is a point x_0 where $\varphi_{\lambda_0} = 0$, it is a minimum of this function, hence $D\varphi_{\lambda_0}(x_0) = 0$. Since the function φ_{λ_0} and the function identical to zero take the same value as well as their first derivatives at x_0 and satisfy the same elliptic equation they must coincide (Alexandrov theorem), a result that contradicts the fact that φ_{λ_0} tends to 1 at infinity.

If we know only that $\mathrm{v} \leq 0$ but $\alpha = r(\gamma)$ we first conformally transform the metric γ to a metric $\gamma' = \gamma\psi^{4n/(n-2)}$ with zero scalar curvature: this is possible by the previous proof for $v \equiv 0$ (original Brill-Cantor theorem). The equation to solve is equivalent to the following equation for $\varphi' = \varphi\psi^{-1}$:

$$\triangle_{\gamma'}\varphi' = \psi^{-(n+2)/(n-2)} v \leqslant 0, \tag{65}$$

whose solution is $\varphi' \geqslant 1$ because φ' cannot attain a minimum at a point of M and φ' tends to 1 at infinity. □

VII. SOLUTION OF THE EQUATION $\triangle_\gamma \varphi - r(\gamma)\varphi = b\varphi^{(n+2)/(n-2)}$

Theorem. If $b \in W^p_{s,\delta+2}, s > n/p, -n/p < \delta < n-2 - n/p, b \geqslant 0$, the equation

$$\triangle_\gamma \varphi - r(\gamma)\varphi = b\varphi^{(n+2)/(n-2)} \tag{66}$$

on the (p,σ,ρ) manifold (M,γ), $\sigma > n/p+2, \rho > n/p$ has a solution $\varphi = 1+u, u \in W^p_{s+2,\delta}, s \leqslant \sigma, \rho > 0$ under one or the other of the following hypotheses:

(1) On the subset of M where $r(\gamma) < 0$ there exists a number $\mu > 0$ such that

$$\sup_{\{x \in M, r(\gamma)(x) < 0\}} \frac{|r(\gamma)|}{b} \leqslant \mu. \tag{67}$$

(2) (M,γ) is in the positive Yamabe class. The solution is unique in both cases.

Proof. (1) The manifold (M,γ) and the function $f(x,y) = r(\gamma)\phi + b(\phi)^{(n+2)/n-2)}$ satisfy the hypothesis (H) spelled out in Appendix B. The equation admits the subsolution $\varphi_- = 0$. A number φ_+ is a supersolution if

$$\varphi_+ \geqslant 1 \quad \text{and} \quad r(\gamma) + b\varphi_+^{4n/(n-2)} \geqslant 0 \quad \text{on } M. \tag{68}$$

The second inequality is a consequence of the first if $r(\gamma) \geqslant 0$.

The hypothesis made on (M,γ) on the subset $r(\gamma) < 0$ insures the existence of the number $\varphi_+ \geqslant \varphi_- \equiv 0$, given by

$$\varphi_+ = \max(1, \mu^{(n-2)/4n}). \tag{69}$$

The existence of a solution ψ, with $0 \leqslant \psi \leqslant \varphi_+$ and $1-\psi \in W^p_{s+2,\delta}$ results from the general theorem. Such a solution can be obtained constructively. We know that $\psi \neq 0$ since it tends to 1 at infinity.

We show that $\psi > 0$ on M by using the Alexandrof theorem as we did in the proof of the Brill-Cantor theorem: if ψ vanishes at a point $x_0 \in M$ this point is a minimum of ψ, hence $D\psi(x_0) = 0$. The functions $\varphi = \psi$ and $\varphi \equiv 0$ both satisfy the elliptic equation

$$\triangle_\gamma \varphi - [r(\gamma) + b\psi^{4n/n-2}]\varphi = 0. \tag{70}$$

They, as well as their gradients, take the same values, zero, at the point x_0, therefore they coincide. This contradicts the fact that ψ tends to 1 at infinity, therefore there exists no point x_0 where $\psi(x_0) = 0$. Hence $\psi > 0$ on M.

(2) If (M,γ) is in the positive Yamabe class we conformally transform it to a manifold (M,γ') such that $r(\gamma') \equiv 0$. The subset of M where $r(\gamma') < 0$ is empty; therefore, $\varphi_+ = 1$ can be chosen as a supersolution. The proof that $\varphi > 0$ on M can be made using simply the maximum principle: a solution $\varphi \in C^2$ of the equation

$$\triangle_\gamma \varphi - b\varphi^{(n+2)/(n-2)} \equiv \triangle_\gamma \varphi - (b\varphi^{4n/(n-2)})\varphi = 0 \tag{71}$$

with $b \geqslant 0$ cannot attain a nonpositive minimum on M without being a constant (which is not possible with φ tending to 1 at infinity except if $b \equiv 0$, in which case $\varphi \equiv 1$).

The uniqueness property in case 2 is simply a consequence of $b \geqslant 0$ and of the increasing property with $\varphi > 0$ of the function $\varphi^{(n+2)/(n-2)}$, together with the fact that the difference of two solutions tends to zero at infinity. The uniqueness in the general case results from the conformal properties. Indeed suppose the equation

$$\triangle_\gamma \varphi - r(\gamma)\varphi = b\varphi^Q, \quad Q = \frac{n+2}{n-2} \tag{72}$$

has two solution φ_1 and φ_2. We deduce from the conformal identity

$$\triangle_\gamma \varphi - r(\gamma)\varphi = -r(\overline{g})\varphi^Q, \quad \overline{g} = \varphi^{2p}\gamma, \quad p = \frac{2}{n-2} \tag{73}$$

that

$$r(\overline{g}) = r(\gamma\varphi_1^{2p}) = r(\gamma\varphi_2^{2p}) = -b. \tag{74}$$

Consider the identity

$$\triangle_{\gamma\varphi_1^{2p}}(\varphi_1^{-1}\varphi_2) - r(\gamma\varphi_1^{2p})\varphi_1^{-1}\varphi_2 \equiv -r(\gamma\varphi_2^{2p})(\varphi_1^{-1}\varphi_2)^Q. \tag{75}$$

It implies, because of the previous equalities,

$$\triangle_{\gamma\varphi_1^{2p}}(u-1) - bu\frac{u^{Q-1}-1}{u-1}(u-1) = 0, \quad u \equiv \varphi_1^{-1}\varphi_2. \tag{76}$$

We have $b \geqslant 0$, $u > 0$, $(u^{Q-1}-1)/(u-1) > 0$ since $u > 0$ and $Q > 1$. We deduce from the fact that $u-1$ tends to zero at infinity that $u-1=0$ on M, i.e., $\varphi_1 \equiv \varphi_2$. □

Remark. By the above theorem, under the hypothesis made, an asymptotically Euclidean manifold (M,γ) is conformal to a metric γ' of given non-positive scalar curvature $r(\gamma')$, and the solution φ of Eq. (66) with $b = -r(\gamma')$ gives the conformal factor. (This result was known to O'Murchadha.)

VIII. SOLUTION OF THE EQUATION $\triangle_\gamma \varphi - r(\gamma)\varphi + a\varphi^{-P} + q\varphi^{-P'} = 0$, $a \geqslant 0$, $q \geqslant 0$

This equation is the conformal expression of the Hamiltonian constraint on a maximal manifold with no unscaled sources. The following theorem has been proved independently in the case $n=3$ in 1979 by Cantor, and Chaljub-

Simon and Choquet-Bruhat (in weighted Holder spaces). We give here a new constructive proof; the corresponding function $f(x,\varphi)$ satisfies the hypothesis H of Appendix B on any interval $[l,\infty)$, $l>0$.

The generalized Brill-Cantor theorem shows that the considered equation can have a solution $\varphi>0$ only if (M,γ) is in the positive Yamabe class, a result in agreement with the fact that the original Hamiltonian constraint on an initial maximal submanifold $(M,\overline{g})$ implies $r(\overline{g})\geqslant 0$.

Theorem. The equation on the (p,σ,ρ) asymptotically Euclidean manifold (M,γ), $\sigma>n/p+2$, $\rho>n/2-2-n/p$ if $p\neq 2$, and $\rho\geqslant -1$ if $p=2$, given by

$$\triangle_\gamma\varphi - r(\gamma)\varphi = -a\varphi^{-P} - q\varphi^{-P'},\quad a\geqslant 0,\quad q\geqslant 0,$$

$$P=(3n-2)/(n-2),\quad P'=n/(n-2),$$

$$a,\quad q\in W^p_{s,\delta+2},\quad \sigma-2\geqslant s>\frac{n}{p},\quad n-2-\frac{n}{p}>\delta>-\frac{n}{p}, \tag{77}$$

has a solution $\varphi>0$, $\varphi-1\in W^p_{s+2,\delta}$ if and only if (M,γ) is in the positive Yamabe class. This solution is such that $\varphi\geqslant 1$. It can be obtained constructively and is unique.

Proof. (1) (''only if'') This part follows from the generalized Brill-Cantor theorem.

(2) (''if'') The manifold (M,γ) is conformal to a manifold (M,γ') with zero scalar curvature, $\gamma'\equiv\psi^{4/(n-2)}\gamma$, $r(\gamma')=0$. Conformal covariance shows that the resolution of the given equation is therefore equivalent to the resolution of an equation of the same type but with no linear term, which, suppressing primes, we write as

$$\triangle_\gamma\varphi = -a\varphi^{-P} - q\varphi^{-P'},\quad a\geqslant 0,\quad q\geqslant 0. \tag{78}$$

This equation admits a constant subsolution $\varphi_-=1$ but no finite constant supersolution. However, it is possible to construct a sequence $u_\nu\in W^p_{s+2,\delta}$ starting from the subsolution $\varphi_-=1$ by solving the equations with $k\geqslant 0$, $k\in W^p_{s,\delta+2}$:

$$\triangle_\gamma u_\nu - k u_\nu = -a(1+u_{\nu-1})^{-P} - q(1+u_{\nu-1})^{-P'} - k u_{\nu-1}. \tag{79}$$

We have $u_\nu\in W^p_{s+2,\delta}\subset C^2_\alpha$ for all α such that $\alpha<\delta+n/p$, hence u_ν tends to zero at infinity and we can use the maximum principle to see that $u_\nu\geqslant 0$. We could choose $k\geqslant Pa+P'q$ and deduce as before from the maximum principle that the sequence u_ν is pointwise increasing, but we do not obtain an upper bound through the maximum principle because we do not have a supersolution. We choose first instead $k=0$ to construct our sequence and write the elliptic estimate, using the fact that $(1+u_\nu)^{-P}\leqslant 1$ since $u_\nu\geqslant 0$,

$$\|u_\nu\|_{W^p_{2,\delta}}\leqslant C\{\|a\|_{W^p_{0,\delta+2}}+\|q\|_{W^p_{0,\delta+2}}\}. \tag{80}$$

The sequence, being uniformly bounded in $W^p_{2,\delta}$, admits a subsequence which converges in the $W^p_{1,\delta'}$ norm, $\delta'<\delta$, to an element $u\in W^p_{2,\delta}$. The rest of the proof is the same as in the general arguments given in Appendix B, except that in the present case, the sequence u_ν is not proven to be monotonic, nor identical to the subsequence which converges. Hence we cannot conclude that the limit u of the subsequence is a solution of Eq. (78).

To obtain a converging sequence, and consequently a solution, we again use Eq. (79), but now with $k\geqslant Pa+P'q$. For Eq. (79) with such a k, the subsequence limit u serves as a supersolution. Therefore, the increasing sequence u_ν is bounded above by u and it converges to it in $W^p_{s+2,\delta}$. We have $\varphi\geqslant 1$. A pointwise upper bound for φ can be deduced from the $W^p_{s+2,\delta}$ norm of $u=\varphi-1$.

Remark. The sequence u_ν and the limit u, bounded in $W^p_{2,\delta}$ norm in terms of the $W^p_{0,\delta+2}$ norms of a and q, are therefore bounded in C^0_α norm in terms of these norms of a and q *if* $p>n/2$.

(3) Uniqueness: the equation with $r(\gamma)=0$ has a unique solution such that φ tends to 1 at infinity because of the monotonicity of the right hand side and the maximum principle. The original equation also has a unique solution. □

IX. SOLUTION FOR SCALED SOURCES

We now prove an existence theorem for the non-linear elliptic equation for φ expressing the Hamiltonian constraint on an arbitrary initial manifold, when there are no unscaled sources.

Theorem (scaled sources). The equation

$$\triangle_\gamma\varphi - r(\gamma)\varphi = f(\cdot,\varphi)\equiv -a\varphi^{-P} - q\varphi^{-P'} + b\varphi^{Q} \tag{81}$$

with $a\geqslant 0$, $q\geqslant 0$, $b\geqslant 0$; $a,q,b\in W^p_{s,\delta+2}$, $s>n/p$, $-n/p<\delta<n-2-n/p$, has a solution $\varphi=1+u$, $u\in W^p_{s+2,\delta}$, $\varphi>0$ which can be obtained constructively, if either (a) or (b) holds:

(a) On the subset where $r(\gamma)<0$

$$|r(\gamma)|\leqslant b. \tag{82}$$

(b) (M,γ) is in the positive Yamabe class.

The solution is unique in either case.

Proof. (a) The solution exists, with the indicated properties, because the equation admits a subsolution φ_-, with $0<\varphi_-$, which is the solution of the equation (from Sec. VII)

$$\triangle_\gamma\varphi_- - r(\gamma)\varphi_- - b\varphi_-^Q = 0. \tag{83}$$

The solution satisfies $\varphi_-\leqslant 1$ under the hypothesis made on $r(\gamma)$ because the equation for φ_- admits then a supersolution equal to 1. The original Lichnerowicz equation (81) admits as supersolution $\varphi_+\geqslant 1$ the solution of the equation (cf. Sec. VIII)

$$\triangle_\gamma \varphi_+ + a\varphi_+^{-P} + q\varphi_+^{-P'} = 0 \tag{84}$$

because we have

$$r(\gamma)\varphi_+ + b\varphi_+^Q \geqslant 0 \quad \text{if} \quad \varphi_+ \geqslant 1 \quad \text{and} \quad r(\gamma) + b \geqslant 0. \tag{85}$$

(b) When (M,γ) is in the positive Yamabe class, the equation is equivalent to an equation of the same type with zero linear term because of conformal covariance. We may then argue existence just as in (a), because the condition on $r(\gamma)$ when it is negative has become vacuous.

The solution tending to 1 at infinity of the equation with $r(\gamma)=0$ is unique because of the monotonicity of f in φ. In the general case one uses the conformal transformation of curvature as in Sec. VI. Take for simplicity of writing q=0. We have now, if φ_i ($i=1$ or 2) is a solution

$$-r(\varphi_i^{2p}\gamma) = -b + a\varphi_i^{-P-Q}; \tag{86}$$

therefore the conformal identity with $u=\varphi_1^{-1}\varphi_2$ gives

$$\triangle_{\varphi_1^{2p}\gamma} u + (b - a\varphi_1^{-(-P+Q)})u = (b - a\varphi_2^{-(P+Q)})u^Q. \tag{87}$$

This equation may be written

$$\triangle_{\varphi_1^{2p}\gamma} u - \left\{ b\left(\frac{u^{Q-1}-1}{u-1}\right) + a\varphi_1^{-(P+Q)}\left(\frac{1-u^{-P-1}}{u-1}\right)\right\} u(u-1) = 0. \tag{88}$$

If $u>0$, $b\geqslant 0$, $a\geqslant 0$ the function u, which tends to 1 at infinity, can only be $u\equiv 1$ on M. □

Remark 1. We see that the condition that (M,γ) be in the positive Yamabe class is not necessary for the existence of a positive solution if $b\not\equiv 0$. However if $b\not\equiv 0$ the Hamiltonian constraint is coupled with the momentum constraint, and its solution is not the whole story.

Remark 2. The condition $b\geqslant -r(\gamma)$ will somewhat be relaxed in the last section but we will require $b>0$.

X. DISCONTINUOUS SOURCES

It is essential for physical applications to admit isolated sources, hence discontinuous functions q. This possibility is included if we extend the previous existence theorem to functions $q\in W^p_{0,\delta+2}$. We also will take $a\in W^p_{0,\delta+2}$ to include the possibility of discontinuous scaled momentum v. We take $d=b\in W^p_{s,\delta+2}$, $s>n/p$. We leave the more general cases for later study.

Theorem. The Lichnerowicz equation with scaled sources,

$$\triangle_\gamma \varphi - r(\gamma)\varphi = f(\cdot,\varphi) \equiv -a\varphi^{-P} - q\varphi^{-P'} + b\varphi^Q, \tag{89}$$

on a (p,σ,ρ), $\sigma>n/p+2$, $\rho>-n/p$ asymptotically Euclidean manifold (M,γ) in the positive Yamabe class has one and only one solution $\varphi>0$, $\varphi-1=u\in W^p_{2,\delta}$ if $a,q \in W^p_{0,\delta+2}$ with $\delta>-\dfrac{n}{p}$, $p>n/2$, $a\geqslant 0$, $q\geqslant 0$, $b\geqslant 0$, $b\in W^p_{s,\delta+2}$, $s>n/p$.

Proof. We first conformally transform the equation to an equation with no linear term. We then proceed as follows. Consider a Cauchy sequence $a_\nu, q_\nu \in W^p_{s,\delta+2}$, $s>n/p$, converging in the $W^p_{0,\delta+2}$ norm to a, q. Denote by $\varphi_\nu = 1+u_\nu$ the solution with coefficients a_ν, q_ν. We know that $u_\nu \in W^p_{s+2,\delta}$ and that there exists numbers $l>0$ [depending only on (M,γ) and b] and $m\geqslant l$ [depending only on (M,γ) and the $W^p_{0,\delta+2}$ norms of a and q] such that $l\leqslant\varphi_\nu\leqslant m$.

The difference $u_\nu - u_\mu$ satisfies the equation

$$\triangle_\gamma(u_\nu-u_\mu) - A_{\mu\nu}(u_\nu-u_\mu) = \varphi_\mu^{-P}(a_\nu-a_\mu) + \varphi_\mu^{-P'}(q_\nu-q_\mu) \tag{90}$$

with

$$A_{\mu\nu} \equiv a_\nu\left(\frac{\varphi_\mu^{-P}-\varphi_\nu^{-P}}{\varphi_\nu-\varphi_\mu}\right) + q_\nu\left(\frac{\varphi_\mu^{-P'}-\varphi_\nu^{-P'}}{\varphi_\nu-\varphi_\mu}\right) + b\left(\frac{\varphi_\nu^{Q}-\varphi_\mu^{Q}}{\varphi_\nu-\varphi_\mu}\right). \tag{91}$$

Recall that for $n=3$ we have $P=7$, $P'=3$ and $Q=5$. The quotients in the above formulas are then polynomials (with coefficients equal to 1) in φ_μ^{-1} and φ_ν^{-1} for the first two, and φ_μ and φ_ν for the third. Therefore, they are on the one hand positive and, on the other hand, uniformly bounded (for any pair ν, μ) because $0<l\leqslant\varphi_\mu$, $\varphi_\nu\leqslant m$. For general n the numbers P, P' and Q are positive rationals, the quotients in the formula are also positive and uniformly bounded. We deduce from this uniform boundedness that there exists a number N such that

$$\|A_{\mu\nu}\|_{W^p_{0,\delta+2}} \leqslant N\{\|a_\nu\|_{W^p_{0,\delta+2}} + \|q_\nu\|_{W^p_{0,\delta+2}} + \|b\|_{W^p_{0,\delta+2}}\}. \tag{92}$$

We infer from this estimate and the positivity of $A_{\mu\nu}$ that the operator $\triangle_\gamma - A_{\mu\nu}$ is injective in $W^p_{2,\delta}$ (see Condition 2 in Theorem 1 of Appendix B). Therefore, there exists a number C depending only on (M,γ), the $W^p_{s,\delta+2}$ norm of b, and the $W^p_{0,\delta+2}$ norms of a and q such that

$$\|u_\nu-u_\mu\|_{W^p_{2,\delta}} \leqslant C\{\|a_\nu-a_\mu\|_{W^p_{0,\delta+2}} + \|q_\nu-q_\mu\|_{W^p_{0,\delta+2}}\}. \tag{93}$$

Since a_ν and q_ν are Cauchy sequences, the same is true of u_ν, because of the above inequality. Hence u_ν converges in $W^p_{2,\delta}$ to a limit $u\in W^p_{2,\delta}$. The convergence is *a fortiori* in C^0_α if $p>n/2$, and for some positive α, since $\delta>-n/p$. The u_ν's are such that $1+u_\nu\geqslant l>0$; therefore also $\varphi=1+u\geqslant l>0$. The function φ satisfies the Lichnerowicz equation (in the sense of generalized derivatives) with scaled sources. □

XI. GENERAL CASES

In the case where there are unscaled sources the coefficient d in the Lichnerowicz equation is negative or zero on a maximal initial manifold M. It can take different signs if M is not maximal. The previous simple method to obtain sub and super solutions does not apply. We will then look for constant sub and super solutions l and m, $0<l\leqslant 1\leqslant m$. We will also obtain a new theorem for the Lichnerowicz equation in the case of scaled sources on a non-maximal submanifold. To make the algebra easier we restrict our study to the important physical case $n=3$. Results along the same lines can likely be obtained for general n. The equation is then

$$\triangle_\gamma\varphi - r\varphi + a\varphi^{-7} + q\varphi^{-3} - d\varphi^5 = 0,$$
$$a\geqslant 0,\ \ q\geqslant 0,\ \ d=b-c,\ \ b\geqslant 0,\ \ c\geqslant 0. \tag{94}$$

The numbers l and m are admissible sub and supersolutions if they satisfy on M the following inequalities:

$$P_x(l^4)\leqslant 0,\ \ P_x(m^4)\geqslant 0,\ \ \text{for all}\ \ x\in M,\ \ 0<l\leqslant 1\leqslant m \tag{95}$$

where P_x is the polynomial

$$P_x(z)\equiv d(x)z^3 + r(x)z^2 - q(x)z - a(x). \tag{96}$$

Remark. In the case of $n>3$ the problem is the study of the sign of the function:

$$F_x(z)\equiv d(x)z^n + r(x)z^{n-1} - q(x)z^{(n-1)/2} - a(x) \tag{97}$$

for numbers $l^{4/(n-2)}$ and $m^{4/(n-2)}$.

Since all the coefficients in P_x tend to zero at infinity the conditions that we will obtain depend on the ratios of their respective decays.

We denote by M_+ the subset of M where $d>0$, by M_- the subset where $d<0$, by M_0 the subset where $d=0$. In the case of isolated sources M_- is a compact subset of M. We study the sign of P_x on these various subsets. The derivative of P_x is

$$dP_x/dz = 3d(x)z^2 + 2r(x)z - q(x). \tag{98}$$

(1) On M_+, $d(x)>0$, the derivative dP_x/dz has 2 roots of opposite signs. The positive root is

$$\zeta_+(x) = \frac{-r(x)+[r^2(x)+3d(x)q(x)]^{1/2}}{3d(x)}\geqslant 0. \tag{99}$$

We have $\zeta_+(x)>0$ if $r(x)<0$, or if $r(x)\geqslant 0$ and $q(x)>0$.

dP_x/dz is equal to $-q(x)\leqslant 0$ for $z=0$ and is negative or zero as long as $z\leqslant\zeta_+(x)$. Therefore P_x decreases from $a(x)\leqslant 0$ for $z=0$ to a minimum for $z=\zeta_+(x)$ and then increases to $+\infty$ when z increases to $+\infty$. Hence P_x has one and only one positive root $z_+(x)$. We have $P_x(z)\geqslant 0$ as long as $z\geqslant z_+(x)$.

There exists $l(x)>0$ such that $P_x(l^4(x))\leqslant 0$ if and only if $z_+(x)>0$. Indeed numbers $l(x)$ and $m(x)$ such that

$$0<l(x)\leqslant z_+(x)\leqslant m(x),\ \ x\in M_+ \tag{100}$$

satisfy

$$P_x(l^4(x))\leqslant 0,\ \ P_x(m^4(x))\geqslant 0. \tag{101}$$

Lemma 1. There exist numbers l_+ and m_+ such that

$$P_x(l_+^4)\leqslant 0,\ \ P_x(m_+^4)\geqslant 0\quad \text{for all}\quad x\in M_+ \tag{102}$$

if and only if

$$\inf_{x\in M_+} z_+(x)>0 \tag{103}$$

and

$$\sup_{x\in M_+} z_+(x)<+\infty. \tag{104}$$

Sufficient conditions for the first inequality are

$$\inf_{x\in M_+}\left\{\frac{-r(x)}{3d(x)}+\left(\frac{r^2(x)}{9d^2(x)}+\frac{q(x)}{3d(x)}\right)^{1/2}\right\}>0 \tag{105}$$

or

$$\inf_{x\in M_+}\frac{a(x)}{d(x)+|r(x)|}>0. \tag{106}$$

For the second inequality they are that $|r(x)|/d(x)$, $q(x)/d(x)$, $a(x)/d(x)$ be uniformly bounded on M_+.

Proof. The necessary condition as well as the first sufficient condition are consequences of the previous study. Sufficient conditions for this first condition to be satisfied are that one of the two terms in the sum has a strictly positive infimum. The second sufficient condition results from the fact (elementary calculus) that $P_x(z)\leqslant 0$ if

$$z\leqslant\min\left(1,\frac{a(x)}{d(x)+|r(x)|}\right).\quad\square \tag{107}$$

Remark. The sufficient conditions will be satisfied on the whole of M_+ if we can split it into two subsets, $M_+\equiv M_1\cup M_2$, such that

$$\inf_{x\in M_1}\frac{a(x)+q(x)}{d(x)+|r(x)|}>0\ \ \text{and}\ \ \inf_{x\in M_2}\frac{-r(x)}{d(x)}>0. \tag{108}$$

This pair of inequalities can be realized when M is compact and $a(x)+q(x)\not\equiv 0$ by a conformal change of choice of the metric γ to a metric γ' having a strictly negative curvature in the complement of M_1 in M. Such a construction can also eventually be made in the asymptotically flat case, by resolution of an adequate Dirichlet problem.

(2) On M_-, $d(x)<0$.

We have $P_x(z)<0$ for all $z>0$, hence no admissible $m(x)$, if $r(x)\leqslant 0$. We therefore suppose $r(x)>0$ for all $x\in M_-$. If $r^2(x)+3q(x)d(x)\leqslant 0$, we have $dP_x/dz\leqslant 0$ for all z; and the polynomial P_x takes non-negative values only if it is identically zero. If $r^2(x)+3q(x)d(x)>0$ the polynomial dP/dz has two positive roots:

$$0\leqslant\zeta_1(x)=\frac{r(x)-\{r^2(x)-3q(x)|d(x)|\}^{1/2}}{3|d(x)|}, \tag{109}$$

$$\zeta_2(x)=\frac{r(x)+\{r^2(x)-3q(x)|d(x)|\}^{1/2}}{3|d(x)|}>0, \quad (110)$$

with $\zeta_1(x)>0$ if and only if $q(x)\neq 0$.

The polynomial P_x decreases for $0\leqslant z\leqslant\zeta_1(x)$, increases for $\zeta_1(x)\leqslant z\leqslant\zeta_2(x)$, and decreases to $-\infty$ for $z\geqslant\zeta_2(x)$. We have $P_x(0)=-a(x)\leqslant 0$. Therefore P_x takes negative values for some $z>0$ if either $a(x)>0$ or $\zeta_1(x)>0$, i.e., $q(x)>0$. The polynomial P_x takes positive values, equivalently admits two positive roots $z_1(x)$ and $z_2(x)$ which are such that

$$\zeta_1(x)\leqslant z_1(x)\leqslant\zeta_2(x)\leqslant z_2(x), \quad (111)$$

if and only if its maximum, attained for $z=\zeta_2(x)$, is positive,

$$P_x(\zeta_2(x))\geqslant 0. \quad (112)$$

We have then $P_x(z)\leqslant 0$ for $0\leqslant z\leqslant z_1(x)$, and $P_x(z)\geqslant 0$ for $z\geqslant z_2(x)$. If

$$r^2(x)+3q(x)d(x)\leqslant 0, \quad (113)$$

the polynomial P_x is always decreasing. It takes positive (i.e., non-negative) values only if it is identically zero.

Lemma 2. Suppose that $r(x)>0$, $r^2(x)-3q(x)d(x)>0$ and $P_x(\zeta_2(x))\geqslant 0$ for all $x\in M_-$. There exist numbers l_- and m_- such that

$$P_x(l_-^4)\leqslant 0,\ P_x(m_-^4)\geqslant 0 \ \text{ for all } \ x\in M_- \quad (114)$$

if the following conditions are satisfied:

$$\inf_{M_-} z_1(x)>0,\quad \sup_{x\in M_-} z_1(x)\leqslant \inf_{x\in M_-} z_2(x), \quad (115)$$

and $|r(x)|/|d(x)|$, $q(x)/|d(x)|$, $a(x)/|d(x)|$ are uniformly bounded on M_-.

Proof. All numbers l_- and m_- such that

$$l_-\leqslant z_1(x),\ \ z_1(x)\leqslant m_-\leqslant z_2(x) \ \text{ for all } \ x\in M_- \quad (116)$$

are such that $P_x(l_-^4)\leqslant 0$, $P_x(m_-^4)\geqslant 0$. These numbers exist, with $l_->0$ and $+\infty\geqslant m_-\geqslant l_-$ under the given conditions.□

(3) On M_0, $d(x)=0$, P_x reduces to a second order polynomial

$$P_x(z)=r(x)z^2-q(x)z-a(x). \quad (117)$$

If $r(x)\leqslant 0$, then $P_x<0$ as soon as $z>0$ except if it is identically zero. We suppose $r(x)>0$. Then P_x admits one positive root $z_0(x)$:

$$z_0(x)=(2r(x))^{-1}\{q^2(x)+4a(x)r(x)\}^{1/2}\geqslant 0. \quad (118)$$

Lemma 3. We suppose that $r(x)>0$ for all $x\in M_0$.

There exist $l_0>0$ and $m_0\geqslant l_0$ such that $P_x(l_0^4)\leqslant 0$ and $P_x(m_0^4)\geqslant 0$ for all $x\in M_0$ if and only if

$$\inf_{x\in M_0}\frac{a(x)}{r(x)}>0 \ \text{ or } \ \inf_{x\in M_0}\frac{q(x)}{r(x)}>0 \quad (119)$$

and

$$\sup_{x\in M_0}\frac{a(x)}{r(x)}<+\infty \ \text{ and } \ \sup_{x\in M_0}\frac{q(x)}{r(x)}<+\infty. \quad (120)$$

Proof. Under one or the other of the first inequalities we have

$$\inf_{x\in M_0} z_0(x)>0. \quad (121)$$

The other ones insure

$$\sup_{x\in M_0} z_0(x)<+\infty. \quad (122)$$

We set

$$l_0^4=\inf_{M_0} z_0(x),\quad m_0^4=\sup_{M_0} z_0(x). \quad (123)$$

All numbers l and m satisfying the following inequalities

$$0<l\leqslant l_0=\inf_{M_0} z_0(x)\leqslant \sup_{x\in M_0} z_0(x)=m_0\leqslant m \quad (124)$$

satisfy $P_x(l^4)\leqslant 0$ and $P_x(m^4)\geqslant 0$ for all $x\in M_0$. □

The following lemma is an immediate consequence of the previous three.

Lemma 4. We suppose that the conditions given in the lemmas 1, 2, and 3 for the existence of l_-, l_+, l_0 and m_-, m_+, m_0 are satisfied. Then there exists l and m such that:

$$0<l\leqslant 1\leqslant m \ \text{ and } \ P_x(l^4)\leqslant 0,\ P_x(m^4)\geqslant 0 \ \text{ for all } \ x\in M \quad (125)$$

if the following inequalities hold:

$$m_+\leqslant m_-,\ \ m_0\leqslant m_-,\ \ m_-\geqslant 1. \quad (126)$$

Proof. Take

$$m=m_-,\quad l=\min(1,l_0,l_+,l_-). \quad (127)$$

Then l and m satisfy the required inequalities for all $x\in M$. They are admissible sub and supersolutions. □

Theorem. On a 3-dimensional asymptotically Euclidean manifold the Lichnerowicz equation

$$\triangle_\gamma\varphi-r\varphi+a\varphi^{-7}+q\varphi^{-3}-d\varphi^5=0,$$

$$a\geqslant 0,\ \ q\geqslant 0,\ \ d=b-c,\ \ b\geqslant 0,\ \ c\geqslant 0, \quad (128)$$

with r, a, q, $d\in W^p_{s,\delta+2}$, $s>n/p$, $-n/p<\delta<n-2-n/p$, admits a solution $\varphi>0$, $\varphi-1\in W^p_{s+2,\delta}$ if the assumptions of Lemma 4 are satisfied.

Corollary. No Unscaled Sources, $d\equiv b>0$. The Lichnerowicz equation has a solution $\varphi>0$, $\varphi-1\in W^p_{s+2,\delta}$ if

(i) The quotients $|r(x)|/b(x)$, $q(x)/b(x)$, $a(x)/b(x)$ are uniformly bounded.

(ii) There is a positive number $\epsilon>0$ such that if $[a(x)+q(x)]/b(x)<\epsilon$, then

$$r(x)<0 \quad \text{and} \quad \frac{|r(x)|}{b(x)}>\epsilon'>0. \tag{129}$$

This last condition can be achieved if $a+q\not\equiv 0$ by a conformal transformation and solution of a Dirichlet problem in the subset of M where $(a+q)/d<\epsilon$, so long as this subset is compact (cf. [1,2]).

XII. UNSCALED SOURCES, CASE $n=3$

We treat in this section the Hamiltonian constraint for unscaled sources in the case $n=3$. The Lichnerowicz equation reads

$$\triangle_\gamma \varphi - r\varphi + a\varphi^{-7} + c\varphi^5 = 0. \tag{130}$$

The functions $a\geqslant 0$ and $c\geqslant 0$ are given on (M,γ).

Theorem. Let (M,γ) be a (p,σ,ρ) asymptotically Euclidean manifold, $\sigma>n/p+2$, $\sigma>-n/p$ with $r>0$. Let a, $c \in W^p_{s,\delta+2}$ be given on (M,γ), $s>n/p$, $\sigma>-n/p$. There exists an open set of values of a and c such that the Lichnerowicz equation with unscaled sources has a solution $\varphi>0$, with $1-\varphi\in W^p_{s+2,\delta}$.

Proof. We look for constant admissible sub and supersolutions l and m such that

$$0<l\leqslant 1\leqslant m,$$

$$P_x(l^4)\geqslant 0, \;\; P_x(m^4)\leqslant 0, \;\; \text{for all} \;\; x\in M, \tag{131}$$

where P is the polynomial,

$$P_x(z)\equiv c(x)z^3 - r(x)z^2 + a(x). \tag{132}$$

(1) Case $c>0$.

We set $z=X^{-1}$ and consider the polynomial which has the same sign as P_x,

$$Q(X)\equiv a\{X^3 - a^{-1}rX + a^{-1}c\}. \tag{133}$$

This polynomial has 3 real roots if

$$4r^3\geqslant 27ac^2. \tag{134}$$

Two of these roots are positive, given by the classical formulas:

$$X_2\equiv\lambda \sin\frac{\theta}{3}, \;\; X_1=\lambda\sin\frac{\theta+2\pi}{3} \tag{135}$$

with

$$\lambda=\frac{2r^{1/2}}{(3a)^{1/2}}, \;\; \sin\theta=\frac{3c(3a)^{1/2}}{2rr^{1/2}}, \;\; 0\leqslant\theta\leqslant\frac{\pi}{2}. \tag{136}$$

The corresponding roots of P_x are

$$z_1\equiv\frac{1}{X_1}\leqslant\frac{1}{X_2}\equiv z_2(x). \tag{137}$$

We have $P_x(z)\geqslant 0$ for $0\leqslant z\leqslant z_1(x)$, $P_x(z)\leqslant 0$ for $z_1(x)\leqslant z\leqslant z_2(x)$.

(2) Case $c(x)=0$.

The polynomial P_x reduces to

$$P_x(z)\equiv -r(x)z^2 + a(x). \tag{138}$$

We have $P_x(z)\geqslant 0$ for $0\leqslant z\leqslant (r^{-1}a)^{1/2}$, and $P_x(z)\leqslant 0$ for $z\geqslant (r^{-1}a)^{1/2}$. Note that $(r^{-1}a)^{1/2}$ is the value for $c=0$ of the previously computed z_1 while the previous z_2 tends to infinity when c tends to zero. The cases $c(x)\geqslant 0$ are thus unified.

The following constants l and m are sub and supersolutions if

$$l\leqslant z_1(x), \;\; z_1(x)\leqslant m\leqslant z_2(x) \;\; \text{for all} \;\; x\in M. \tag{139}$$

They exist, satisfying the required properties $0<l\leqslant 1\leqslant m$, if

$$\inf_{x\in M} z_1(x)>0, \;\; \inf_{x\in M} z_2(x)\geqslant \max\{1, \sup_{x\in M} z_1(x)\}. \tag{140}$$

One then takes

$$l=\min\{1, \inf_{x\in M} z_1(x)\}, \;\; m=\inf_{x\in M} z_2(x). \tag{141}$$

We give below sufficient conditions to satisfy the various inequalities, using the expressions

$$z_1=\frac{2}{\sqrt{3}}\frac{r^{-1}a}{\sin((\theta+2\pi)/3)}, \;\; z_2=\frac{2}{\sqrt{3}}\frac{r^{-1}a}{\sin(\theta/3)},$$

$$\text{with} \;\; 0\leqslant\theta\leqslant\frac{\pi}{2}. \tag{142}$$

The functions $\sin(\theta/3)$ and $\sin((\theta+2\pi)/3)$ are respectively increasing and decreasing when θ increases from 0 to $\pi/2$. Denote by $\theta_{\min}$ and $\theta_{\max}$ the infimum and supremum of θ on M defined as solutions between 0 and $\pi/2$ of the equations

$$\sin\theta_{\min}=\inf_M\frac{3c(3a)^{1/2}}{2rr^{1/2}}, \;\; \sin\theta_{\max}=\sup_M\frac{3c(3a)^{1/2}}{2rr^{1/2}}. \tag{143}$$

Therefore

$$\inf_M z_1\geqslant\frac{2}{\sqrt{3}}\inf_M(r^{-1}a)\{\sin((\theta_{\min}+2\pi)/3)\}^{-1}$$

$$\sup_M z_1\leqslant\frac{2}{\sqrt{3}}\sup_M(r^{-1}a)\{\sin((\theta_{\max}+2\pi)/3)\}^{-1}$$

$$\inf_M z_2\geqslant\frac{2}{\sqrt{3}}\inf_M(r^{-1}a)\{\sin\theta_{\max}\}^{-1}. \tag{144}$$

We find by elementary calculus that

$$0\leqslant\sin((\theta_{\max}+2\pi)/3)-\sin(\theta_{\max}/3)=\sqrt{3}\cos((\theta_{\max}+\pi)/3)\leqslant\frac{\sqrt{3}}{2}. \tag{145}$$

The minimum zero is attained for $\theta_{\max}=\pi/2$, i.e.,

$$\sup_M\frac{3c(3a)^{1/2}}{2rr^{1/2}}=1. \tag{146}$$

To insure the existence of constants l and m satisfying the required properties we suppose

$$l_-=\frac{2}{\sqrt{3}}\inf_M(r^{-1}a)\{\sin((\theta_{\min}+2\pi)/3)\}^{-1}>0, \tag{147}$$

and we set

$$l=\min\{l_-,1\}. \tag{148}$$

We suppose also

$$m_+\equiv\frac{2}{\sqrt{3}}\inf_M(r^{-1}a)\{\sin(\theta_{\max}/3)\}^{-1}\geqslant 1 \tag{149}$$

and

$$m_-\equiv\frac{2}{\sqrt{3}}\sup_M(r^{-1}a)\{\sin((\theta_{\max}+2\pi)/3)\}^{-1}\leqslant m_+. \tag{150}$$

The condition $m_-\leqslant m_+$ can be satisfied for an open set of values of the coefficients c, a (given r) due to the previous elementary study. We can take for m any number between $\max\{1,m_-\}$ and m_+. The numbers l and m so chosen are admissible sub and supersolutions of the Lichnerowicz equation. The existence of a solution φ with the required properties results from the general theorem, given in Sec. V. □

XIII. COUPLED SYSTEM

In the conformal method the momentum and the Hamiltonian constraints decouple when the initial manifold M has constant mean extrinsic curvature and the unscaled sources have a momentum $N=0$. The theorems of the previous sections are then sufficient to give existence, non-existence or uniqueness theorems of the systems of constraints. We will in the next sections study cases where one of these hypothesis does not hold; hence the constraints do not decouple.

XIV. IMPLICIT FUNCTION THEOREM METHOD

The use of the implicit function theorem is the simplest way of proving existence of solutions of equations in the neighborhood of a given one. It works as follows.

Let U and V be open sets of Banach spaces X and Y and let $\mathcal{F}$ be a C^1 mapping from $U\times V$ into another Banach space Z:

$$\mathcal{F}:U\times V\to Z \text{ by } (x,y)\mapsto\mathcal{F}(x,y). \tag{151}$$

Suppose that the partial derivative of $\mathcal{F}$ with respect to y at a point $(x_0,y_0)\in U\times V$, $\mathcal{F}'_y(x_0,y_0)$, is an isomorphism from Y onto Z; then there exists a neighborhood W of x_0 in U such that the equation

$$\mathcal{F}(x,y)=0 \tag{152}$$

has a solution $y\in V$ for each $x\in W$.

We consider the quantities q and v (scaled sources) together with N and u, a traceless symmetric 2-tensor as given on the asymptotically Euclidean manifold (M,γ), with q, v, $1-N^{-1}\in W^p_{s,\delta+2}$, $u\in W^p_{s+1,\delta+1}$. We will discuss the existence of φ and β as we perturb J and τ away from zero. The points x,y and the Banach spaces X, Y, and Z are as follows:

$$x\equiv(\tau,J)\in X\equiv W^p_{s+1,\delta+1}\times W^p_{s,\delta+2},$$

$$y\equiv(\beta,\varphi-1)\in V\equiv Y\cap\{\varphi>0\},\quad Y\equiv W^p_{s+2,\delta}\times W^p_{s+2,\delta},$$

$$Z\equiv W^p_{s,\delta+2}\times W^p_{s,\delta+2}. \tag{153}$$

The mapping $\mathcal{F}$ is given by

$$\mathcal{F}(x,y)\equiv(\mathcal{H}(\tau,J;\varphi,\beta),\ \mathcal{M}(\tau,J;\varphi,\beta)) \tag{154}$$

where $\mathcal{H}$ and $\mathcal{M}$ are the left hand sides of the conformal formulation of the constraints, $\mathcal{H}(x,y)\equiv\triangle_\gamma\varphi-f(\cdot;\tau;\beta,\varphi)$, $\mathcal{M}(x,y)\equiv\overline{\nabla}\cdot((2N)^{-1}\mathcal{L}\beta)-h(\cdot;\tau,J;\varphi)$.

The multiplication properties of weighted Sobolev spaces show that $\mathcal{F}$ is a C^1 mapping from $X\times V$ into Z if $s>n/p$ and $\delta>-n/p$. The partial derivative $\mathcal{F}'_y$ at a point (x,y) is the linear mapping from Y into Z given by

$$(\delta\beta,\delta\varphi)\mapsto(\mathcal{H}'_y,\mathcal{M}'_y)\cdot(\delta\beta,\delta\varphi) \tag{155}$$

with [A is given by Eq. (16)]

$$\mathcal{H}'_y\cdot(\delta\beta,\delta\varphi)\equiv\triangle_\gamma\delta\varphi-\alpha\,\delta\varphi+2k_n\varphi^{-P}2N^{-1}A\cdot\mathcal{L}\delta\beta,$$

$$\alpha\equiv r+P\varphi^{-P-1}a(\beta)+P'q\varphi^{-P'-1}+dQ\varphi^{Q-1},$$

$$\mathcal{M}'_y\cdot(\delta\beta,\delta\varphi)\equiv\nabla\cdot(2N^{-1}\mathcal{L}\delta\beta)-\lambda\,\delta\varphi,$$

$$\lambda\equiv\frac{2(n-1)}{(n-2)}\varphi^{(n+2)/(n-2)}D\tau+\frac{2(n+2)}{n-2}\times\varphi^{(n+6)/(n-2)}J. \tag{156}$$

Theorem. Specify on the asymptotically Euclidean manifold (M,γ) a traceless tensor $u\in W^p_{s+1,\delta+1}$, a scalar $N>0$ with $N-1\in W^p_{s+2,\delta}$, and scaled and unscaled sources q, v, $c\in W^p_{s,\delta+2}$, $s>n/p$, $-n/p<\delta<n-2-n/p$. Let (β_0,φ_0) be a solution of the corresponding constraints with $D\tau_0=0$ (hence $\tau_0=0$ since $\tau_0\in W^p_{s+1,\delta+1}$), and $J_0=0$. Suppose that on M

$$\alpha_0 \equiv r + P\varphi_0^{-P-1} a(\beta_0) + P'\varphi_0^{-P'-1} q - cQ\varphi_0^{Q-1} \geqslant 0. \tag{157}$$

Then there exists a neighborhood U of (τ_0, J_0) in X such that the coupled constraints have one and only one solution (β,φ), $\varphi>0$, $(\beta,1-\varphi)\in Y$.

Proof. Under the hypotheses that we have made the partial derivative $\mathcal{F}'_y(x_0,y_0)$ is an isomorphism from Y onto Z because the system of linear elliptic equations

$$\bar{\nabla}\cdot\{(2N)^{-1}\mathcal{L}\delta\beta\} = h$$

$$\triangle_\gamma \delta\varphi - \alpha_0 \delta\varphi = -2k_n \varphi^{-P}(2N^{-1}A)\cdot\mathcal{L}\delta\beta + k \tag{158}$$

has one and only one solution $(\delta\varphi,\delta\beta)\in Y$ for any pair $(h,k)\in Z$. □

Corollary. The conclusion of the theorem holds if (M,γ) is in the positive Yamabe class and $d\geqslant 0$ (realized in particular if all sources are scaled) without having to consider the sign of α_0.

Proof. If (M,γ) is in the positive Yamabe class we can choose (M,γ') in the same conformal class and such that $r(\gamma')=0$. To the data N, u, q, v correspond data N', u', q', v' and to the solution β_0,φ_0 corresponds a solution of the transformed conformal constraints. The corresponding α'_0 is positive and the conclusion of the theorem applies to the transformed system, hence also to the original system. □

XV. CONSTRUCTIVE METHOD WITH SCALED SOURCES

We will give in the next two sections another method to obtain solutions of the coupled system. It will give new results for unscaled sources on a maximal manifold. It is possible, though not proven, that the hypotheses we make in the case of scaled sources on a non-maximal manifold imply that this manifold is in the positive Yamabe class.

Lemma 1. The equation

$$\triangle_\gamma\varphi = f(\cdot,\varphi) \equiv r\varphi - a\varphi^{-P} - q\varphi^{-P'} + b\varphi^{Q} \tag{159}$$

with r, a, q, b satisfying the hypothesis of the theorem in Sec. IX admits as a supersolution the solution $\Phi(A)$, $1-\Phi(A)\in W^p_{s+2,\delta}$, of the equation

$$\triangle_\gamma\varphi = f_A(\cdot,\varphi) \equiv -A\varphi^{-P} - q\varphi^{-P'} \tag{160}$$

if $a\leqslant A$, with A a given function in $W^p_{s,\delta+2}$.

Proof. The function $\Phi(A)$ exists by the theorem in Sec. IX. It satisfies

$$\triangle_\gamma\Phi - f(\cdot,\Phi) = f_A(\cdot,\Phi) - f(\cdot,\Phi) \leqslant (a-A)\Phi^{-P} \leqslant 0; \tag{161}$$

hence it is a supersolution. □

Theorem. Under the conditions on r and b given in the theorem of Sec. IX there exists a number $\epsilon>0$ such that if

$$\|D\tau\|_{W^p_{0,\delta+2}} \leqslant \epsilon, \quad n>p, \quad \delta>-n/p, \tag{162}$$

the coupled constraints admit a solution (β,φ) with β, $1-\varphi\in W^p_{s+2,\delta}$.

Proof. We will construct a sequence (φ_ν,β_ν) by the inductive algorithm

$$\begin{aligned}\triangle_\gamma\varphi_\nu &= f(\cdot,\beta_{\nu-1},\varphi_\nu) \equiv r\varphi_\nu - a(\beta_{\nu-1})\varphi_\nu^{-P} - q\varphi_\nu^{-P'} \\ &\quad + b\varphi_\nu^{Q}, \\ \nabla_i\{(2N^{-1})(\mathcal{L}\beta_\nu)^{ij}\} &= h^j(\cdot,\varphi_\nu) \\ &\equiv \nabla_i\{(2N)^{-1}u^{ij}\} \\ &\quad + \frac{n-1}{n}\varphi_\nu^{2n/(n-2)}\nabla^j\tau + v^j\end{aligned} \tag{163}$$

with

$$a(\beta) \equiv k_n(2N)^{-2}|-u+\mathcal{L}\beta|^2, \quad k_n \equiv \frac{n-2}{4(n-1)}. \tag{164}$$

The equations for the φ_ν's admit all the same subsolution φ_-, which depends only on r and b. They admit the same supersolution $\Phi(A)$ if there exists $A\in W^p_{s,\delta+2}$ such that $a(\beta_{\nu-1})\leqslant A$ for all ν.

We start for instance from $\beta_0=0$ and choose A such that

$$A > a(0) \equiv k_n(2N)^{-2}|u|^2. \tag{165}$$

Suppose that $\beta_{\nu-1}\in W^p_{s,\delta+2}$ and that $a(\beta_{\nu-1})<A$. Then φ_ν exists, $1-\varphi_\nu\in W^p_{s+2,\delta}$, and $\varphi_-\leqslant\varphi_\nu\leqslant\Phi(A)$. Also, $\beta_\nu\in W^p_{s+2,\delta}$ exists, and there is a constant C (cf. Appendix A) depending only on (M,γ) such that

$$\|\beta_\nu\|_{W^p_{2,\delta}} \leqslant C\left\{\frac{n}{n-1}\|D\tau\|_{W^p_{0,\delta+2}}\sup_M\Phi(A)^{2n/(n-2)} + \|P\|_{W^p_{0,\delta+2}}\right\}, \tag{166}$$

where P is the given vector

$$P \equiv \nabla\cdot\{(2N)^{-1}u\} + v. \tag{167}$$

The weighted Sobolev multiplication theorem and the expression for $a(\beta)$ imply that if $\beta_\nu\in W^p_{2,\delta}$, $n/p<1$, $\delta>-n/p$, then

$$a(\beta_\nu)\in W^p_{1,\delta'+2} \quad \text{for all} \quad \delta' \quad \text{such that} \quad \delta'<\delta+\left(\delta+\frac{n}{p}\right), \tag{168}$$

and there exists a number C such that

$$\|a(\beta_\nu)\|_{W^p_{1,\delta'+2}} \leqslant C\{\|u\|^2_{W^p_{1,\delta+1}} + \|\beta_\nu\|^2_{W^p_{2,\delta}}\}. \tag{169}$$

By the weighted Sobolev inclusion theorem, there exists then another constant C such that

$$\|a(\beta_\nu)\|_{C^0_{\delta''}}\leqslant C\|a(\beta_\nu)\|_{W^p_{1,\delta'+2}} \tag{170}$$

for all $\delta''<\delta'+2+n/p$, hence also for all $\delta''<2\delta+2+n/p$.

Since $\delta+n/p>0$ there exists a number α such that

$$\delta+2+\frac{n}{p}<\alpha<2\delta+2+\frac{n}{p}. \tag{171}$$

We choose for A a function of the form, with μ some positive constant,

$$A=\mu/\sigma^\alpha, \tag{172}$$

where $\sigma\equiv 1+d^2$ (see Sec. III). We have $A\in W^p_{0,\delta+2}$ since $\alpha>\delta+2+n/p$. For such a function A, the inequality $a(\beta_\nu)\leqslant A$ is equivalent to

$$\sigma^\alpha a(\beta_\nu)\leqslant\mu,\ \text{ i.e., }\ \|a(\beta_\nu)\|_{W^2_0,\alpha}\leqslant\mu. \tag{173}$$

Using the previous estimates we see that a sufficient condition to insure on M the inequality $a(\beta_\nu)\leqslant A$ is to have some number depending only on (M,γ) and N, denoted by C, that has the property

$$\|D\tau\|^2_{W^p_{0,\delta+2}}\sup_M\Phi_\mu^{4n/(n-2)}+\|v\|^2_{W^p_{0,\delta+2}}+\|u\|^2_{W^p_{1,\delta+1}}\leqslant C\mu, \tag{174}$$

where we have set $\Phi_\mu\equiv\Phi(\sigma^{-\alpha}\mu)$.

We choose μ large enough to have

$$C\mu>S,\ \ S\equiv\|v\|^2_{W^p_{0,\delta+2}}+\|u\|^2_{W^p_{0+1,\delta+1}}. \tag{175}$$

The inequality obtained above shows that we can construct $\varphi_{\nu+1}$ and hence $\beta_{\nu+1}$, enjoying the same properties as φ_ν, β_ν if $D\tau$ is sufficiently small in $W^p_{0,\delta+2}$ norm. The existence of a solution (φ,β) of the coupled constraints as limit of a subsequence is proved by a compactness argument and elliptic regularity as in the case of the Hamiltonian constraint with a given β. □

Remark. The number ϵ depends on the choice of μ and the function Φ_μ. Neither of those depends on r or on b, i.e., on τ. However, our theorem imposes a restriction on the size of τ on the subset of the manifold M where $r<0$, since it supposes that $b\equiv(n-2)/4n\tau^2\geqslant -r$. This could lead to a difficulty, pointed out by O'Murchadha, on an asymptotically Euclidean manifold where r would be too negative. Indeed it is known that in the case $p=2$, $\delta=-1$ there exists a constant C_P such that the following Poincaré estimate gives an upper bound of $|\tau|$ in terms of $|D\tau|$:

$$\|\tau\|_{H_{0,-1}}\leqslant C_P\|D\tau\|_{H_{0,0}}. \tag{176}$$

If we suppose that an analogous inequality holds for $p>n$, $\delta>-n/p$, i.e, that there exists a constant C_P such that

$$\|\tau\|_{W^p_{1,\delta}}\leqslant C_P\|D\tau\|_{W^p_{0,\delta+1}}, \tag{177}$$

then by using the Sobolev embedding theorem:

$$\sup_M\sigma^{2\alpha}\tau^2\leqslant CC_p^2\|D\tau\|^2_{W^p_{0,\delta+1}},\ \text{ for all }\ \alpha<\delta+\frac{n}{p}. \tag{178}$$

This inequality implies that the condition $b\geqslant -r$ can be satisfied only if r satisfies the condition

$$-r\sigma^{2\alpha}\leqslant\frac{4n}{n-2}CC_P^2\epsilon. \tag{179}$$

We can estimate the value of ϵ as follows, considering for simplicity the vacuum case $q=0=v=0$. The supersolution Φ_μ satisfies the equation

$$\triangle_\gamma\Phi_\mu=-A\Phi_\mu^{-P}. \tag{180}$$

We know that $\Phi_\mu\geqslant 1$; therefore $A\Phi_\mu^{-P}\leqslant A$ and $\Phi_\mu\leqslant\Psi_\mu$, where Ψ_μ is the solution with $\Psi_\mu-1\in W^p_{s+2,\delta}$ of the equation

$$\triangle_\gamma\Psi_\mu=-A\equiv-\mu/\sigma^\alpha. \tag{181}$$

Obviously $\Psi_\mu=1+\mu w_1$, where w_1 depends only on (M,γ), satisfies the equation

$$\triangle_\gamma w_1=-1/\sigma^\alpha \tag{182}$$

and tends to zero at infinity. The inequality to satisfy is then

$$\|D\tau\|^2_{W^p_{0,\delta+2}}\leqslant(C\mu-S)(1+C_1\mu)^{-4n/(n-2)},$$

$$\text{with }\ C_1=\sup_M w_1. \tag{183}$$

The right hand side is maximum for a finite value $\mu=\mu_0$ with μ_0 given by

$$\mu_0=\frac{(n-2)C+4nC_1S}{(3n+2)CC_1}. \tag{184}$$

(Note that $\mu_0>S/C$ if $n>2$.) We find therefore

$$\epsilon=\left\{\frac{4n}{n+2}\left(1+\frac{C_1S}{C}\right)\right\}^{-4n/(n-2)}\frac{n-2}{3n+2}\left(S+\frac{C}{C_1}\right). \tag{185}$$

It is an open problem to prove the analogue of the Poincaré inequality in $W^p_{0,\delta}$ and to decide whether the restriction imposed on r implies that (M,γ) is in the positive Yamabe class or not. The conclusion may be (cf. [7]) related to the existence of apparent horizons as in the proof by Schoen and Yau [13] of the positive energy theorem.

XVI. COUPLED CONSTRAINTS WITH UNSCALED SOURCES

We treat in this section the system of constraints for unscaled sources on a maximal submanifold in the case $n=3$. This system is coupled if the given initial momentum J does not vanish. The equations are

$$\triangle_\gamma \varphi - r\varphi + a(\beta)\varphi^{-7} + c\varphi^5 = 0$$

$$\nabla_i\{(2N)^{-1}(\mathcal{L}\beta)^{ij}\} = \nabla_i\{(2N)^{-1}u^{ij}\} + \varphi^{10}J^j. \quad (186)$$

The functions $c \geq 0$ and $N > 0$ and the tensor u and vector J are given on (M,γ). We denote by β_0 the solution of the equation:

$$\nabla_i\{(2N)^{-1}(\mathcal{L}\beta_0)^{ij}\} = \nabla_i\{(2N)^{-1}u^{ij}\}. \quad (187)$$

We have the following straightforward result.

Theorem. We suppose that the given quantities r, c, $J \in W^p_{s,\delta+2}$, $u \in W^p_{s+1,\delta+1}$, $1-N \in W^p_{s+2,\delta}$, $p>n$, $\delta > -n/p$ are such that there exist positive functions A_-, $A_+ \in W^p_{s,\delta+2}$ with

$$A_- < a(\beta_0) < A_+ \quad (188)$$

and such that the equation

$$\triangle_\gamma \varphi - r\varphi + A_+\varphi^{-7} + c\varphi^5 = 0 \quad (189)$$

admits as supersolution the constant $m_{A_+} \geq 1$, and that the analogous equation constructed with A_- admits as subsolution the constant $l_{A_-} \leq 1$, $l_{A_-} > 0$. Then there exists $\epsilon > 0$ such that

$$\|J\|_{W^p_{0,\delta+2}} \leq \epsilon \quad (190)$$

implies that the coupled constraint equations have a solution (β,φ) with $\varphi > 0$ and β, $1-\varphi \in W^p_{s+2,\delta}$.

Proof. It is elementary to check that l_{A_-} and m_{A_+} are admissible sub and supersolutions of the Lichnerowicz equation constructed with $a(\beta)$ if

$$A_- \leq a(\beta) \leq A_+. \quad (191)$$

In this case the equation has a solution φ with $l_{A_-} \leq \varphi \leq m_{A_+}$.

The momentum current being independent of other quantities its bound does not affect other estimates.

We will construct a sequence φ_ν, β_ν as in the previous section. We now have to show $A_- < a(\beta_{\nu-1}) < A_+$ implies the same inequalities for $a(\beta_\nu)$ if $\varphi_{\nu-1} \leq m_{A_+}$ and $\|J\|_{W^p_{s,\delta+2}}$ is small enough. We use the fact that (the notation $|\ \ |$ means here the γ norm):

$$\left|\{a(\beta_0)\}^{1/2} - (16N)^{-1}|\mathcal{L}(\beta-\beta_0)\|\right. \leq \{a(\beta)\}^{1/2} \leq \{a(\beta_0)\}^{1/2} + (16N)^{-1}|\mathcal{L}(\beta-\beta_0)|. \quad (192)$$

We deduce from the momentum constraint satisfied by β_ν the elliptic estimate

$$\|\beta_\nu - \beta_0\|_{W^p_{2,\delta}} \leq C m^{10}_{A_+}\|J\|_{W^p_{0,\delta+2}}. \quad (193)$$

The proof is then completed along the same lines of previous proofs. □

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APPENDIX A: ELLIPTIC LINEAR SYSTEMS ON MANIFOLDS EUCLIDEAN AT INFINITY

For the convenience of the reader we recapitulate here some known facts.

A linear differential operator of order m from sections u of a tensor bundle E over a smooth Riemannian manifold (M,γ) into sections of another such bundle F reads

$$Lu \equiv \sum_{k=0}^{m} a_k D^k u \quad (A1)$$

with a_k a linear map from tensor fields to tensor fields given also by tensor fields over M.

The principal symbol of the operator L at a point $x \in M$, for a covector ξ at x, is the linear map from E_x to F_x determined by the contraction of a_m with $(\otimes \xi)^m$. The operator is said to be elliptic if for each $x \in M$ and $\xi \in T^*_x M$ its principal symbol is an isomorphism from E_x onto F_x for all $\xi \neq 0$.

Example. The conformal Laplace operator for a metric γ on M acting from vector fields β into vector fields is

$$\nabla_i(\mathcal{L}\beta)^{ij} \equiv \nabla_i\left(\nabla^i\beta^j + \nabla^j\beta^i - \frac{2}{n}\gamma^{ij}\nabla_k\beta^k\right). \quad (A2)$$

Its principal symbol at x, with $\xi \in T_x M$, is the linear mapping from covariant vectors b into covariant vectors a given by

$$\xi^i\xi_i b_j + \xi^i\xi_j b_i - \frac{2}{n}\xi_j\xi^k b_k = a_j. \quad (A3)$$

This linear mapping is an isomorphism if $\xi \neq 0$ because

$$(\xi^i\xi_i)(b^j b_j) + \left(1 - \frac{2}{n}\right)(\xi^i b_i)^2 > 0 \quad (A4)$$

if $\xi \neq 0$ and $b \neq 0$. The conformal Killing operator is elliptic.

Theorem. Let (M,e) be a (complete) Riemannian manifold Euclidean at infinity. Let

$$Lu \equiv \sum_{k=0}^{m} a_k D^k u \quad (A5)$$

be an elliptic operator on (M,e). Suppose the coefficients of L satisfy the following hypotheses.

(1) There is a C^∞ tensor field A_m on M, constant in each end of (M,e) such that for some p with $1<p<+\infty$

$$a_m - A_m \in W^p_{s_m,\delta_m}, \quad s_m > \frac{n}{p}+1, \quad \delta_m > -\frac{n}{p};$$

and

(2)

$$a_k \in W^p_{s_k,\delta_k}, \quad s_k > \frac{n}{p}+k = m+1, \quad \delta_k > m-k-\frac{n}{p}, \quad 0 \leqslant k.$$

Then for each s such that $s_k+m \geqslant s \geqslant m$ the operator L maps $W^p_{s,\delta}$ into $W^p_{s-m,\delta+m}$ with finite dimensional kernel and closed range if

$$-\frac{n}{p} < \delta < -m+n-\frac{n}{p}. \tag{A6}$$

If, moreover, L is injective on $W^p_{s,\delta}$ then it is an isomorphism and there is a number C such that for each u in $W^p_{0,\delta}$ the following inequality holds:

$$\|u\|_{W^p_{s,\delta}} \leqslant C \|Lu\|_{W^p_{s-m,\delta+m}}. \tag{A7}$$

This theorem applies to the Poisson operator $\triangle - k$ under the hypothesis indicated in the theorem in Appendix B.

APPENDIX B: SOLUTION OF $\triangle_\gamma \varphi \equiv f(x,\varphi)$ ON AN ASYMPTOTICALLY EUCLIDEAN (M,γ)

Let $\triangle_\gamma$ denote the Laplace operator on scalar functions on (M,γ). Let f be a real valued function on $M\times I$, with I an interval of R, given by $(x,y)\mapsto f(x,y)$. We will show that the sub and supersolution method used by one of us (J.I.) in the case of a compact manifold can be extended to asymptotically Euclidean ones. Recall that (M,γ) is a (p,σ,ρ) asymptotically Euclidean manifold M of dimension n, if $\gamma - e \in W^p_{\sigma,\rho}$ with (M,e) Euclidean at infinity and $\rho > -n/p$, $\sigma > n/p+1$.

1. Linear equations

Definition. Suppose M is not compact. We say that f tends to a value $c\in\mathbb{R}$ at infinity if for any $\epsilon \geqslant 0$ there exists a compact S such that

$$\sup_{M-S} |f-c| \leqslant \epsilon. \tag{B1}$$

Lemma 1 (maximum principle). Let (M,γ) be an asymptotically Euclidean manifold. Suppose that a C^2 function φ on M satisfies an inequality of the form

$$\triangle_\gamma \varphi + a\cdot D\varphi - h\varphi \leqslant 0 \tag{B2}$$

with a dot denoting the scalar product in the metric γ, while a and h are respectively a vector field and a function on M, both bounded. Suppose that $h \geqslant 0$ on M. Then (a) If φ tends to $c>0$ at infinity then there exists a number $l>0$ such that $\varphi \geqslant l$ on M. (b) If φ tends to $c=0$ at infinity then $\varphi \geqslant 0$ on M.

Proof. One knows by the classical maximum principle that if φ attains a nonpositive minimum λ at a point of M then $\varphi \equiv \lambda$ on M. Also, if D is a bounded domain of M with smooth boundary ∂D and if the function φ attains a nonpositive minimum in $D\cup\partial D$ this minimum must be attained on the boundary ∂D.

(a) Choose $\epsilon \geqslant 0$ so small that $\epsilon < c$. If φ tends to c at infinity there is a compact S such that $\varphi \geqslant c-\epsilon > 0$ on $M-K$. Imbed S in a relatively compact domain D with smooth boundary ∂D. On ∂D, φ takes positive values, therefore φ does not attain a nonpositive minimum on the compact set $D\cup\partial D$; it attains a positive minimum c'. The number l is the smaller of the two positive numbers $c-\epsilon$ and c'.

(b) Suppose that φ takes a negative value α on M. Choose $\epsilon < |\alpha|$. There is a compact S such that

$$\sup_{M-S} |f| \leqslant \epsilon. \tag{B3}$$

Take a relatively compact open set D containing S. If φ takes a nonpositive minimum it is on the boundary ∂D, i.e., in $M-S$, which contradicts the fact that the absolute value of this minimum is necessarily greater than or equal to $|\alpha|$, itself greater than ϵ, which is the maximum of $|\varphi|$ in $M-S$. □

Theorem. Let (M,γ) be an (p,σ,ρ) asymptotically Euclidean manifold. Let $k\in W^p_{s,\delta+2}$, $\delta > -n/p$ be given. The operator $\triangle_\gamma - k$ is injective on $W^p_{s+2,\delta}$ if either

$$(1) \quad k \geqslant 0, \quad s > \frac{n}{p}, \tag{B4}$$

$$(2) \quad \int_M \{|Df|^2 + kf^2\}\mu_\gamma > 0 \quad \text{for all} \quad f\in C_0^\infty, \quad f \not\equiv 0,$$

$$s \geqslant 0 \quad \text{and} \quad \delta > \frac{n}{2} - \frac{n}{p} - 1 \quad \text{if} \quad p\neq 2, \quad \delta \geqslant -1 \quad \text{if} \quad p=2. \tag{B5}$$

Corollary. Under the hypotheses 1 or 2 the operator $\triangle_\gamma - k$ is an isomorphism from $W^p_{s+2,\delta}$ onto $W^p_{s,\delta+2}$ if $s \leqslant \sigma - 1 - n/p$, $-n/p < \delta < -n/p + n - 2$.

Proof. (1) If $s > n/p$, a solution in $W^p_{s+2,\delta}$ with $\delta > -n/p$ is in C^2_α for some positive α. The difference $\gamma - e$ is in C^1_β for some positive β. The maximum principle applies and shows that $u\equiv 0$ on M.

(2) The solution $u\in W^p_{2,\delta}$ is not necessarily C^2. To prove that $u\equiv 0$ we will multiply by u the equation and integrate on M.

If $u\in C_0^\infty$, then

$$\int_M u\triangle_\gamma u\mu_\gamma = -\int_M Du.Du\mu_\gamma. \tag{B6}$$

We can estimate the integrals in the above formula in terms of the Sobolev norm. In the case $p=2$ we have

$$\int_M u\triangle_\gamma u\mu_\gamma \leq \|u\|_{H_{0,\delta}}\|\triangle_\gamma u\|_{H_{0,\delta+2}}\sup_M(1+d^2)^{-(\delta+1)}, \tag{B7}$$

which is a bounded quantity whenever $\delta+1\geq 0$.

In the case $p\neq 2$ we have

$$\int_M u\triangle_\gamma u\mu_\gamma \leq \|u\|_{W^p_{0,\delta}}\|\triangle_\gamma u\|_{W^p_{0,\delta+2}}\|(1+d^2)^{-(\delta+1)}\|_{L^{p'}},$$

$$p'=\frac{p}{p-2}. \tag{B8}$$

The considered $L^{p'}$ norm is bounded if

$$2p'(\delta+1)>n, \text{ i.e., } \delta+1>\frac{n}{2}-\frac{n}{p}. \tag{B9}$$

The same kind of estimate applies to the integral of $Du\cdot Du$.

The density of C_0^∞ in the weighted Sobolev spaces shows then that a solution $u\in W^p_{2,\delta}$ satisfies the equality

$$\int_M u\triangle_\gamma u\mu_\gamma=-\int_M Du\cdot Du\mu_\gamma=\int_M ku^2\mu_\gamma. \tag{B10}$$

Therefore, under the hypothesis made, we have $Du=0$; hence $u=$constant and $u=0$ because u tends to zero at infinity. □

The corollary is a consequence of the general theorem on elliptic systems on an asymptotically Euclidean manifold, recalled in Appendix A.

If $n>2$ the inequality $\delta<-n/p+n-2$ is compatible with Eq. (B9) if $p\neq 2$ [respectively with $\delta\geq -1$ if $p=2$].

Remark. Under the hypothesies made on (M,γ) and k, the solution $u\in W^p_{s+2,\delta}$ of an equation

$$\triangle_\gamma u-ku=v \tag{B11}$$

with $v\in W^p_{s,\delta'+2}$ for some δ' such that $\delta\leq\delta'<-n/p+n-2$ is in $W^p_{s+2,\delta'}$ if $p>n/2$. Indeed $u\in W^p_{s+2,\delta}$ and $k\in W^p_{s,\delta+2}$ imply that $ku\in W^p_{s,\delta''+2}$, since $s<2s+2-n/p$ if $p>n/2$, $\delta''<\delta+(\delta+n/p)$. Since u satisfies

$$\triangle_\gamma u=ku+v \quad \in W^p_{s,\ \inf(\delta'',\delta')+2}, \tag{B12}$$

we have

$$u\in W^p_{s+2,\ \inf(\delta'',\delta')}. \tag{B13}$$

An induction argument shows that $u\in W^p_{s+2,\delta'}$.

2. Non-linear equations

We suppose that the function f is smooth in y and $W^p_{s,\beta+2}$ in x. To make it more transparent we take f as a finite sum of products of functions of x by functions of y, as it appears in the Hamiltonian constraint:

$$f(x,y)\equiv\sum_{P=0}^{Q} a_P(x)b_P(y). \tag{B14}$$

We make the following hypothesis:

Hypothesis $H(W^p_{s,\delta})$.

(1) There exists an interval $I\subset\mathbb{R}$ such that the b's are smooth functions of $y\in I$.

(2) The a's are functions on M in $W^p_{s,\delta+2}$.

Lemma 1. Under the hypothesis $H(W^p_{s,\delta})$ the function on M given by $x\mapsto f(x,\varphi(x))$, denoted in the sequel $f(x,\varphi)$, has the following properties when φ is continuous and takes its values in a closed interval $[l,m]\subset I$:

(1) $f(x,\varphi)\in W^p_{0,\delta+2}$ if $s\geq 0$.

(2) If $s>n/p$, $\delta>-n/p$ and $D\varphi\in W^p_{s'-1,\delta'+1}$ with $\delta'>-n/p$, $s\geq s'>n/p$ then $f(x,\varphi)\in W^p_{s',\delta+2}$.

Proof. Part 1 is trivial. To prove part 2 one uses the calculus derivation formulas and the multiplication properties of weighted Sobolev spaces. □

Definitions. A C^2 function φ_- on M is called a subsolution of $\triangle_\gamma\varphi=f(x,\varphi)$ if it is such that on M,

$$\triangle_\gamma\varphi_-\geq f(x,\varphi_-). \tag{B15}$$

A C^2 function φ_+ is called a supersolution if on M

$$\triangle_\gamma\varphi_+\leq f(x,\varphi_+). \tag{B16}$$

Theorem 1 (existence). Let (M,γ) be a (p,σ,ρ) asymptotically Euclidean manifold $\sigma>n/p+2$ and $f(x,y)$ a function satisfying the hypothesis $H(W^p_{s,\delta})$ with $s>n/p$, $\delta>-n/p$. Suppose the equation $\triangle_\gamma\varphi=f(x,\varphi)$ admits a subsolution φ_- and a supersolution φ_+ such that on M

$$l\leq\varphi_-\leq\varphi_+\leq m, \ [l,m]\subset I \tag{B17}$$

and

$$\lim_\infty\varphi_-\leq 1, \ \lim_\infty\varphi_+\geq 1. \tag{B18}$$

Suppose that $D\varphi_-, D\varphi_+\in W^p_{s'-1,\delta'+1}$, $s'\geq s$, $\delta'>-n/p$. Then the equation admits a solution φ such that

$$\varphi_-\leq\varphi\leq\varphi_+, \ 1-\varphi\in W^p_{s+2,\delta} \text{ with } \delta\leq\beta$$

$$\text{and } -\frac{n}{p}<\delta<n-2-\frac{n}{p}. \tag{B19}$$

Proof. We construct a solution by induction, starting from φ_-.

Let k be a positive function on M such that $k\in W^p_{s,\delta+2}$ and at each point $x\in M$

$$k(x)\geq\sup_{l\leq y\leq m} f'_y(x,y). \tag{B20}$$

Such a function exists by the hypothesis made on f.

We set $\varphi_1=1+u_1$. The linear elliptic equation for u_1

$$\triangle_\gamma u_1-ku_1=f(x,\varphi_-)-k(\varphi_--1) \tag{B21}$$

has one solution $u_1 \in W^p_{s+2,\delta} \subset C^2$, since the right hand side is in $W^p_{s,\delta+2}$ and the operator on the left is injective from $W^p_{s+2,\delta}$ into $W^p_{s,\delta+2}$ under the hypothesis made on s and δ. The function φ_1 tends to 1 at infinity.

We deduce from the equality and the inequalities satisfied respectively by φ_1 and φ_- the following inequality:

$$\triangle_\gamma(\varphi_1-\varphi_-)-k(\varphi_1-\varphi_-)\leqslant 0; \tag{B22}$$

hence, by the maximum principle lemma, since $\varphi_1-\varphi_-$ tends to $c\geqslant 0$ at infinity,

$$\varphi_1 \geqslant \varphi_- \quad \text{on} \quad M. \tag{B23}$$

Also,

$$\triangle_\gamma(\varphi_+-\varphi_1)-k(\varphi_+-\varphi_1) \leqslant f(x,\varphi_+)-f(x,\varphi_-)-k(\varphi_+-\varphi_-), \tag{B24}$$

and

$$f(x,\varphi_+)-f(x,\varphi_-)=(\varphi_+-\varphi_-) \times \int_0^1 f'_y(x,\varphi_-+t(\varphi_+-\varphi_-))dt. \tag{B25}$$

By the hypothesis made on k, φ_+, and φ_1 we have on M

$$\triangle_\gamma(\varphi_+-\varphi_1)-k(\varphi_+-\varphi_1)\leqslant 0, \tag{B26}$$

hence

$$\varphi_1 \leqslant \varphi_+. \tag{B27}$$

The induction formula is, with $\varphi_n = 1+u_n$:

$$\triangle_g u_n - k u_n = f(x,\varphi_{n-1}) - k u_{n-1}. \tag{B28}$$

We suppose that φ_p exists for $0\leqslant p \leqslant n-1$ with $\varphi_0=\varphi_-$ and $u_p \in W^p_{s+2,\delta}$ for $1 \leqslant p \leqslant n-1$ and that for these p's

$$\varphi_- \leqslant \varphi_{p-1} \leqslant \varphi_p \leqslant \varphi_+. \tag{B29}$$

The elliptic theory shows $u_n \in W^p_{s+2,\delta}$ exists. The functions φ_n are continuous, even C^2 since $s>n/p$, and tend to 1 at infinity. The equality resulting from the equations satisfied by φ_p when $p\leqslant n-1$ gives

$$\triangle_g \varphi_{n-1} - k\varphi_{n-1} = f(x,\varphi_{n-2}) - k\varphi_{n-2}, \tag{B30}$$

$$\triangle_g \varphi_p - k\varphi_p = f(x,\varphi_{p-1}) - k\varphi_{p-1}. \tag{B31}$$

One deduces then from the maximum principle lemma that on M

$$\varphi_{n-1} \leqslant \varphi_n. \tag{B32}$$

Analogously one uses the maximum principle and the inequality deduced from the equation and inequality satisfied by φ_n and φ_+,

$$\triangle_\gamma(\varphi_n-\varphi_+)-k(\varphi_n-\varphi_+) \geqslant f(x,\varphi_{n-1})-f(x,\varphi_+) -k(\varphi_{n-1}-\varphi_+), \tag{B33}$$

to show that $\varphi_{n-1}\leqslant\varphi_+$ implies $\varphi_n \leqslant \varphi_+$.

The sequence of continuous functions φ_n has been shown to be pointwise increasing and bounded. It is therefore converging at each point $x\in M$ to a limit $\varphi(x)=1+u(x)$, with $\varphi_- \leqslant \varphi \leqslant \varphi_+$.

To show that φ is a solution of the given equation and $\varphi - 1 \in W^p_{s+2,\delta}$ we proceed as follows. Since the φ_n are continuous and take their values in the interval $[l,m]$, the functions $f(x,\varphi_n)-ku_n$ belong to $W^p_{0,\delta+2}$ with uniformly bounded norms. The linear elliptic inequality [following from Eq. (B28)]

$$\|u_{n+1}\|_{W^p_{2,\delta}} \leqslant C\|f(x,\varphi_n)-ku_n)\|_{W^p_{0,\delta+2}} \tag{B34}$$

shows that the sequence u_n is uniformly bounded in the $W^p_{2,\delta}$ norm. Since $W^p_{2,\delta}$ is compactly embedded in $W^p_{1,\delta'}$ for any $\delta'<\delta$, there is a subsequence, still denoted u_n, which converges in $W^p_{1,\delta'}$ norm to a function $u\in W^p_{2,\delta}$.

The functions $f(x,\varphi_n)$ converge to $f(x,\varphi)$ in the $W^p_{1,\delta'+2}$ norm because of the inequality, which is satisfied if $s>n/p$, $\delta'>-n/p$,

$$\|f(x,\varphi)-f(x,\varphi_n)\|_{W^p_{1,\delta+2}} \leqslant C\|\varphi-\varphi_n\|_{W^p_{1,\delta'}}\|F_1\|_{W^p_{s,\delta+2}}, \tag{B35}$$

where C depends only on (M,e) and $F_1 \in W^p_{s,\delta+2}$ is a function on M, which exists by the hypothesis on f, such that

$$F_1(x) \geqslant \sup_{y\in[l,m]} |f'_y(x,y)|. \tag{B36}$$

These convergences imply that the limit $\varphi=1+u$ satisfies the equation in a generalized sense. From the linear theory, we find that the equation satisfied by u and the fact that (cf. above lemma) that $f(x,\varphi)\in W^p_{1,\delta+2}$ (also $\in W^p_{2,\delta+2}$ since $u \in W^p_{2,\delta}$) that $u\in W^p_{3,\delta}$. An induction argument completes the proof that $u \in W^p_{s+2,\delta}$.

The theorem holds with $s=0$ if f is an increasing function of y. An example is treated in Sec. X. □

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