Hawking radiation in string theory and the string phase of black holes

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The quantum string emission by black holes is computed in the framework of the ''string analog model'' (or thermodynamical approach), which is well suited to combine quantum-field theory (QFT) and string theory in curved backgrounds (particularly here, as black holes and strings possess intrinsic thermal features and temperatures). The QFT-Hawking temperature T_H is upper bounded by the string temperature T_S in the black hole background. The black hole emission spectrum is an incomplete gamma function of $(T_H - T_S)$. For $T_H \ll T_S$, it yields the QFT-Hawking emission. For $T_H \to T_S$, it shows that highly massive string states dominate the emission and undergo a typical string phase transition to a *microscopic* ''minimal'' black hole of mass $M_{\min}$ or radius $r_{\min}$ (inversely proportional to T_S) and string temperature T_S. The string back reaction effect [self-consistent black hole solution of the semiclassical Einstein equations with mass M_+ (radius r_+) and temperature T_+] is computed. Both the QFT and string black hole regimes are well defined and bounded: $r_{\min} \leq r_+ \leq r_S$, $M_{\min} \leq M_+ \leq M$, $T_H \leq T_+ \leq T_S$. The string ''minimal'' black hole has a life time $\tau_{\min} \simeq (k_B c / G\hbar)\, T_S^{-3}$.

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I. INTRODUCTION AND RESULTS

In the context of quantum field theory (QFT) in curved spacetime, black holes have an intrinsic Hawking temperature Ref. [1] given by

$$T_H = \frac{\hbar c}{4\pi k_B} \frac{(D-3)}{r_S}, \quad r_S \equiv L_{cl},$$

r_S being the Schwarzchild's radius (classical length L_{cl}).

In the context of quantum string theory in curved spacetime, quantum strings in black hole spacetimes have an intrinsic temperature given by

$$T_S = \frac{\hbar c}{4\pi k_B} \frac{(D-3)}{L_q}, \quad L_q = \frac{b L_S (D-3)}{4\pi}, \quad L_S \equiv \sqrt{\frac{\hbar \alpha'}{c}},$$

which is the same as the string temperature in flat spacetime (see Ref. [2] and Sec. III in this paper).

The QFT-Hawking temperature T_H is a measure of the Compton length of the black hole, and thus, of its ''quantum size,'' or quantum property in the semiclassical-QFT regime. The Compton length of a quantum string is a direct measure of its size L_q. The string temperature T_S is a measure of the string mass, and thus inversely proportional to L_q.

The $\mathcal{R}$ transform over a length introduced in Ref. [3] is given by

$$\tilde{L}_{cl} = \mathcal{R} L_{cl} = L_q,$$

$$\tilde{L}_q = \mathcal{R} L_q = L_{cl}.$$

Under the $\mathcal{R}$ operation, $\tilde{T}_H = T_S$ and $\tilde{T}_S = T_H$. This relation also holds for the respective QFT-Hawking temperature and string temperature in de Sitter space, Ref. [3].

In this paper, we investigate the issue of Hawking radiation and the back reaction effect on a black hole in the context of string theory. In principle, this question should be properly addressed in the context of string field theory. Because of the lack of a tractable framework for it, we work here in the framework of the string analog model (or thermodynamical approach). This is a suitable approach for cosmology and black holes in order to combine QFT and string study, and to go further in the understanding of quantum gravity effects. The thermodynamical approach is particularly appropriated and natural for black holes, as Hawking radiation and the string gas [4,5] possess intrinsic thermal features and temperatures.

In this approach, the string is a collection of fields Φ_n coupled to the curved background, and whose masses m_n are given by the degenerate string mass spectrum in the curved space considered. Each field Φ_n appears as many times the degeneracy of the mass level $\rho(m)$. (Although the fields Φ_n do not interact among themselves, they do with the black hole background).

In black hole spacetimes, the mass spectrum of strings is the same as in flat spacetime Ref. [2], therefore the higher masses string spectrum satisfies Eq. (4) (a and b being constants, depending on the model, and on the number of space dimensions).

We consider the canonical partition function ($\ln Z$) for the higher excited quantum string states of open strings (which may be or may be not supersymmetric) in the asymptotic (flat) black hole region. The gas of strings is at thermal equilibrium with the black hole at the Hawking temperature T_H, it follows that the canonical partition function, [Eq. (5)] is well defined for Hawking temperatures satisfying the condition, $T_H < T_S$, where T_S represents a maximal or critical value temperature. This limit implies a minimum horizon radius

$$r_{\rm min} = \frac{b(D-3)}{4\pi} L_S$$

and a minimal mass for the black hole (BH):

$$M_{\rm min} = \frac{c^2(D-2)}{16\pi G} A_{D-2} r_{\rm min}^{D-3}$$

$$\left(M_{\rm min}(D=4) = \frac{b}{8\pi G}\sqrt{\hbar c^3 \alpha'} \right).$$

We compute the thermal quantum string emission of very massive particles by a D-dimensional Schwarzschild BH. This highly massive emission, corresponding to the higher states of the string mass spectrum, is naturally expected in the last stages of BH evaporation.

In the context of QFT, BH emit particles with a Planckian (thermal) spectrum at temperature T_H. The quantum BH emission $\sigma_q(k,D)$ is related to the classical absorption cross section through the Hawking formula Eq. (17). The classical total absorption spectrum $\sigma_A(k,D)$, Ref. [6], is entirely oscillatory as a function of the energy. This is exclusive to the black hole (other absorptive bodies do not show this property).

In the context of the string analog model, the quantum emission by the BH is given by Eq. (27), $\sigma_q(m,D)$ being the quantum emission for an individual quantum field with mass m in the string mass spectrum. m_0 is the lowest mass from which the asymptotic expression for $\rho(m)$ is still valid. We find $\sigma_{\rm string}(D)$ as given by [Eq. (30)] (open strings). It consists of two terms: the first term is characteristic of a quantum thermal string regime, dominant for T_H close to T_S; the second term, in terms of the exponential-integral function E_i is dominant for $T_H \ll T_S$ from which the QFT Hawking radiation is recovered (semiclassical QFT regime).

The computed $\sigma_{\rm string}(D)$ shows the following: At the first stages, the BH emission is in the lighter particle masses at the Hawking temperature T_H as described by the semiclassical QFT regime [second term in Eq. (30)]. As evaporation proceeds, the temperature increases, the BH radiates the higher massive particles in the string regime [as described by the first term of Eq. (30)]. For $T_H \to T_S$, the BH enters its quantum string regime $r_S \to r_{\rm min}$, $M \to M_{min}$. That is, ''the BH becomes a string,'' in fact it is more than that, as [Eq. (30)] accounts for the back reaction effect too: The first term is characteristic of a Hagedorn-type singularity Ref. [5], and the partition function here has the same behavior as this term. Its meaning is the following: At the late stages, the emitted BH radiation (highly massive string gas) dominates and undergoes a Carlitz-type phase transition Ref. [5] at the temperature T_S into a condensed finite energy state. Here such a state (almost all the energy concentrated in one object) is a *microscopic* (or ''minimal'') BH of size $r_{\rm min}$, (mass $M_{\rm min}$), and temperature T_S. The last stage of the BH radiation, properly taken into account by string theory, makes such a phase transition possible. Here the T_S scale is in the Planck energy range and the transition is to a state of string size L_S. The precise detailed description of such a phase transition and such a final state deserve investigation. A phase transition of this kind has been considered in Ref. [7]. Our results here support and provide a precise picture to some issues of BH evaporation discussed there in terms of purely thermodynamical considerations.

We also describe the (perturbative) back reaction effect in the framework of the semiclassical Einstein equations (c-number gravity coupled to quantum string matter) with the vacuum expectation value (VEV) of the energy momentum tensor of the quantum string emission as a source. In the context of the analog model, such stress tensor VEV is given by Eq. (32), where $\langle T_\mu^\nu(r,m)\rangle$ is the VEV of the QFT stress tensor of individual quantum fields of mass m in the higher excited string spectrum. The solution to the semiclassical Einstein equations is given by [Eqs. (48), (51), and (55)] ($D=4$):

$$r_+ = r_S\left(1 - \frac{4}{21}\frac{\mathcal{A}}{r_S^6}\right), \quad M_+ = M\left(1 - \frac{4}{21}\frac{\mathcal{A}}{r_S^6}\right),$$

$$T_+ = T_H\left(1 + \frac{1}{3}\frac{\mathcal{A}}{r_S^6}\right).$$

The string form factor $\mathcal{A}$ is given by [Eq. (57)], it is finite and positive. For $T_H \ll T_S$, the back reaction effect in the QFT-Hawking regime is consistently recovered. Algebraic terms in $(T_H - T_S)$ are entirely stringy. In both cases, the relevant ratio $\mathcal{A}/r_S^6$ entering in the solution (r_+, M_+, T_+) is negligible.

The string back reaction solution shows that the BH radius and mass decrease, and the BH temperature increases, as it should be. But here the BH radius is bounded from below (by $r_{\rm min}$ and the temperature does not blow up (as it is bounded by T_S)). The ''mass loss'' and ''time life'' are

$$-\left(\frac{dM}{dt}\right)_+ = -\left(\frac{dM}{dt}\right)\left(1 + \frac{20}{21}\frac{\mathcal{A}}{r_S^6}\right), \quad \tau_+ = \tau_H\left(1 - \frac{8}{7}\frac{\mathcal{A}}{r_S^6}\right).$$

The lifetime of the string black hole is $\tau_{\rm min} = (K_B c/G\hbar)T_S^{-3}$.

The string back reaction effect is finite and consistently describes both the QFT regime (BH of mass M and temperature T_H) and the string regime (BH of mass $M_{\rm min}$ and temperature T_S). Both regimes are bounded as in string theory we have

$$r_{\rm min} \leq r_+ \leq r_S, \quad M_{\rm min} \leq M_+ \leq M,$$

$$\tau_{\rm min} \leq \tau_+ \leq \tau_H, \quad T_H \leq T_+ \leq T_S.$$

This paper is organized as follows: In Sec. II we consider quantum strings in the BH geometry and derive the bounds

TABLE I. Density of mass levels $\rho(m)\sim m^{-a}\exp\{b\sqrt{\alpha' c/\hbar}m\}$. For open strings $\alpha'(c/\hbar)m^2\simeq n$; for closed strings $\alpha'(c/\hbar)m^2\simeq 4n$.

Dimension	String theory	a	b	$k_B T_S/c^2$
D	Open bosonic closed	$(D-1)/2$ D	$2\pi\sqrt{\frac{D-2}{6}}$	$\left[2\pi\sqrt{\frac{(D-2)}{6}\left(\frac{\alpha' c}{\hbar}\right)}\right]^{-1}$
26 (critical)	Open bosonic closed	25/2 26	4π	$\left(4\pi\sqrt{\frac{\alpha' c}{\hbar}}\right)^{-1}$
10 (critical)	Open superstring Closed superstring (type II)	9/2 10	$\pi 2\sqrt{2}$	$\left[\pi 2\sqrt{2\left(\frac{\alpha' c}{\hbar}\right)}\right]^{-1}$
	Heterotic	10	$\pi(2+\sqrt{2})$	$\left[\pi(1+\sqrt{2})\sqrt{2\left(\frac{\alpha' c}{\hbar}\right)}\right]^{-1}$

imposed by string theory on the quantum size and temperature of the BH. In Sec. III we compute the quantum string emission by the BH. In Sec. IV we compute its back reaction effect. Section V presents conclusions and remarks.

Such a phase transition takes into account (in the thermodynamical description) the back reaction of the string emission on the black hole. It is clear that at such a stage, the validity of the semiclassical approximation breaks down. Semiclassical in this context means that quantum matter is coupled to c-number gravity, but for black holes with masses of the order of the Planck mass, a full quantum gravity description is needed.

II. QUANTUM STRINGS IN THE BLACK HOLE SPACE TIME

The D-dimensional Schwarzschild black hole metric reads

$$ds^2=-a(r)c^2dt^2+a^{-1}(r)dr^2+r^2d\Omega_{D-2}^2, \tag{1}$$

where

$$a(r)=1-\left(\frac{r_S}{r}\right)^{D-3}, \quad r_S=\left(\frac{16\pi GM}{c^2(D-2)A_{D-2}}\right)^{1/(D-3)},$$

$$A_{D-2}=\frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}. \tag{2}$$

G is the Newton gravitational constant. For $D=4$ one has

$$r_S=\frac{2GM}{c^2}. \tag{3}$$

As it is known, the BH — according to its specific heat being negative — increases its temperature in its quantum emission process (M decreases). Also, it would seem that, if the BH would evaporate completely ($M=0$), the QFT-Hawking temperature T_H would become infinite. However, at this limit, and more precisely when $M\sim M_{PL}$, the fixed classical background approximation for the BH geometry breaks down, and the back reaction effect of the radiation matter on the BH must be taken into account. In Sec. IV, we will take into account this back reaction effect in the framework of string theory.

First, we will consider quantum strings in the fixed BH background. We will see that even in this approximation, quantum string theory cannot only retard the catastrophic process but, furthermore, provides nonzero lower bounds for the BH mass (M) or horizon (r_S), and a finite (maximal) value for the BH temperature T_H as well.

The Schwarzschild black hole spacetime is asymptotically flat. Black hole evaporation — and any ''slow down'' of this process — will be measured by an observer which is at this asymptotic region. In Ref. [2] it has been found that the mass spectrum of quantum string states coincides with the one in Minkowski space. Critical dimensions are the same as in Ref. [2] ($D=26$, open and closed bosonic strings; $D=10$ super and heterotic strings). Therefore, the asymptotic string mass density of levels in black hole spacetimes will read as in Minkowski space

$$\rho(m)\sim\left(\sqrt{\frac{\alpha' c}{\hbar}}m\right)^{-a}e^{b\sqrt{\alpha' c/\hbar}m}, \tag{4}$$

where $\alpha'\equiv c^2/2\pi T$ (T is the string tension) has dimensions of (linear mass density)$^{-1}$; constants a/b depend on the dimensions and on the type of string [8]. For a noncompactified space time these coefficients are given in Table I.

In this paper, strings in a BH spacetime are studied in the framework of the string analog model. In this model, one considers the strings as a collection of quantum fields $\phi_1,\ldots,\phi_n$, whose masses are given by the string mass spectrum [$\alpha'(c/\hbar)m^2\simeq n$, for open strings and large n in flat spacetime]. Each field of mass m appears as many times as the degeneracy of the mass level ; for higher excited modes this is described by $\rho(m)$ [Eq. (4)]. Although quantum fields do not interact among themselves, they do with the BH background.

In the asymptotic (flat) BH region, the thermodynamical behavior of the higher excited quantum string states of open strings, for example, is deduced from the canonical partition function [5]:

$$\ln Z = \frac{V}{(2\pi)^d}\sqrt{\frac{\alpha' c}{\hbar}}\int_{m_0}^{\infty} dm\rho(m) \times \int d^dk \ln\left\{\frac{1+\exp[-\beta_H(m^2c^4+k^2\hbar^2c^2)^{1/2}]}{1-\exp[-\beta_H(m^2c^4+k^2\hbar^2c^2)^{1/2}]}\right\} \tag{5}$$

(d is the number of spatial dimensions) where supersymmetry has been considered for the sake of generality; $\rho(m)$ is the asymptotic mass density given by [Eq. (4)]; $\beta_H = (k_B T_H)^{-1}$ where T_H is the BH Hawking temperature; m_0 is the lowest mass for which $\rho(m)$ is valid.

For the higher excited string modes, i.e., the masses of the BH and the higher string modes satisfy the condition

$$\beta_H mc^2 = \frac{4\pi mc}{(D-3)\hbar}\left[\frac{16\pi GM}{c^2(D-2)A_{D-2}}\right]^{1/(D-3)} \gg 1, \tag{6a}$$

which reads for $D=4$

$$\beta_H mc^2 = \frac{8\pi GMm}{\hbar c} \gg 1 \tag{6b}$$

[condition Eq. (6b) will be considered later in Sec. IV]. The leading contribution to the right-hand side (rhs) of Eq. (5) will give as a canonical partition function

$$\ln Z \simeq \frac{2V_{D-1}\left(\alpha'\frac{c}{\hbar}\right)^{-(a-1)/2}}{(2\pi\beta_H\hbar^2)^{(D-1)/2}} \times \int_{m_0}^{\infty} dm\, m^{-a+(D-1)/2} e^{-(\beta_H-\beta_S)mc^2}, \tag{7}$$

where $\beta_S = (k_B T_S)^{-1}$, T_S being [Eq. (4)]

$$T_S = \frac{c^2}{k_B b\left(\frac{\alpha' c}{\hbar}\right)^{1/2}}, \tag{8}$$

the string temperature (Table I). For open bosonic strings one divides by 2 the rhs of [Eq. (7)] (leading contributions are the same for bosonic and fermionic sector).

From [Eq. (7)] we see that the definition of $\ln Z$ implies the following condition on the Hawking temperature:

$$T_H < T_S. \tag{9}$$

Furthermore, as T_H depends on the BH mass M, or on the horizon r_S, [Eqs. (6a), (6b), and (3)], the above condition will lead to further conditions on the horizon. Then T_S represents a critical value temperature: $T_S \equiv T_{cr}$. In order to see this more clearly, we rewrite T_S in terms of the quantum string length scale

$$L_S = \left(\frac{\hbar\alpha'}{c}\right)^{1/2}, \tag{10}$$

namely,

$$T_S = \frac{\hbar c}{b k_B L_S}. \tag{11}$$

From Eq. (9), and with the help of Eqs. (6a), (6b), and (11), we deduce

$$r_S > \frac{b(D-3)}{4\pi} L_S, \tag{12}$$

which shows that (first quantized) string theory provides a lower bound, or *minimum radius*, for the BH horizon.

Taking into account Eqs. (2) and (12), we have the following condition on the BH mass:

$$M > \frac{c^2(D-2)A_{D-2}}{16\pi G}\left[\frac{b(D-3)}{4\pi}L_S\right]^{D-3}. \tag{13}$$

Therefore, there is a *minimal* BH *mass* given by string theory.

For $D=4$ we have

$$r_S > \frac{b}{4\pi}L_S, \tag{14}$$

$$M > \frac{c^2 b}{8\pi G}L_S. \tag{15}$$

These lower bounds obviously satisfy Eq. (3), and Eq. (15) can be rewritten as

$$M > \frac{bM_{PL}^2}{8\pi M_S},$$

where $M_S = \hbar/L_S c$ is the string mass scale (L_S is the reduced Compton wavelength) and $M_{PL} \equiv (\hbar c/G)^{1/2}$ is the Planck mass. The minimal BH mass is then [Eqs. (10) and (15)]

$$M_{\min} = \frac{b}{8\pi G}\sqrt{\hbar c^3\alpha'}.$$

It is appropriate, at this point, to make use of the $\mathcal{R}$ or dual transformation over a length introduced in Ref. [3]. This operation is

$$\tilde{L}_{cl} = \mathcal{R}L_{cl} = \mathcal{L}_{\mathcal{R}}L_{cl}^{-1} = L_q \quad \text{and} \quad \tilde{L}_q = \mathcal{R}L_q = \mathcal{L}_{\mathcal{R}}L_q^{-1} = L_{cl}, \tag{16a}$$

where $\mathcal{L}_{\mathcal{R}}$ has dimensions of $(\text{length})^2$; and it is given by $\mathcal{L}_{\mathcal{R}} = L_{cl}L_q$.

In our case, L_{cl} is the classical Schwarzschild radius, and $L_q \equiv r_{\min} = [b(D-3)L_S]/4\pi$ [Eq. (12)]. The $\mathcal{R}$ transformation links classical lengths to quantum string lengths. For the BH, the QFT-Hawking temperature is

$$T_H = \frac{\hbar c(D-3)}{4\pi k_B L_{cl}}, \tag{16b}$$

while the string temperature is

$$T_S = \frac{\hbar c(D-3)}{4\pi k_B L_q}. \tag{16c}$$

Under the $\mathcal{R}$ operation we have

$$\tilde{T}_H = T_S \quad \text{and} \quad \tilde{T}_S = T_H, \tag{16d}$$

which are valid for all D. From the above equations we can read as well

$$\tilde{T}_H \tilde{T}_S = T_S T_H.$$

It is interesting to express T_H and T_S in terms of their respective masses

$$T_H = \frac{\hbar c(D-3)}{4\pi k_B}\left(\frac{16\pi GM}{c^2(D-2)A_{D-2}}\right)^{-1/(D-3)},$$

$$\left(T_H(D=4) = \frac{\hbar c^3}{8\pi k_B GM}\right),$$

and

$$T_S = \frac{c^2 M_S}{b k_B}.$$

III. THERMAL QUANTUM STRING EMISSION FOR A SCHWARZSCHILD BLACK HOLE

As it is known, thermal emission of massless particles by a black hole has been considered in the context of QFT [1,9,10]. Here, we are going to deal with thermal emission of high massive particles which correspond to the higher excited modes of a string. The study will be done in the framework of the string analog model.

For a static D-dimensional black hole, the quantum emission cross section $\sigma_q(k,D)$ is related to the total classical absorption cross section $\sigma_A(k,D)$ through the Hawking formula [1]

$$\sigma_q(k,D) = \frac{\sigma_A(k,D)}{e^{E(k)\beta_H}-1}, \tag{17}$$

where $E(k)$ is the energy of the particle (of momentum: $p = \hbar k$) and $\beta_H = (k_B T_H)^{-1}$, being T_H Hawking temperature [Eq. (16b)]. The total absorption cross section $\sigma_A(k,D)$ in [Eq. (17)] has two terms Ref. [6], one is an isotropic k-independent part, and the other has an oscillatory behavior, as a function of k, around the optical geometric constant value with decreasing amplitude and constant period. Here we will consider only the isotropic term, which is the more relevant in our case. For a D-dimensional black hole space time, this is given by (see, for example, Ref. [2])

$$\sigma_A(k,D) = a(D) r_S^{D-2}, \tag{18}$$

where r_S is the horizon [Eqs. (2) and (3)] and

$$a(D) = \frac{\pi^{(D-2)/2}}{\Gamma[(D-2)/2]}\left(\frac{D-1}{D-3}\right)\left(\frac{D-1}{2}\right)^{2/(D-3)}. \tag{19}$$

We notice that $\rho(m)$ [Eq. (4)] depends only on the mass, therefore, we could consider, in our formalism, the emitted high mass spectrum as spinless. On the other hand, as we are dealing with a Schwarzschild black hole (angular momentum equal to zero), spin considerations can be overlooked. Emission is larger for spinless particles Ref. [11]. The number of scalar field particles of mass m emitted per unit time is

$$\langle n(m)\rangle = \int_0^\infty \langle n(k)\rangle d\mu(k), \tag{20}$$

where $d\mu(k)$ is the number of states between k and $k+dk$:

$$d\mu(k) = \frac{V_d}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}k^{d-1}dk, \tag{21}$$

and $\langle n(k)\rangle$ is now related to the quantum cross section σ_q [Eqs. (17) and (18)] through the equation

$$\langle n(k)\rangle = \frac{\sigma_q(k,D)}{r_S^{D-2}}. \tag{22}$$

Considering the isotropic term for σ_q [Eqs. (18) and (19)], we have

$$\langle n(k)\rangle = \frac{a(D)}{e^{E(k)\beta_H}-1}, \tag{23}$$

where $\beta_H = 1/k_B T_H$, T_H being the BH temperature [Eq. (16b)]. From Eqs. (20) and (23), $\langle n(m)\rangle$ will be given by

$$\langle n(m)\rangle = F(D,\beta_H) m^{(D-3)/2}(mc^2\beta_H+1)e^{-\beta_H mc^2}, \tag{24}$$

where

$$F(D,\beta_H) \equiv \frac{V_{D-1}a(D)}{(2\pi)^{(D-1)/2}}\frac{(c^2)^{(D-3)/2}}{\beta_H^{(D+1)/2}(\hbar c)^{(D-1)}}$$
$$\equiv A(D)\beta_H^{-(D+1)/2}. \tag{25}$$

The large argument $\beta_H mc^2 \gg 1$, i.e., [Eqs. (6a) and (6b)], and the leading approximation have been considered in performing the k integration.

The quantum thermal emission cross section for particles of mass m is defined as

$$\sigma_q(m,D) = \int \sigma_q(k,D) d\mu(k) \tag{26a}$$

and with the help of Eq. (22) we have

$$\sigma_q(m,D) = r_S^{D-2}\langle n(m)\rangle, \tag{26b}$$

where $\langle n(m)\rangle$ is given by Eq. (24).

In the string analog model, the string quantum thermal emission by a BH will be given by the cross section

$$\sigma_{\rm string}(D)=\sqrt{\frac{\alpha' c}{\hbar}}\int_{m_0}^{\infty}\sigma_q(m,D)\rho(m)dm, \qquad (27)$$

where $\rho(m)$ is given by Eq. (4), and $\sigma_q(m,D)$ by Eqs. (26) and (24); m_0 is the lowest string field mass for which the asymptotic value of the density of mass levels, $\rho(m)$, is valid. For arbitrary D and a, we have from Eqs. (27), (26b), (24), and (4)

$$\sigma_{\rm string}(D)=F(D,\beta_H)r_S^{D-2}\left(\sqrt{\frac{\alpha' c}{\hbar}}\right)^{-a+1}\times I_D(m,\beta_H-\beta_{cr},a), \qquad (28)$$

where $F(D,\beta_H)$ is given by Eq. (25),

$$\beta_{cr}\equiv\beta_S=(k_BT_S)^{-1}=\frac{b}{c^2}\sqrt{\frac{\alpha' c}{\hbar}} \qquad (29a)$$

and

$$I_D(m,\beta_H-\beta_{cr},a)\equiv\int_{m_0}^{\infty}m^{-a+(D-3)/2}\times(mc^2\beta_H+1)e^{-(\beta_H-\beta_{cr})mc^2}dm. \qquad (29b)$$

After a straightforward calculation we have

$$I_D(m,\beta_H-\beta_S,a)=\frac{c^2\beta_H}{[(\beta_H-\beta_S)c^2]^{-a+(D+1)/2}}\times\Gamma\left(-a+\frac{D+1}{2},(\beta_H-\beta_S)c^2m_0\right)+\frac{1}{[(\beta_H-\beta_S)c^2]^{-a+(D-1)/2}}\times\Gamma\left(-a+\frac{D-1}{2},(\beta_H-\beta_S)c^2m_0\right), \qquad (29c)$$

where $\Gamma(x,y)$ is the incomplete gamma function. For open strings, $a=(D-1)/2$ (D is the noncompact dimensions), we have

$$\sigma_{\rm string}^{\rm (open)}(D)=A(D)\beta_H^{-(D+1)/2}r_S^{D-2}\left(\frac{c^2\beta_S}{b}\right)^{-(D-3)/2}\times\left\{\frac{\beta_H}{\beta_H-\beta_S}e^{-(\beta_H-\beta_S)c^2m_0}-E_i[-(\beta_H-\beta_S)c^2m_0]\right\}, \qquad (30)$$

where E_i is the exponential-integral function, and we have used Eqs. (25) and (29a).

When T_H approaches the limiting value T_S, and as $E_i(-x)\sim C+\ln x$ for small x, we have from Eq. (30)

$$\sigma_{\rm string}^{\rm (open)}(D)=A(D)\beta_S^{-(D+1)/2}r_{\rm min}^{D-2}\left(\frac{c^2}{b}\beta_S\right)^{-(D-3)/2}\times\left\{\frac{\beta_S}{\beta_H-\beta_S}-C-\ln((\beta_H-\beta_S)m_0c^2)\right\}$$
$$=B(D)\beta_S^{-1}\left\{\frac{\beta_S}{\beta_H-\beta_S}-C-\ln((\beta_H-\beta_S)c^2m_0)\right\},$$

where

$$r_{\rm min}=\frac{\hbar c(D-3)\beta_S}{4\pi}$$

and

$$B(D)\equiv A(D)\left(\frac{\hbar c(D-3)}{4\pi}\right)^{D-2}\left(\frac{c^2}{b}\right)^{-(D-3)/2}. \qquad (30')$$

For $\beta_H\rightarrow\beta_S$ the dominant term is

$$\sigma_{\rm string}^{\rm (open)}(D)\overset{T_H\rightarrow T_S}{\simeq}B(D)\frac{1}{(\beta_H-\beta_S)} \qquad (31a)$$

for any dimension.

For $\beta_H\gg\beta_S$, i.e., $T_H\ll T_S$,

$$\sigma_{\rm string}^{\rm (open)}(D)\overset{T_H\ll T_S}{\simeq}A(D)\beta_H^{-(D+1)/2}r_S^{D-2}\left(\frac{c^2}{b}\beta_S\right)^{-(D-3)/2}\times e^{-\beta_Hc^2m_0}\left(1+\frac{1}{\beta_Hc^2m_0}\right)$$
$$\simeq B(D)\beta_H^{(D-5)/2}\beta_S^{-(D-3)/2}e^{-\beta_Hc^2m_0} \qquad (31b)$$

as $E_i(-x)\sim e^{-x}/x+\cdots$ for large x. For $D=4$,

$$\sigma_{\rm string}^{\rm (open)}(4)\simeq B(4)\left(\frac{1}{\beta_H\beta_S}\right)^{1/2}e^{-\beta_Hc^2m_0}.$$

At this point, and in order to interpret the two different behaviors, we compare them with the corresponding behaviors for the partition function [Eq. (7)]. For open strings [$a=(D-1)/2$] $\ln Z$ is equal to

$$\ln\mathcal{Z}_{\rm open}\simeq\frac{2V_{D-1}\left(\frac{\alpha' c}{\hbar}\right)^{-(D-3)/4}}{(2\pi\beta_H\hbar^2)^{(D-1)/2}}\frac{1}{(\beta_H-\beta_S)c^2}e^{-(\beta_H-\beta_S)m_0c^2}.$$

For $\beta_H\rightarrow\beta_S$:

$$\ln \mathcal{Z}_{\rm open} \simeq \frac{2V_{D-1}\left(\frac{\alpha' c}{\hbar}\right)^{-(D-3)/4}}{(2\pi\beta_H\hbar^2)^{(D-1)/2}}\frac{1}{(\beta_H-\beta_S)c^2} \quad \beta_H \to \beta_S,$$

and for $\beta_H \gg \beta_S$:

$$\ln \mathcal{Z}_{\rm open} \simeq \frac{2V_{D-1}\left(\frac{\alpha' c}{\hbar}\right)^{-(D-3)/4}}{(2\pi\hbar^2)^{(D-1)/2}c^2\beta_H^{(D+1)/2}} e^{-\beta_H m_0 c^2} \quad \beta_H \gg \beta_S.$$

The singular behavior for $\beta_H \to \beta_S$, and all D, is typical of a string system with intrinsic Hagedorn temperature, and indicates a string phase transition (at $T=T_S$) to a condensed finite energy state (Ref. [5]). This would be the minimal black hole, of mass $M_{\rm min}$ and temperature T_S.

IV. QUANTUM STRING BACK REACTION IN BLACK HOLE SPACETIMES

When we consider quantized matter on a classical background, the dynamics can be described by the following Einstein equations:

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = \frac{8\pi G}{c^4}\langle \tau_\mu^\nu \rangle.$$

The space time metric $g_{\mu\nu}$ generates a nonzero vacuum expectation value of the energy momentum tensor $\langle \tau_\mu^\nu \rangle$, which in turn, acting as a source, modifies the former background. This is the so-called back reaction problem, which is a semiclassical approach to the interaction between gravity and matter.

Our aim here is to study the back reaction effect of higher massive (open) string modes [described by $\rho(m)$, Eq. (4)] in black hole space times. This will give us an insight on the last stage of black hole evaporation. Back reaction effects of massless quantum fields in these equations were already investigated [12–14].

As we are also interested in establishing the differences, and partial analogies, between string theory and the usual quantum field theory for the back reaction effects in black hole space times, we will consider a four-dimensional physical black hole.

The question now is how to write the appropriate energy-momentum tensor $\langle \tau_\mu^\nu \rangle$ for these higher excited string modes. For this purpose, we will consider the framework of the string analog model. In the spirit of this model, the VEV of the stress tensor $\langle \tau_\mu^\nu \rangle$ for the string higher excited modes is defined by

$$\langle \tau_\mu^\nu(r) \rangle = \frac{\int_{m_0}^\infty \langle T_\mu^\nu(r,m)\rangle \langle n(m)\rangle \rho(m)dm}{\int_{m_0}^\infty \langle n(m)\rangle \rho(m)dm}, \tag{32}$$

where $\langle T_\mu^\nu(r,m)\rangle$ is the Hartle-Hawking vacuum expectation value of the stress tensor of an individual quantum field, and the r dependence of $\langle T_\mu^\nu \rangle$ preserves the central gravitational character of the problem; $\rho(m)$ is the string mass density of levels [Eq. (4)] and $\langle n(m)\rangle$ is the number of field particles of mass m emitted per unit time, [Eq. (24)].

For a static spherically symmetric metric

$$ds^2 = g_{00}(r)c^2dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega_2^2, \tag{33}$$

where $g_{00}(r)<0$ $(r>r_S)$ for a Schwarzschild black hole solution, the semiclassical Einstein equations [Eq. (32)] read

$$\frac{8\pi G}{c^4}\langle \tau_r^r \rangle = g_{rr}^{-1}\left(\frac{1}{r}\frac{d\ln g_{00}}{dr} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{34a}$$

$$\frac{8\pi G}{c^4}\langle \tau_0^0 \rangle = g_{rr}^{-1}\left(\frac{1}{r^2} - \frac{1}{r}\frac{d\ln g_{rr}}{dr}\right) - \frac{1}{r^2}. \tag{34b}$$

For Schwarzschild boundary conditions

$$g_{00}(r_\infty)g_{rr}(r_\infty) = -1, \tag{35a}$$

where

$$g_{rr}(r_\infty) = \left(1 - \frac{r_S}{r_\infty}\right)^{-1}. \tag{35b}$$

The solution to Eqs. (34a) and (34b) is given by

$$g_{rr}^{-1}(r) = 1 - \frac{2GM}{c^2 r} + \frac{8\pi G}{c^4 r}\int_{r_\infty}^r \langle \tau_0^0(r')\rangle r'^2 dr', \tag{36a}$$

$$g_{00}(r) = -g_{rr}^{-1}(r)\exp\left\{\frac{8\pi G}{c^4}\int_{r_\infty}^r (\langle \tau_r^r(r')\rangle - \langle \tau_0^0(r')\rangle) \times r' g_{rr}(r')dr'\right\}, \tag{36b}$$

where M is the black hole mass measured from r_∞ (r_∞ may be infinite or the radius of a cavity where the black hole is put inside to maintain the thermal equilibrium).

In order to write $\langle T_\mu^\mu(m,r)\rangle$ for an individual quantum field in the framework of the analog model, we notice that $\rho(m)$ [Eq. (4)] depends only on m; therefore, we will consider for simplicity the vacuum expectation value of the stress tensor for a massive scalar field.

For the Hartle-Hawking vacuum (black-body radiation at infinity in equilibrium with a black hole at the temperature T_H), and when the (reduced) Compton wavelength of the massive particle ($\lambda = \hbar/mc$) is much smaller than the Schwarzschild radius (r_S)

$$\frac{\hbar c}{2GMm} \ll 1 \tag{37}$$

[the same condition as that of Eq. (6b)]. $\langle T_r^r \rangle$ and $\langle T_0^0 \rangle$ for the background BH metric [Eq. (1), $D=4$] read [13]

$$\frac{8\pi G}{c^4}\langle T^r_r\rangle = \frac{A}{r^8}F_1\left(\frac{r_S}{r}\right), \tag{38a}$$

$$\frac{8\pi G}{c^4}\langle T^0_0\rangle = \frac{A}{r^8}F_2\left(\frac{r_S}{r}\right), \tag{38b}$$

where

$$A = \frac{M^2 L^6_{PL}}{1260\pi m^2}, \tag{38c}$$

$$F_1\left(\frac{r_S}{r}\right) = 441 - \zeta 2016 + \frac{r_S}{r}(-329 + \zeta 1512) + O(m^{-4}), \tag{38d}$$

$$F_2\left(\frac{r_S}{r}\right) = -1125 + \zeta 5040 + \frac{r_S}{r}(1237 - \zeta 5544) + O(m^{-4}). \tag{38e}$$

M and m are the black hole and the scalar field masses respectively, ζ (a numerical factor) is the scalar coupling parameter ($-\zeta R\phi^2/2$; R is the scalar curvature, ϕ is the scalar field) and $L_p \equiv (\hbar G/c^3)^{1/2}$ is the Planck length.

From Eqs. (32), (38a), and (38b) the VEV of the string stress tensor will read

$$\frac{8\pi G}{c^4}\langle \tau^r_r\rangle = \frac{\mathcal{A}}{r^8}F_1\left(\frac{r_S}{r}\right), \tag{39a}$$

$$\frac{8\pi G}{c^4}\langle \tau^0_0\rangle = \frac{\mathcal{A}}{r^8}F_2\left(\frac{r_S}{r}\right), \tag{39b}$$

where

$$\mathcal{A} = \frac{M^2 L^6_{PL}}{1260\pi}\,\frac{\int_{m_0}^{\infty} m^{-2}\langle n(m)\rangle \rho(m)\,dm}{\int_{m_0}^{\infty}\langle n(m)\rangle \rho(m)\,dm}. \tag{40}$$

We return now to Eqs. (36a) and (36b) which, with the help of Eqs. (39a), (39b), and (3), can be rewritten as

$$g^{-1}_{rr}(r) = 1 - \frac{r_S}{r} + \frac{\mathcal{A}}{r}\int_{r_\infty}^{r} F_2\left(\frac{r_S}{r}\right)\frac{1}{r'^6}\,dr', \tag{41}$$

$$g_{00}(r) = -g^{-1}_{rr}(r)\exp\left\{\mathcal{A}\int_{r_\infty}^{r}\left[F_1\left(\frac{r_S}{r}\right) - F_2\left(\frac{r_S}{r}\right)\right]\times\frac{g_{rr}(r')}{r'^7}\,dr'\right\}. \tag{42}$$

A Schwarzschild black-body configuration

$$g_{00}(r) = -g^{-1}_{rr}(r) \tag{43}$$

is obtained when [Eq. (42)]

$$F_1\left(\frac{r_S}{r}\right) - F_2\left(\frac{r_S}{r}\right) \equiv (1566 - \zeta 7056)\left(1 - \frac{r_S}{r}\right) = 0, \tag{44}$$

i.e., for $\zeta = \frac{87}{392}$ for all r.

Then from Eqs. (41) and (38e), we obtain

$$g^{-1}_{rr} = 1 - \frac{r_S}{r} - \frac{\mathcal{A}}{21 r^6}\left[23\left(\frac{r_S}{r}\right) - 27\right]. \tag{45}$$

From the above equation it is clear that the quantum matter back reaction modifies the horizon r_+, which will no longer be equal to the classical Schwarzschild radius r_S. The new horizon will satisfy

$$g^{-1}_{rr} = 0, \tag{46a}$$

i.e.,

$$r^7_+ - r_S r^6_+ + \mathcal{A}\frac{27}{21} r_+ - \mathcal{A}\frac{23}{21} r_S = 0. \tag{46b}$$

In the approximation we are dealing with [$O(m^{-4})$, i.e., $\mathcal{A}^2 \ll \mathcal{A}$], the solution will have the form

$$r_+ \simeq r_S(1+\epsilon), \quad \epsilon \ll 1. \tag{47}$$

From Eq. (46b) we obtain

$$r_+ \simeq r_S\left(1 - \frac{4\mathcal{A}}{21 r^6_S}\right), \tag{48}$$

which shows that the horizon decreases.

Let us consider now the surface gravity, which is defined as

$$k(r_+) = \frac{c^2}{2}\frac{dg^{-1}_{rr}}{dr}\Big|_{r=r_+} \tag{49}$$

[in the absence of back reaction, $k(r_+) = k(r_S)$ given by $k = c^2/2r_s$ for $D=4$].

From Eqs. (45), (48), and (49) we get

$$k(r_+) = \frac{c^2}{2 r_S}\left(1 + \frac{1}{3}\frac{\mathcal{A}}{r^6_S}\right). \tag{50}$$

The black hole temperature will then be given by

$$T_+ = \frac{\hbar\kappa(r_+)}{2\pi k_B c} \simeq T_H\left(1 + \frac{1}{3}\frac{\mathcal{A}}{r^6_S}\right), \tag{51}$$

where $T_H = \hbar c/4\pi k_B r_S$. [Eqs. (16b) and (6b) for $D=4$]. The black hole temperature increases due to the back reaction.

Due to the quantum emission the black hole suffers a loss of mass. The mass loss rate is given by a Stefan-Boltzman relation. Without back reaction, we have

$$-\left(\frac{dM}{dt}\right)=\sigma 4\pi r_S^2 T_H^4, \tag{52}$$

where σ is a constant. When the back reaction is considered, we will have

$$-\left(\frac{dM}{dt}\right)_+=\sigma 4\pi r_+^2 T_+^4, \tag{53}$$

where r_+ is given by Eq. (48) and T_+ is given by Eq. (51). Inserting these values into the above equation we obtain

$$-\left(\frac{dM}{dt}\right)_+\simeq-\left(\frac{dM}{dt}\right)\left(1+\frac{20\mathcal{A}}{21r_S^6}\right). \tag{54}$$

On the other hand, the modified black hole mass is given by

$$M_+\equiv\frac{c^2}{2G}r_+\simeq M\left(1-\frac{4\mathcal{A}}{21r_S^6}\right), \tag{55}$$

which shows that the mass decreases.

From Eqs. (54) and (55), we calculate the modified lifetime of the black hole due to the back reaction

$$\tau_+\simeq\tau_H\left(1-\frac{8\mathcal{A}}{7r_S^6}\right). \tag{56}$$

We see that $\tau_+<\tau_H$ since $\mathcal{A}>0$.

We come back to the string back reaction ‘‘form factor’’ $\mathcal{A}$ [Eq. (40)] which can be rewritten as

$$\mathcal{A}=\frac{M^2L_{PL}^6}{1260\pi}\,\frac{N}{De}, \tag{57}$$

where

$$N=\int_{m_0}^{\infty}m^{-a+(D-7)/2}(mc^2\beta_H+1)e^{-(\beta_H-\beta_S)mc^2}dm \tag{58}$$

and [Eq. (29)]

$$De=I_D(m,\beta_H-\beta_S,a), \tag{59}$$

where use of Eqs. (24) and (4) has been made (the common factors for numerator and denominator cancelled out).

For arbitrary D and a, N is given by

$$\begin{aligned}N=&\frac{c^2\beta_H}{[(\beta_H-\beta_S)c^2]^{-a+(D-3)/2}}\\&\times\Gamma\left(-a+\frac{D-3}{2},(\beta_H-\beta_S)c^2m_0\right)\\&+\frac{1}{[(\beta_H-\beta_S)c^2]^{-a+(D-5)/2}}\\&\times\Gamma\left(-a+\frac{D-5}{2},(\beta_H-\beta_S)c^2m_0\right).\end{aligned} \tag{60}$$

In particular, for open strings $[a=(D-1)/2]$ we have for N and De

$$\begin{aligned}N=&c^4\beta_H(\beta_H-\beta_S)\left[E_i(-(\beta_H-\beta_S)m_0c^2)\right.\\&\left.+\frac{e^{-(\beta_H-\beta_S)m_0c^2}}{(\beta_H-\beta_S)m_0c^2}\right]-\frac{(\beta_H-\beta_S)^2c^4}{2}\\&\times\left[E_i(-(\beta_H-\beta_S)m_0c^2)+e^{-(\beta_H-\beta_S)m_0c^2}\right.\\&\left.\times\left(\frac{1}{(\beta_H-\beta_S)m_0c^2}-\frac{1}{(\beta_H-\beta_S)^2m_0^2c^4}\right)\right]\end{aligned} \tag{61}$$

and

$$De=\frac{\beta_H}{\beta_H-\beta_S}e^{-(\beta_H-\beta_S)c^2m_0}-E_i(-(\beta_H-\beta_S)c^2m_0). \tag{62}$$

For $\beta_H\to\beta_S$ ($M\to M_{\min},r_S\to r_{\min}$), we have for the open string form factor

$$\mathcal{A}_{\rm open}\simeq\frac{M_{\min}^2L_{PL}^6(\beta_H-\beta_S)}{1260\pi\beta_S}\left(\frac{1}{2m_0^2}+\frac{c^2\beta_S}{m_0}\right). \tag{63}$$

Although the string analog model is in the spirit of the canonical ensemble—all (higher) massive string fields are treated equally — we will consider too, for the sake of completeness, the string ‘‘form factor’’ $\mathcal{A}$ for closed strings.

For $a=D$ (D is the noncompact dimensions), from Eqs. (29b), (29c), and (59), we have the following expressions:

$$\begin{aligned}De=&I_D(m,\beta_H-\beta_S,D)\\=&\frac{c^2\beta_H}{[(\beta_H-\beta_S)c^2]^{-(D-1)/2}}\\&\times\Gamma\left(-\frac{D-1}{2},(\beta_H-\beta_S)m_0c^2\right)\\&+\frac{1}{[(\beta_H-\beta_S)c^2]^{-(D+1)/2}}\Gamma\left(-\frac{D+1}{2},(\beta_H-\beta_S)m_0c^2\right)\end{aligned} \tag{64}$$

and [Eq. (58)]

$$N=\frac{c^2\beta_H}{[(\beta_H-\beta_S)c^2]^{-(D+3)/2}}\Gamma\left(-\frac{D+3}{2},(\beta_H-\beta_S)c^2m_0\right)+\frac{1}{[(\beta_H-\beta_S)c^2]^{-(D+5)/2}}\Gamma\left(-\frac{D+5}{2},(\beta_H-\beta_S)c^2m_0\right). \tag{65}$$

For $\beta_H\to\beta_S$ and D even, we have then

$$N=c^2\beta_S[(\beta_H-\beta_S)c^2]^{(D+3)/2}\Gamma\left(-\frac{D+3}{2}\right)+\frac{c^2\beta_S}{(m_0)^{(D+3)/2}[(D+3)/2]}+[(\beta_H-\beta_S)c^2]^{(D+5)/2}\times\Gamma\left(-\frac{D+5}{2}\right)+\frac{1}{[(D+5)/2]m_0^{(D+5)/2}} \tag{66}$$

and

$$De=c^2\beta_S[(\beta_H-\beta_S)c^2]^{(D-1)/2}\Gamma\left(-\frac{D-1}{2}\right)+\frac{c^2\beta_S}{(m_0)^{(D-1)/2}[(D-1)/2]}+[c^2(\beta_H-\beta_S)]^{(D+1)/2}\times\Gamma\left(-\frac{D+1}{2}\right)+\frac{1}{m_0^{(D+1)/2}\left(\frac{D+1}{2}\right)}. \tag{67}$$

Therefore, from Eqs. (57), (66), and (67), $\mathcal{A}_{\text{closed}}$ is given by

$$\left(\frac{N}{De}\right)_{\text{closed}}=\frac{\dfrac{c^2\beta_S}{(m_0)^{(D+3)/2}\left(\frac{D+3}{2}\right)}+\dfrac{1}{\left(\frac{D+5}{2}\right)(m_0)^{(D+5)/2}}}{\dfrac{c^2\beta_S}{m_0^{(D-1)/2}\left(\frac{D-1}{2}\right)}+\dfrac{1}{m_0^{(D+1)/2}\left(\frac{D+1}{2}\right)}} \tag{68}$$

and for $D=4$, we have for $\beta_H\to\beta_S$ ($M\to M_{\min}$, $r_S\to r_{\min}$)

$$\mathcal{A}_{\text{closed}}=\frac{M_{\min}^2L_{PL}^6}{1260\pi m_0^2}\left(\frac{\dfrac{c^2\beta_S}{7}+\dfrac{1}{9m_0}}{\dfrac{c^2\beta_S}{3}+\dfrac{1}{5m_0}}\right). \tag{69}$$

From Eqs. (63) and (68), we now evaluate the number $\mathcal{A}/r_S^6$ appearing in the expressions for r_+ [Eq. (48)], T_+ [Eq. (51)], M_+ [Eq. (55)], and τ_+ [Eq. (56)], for the two opposite limiting regimes $\beta_H\to\beta_S$ and $\beta_H\gg\beta_S$:

$$\left(\frac{\mathcal{A}_{\text{open}}}{r_{\min}^6}\right)_{\beta_H\to\beta_S}\simeq\frac{(\beta_H-\beta_S)}{\beta_S}\frac{16}{315}\left(\frac{\pi}{b}\right)^3\left(\frac{M_S}{M_{PL}}\right)^2\left(\frac{M_S}{m_0}\right)\simeq(\beta_H-\beta_S)\frac{16}{315}\left(\frac{\pi}{b}\right)^3\left(\frac{M_S}{M_{PL}}\right)^2\frac{M_S^2c^2}{bm_0}\ll1, \tag{70}$$

$$\left(\frac{\mathcal{A}_{\text{closed}}}{r_{\min}^6}\right)_{\beta_H\to\beta_S}\simeq\frac{16}{735b}\left(\frac{\pi}{b}\right)^3\left(\frac{M_S}{M_{PL}}\right)^2\left(\frac{M_S}{m_0}\right)^2\ll1. \tag{71}$$

In the opposite (semiclassical) regime $\beta_H\gg\beta_S$, i.e., $T_H\ll T_S$, we have from Eqs. (61), (62), (64), and (65)

$$\left(\frac{N}{De}\right)^{\text{open}}_{\beta_H\gg\beta_S}\simeq\frac{1}{m_0^2}\simeq\left(\frac{N}{De}\right)^{\text{closed}}_{\beta_H\gg\beta_S} \tag{72}$$

as

$$\beta_Hm_0c^2=8\pi\left(\frac{M}{M_{PL}}\right)\left(\frac{m_0}{M_{PL}}\right)\gg1 \tag{73}$$

($m_0,M\gg M_{PL}$). Then, from [Eq. (57)]

$$\left(\frac{\mathcal{A}}{r_S^6}\right)^{\text{open / closed}}_{\beta_H\gg\beta_S}\simeq\frac{1}{80640\pi}\left(\frac{M_{PL}}{M}\right)^4\left(\frac{M_{PL}}{m_0}\right)^2\ll1. \tag{74}$$

That is, in this regime, we consistently recover $r_+\simeq r_S$, $T_+\simeq T_H$, $M_+\simeq M$, and $\tau_+\simeq\tau_H=k_Bc/6\sigma G\hbar T_H^3$.

V. CONCLUSIONS

We have suitably combined QFT and quantum string theory in the black hole background in the framework of the string analog model (or thermodynamical approach). We have computed the quantum string emission by a black hole and the back reaction effect on the black hole in the framework of this model. A clear and precise picture of the black hole evaporation emerges. The QFT semiclassical regime and the quantum string regime of black holes have been identified and described.

The Hawking temperature T_H is the intrinsic black hole temperature in the QFT semiclassical regime. The intrinsic string temperature T_S is the black hole temperature in the quantum string regime. The two regimes can be mapped one onto another by the $\mathcal{R}$ transform.

String theory properly describes black hole evaporation: because of the emission, the semiclassical BH becomes a string state (the ‘‘minimal’’ BH), and the emitted string gas becomes a condensed microscopic state (the ‘‘minimal’’ BH) due to a phase transition. The last stage of the radiation in string theory makes such a transition possible.

The phase transition undergone by the string gas at the critical temperature T_S represents (in the thermodynamical framework) the back reaction effect of the string emission on the BH.

Cosmological evolution goes from a quantum string phase to a QFT and classical phase. Black hole evaporation goes from a QFT semiclassical phase to a string phase. The Hawking temperature, which we know as the black hole temperature in the QFT semiclassical regime, becomes the string temperature for the ‘‘string black hole’’ in the quantum string regime.

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