General definition of ''conserved quantities'' in general relativity and other theories of gravity

Robert M. Wald and Andreas Zoupas
Enrico Fermi Institute and Department of Physics, University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637-1433
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In general relativity, the notion of mass and other conserved quantities at spatial infinity can be defined in a natural way via the Hamiltonian framework: Each conserved quantity is associated with an asymptotic symmetry and the value of the conserved quantity is defined to be the value of the Hamiltonian which generates the canonical transformation on phase space corresponding to this symmetry. However, such an approach cannot be employed to define ''conserved quantities'' in a situation where symplectic current can be radiated away (such as occurs at null infinity in general relativity) because there does not, in general, exist a Hamiltonian which generates the given asymptotic symmetry. (This fact is closely related to the fact that the desired ''conserved quantities'' are not, in general, conserved.) In this paper we give a prescription for defining ''conserved quantities'' by proposing a modification of the equation that must be satisfied by a Hamiltonian. Our prescription is a very general one, and is applicable to a very general class of asymptotic conditions in arbitrary diffeomorphism covariant theories of gravity derivable from a Lagrangian, although we have not investigated existence and uniqueness issues in the most general contexts. In the case of general relativity with the standard asymptotic conditions at null infinity, our prescription agrees with the one proposed by Dray and Streubel from entirely different considerations.

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I. INTRODUCTION

Notions of energy and angular momentum have played a key role in analyzing the behavior of physical theories. For theories of fields in a fixed, background spacetime, a locally conserved stress-energy-momentum tensor, T_{ab}, normally can be defined. If the background spacetime has a Killing field k^a, then $J^a = T^a{}_b k^b$ is a locally conserved current. If Σ is a Cauchy surface, then $q = \int_\Sigma J^a d\Sigma_a$ defines a conserved quantity associated with k^a; if Σ is a timelike or null surface, then $\int_\Sigma J^a d\Sigma_a$ has the interpretion of the flux of this quantity through Σ.

However, in diffeomorphism covariant theories such as general relativity, there is no notion of the local stress-energy tensor of the gravitational field, so conserved quantities (which clearly must include gravitational contributions) and their fluxes cannot be defined by the above procedures, even when Killing fields are present. Nevertheless, in general relativity, for asymptotically flat spacetimes, conserved quantities associated with asymptotic symmetries have been defined at spatial and null infinity.

A definition of mass-energy and radiated energy at null infinity, $\mathcal{I}$, was first given about 40 years ago by Trautman [1] and Bondi *et al.* [2]. This definition was arrived at via a detailed study of the asymptotic behavior of the metric, and the main justification advanced for this definition has been its agreement with other notions of mass in some simple cases as well as the fact that the radiated energy is always positive (see, e.g., [3,4] for further discussion of the justification for this definition). A number of inequivalent definitions of quantities associated with general Bondi-Matzner-Sachs (BMS) asymptotic symmetries at null infinity have been proposed over the years, but it was not until the mid 1980s that Dray and Streubel [5] gave a general definition that appears to have fully satisfactory properties [6]. This definition generalized a definition of angular momentum given by Penrose [7] that was motivated by twistor theory.

In much of the body of work on defining ''conserved quantities'' at null infinity, little contact has been made with the Hamiltonian formulation of general relativity. An important exception is the work of Ashtekar and Streubel [8] (see also [9]), who noted that BMS transformations correspond to canonical transformations on the radiative phase space at $\mathcal{I}$. They identified the Hamiltonian generating these canonical transformations as representing the net flux of the ''conserved quantity'' through $\mathcal{I}$. They then also obtained a local flux formula under some additional assumptions not related to the canonical framework (in particular, by their choice of topology they, in effect, imposed the condition that the local flux formula contain no ''second derivative terms''). However, they did not attempt to derive a local expression for the ''conserved quantity'' itself within the Hamiltonian framework, and, indeed, until the work of [5] and [6], it was far from clear that, for arbitrary BMS generators, their flux formula corresponded to a quantity that could be locally defined on cross sections of $\mathcal{I}$.

The status of the definition of ''conserved quantities'' at null infinity contrasts sharply with the situation at spatial infinity, where formulas for conserved quantities have been derived in a clear and straightforward manner from the Hamiltonian formulation of general relativity [10,11]. As will be reviewed in Secs. II and III below, for a diffeomorphism covariant theory derived from a Lagrangian, if one is given a spacelike slice Σ and a vector field ξ^a representing ''time evolution,'' then the Hamiltonian generating this time evolution—if it exists—must be purely a ''surface term'' when evaluated on solutions (i.e., ''on shell''). It can be shown that if Σ extends to spatial infinity in a suitable manner and if ξ^a is a suitable infinitesimal asymptotic symmetry, then a Hamiltonian does exist (see ''case I'' of Sec. IV below). The value of this Hamiltonian ''on shell'' then can be interpreted as being the conserved quantity conjugate to ξ^a. One thereby directly obtains formulas for the Arnowitt-Deser-Misner (ADM) mass, momentum, and angular momentum as limits as one approaches spatial infinity of surface integrals over two-spheres.

It might seem natural to try a similar approach at null infinity: Let Σ be a spacelike slice which is asymptotically null in the sense that in the unphysical spacetime its boundary is a cross section, $\mathcal{C}$, of $\mathcal{I}$. Let the vector field ξ^a be an infinitesimal BMS asymptotic symmetry. Then, when evaluated on solutions, the Hamiltonian generating this time evolution—if it exists—must again be purely a ''surface term'' on Σ; i.e., it must be expressible as an integral of a local expression over the cross section $\mathcal{C}$. This expression would then provide a natural candidate for the value of the ''conserved quantity'' conjugate to ξ^a at ''time'' $\mathcal{C}$.

As we shall see in Sec. III below, the above proposal works if ξ^a is everywhere tangent to $\mathcal{C}$. However, if ξ^a fails to be everywhere tangent to $\mathcal{C}$, then it is easy to show that no Hamiltonian generating the time evolution exists. The obstruction to defining a Hamiltonian arises directly from the possibility that symplectic current can escape through $\mathcal{C}$.

The main purpose of this paper is to propose a general prescription for defining ''conserved quantities'' in situations where a Hamiltonian does not exist. This proposal consists of modifying the equation that a Hamiltonian must satisfy via the addition of a ''correction term'' involving a symplectic potential that is required to vanish whenever the background spacetime is stationary. If such a symplectic potential exists and is unique—and if a suitable ''reference solution'' can be chosen to fix the arbitrary constant in the definition of the ''conserved quantity''—we obtain a unique prescription for defining a ''conserved quantity'' associated with any infinitesimal asymptotic symmetry. In the case of asymptotically flat spacetimes at null infinity in vacuum general relativity, we show in Sec. V that existence and uniqueness do hold, and that this prescription yields the quantities previously obtained in [5].

In Sec. II, we review some preliminary material on the diffeomorphism covariant theories derived from a Lagrangian. In Sec. III, we investigate the conditions under which a Hamiltonian exists. In Sec. IV, we present, in a very general setting, our general proposal for the definition of ''conserved quantities'' associated with infinitesimal asymptotic symmetries. This general proposal is then considered in the case of asymptotically flat spacetimes at null infinity in general relativity in Sec. V, where it is shown to yield the results of [5]. Some further applications are briefly discussed in Sec. VI.

II. PRELIMINARIES

In this paper, we will follow closely both the conceptual framework and the notational conventions of [12] and [13]. Further details of most of what is discussed in this section can be found in those references.

On an n-dimensional manifold, M, we consider a theory of dynamical fields, collectively denoted ϕ, which consist of a Lorentzian metric, g_{ab}, together with other tensor fields, collectively denoted as ψ. To proceed, we must define a space, $\mathcal{F}$, of ''kinematically allowed'' field configurations, $\phi=(g_{ab},\psi)$ on M. A precise definition of $\mathcal{F}$ would involve the specification of smoothness properties of ϕ, as well as possible additional restrictions on g_{ab} (such as global hyperbolicity or the requirement that a given foliation of M by hypersurfaces be spacelike) and asymptotic conditions on ϕ (such as the usual asymptotic flatness conditions on fields at spatial and/or null infinity in general relativity). The precise choice of $\mathcal{F}$ that would be most suitable for one's purposes would depend upon the specific theory and issues being considered. In this section and the next section, we will merely assume that a suitable $\mathcal{F}$ has been defined in such a way that the integrals occurring in the various formulas below converge. In Sec. IV, we will impose a general set of conditions on $\mathcal{F}$ that will ensure convergence of all relevant integrals. In Sec. V, we will verify that asymptotically flat spacetimes at null infinity in vacuum general relativity satisfy these conditions.

We assume that the equations of motion of the theory arise from a diffeomorphism covariant n-form Lagrangian density [13]

$$\mathbf{L}=\mathbf{L}(g_{ab};R_{abcd},\nabla_a R_{bcde},\ldots;\psi,\nabla_a\psi,\ldots) \qquad (1)$$

where ∇_a denotes the derivative operator associated with g_{ab}, and R_{abcd} denotes the Riemann curvature tensor of g_{ab}. [An arbitrary (but finite) number of derivatives of R_{abcd} and ψ are permitted to appear in $\mathbf{L}$.] Here and below we use boldface letters to denote differential forms on spacetime and, when we do so, we will suppress the spacetime indices of these forms. Variation of $\mathbf{L}$ yields

$$\delta\mathbf{L}=\mathbf{E}(\phi)\delta\phi+d\boldsymbol{\theta}(\phi,\delta\phi) \qquad (2)$$

where no derivatives of $\delta\phi$ appear in the first term on the right side. The Euler-Lagrange equations of motion of the theory are then simply $\mathbf{E}=0$. Note that—when the variation is performed under an integral sign—the term $\boldsymbol{\theta}$ corresponds to the boundary term that arises from the integrations by parts needed to remove derivatives from $\delta\phi$. We require that $\boldsymbol{\theta}$ be locally constructed out of ϕ and $\delta\phi$ in a covariant manner. This restricts the freedom in the choice of $\boldsymbol{\theta}$ to[1]

$$\boldsymbol{\theta}\rightarrow\boldsymbol{\theta}+d\mathbf{Y} \qquad (3)$$

where $\mathbf{Y}$ is locally constructed out of ϕ and $\delta\phi$ in a covariant manner.

The presymplectic current $(n-1)$-form, $\boldsymbol{\omega}$—which is a local function of a field configuration, ϕ, and two linearized perturbations, $\delta_1\phi$ and $\delta_2\phi$ off of ϕ—is obtained by taking an antisymmetrized variation of $\boldsymbol{\theta}$:

[1]If we change the Lagrangian by $\mathbf{L}\rightarrow\mathbf{L}+d\mathbf{K}$, the equations of motion are unaffected. Under such a change in the Lagrangian, we have $\boldsymbol{\theta}\rightarrow\boldsymbol{\theta}+\delta\mathbf{K}$. Thus, if such changes in the Lagrangian are admitted, we will have this additional ambiguity in $\boldsymbol{\theta}$. However, this ambiguity does not affect the definition of the presymplectic current form [see Eq. (4) below] and will not affect our analysis.

$$\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)=\delta_1\boldsymbol{\theta}(\phi,\delta_2\phi)-\delta_2\boldsymbol{\theta}(\phi,\delta_1\phi). \tag{4}$$

On account of the ambiguity (3) in the choice of $\boldsymbol{\theta}$, we have the ambiguity

$$\boldsymbol{\omega}\rightarrow\boldsymbol{\omega}+d[\delta_1\mathbf{Y}(\phi,\delta_2\phi)-\delta_2\mathbf{Y}(\phi,\delta_1\phi)] \tag{5}$$

in the choice of $\boldsymbol{\omega}$.

Now let Σ be a closed, embedded $(n-1)$-dimensional submanifold without boundary; we will refer to Σ as a *slice*. The presymplectic form, Ω_Σ, associated with Σ is a map taking field configurations, ϕ, together with a pair of linearized perturbations off of ϕ, into the real numbers—i.e., it is a two-form on $\mathcal{F}$—defined by integrating[2] $\boldsymbol{\omega}$ over Σ:

$$\Omega_\Sigma(\phi,\delta_1\phi,\delta_2\phi)=\int_\Sigma\boldsymbol{\omega}. \tag{6}$$

Although this definition depends, in general, upon the choice of Σ, if $\delta_1\phi$ and $\delta_2\phi$ satisfy the linearized field equations and Σ is required to be a Cauchy surface, then Ω_Σ does not depend upon the choice of Σ, provided that Σ is compact or suitable asymptotic conditions are imposed on the dynamical fields [12]. The ambiguity (5) in the choice of $\boldsymbol{\omega}$ gives rise to the ambiguity

$$\Omega_\Sigma(\phi,\delta_1\phi,\delta_2\phi)\rightarrow\Omega_\Sigma(\phi,\delta_1\phi,\delta_2\phi)+\int_{\partial\Sigma}[\delta_1\mathbf{Y}(\phi,\delta_2\phi)-\delta_2\mathbf{Y}(\phi,\delta_1\phi)] \tag{7}$$

in the presymplectic form Ω_Σ. In this equation, by the integral over $\partial\Sigma$, we mean a limiting process in which the integral is first taken over the boundary, ∂K, of a compact region, K, of Σ (so that Stokes' theorem can be applied[3]), and then K approaches all of Σ in a suitably specified manner. (Note that since Σ is a slice, by definition it does not have an actual boundary in the spacetime.) Thus, for example, if Σ is an asymptotically flat spacelike slice in an asymptotically flat spacetime, the integral on the right side of Eq. (7) would correspond to the integral over a two-sphere on Σ in the asymptotically flat region in the limit as the radius of the two-sphere approaches infinity. Of course, the right side of Eq. (7) will be well defined only if this limit exists and is independent of any of the unspecified details of how the compact region, K, approaches Σ. In Sec. IV below, we will make some additional assumptions that will ensure that integrals over ''$\partial\Sigma$'' of certain quantities that we will consider are well defined.

[2]If Σ is spacelike, the orientation of Σ relative to the spacetime orientation $\boldsymbol{\epsilon}_{a_1\ldots a_n}$ is chosen to be $v^{a_1}\boldsymbol{\epsilon}_{a_1\ldots a_n}$ where v^a is a future-directed timelike vector.

[3]We choose the orientation of ∂K to be the one specified by Stokes' theorem; i.e., we dot the first index of the orientation form on K into an outward pointing vector.

Given the presymplectic form, Ω_Σ, we can factor $\mathcal{F}$ by the orbits of the degeneracy subspaces of Ω_Σ to construct a phase space, Γ, in the manner described in [12]. This phase space acquires directly from the presymplectic form Ω_Σ on $\mathcal{F}$ a nondegenerate symplectic form, Ω. One also obtains by this construction a natural projection from $\mathcal{F}$ to Γ. Now, a complete vector field ξ^a on M naturally induces the field variation $\mathcal{L}_\xi\phi$ on fields $\phi\in\mathcal{F}$. If ξ^a is such that $\mathcal{L}_\xi\phi$ corresponds to a tangent field on $\mathcal{F}$ (i.e., if the diffeomorphisms generated by ξ^a map $\mathcal{F}$ into itself), then we may view $\delta_\xi\phi=\mathcal{L}_\xi\phi$ as the dynamical evolution vector field corresponding to the notion of ''time translations'' defined by ξ^a. If, when restricted to the solution submanifold,[4] $\bar{\mathcal{F}}$, of $\mathcal{F}$, this time evolution vector field on $\mathcal{F}$ consistently projects to phase space, then one has a notion of time evolution associated with ξ^a on the ''constraint submanifold,'' $\bar{\Gamma}$, of Γ, where $\bar{\Gamma}$ is defined to be the image of $\bar{\mathcal{F}}$ under the projection of $\mathcal{F}$ to Γ. If this time evolution vector field on $\bar{\Gamma}$ preserves the pullback to $\bar{\Gamma}$ of Ω, it will be generated by a Hamiltonian, H_ξ [12]. (As argued in the Appendix of [12], this will be the case when Σ is compact; see Sec. III below for some general results in the noncompact case.) Thus, this construction provides us with the notion of a Hamiltonian, H_ξ, conjugate to a vector field ξ^a on M.

However, a number of complications arise in the above construction. In particular, in order to obtain a consistent projection of $\mathcal{L}_\xi\phi$ from $\bar{\mathcal{F}}$ to $\bar{\Gamma}$, it is necessary to choose ξ^a to be ''field dependent,'' i.e., to depend upon ϕ. As explained in [12], this fact accounts for why, in a diffeomorphism covariant theory, the Poisson bracket algebra of constraints does not naturally correspond to the Lie algebra of infinitesimal diffeomorphisms. However, these complications are not relevant to our present concerns. To avoid dealing with them, we prefer to work on the original field configuration space $\mathcal{F}$ with its (degenerate) presymplectic form, Ω_Σ, rather than on the phase space, Γ. The notion of a Hamiltonian, H_ξ, on $\mathcal{F}$ can be defined as follows:

Definition. Consider a diffeomorphism covariant theory within the above framework, with field configuration space $\mathcal{F}$ and solution submanifold $\bar{\mathcal{F}}$. Let ξ^a be a vector field on the spacetime manifold, M, let Σ be a slice of M, and let Ω_Σ denote the presymplectic form (6). (If the ambiguity (5) in the choice of $\boldsymbol{\omega}$ gives rise to an ambiguity in Ω_Σ [see Eq. (7)], then we assume that a particular choice of Ω_Σ has been made.) Suppose that $\mathcal{F}$, ξ^a, and Σ have been chosen so that the integral $\int_\Sigma\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi)$ converges for all $\phi\in\bar{\mathcal{F}}$ and all tangent vectors $\delta\phi$ to $\bar{\mathcal{F}}$ at ϕ. Then a function $H_\xi:\mathcal{F}\rightarrow\mathbb{R}$ is said to be a *Hamiltonian conjugate to* ξ^a on slice Σ if for all $\phi\in\bar{\mathcal{F}}$ and all field variations $\delta\phi$ tangent to $\mathcal{F}$ (but not necessarily tangent to $\bar{\mathcal{F}}$) we have

$$\delta H_\xi=\Omega_\Sigma(\phi,\delta\phi,\mathcal{L}_\xi\phi)=\int_\Sigma\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi). \tag{8}$$

[4]The solution submanifold, $\bar{\mathcal{F}}$, is sometimes referred to as the ''covariant phase space'' [9].

Note that if a Hamiltonian conjugate to ξ^a on slice Σ exists, then—assuming that $\bar{\mathcal{F}}$ is connected—its value on $\bar{\mathcal{F}}$ is uniquely determined by Eq. (8) up to the addition of an arbitrary constant. In many situations, this constant can be fixed in a natural way by requiring H_ξ to vanish for a natural reference solution, such as Minkowski spacetime. On the other hand, the value of H_ξ off of $\bar{\mathcal{F}}$ is essentially arbitrary, since Eq. (8) fixes only the ''field space gradient'' of H_ξ in directions off of $\bar{\mathcal{F}}$ at points of $\bar{\mathcal{F}}$.

If a Hamiltonian conjugate to ξ^a on slice Σ exists, then its value provides a natural definition of a conserved quantity associated with ξ^a at ''time'' Σ. However, in many cases of interest—such as occurs in general relativity when, say, ξ^a is an asymptotic time translation and the slice Σ goes to null infinity—no Hamiltonian exists. In the next section, we shall analyze the conditions under which a Hamiltonian exists. In Sec. IV, we shall propose a definition of the ''conserved quantity'' conjugate to ξ^a on a slice Σ when no Hamiltonian exists.

III. EXISTENCE OF A HAMILTONIAN

When does a Hamiltonian conjugate to ξ^a on slice Σ exist? To analyze this issue, it is very useful to introduce the Noether current $(n-1)$-form associated with ξ^a, defined by

$$\mathbf{j}=\boldsymbol{\theta}(\phi,\mathcal{L}_\xi\phi)-\xi\cdot\mathbf{L} \tag{9}$$

where the ''$\cdot$'' denotes the contraction of the vector field ξ^a into the first index of the differential form $\mathbf{L}$. One can show (see the Appendix of [14]) that for a diffeomorphism covariant theory, $\mathbf{j}$ always can be written in the form

$$\mathbf{j}=d\mathbf{Q}+\xi^a\mathbf{C}_a, \tag{10}$$

where $\mathbf{C}_a=0$ when the equations of motion hold; i.e., $\mathbf{C}_a$ corresponds to ''constraints'' of the theory. Equation (10) defines the Noether charge $(n-2)$-form, $\mathbf{Q}$. It was shown in [13] that the Noether charge always takes the form

$$\mathbf{Q}=\mathbf{X}^{ab}(\phi)\nabla_{[a}\xi_{b]}+\mathbf{U}_a(\phi)\xi^a+\mathbf{V}(\phi,\mathcal{L}_\xi\phi)+d\mathbf{Z}(\phi,\xi). \tag{11}$$

From Eqs. (2), (4), and (9), it follows immediately that for $\phi\in\bar{\mathcal{F}}$ but $\delta\phi$ arbitrary (i.e., $\delta\phi$ tangent to $\mathcal{F}$ but not necessarily tangent to $\bar{\mathcal{F}}$), the variation of $\mathbf{j}$ satisfies

$$\delta\mathbf{j}=\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi)+d(\xi\cdot\boldsymbol{\theta}). \tag{12}$$

Thus, we obtain

$$\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi)=\xi^a\delta\mathbf{C}_a+d(\delta\mathbf{Q})-d(\xi\cdot\boldsymbol{\theta}). \tag{13}$$

Consequently, if there exists a Hamiltonian, H_ξ, conjugate to ξ^a on Σ, then for all $\phi\in\bar{\mathcal{F}}$ and all $\delta\phi$ it must satisfy the equation

$$\delta H_\xi=\int_\Sigma\xi^a\delta\mathbf{C}_a+\int_{\partial\Sigma}[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}] \tag{14}$$

where the integral over $\partial\Sigma$ has the meaning explained below Eq. (7). Note that for field variations which are ''on shell,'' i.e., such that $\delta\phi$ satisfies the linearized equations of motion, we have

$$\delta H_\xi=\int_{\partial\Sigma}[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}]. \tag{15}$$

Consequently, if H_ξ exists, it is given purely as a ''surface term'' (i.e., an integral over $\partial\Sigma$) when evaluated on $\bar{\mathcal{F}}$.

Equation (14) gives rise to an obvious necessary condition for the existence of H_ξ: Let $\phi\in\bar{\mathcal{F}}$ (i.e., ϕ is a solution to the field equations) and let $\delta_1\phi$ and $\delta_2\phi$ be tangent to $\bar{\mathcal{F}}$ (i.e., $\delta_1\phi$ and $\delta_2\phi$ satisfy the linearized field equations). Let $\phi(\lambda_1,\lambda_2)$ be a two-parameter family with $\phi(0,0)=\phi$, $\partial\phi/\partial\lambda_1(0,0)=\delta_1\phi$, and $\partial\phi/\partial\lambda_2(0,0)=\delta_2\phi$. Then, if Eq. (14) holds, by equality of mixed partial derivatives, we must have

$$\begin{aligned}0&=(\delta_1\delta_2-\delta_2\delta_1)H_\xi\\&=-\int_{\partial\Sigma}\xi\cdot[\delta_1\boldsymbol{\theta}(\phi,\delta_2\phi)-\delta_2\boldsymbol{\theta}(\phi,\delta_1\phi)]\\&=-\int_{\partial\Sigma}\xi\cdot\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi).\end{aligned} \tag{16}$$

Conversely, if Eq. (16) holds, then—assuming that $\bar{\mathcal{F}}$ is simply connected (and has suitable differentiable properties)—it will be possible to define H_ξ on $\bar{\mathcal{F}}$ so that Eq. (14) holds whenever $\delta\phi$ is tangent to $\bar{\mathcal{F}}$.

To show this, on each connected component of $\bar{\mathcal{F}}$ choose a ''reference solution'' $\phi_0\in\bar{\mathcal{F}}$ and define $H_\xi=0$ at ϕ_0. Let $\phi\in\bar{\mathcal{F}}$ and let $\phi(\lambda)$ for $\lambda\in[0,1]$ be a smooth, one-parameter family of solutions that connects ϕ_0 to ϕ. Define

$$H_\xi[\phi]=\int_0^1 d\lambda\int_{\partial\Sigma}[\delta\mathbf{Q}(\lambda)-\xi\cdot\boldsymbol{\theta}(\lambda)]. \tag{17}$$

This definition will be independent of the choice of path $\phi(\lambda)$ when Eq. (16) holds since, by simple connectedness, any other path $\phi'(\lambda)$ will be homotopic to $\phi(\lambda)$ and one can apply Stokes' theorem to the two-dimensional submanifold spanned by this homotopy. This defines H_ξ on $\bar{\mathcal{F}}$. However, if H_ξ is defined on $\bar{\mathcal{F}}$, there is no obstruction to extending H_ξ to $\mathcal{F}$ so that Eq. (14) holds on $\bar{\mathcal{F}}$ for all $\delta\phi$ tangent to $\mathcal{F}$ (i.e., including $\delta\phi$ that are not tangent to $\bar{\mathcal{F}}$), since the additional content of that equation merely fixes the first derivative of H_ξ in the ''off shell'' directions of field space.

Therefore, the necessary and sufficient condition for the existence of a Hamiltonian conjugate to ξ^a on Σ is that for all solutions $\phi\in\bar{\mathcal{F}}$ and all pairs of linearized solutions $\delta_1\phi,\delta_2\phi$ tangent to $\bar{\mathcal{F}}$, we have

$$\int_{\partial\Sigma}\xi\cdot\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)=0. \tag{18}$$

Note that since this condition refers only to the ''covariant phase space'' $\bar{\mathcal{F}}$, we shall in the following restrict attention to entirely $\bar{\mathcal{F}}$ and use Eq. (15) for H_ξ [even though the ''off shell'' volume integral in Eq. (14) is crucial to justifying the interpretation of H_ξ as the generator of dynamics conjugate to ξ^a].

Note that there are two situations where Eq. (18) will automatically hold: (i) if the asymptotic conditions on ϕ are such that $\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)$ goes to zero sufficiently rapidly that the integral of $\xi\cdot\boldsymbol{\omega}$ over ∂K vanishes in the limit as K approaches Σ; (ii) if ξ^a is such that K can always be chosen so that ξ^a is tangent to ∂K, since then the pullback of $\xi\cdot\boldsymbol{\omega}$ to ∂K vanishes. In two these cases, a Hamiltonian conjugate to ξ^a will exist on Σ. However, if these conditions do not hold, then in general no Hamiltonian will exist.

We turn, now, to giving a general prescription for defining ''conserved quantities,'' even when no Hamiltonian exists.

IV. GENERAL DEFINITION OF ''CONSERVED QUANTITIES''

In this section, we will propose a definition of conserved quantities under very general assumptions about asymptotic conditions ''at infinity.'' We begin by specifying these assumptions.

We shall assume that the desired asymptotic conditions in the given diffeomorphism covariant theory under consideration are specified by attaching a boundary, $\mathcal{B}$, to the spacetime manifold, M, and requiring certain limiting behavior of the dynamical fields, ϕ, as one approaches $\mathcal{B}$. We shall assume that $\mathcal{B}$ is an $(n-1)$-dimensional manifold, so that $M\cup\mathcal{B}$ is an n-dimensional manifold with boundary.[5] In cases of interest, $M\cup\mathcal{B}$ will be equipped with additional nondynamical structure (such as a conformal factor on $M\cup\mathcal{B}$ or certain tensor fields on $\mathcal{B}$) that will enter into the specification of the limiting behavior of ϕ and thereby be part of the specification of the field configuration space, $\mathcal{F}$, and the covariant phase space, $\bar{\mathcal{F}}$. We will refer to such fixed, nondynamical structure as the ''universal background structure'' of $M\cup\mathcal{B}$.

We now state our two main assumptions concerning the asymptotic conditions on the dynamical fields, ϕ, and the asymptotic behavior of the allowed hypersurfaces, Σ: (1) We assume that $\mathcal{F}$ has been defined so that for all $\phi\in\bar{\mathcal{F}}$ and for all $\delta_1\phi,\delta_2\phi$ tangent to $\bar{\mathcal{F}}$, the $(n-1)$-form $\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi)$ defined on M extends continuously[6] to $\mathcal{B}$. (2) We restrict consideration to slices, Σ, in the ''physical spacetime,'' M, which extend smoothly to $\mathcal{B}$ in the ''unphysical spacetime,'' $M\cup\mathcal{B}$, such that this extended hypersurface intersects $\mathcal{B}$ in a smooth $(n-2)$-dimensional submanifold, denoted $\partial\Sigma$. Following terminology commonly used for null infinity, we shall refer to $\partial\Sigma$ as a ''cross section'' of $\mathcal{B}$. We also shall assume that $\Sigma\cup\partial\Sigma$ is compact—although it would be straightforward to weaken this assumption considerably, since only the behavior of Σ near $\mathcal{B}$ is relevant to our considerations.

An important immediate consequence of the above two assumptions is that the integral (6) defining Ω_Σ always converges, since it can be expressed as the integral of a continuous $(n-1)$-form over the compact $(n-1)$-dimensional hypersurface $\Sigma\cup\partial\Sigma$.

We turn, now, to the definition of infinitesimal asymptotic symmetries. Let ξ^a be a complete vector field on $M\cup\mathcal{B}$ (so that, in particular, ξ^a is tangent to $\mathcal{B}$ on $\mathcal{B}$). We say that ξ^a is a *representative of an infinitesimal asymptotic symmetry* if its associated one-parameter group of diffeomorphisms maps $\bar{\mathcal{F}}$ into $\bar{\mathcal{F}}$, i.e., if it preserves the asymptotic conditions specified in the definition of $\bar{\mathcal{F}}$. Equivalently, ξ^a is a representative of an infinitesimal asymptotic symmetry if $\mathcal{L}_\xi\phi$ (which automatically satisfies the linearized field equations [12]) satisfies all of the asymptotic conditions on linearized solutions arising from the asymptotic conditions imposed upon $\phi\in\bar{\mathcal{F}}$, i.e., if $\mathcal{L}_\xi\phi$ corresponds to a vector tangent to $\bar{\mathcal{F}}$.

If ξ^a is a representative of an infinitesimal asymptotic symmetry, then the integral appearing on the right side of Eq. (15), namely

$$I=\int_{\partial\Sigma}[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}], \tag{19}$$

always is well defined via the limiting procedure described below Eq. (7), and, indeed, I depends only on the cross section $\partial\Sigma$ of $\mathcal{B}$, not on Σ. To see this,[7] let K_i be a nested sequence of compact subsets of Σ such that ∂K_i approaches $\partial\Sigma$, and let

$$I_i=\int_{\partial K_i}[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}]. \tag{20}$$

Then, since ''on shell'' we have

$$\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi)=d[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}] \tag{21}$$

[see Eq. (13) above], we have, by Stokes' theorem for $i\geqslant j$,

$$I_i-I_j=\int_{\Sigma_{ij}}d[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}]=\int_{\Sigma_{ij}}\boldsymbol{\omega}(\phi,\delta\phi,\mathcal{L}_\xi\phi) \tag{22}$$

where Σ_{ij} denotes $K_i\backslash K_j$, i.e., the portion of Σ lying between ∂K_i and ∂K_j. As a direct consequence of our assump-

[5]The assumption that $\mathcal{B}$ is an $(n-1)$-dimensional manifold structure is not essential in cases where $\boldsymbol{\omega}$ vanishes at $\mathcal{B}$ (see ''case I'' below). In particular, there should be no difficulty in extending our framework to definitions of asymptotic flatness at spatial infinity in which $\mathcal{B}$ is comprised by a single point [15].

[6]It should be emphasized that we require that the full $\boldsymbol{\omega}$ extend continuously to $\mathcal{B}$—not merely its pullback to hypersurfaces that approach $\mathcal{B}$.

[7]A similar argument has previously been given to show that the ''linkage formulas'' are well defined (see [16,17]).

tions that $\boldsymbol{\omega}$ extends continuously to $\mathcal{B}$ and that $\Sigma \cup \partial\Sigma$ is compact, it follows that $\{I_i\}$ is a Cauchy sequence, and hence it has a well-defined limit, I, as $i \to \infty$. Note that this limit always exists despite the fact that there is no guarantee that the differential forms $\mathbf{Q}$ or $\boldsymbol{\theta}$ themselves extend continuously to $\mathcal{B}$. A similar argument establishes that this limit is independent of Σ; i.e., for a slice $\tilde{\Sigma}$ such that $\partial\tilde{\Sigma} = \partial\Sigma$, a similarly defined sequence $\{\tilde{I}_i\}$ of integrals on $\tilde{\Sigma}$ will also converge to I.

Let ξ^a and ξ'^a be representatives of infinitesimal asymptotic symmetries. We say that ξ^a is *equivalent* to ξ'^a if they coincide on $\mathcal{B}$ and if, for all $\phi \in \bar{\mathcal{F}}$, $\delta\phi$ tangent to $\bar{\mathcal{F}}$, and for all $\partial\Sigma$ on $\mathcal{B}$, we have $I = I'$, where I is given by Eq. (19) and I' is given by the same expression with ξ^a replaced by ξ'^a. The *infinitesimal asymptotic symmetries* of the theory are then comprised by the equivalence classes of the representatives of the infinitesimal asymptotic symmetries.

Now consider an infinitesimal asymptotic symmetry, represented by the vector field ξ^a, and let Σ be a slice in the spacetime with boundary $\partial\Sigma$ on $\mathcal{B}$. We would like to define a conserved quantity $H_\xi : \bar{\mathcal{F}} \to \mathbb{R}$ associated with ξ^a at ''time'' Σ via Eq. (15). As we have seen above, the right side of Eq. (15) is well defined under our asymptotic assumptions, but as discussed in the previous section, in general, there does not exist an H_ξ which satisfies this equation. The analysis naturally breaks up into the following two cases:

Case I. Suppose that the continuous extension of $\boldsymbol{\omega}$ to $\mathcal{B}$ has vanishing pullback to $\mathcal{B}$. Then by Eq. (18), H_ξ exists for all infinitesimal asymptotic symmetries (assuming that $\bar{\mathcal{F}}$ is simply connected and has suitable differentiable properties) and is independent of the choice of representative ξ^a. Furthermore, if $\partial\Sigma_1$ and $\partial\Sigma_2$ are cross sections of $\mathcal{B}$ that bound a region $\mathcal{B}_{12} \subset \mathcal{B}$, we have,[8] by Eqs. (15) and (21),

$$\delta H_\xi|_{\partial\Sigma_2} - \delta H_\xi|_{\partial\Sigma_1} = -\int_{\mathcal{B}_{12}} \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi) = 0. \quad (23)$$

Thus, δH_ξ is independent of choice of cross section within the same homology class. If the arbitrary constant (for each cross section) in H_ξ is fixed in such a way that there is a ''reference solution'' for which $H_\xi = 0$ on all cross sections (see below), then on all solutions H_ξ will be independent of the choice of cross section within the same homology class. Thus, in this case, not only does H_ξ exist, but it truly corresponds to a conserved quantity; i.e., its value is independent of ''time,'' Σ.

Case II. Suppose that the continuous extension of $\boldsymbol{\omega}$ to $\mathcal{B}$ does not, in general, have vanishing pullback to $\mathcal{B}$. Then, in general, there does not exist an H_ξ satisfying Eq. (15). One exception is the case where ξ^a and $\partial\Sigma$ are such that ξ^a is everywhere tangent to $\partial\Sigma$. In this case, if ξ^a is tangent to cross sections $\partial\Sigma_1$ and $\partial\Sigma_2$, we have

$$\delta H_\xi|_{\partial\Sigma_2} - \delta H_\xi|_{\partial\Sigma_1} = -\int_{\mathcal{B}_{12}} \boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi). \quad (24)$$

Since the right side of this equation is nonvanishing in general, we see that even when ξ^a is tangent to cross sections so that H_ξ exists, H_ξ will not be conserved.

Case I arises in general relativity for spacetimes which are asymptotically flat at spatial infinity as defined in [18], and our prescription for defining H_ξ corresponds to that given in [10] and [11]; see [13] for explicit details of how Eq. (15) gives rise to the usual expression for ADM mass when ξ^a is an asymptotic time translation. As we shall discuss in detail in the next section, case II arises in general relativity for spacetimes which are asymptotically flat at null infinity.

The main purpose of this paper is to provide a general definition of a ''conserved quantity'' conjugate to an arbitrary infinitesimal asymptotic symmetry ξ^a in case II. In the following, we will restrict attention to this case, and we will denote the quantity we seek as $\mathcal{H}_\xi$ to distinguish it from a true Hamiltonian H_ξ. As we have seen, in this case an attempt to define $\mathcal{H}_\xi$ by Eq. (15) fails the consistency check (16) and thus does not define any quantity. However, consider the following simple modification of Eq. (15): On $\mathcal{B}$, let $\boldsymbol{\Theta}$ be a symplectic potential for the pullback, $\bar{\boldsymbol{\omega}}$, of the (extension of the) symplectic current form $\boldsymbol{\omega}$ to $\mathcal{B}$, so that on $\mathcal{B}$ we have, for all $\phi \in \bar{\mathcal{F}}$ and $\delta_1\phi, \delta_2\phi$ tangent to $\bar{\mathcal{F}}$,

$$\bar{\boldsymbol{\omega}}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \boldsymbol{\Theta}(\phi, \delta_2\phi) - \delta_2 \boldsymbol{\Theta}(\phi, \delta_1\phi). \quad (25)$$

We require that $\boldsymbol{\Theta}$ be locally constructed[9] out of the dynamical fields, ϕ, and their derivatives (or limits of such quantities to $\mathcal{B}$) as well as any fields present in the ''universal background structure.'' In the case where $\mathbf{L}$ (and, hence $\boldsymbol{\omega}$) is an analytic function[10] of its variables [see Eq. (1)], we also require that $\boldsymbol{\Theta}$ depend analytically on the dynamical fields; more precisely, if $\phi(\lambda)$ is a one-parameter family of fields on M that depends analytically on λ and satisfies suitable uniformity conditions[11] near $\mathcal{B}$, we require that the corre-

[8]We define the orientation of $\mathcal{B}$ to be that obtained by dotting the first index of the orientation of M into an outward pointing vector. The orientation of $\partial\Sigma$ was previously specified in footnotes 2 and 3. The signs in Eq. (24) to correspond to the case where $\partial\Sigma_2$ lies to the future of $\partial\Sigma_1$.

[9]More precisely, by ''locally constructed'' we mean the following: Suppose that $\chi: M \cup \mathcal{B} \to M \cup \mathcal{B}$ is a diffeomorphism which preserves the universal background structure. Suppose $(\phi, \delta\phi)$ and $(\phi', \delta\phi')$ are such that there exists an open (in $M \cup \mathcal{B}$) neighborhood, $\mathcal{O}$, of $p \in \mathcal{B}$ such that for all $x \in M \cap \mathcal{O}$ we have $\phi = \chi_* \phi'$ and $\delta\phi = \chi_* \delta\phi'$, where χ_* denotes the pullback map on tensor fields associated with the diffeomorphism χ. Then we require that at p we have $\boldsymbol{\Theta} = \chi_* \boldsymbol{\Theta}'$.

[10]The condition that $\mathbf{L}$ be an analytic function of its variables (as occurs in essentially all theories ever seriously considered) has nothing to do with any smoothness or analyticity conditions concerning the behavior of the dynamical fields themselves on M. We do not impose any analyticity conditions on the dynamical fields.

[11]For the case of asymptotically flat spacetimes at null infinity, $\mathcal{I}$, a suitable uniformity condition would be to require the unphysical fields to vary analytically with λ at $\mathcal{I}$.

sponding $\mathbf{\Theta}(\lambda)$ also depend analytically on λ. If any arbitrary choices are made in the specification of the background structure (such as a choice of conformal factor in the definition of null infinity in general relativity), then we demand that $\mathbf{\Theta}$ be independent of such choices (so, in particular, in the case of null infinity, $\mathbf{\Theta}$ is required to be conformally invariant). Our proposal is the following: Let $\mathcal{H}_\xi$ satisfy[12]

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma}[\delta\mathbf{Q} - \xi\cdot\boldsymbol{\theta}] + \int_{\partial\Sigma}\xi\cdot\mathbf{\Theta}. \tag{26}$$

Then it is easily seen that this formula satisfies the consistency check (16) and, thus, defines a ''conserved quantity'' $\mathcal{H}_\xi$ up to an arbitrary constant. Finally, let this arbitrary constant be fixed by requiring that $\mathcal{H}_\xi$ vanish (for all infinitesimal asymptotic symmetries ξ^a and all cross sections $\partial\Sigma$) on a suitably chosen ''reference solution'' $\phi_0 \in \overline{\mathcal{F}}$. We will specify below the necessary conditions that must be satisfied by ϕ_0.

However, the above proposal fails to define a unique prescription because the choice of symplectic potential $\mathbf{\Theta}$ is ambiguous up to[13]

$$\mathbf{\Theta}(\phi,\delta\phi) \to \mathbf{\Theta}(\phi,\delta\phi) + \delta\mathbf{W}(\phi) \tag{27}$$

where $\mathbf{W}$ is an $(n-1)$-form on $\mathcal{B}$ locally constructed out of the dynamical fields ϕ as well as the universal background structure defined on $\mathcal{B}$, with $\mathbf{W}$ independent of any arbitrary choices made in the specification of the background structure. Thus, in order to obtain a prescription which defines $\mathcal{H}_\xi$, we must specify an additional condition or conditions which uniquely select $\mathbf{\Theta}$.

An additional requirement on $\mathbf{\Theta}$ can be motivated as follows. We have already seen from Eq. (24) above that $\mathcal{H}_\xi$ cannot, in general, be conserved; i.e., there must be a nonzero flux, $\mathbf{F}_\xi$, on $\mathcal{B}$ associated with this ''conserved quantity.'' This is to be expected on account of the possible presence of radiation at $\mathcal{B}$. However, it seems natural to demand that $\mathbf{F}_\xi$ vanish (and, thus, that $\mathcal{H}_\xi$ be conserved) in the case where no radiation is present at $\mathcal{B}$. Such a case should occur when ϕ is a stationary solution, i.e., when there exists a nonzero infinitesimal asymptotic symmetry represented by an exact symmetry t^a—so that $\mathcal{L}_t\phi = 0$ in M—and t^a is timelike in M in a neighborhood of $\mathcal{B}$. Hence, we wish to require that $\mathbf{F}_\xi$ vanish on $\mathcal{B}$ for all ξ^a for stationary solutions.

To see what condition on $\mathbf{\Theta}$ will ensure that this holds, we note that from Eq. (26) it follows immediately that

[12]Here it should be noted that the new term on the right side of this equation is an ordinary integral over the surface $\partial\Sigma$ of $\mathcal{B}$, whereas, as explained above, the first term in general is defined only as an asymptotic limit.

[13]Note that the ambiguity in $\mathbf{\Theta}$ is of an entirely different nature than the ambiguity (3) in $\boldsymbol{\theta}$. The quantity $\boldsymbol{\theta}$ is defined from the Lagrangian $\mathbf{L}$ (*before* $\boldsymbol{\omega}$ has been defined) and its ambiguity arises from Eq. (2). The quantity $\mathbf{\Theta}$ is defined from $\boldsymbol{\omega}$ and its ambiguity arises from Eq. (25).

$$\delta\mathcal{H}_\xi|_{\partial\Sigma_2} - \delta\mathcal{H}_\xi|_{\partial\Sigma_1} = -\int_{\mathcal{B}_{12}}\delta\mathbf{F}_\xi \tag{28}$$

where the variation of the flux $(n-1)$-form, $\mathbf{F}_\xi$, on $\mathcal{B}$ is given by

$$\delta\mathbf{F}_\xi = \overline{\boldsymbol{\omega}}(\phi,\delta\phi,\mathcal{L}_\xi\phi) + d[\xi\cdot\mathbf{\Theta}(\phi,\delta\phi)]. \tag{29}$$

Here the first term in this equation arises from taking ''d'' of the integrand of the first term in Eq. (26) [using Eq. (21) above], whereas the second term is just the ''d'' of the integrand of the second term in Eq. (26). However, we have

$$\begin{aligned} d[\xi\cdot\mathbf{\Theta}(\phi,\delta\phi)] &= \mathcal{L}_\xi\mathbf{\Theta}(\phi,\delta\phi) \\ &= -\overline{\boldsymbol{\omega}}(\phi,\delta\phi,\mathcal{L}_\xi\phi) + \delta\mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi). \end{aligned} \tag{30}$$

Thus, we obtain

$$\delta\mathbf{F}_\xi = \delta\mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi). \tag{31}$$

We now impose the requirement that $\mathbf{\Theta}(\phi,\delta\phi)$ vanish whenever ϕ is stationary (even when $\delta\phi$ is non-stationary). We also explicitly assume that the reference solution, ϕ_0 (on which $\mathcal{H}_\xi$ vanishes for all cross sections and hence $\mathbf{F}_\xi = 0$), is stationary. Since both $\mathbf{\Theta}$ and $\mathbf{F}_\xi$ vanish on ϕ_0, we obtain from Eq. (31) the remarkably simple formula

$$\mathbf{F}_\xi = \mathbf{\Theta}(\phi,\mathcal{L}_\xi\phi). \tag{32}$$

It then follows immediately (as a consequence of our choice of $\mathbf{\Theta}$) that $\mathbf{F}_\xi$ vanishes (for all ξ) on stationary solutions, as we desired. Equation (32) also implies an additional desirable property of $\mathbf{F}_\xi$: We have $\mathbf{F}_\xi = 0$ whenever ξ^a is an exact symmetry—i.e., whenever $\mathcal{L}_\xi\phi = 0$—regardless of whether radiation may be present.

If a symplectic potential $\mathbf{\Theta}$ satisfying our above condition exists and is unique, then Eq. (26) together with the requirement that $\mathcal{H}_\xi$ vanish (for all cross sections and all ξ^a) on a particular, specified solution, ϕ_0, uniquely determines $\mathcal{H}_\xi$. However, there remains a potential difficulty in specifying ϕ_0: If $\phi_0 \in \overline{\mathcal{F}}$, then we also have $\psi_*\phi_0 \in \overline{\mathcal{F}}$, where $\psi: M\cup\mathcal{B} \to M\cup\mathcal{B}$ is any diffeomorphism generated by a representative of an infinitesimal asymptotic symmetry. Since we have no meaningful way of distinguishing between ϕ_0 and $\psi_*\phi_0$, if we demand that $\mathcal{H}_\xi$ vanish on ϕ_0, we must also demand that it vanish on $\psi_*\phi_0$. However, this overdetermines $\mathcal{H}_\xi$ (so that no solution exists) unless the following consistency condition holds: Let η^a be a representative of an infinitesimal asymptotic symmetry and consider the field variation about ϕ_0 given by $\delta\phi = \mathcal{L}_\eta\phi_0$. Since this corresponds to the action of an infinitesimal asymptotic symmetry on ϕ_0, under this field variation we must have $\delta\mathcal{H}_\xi = 0$. On the other hand, $\delta\mathcal{H}_\xi$ is specified by Eq. (26). Since under this field variation we have

$$\delta\mathbf{Q}[\xi] = \mathcal{L}_\eta\mathbf{Q}[\xi] - \mathbf{Q}[\mathcal{L}_\eta\xi] \tag{33}$$

and since, by assumption, $\mathbf{\Theta}$ vanishes at ϕ_0, we find that the consistency requirement on ϕ_0 is that for all representatives ξ^a and η^a of infinitesimal asymptotic symmetries and for all cross sections $\partial\Sigma$, we must have

$$0=\int_{\partial\Sigma}\{\mathcal{L}_\eta\mathbf{Q}[\xi]-\mathbf{Q}[\mathcal{L}_\eta\xi]-\xi\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\eta\phi_0)\}. \quad (34)$$

From Eqs. (21) and (25) together with the vanishing of $\mathbf{\Theta}$ at ϕ_0, it follows that the right side of Eq. (34) is independent of cross section and thus need only be checked at one cross section. In addition, Eq. (34) manifestly holds when η^a is an exact symmetry of ϕ_0—i.e., when $\mathcal{L}_\eta\phi_0=0$—since $\delta\phi=0$ in that case. Using

$$\mathcal{L}_\eta\mathbf{Q}=d(\eta\cdot\mathbf{Q})+\eta\cdot d\mathbf{Q}=d(\eta\cdot\mathbf{Q})+\eta\cdot\mathbf{j} \quad (35)$$

together with Eq. (9), we may rewrite Eq. (34) in the form

$$0=\int_{\partial\Sigma}\{\eta\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\xi\phi_0)-\xi\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\eta\phi_0) -\eta\cdot(\xi\cdot\mathbf{L})-\mathbf{Q}[\mathcal{L}_\eta\xi]\}. \quad (36)$$

[Here, the integral over $\partial\Sigma$ is to be interpreted as an asymptotic limit, with the limit guaranteed to exist by the argument given above. If $\mathbf{L}$ extends continuously to $\mathcal{B}$, then the term $\eta\cdot(\xi\cdot\mathbf{L})$ makes no contribution to the integral since both η^a and ξ^a are tangent to $\mathcal{B}$.] Since Eq. (36) is manifestly antisymmetric in η^a and ξ^a, it follows that the consistency condition also is automatically satisfied whenever ξ^a is an exact symmetry of ϕ_0. However, if both η^a and ξ^a are asymptotic symmetries that are not exact symmetries of ϕ_0, then Eq. (34) [or, equivalently, Eq. (36)] yields a nontrivial condition that must be satisfied by ϕ_0.

To summarize, we propose the following prescription for defining ''conserved quantities'' in case II: Let $\mathbf{\Theta}$ be a symplectic potential on $\mathcal{B}$ [see Eq. (25) above] which is locally constructed out of the dynamical fields and background structure (and is an analytic function of the dynamical fields when $\mathbf{L}$ is analytic), is independent of any arbitrary choices made in specifying the background structure, and is such that $\mathbf{\Theta}(\phi,\delta\phi)$ vanishes for all $\delta\phi$ tangent to $\bar{\mathcal{F}}$ whenever $\phi\in\bar{\mathcal{F}}$ is stationary. [If it exists, such a $\mathbf{\Theta}$ is unique up to addition of a term $\delta\mathbf{W}$ where $\mathbf{W}$ is locally constructed out of the dynamical fields and background structure (and is analytic in the dynamical fields when $\mathbf{L}$ is analytic), is independent of any arbitrary choices made in specifying the background structure, and is such that $\delta\mathbf{W}$ vanishes for all $\delta\phi$ tangent to $\bar{\mathcal{F}}$ whenever ϕ is stationary.] Let ϕ_0 be a stationary solution that satisfies Eq. (34) [or, equivalently, Eq. (36)] for all infinitesimal asymptotic symmetries η^a and ξ^a. Then we define $\mathcal{H}_\xi$ by Eq. (26) together with the requirement that $\mathcal{H}_\xi$ vanish on ϕ_0. To the extent that a $\mathbf{\Theta}$ satisfying the above requirements exists and is unique, and to the extent that a stationary ϕ_0 satisfying Eq. (34) exists, this defines a prescription for defining ''conserved quantities'' associated with asymptotic symmetries. This prescription automatically gives rise to the flux formula (32), so that the flux vanishes whenever ϕ is stationary or ξ^a is an exact symmetry.

In the next section, we analyze what this general prescription yields for the case of asymptotically flat spacetimes at null infinity in vacuum general relativity.

V. ''CONSERVED QUANTITIES'' AT NULL INFINITY IN GENERAL RELATIVITY

In vacuum general relativity, the manifold M is taken to be 4-dimensional and the only dynamical field, ϕ, is the spacetime metric, g_{ab}. We shall write the varied field as

$$\gamma_{ab}\equiv\delta g_{ab}. \quad (37)$$

The Einstein-Hilbert Lagrangian of general relativity is

$$\mathbf{L}=\frac{1}{16\pi}R\boldsymbol{\epsilon} \quad (38)$$

where R denotes the scalar curvature of g_{ab} and $\boldsymbol{\epsilon}$ is the spacetime volume form associated with g_{ab}. The presymplectic potential 3-form $\boldsymbol{\theta}$ is given by

$$\theta_{abc}=\frac{1}{16\pi}\epsilon_{dabc}v^d \quad (39)$$

where

$$v^a=g^{ae}g^{fh}[\nabla_f\gamma_{eh}-\nabla_e\gamma_{fh}] \quad (40)$$

where ∇_a is the derivative operator associated with g_{ab}. The corresponding presymplectic current 3-form is [19]

$$\omega_{abc}=\frac{1}{16\pi}\epsilon_{dabc}w^d \quad (41)$$

where

$$w^a=P^{abcdef}[\gamma_{2bc}\nabla_d\gamma_{1ef}-\gamma_{1bc}\nabla_d\gamma_{2ef}] \quad (42)$$

with

$$P^{abcdef}=g^{ae}g^{fb}g^{cd}-\frac{1}{2}g^{ad}g^{be}g^{fc}-\frac{1}{2}g^{ab}g^{cd}g^{ef} -\frac{1}{2}g^{bc}g^{ae}g^{fd}+\frac{1}{2}g^{bc}g^{ad}g^{ef}. \quad (43)$$

Finally, the Noether charge 2-form associated with a vector field ξ^a is given by [13]

$$Q_{ab}[\xi]=-\frac{1}{16\pi}\epsilon_{abcd}\nabla^c\xi^d. \quad (44)$$

We wish to consider spacetimes that are asymptotically flat at future and/or past null infinity. For definiteness, we will consider future null infinity. [Sign changes would occur in several formulas when we consider past null infinity on account of our orientation convention on $\mathcal{B}$ (see footnote 8).]

We denote future null infinity by $\mathcal{I}$ and adopt the standard definition of asymptotic flatness there (see, e.g., [21]). The key ingredient of this definition is that there exist a smooth[14] metric, $\tilde{g}_{ab}$, on $M\cup\mathcal{I}$ and a smooth function, Ω, on $M\cup\mathcal{I}$ such that $\Omega>0$ on M, $\Omega=0$ on $\mathcal{I}$, and $\tilde{\nabla}_a\Omega$ is null[15] and nonvanishing everywhere on $\mathcal{I}$, and such that throughout M we have

$$\tilde{g}_{ab}=\Omega^2 g_{ab}. \tag{45}$$

We also assume that $\mathcal{I}$ has topology $S^2\times\mathbb{R}$. In the following all indices will be raised and lowered using the ''unphysical metric,'' $\tilde{g}_{ab}$. We write

$$n_a=\tilde{\nabla}_a\Omega. \tag{46}$$

(Here $\tilde{\nabla}_a$ denotes the derivative operator associated with $\tilde{g}_{ab}$, although, of course, since Ω is a scalar, $\nabla_a\Omega$ is independent of the choice of derivative operator.) We may use the freedom $\Omega\to\omega\Omega$ with ω a smooth, strictly positive function on $M\cup\mathcal{I}$ to assume, without loss of generality, that the Bondi condition

$$\tilde{\nabla}_a n_b|_{\mathcal{I}}=0 \tag{47}$$

holds. An immediate consequence of Eq. (47) is that on $\mathcal{I}$ we have $\tilde{\nabla}_a(n^b n_b)=2n^b\tilde{\nabla}_a n_b=0$, so, in the Bondi gauge,

$$n^a n_a=O(\Omega^2). \tag{48}$$

Without loss of generality (see, e.g., [17]), we also may assume that the conformal factor, Ω, on $M\cup\mathcal{I}$ and the unphysical metric, $\tilde{g}_{ab}$, on $\mathcal{I}$ are universal quantities; i.e., they may be assumed to be independent of the physical metric, g_{ab}, on M. Without loss of generality, we may (by use of freedom remaining in the choice of Ω) take the universal unphysical metric $\tilde{g}^0_{ab}$, on $\mathcal{I}$ to be such that the induced spatial metric on all cross sections of $\mathcal{I}$ is that of a round two-sphere of scalar curvature k. In the following, we will fix an allowed choice of Ω on $M\cup\mathcal{I}$ and a choice of k. We will then take[16] $\mathcal{F}$ to consist of metrics, g_{ab}, on M such that $\Omega^2 g_{ab}$ extends smoothly to $\mathcal{I}$ and equals $\tilde{g}^0_{ab}$ there, and such that the Bondi condition (47) holds on $\mathcal{I}$. It may then be checked that the general notion of infinitesimal asymptotic symmetries given in the previous section corresponds to the usual notion of infinitesimal BMS symmetries; indeed, our general definition of infinitesimal asymptotic symmetries corresponds closely to the definition of infinitesimal BMS symmetries[17] given in [17].

It follows immediately from our conditions on $\mathcal{F}$ that the unphysical perturbed metric

$$\tilde{\gamma}_{ab}\equiv\Omega^2\gamma_{ab} \tag{49}$$

extends smoothly to $\mathcal{I}$ and vanishes there, so it can be written in the form

$$\tilde{\gamma}_{ab}=\Omega\,\tau_{ab} \tag{50}$$

where τ_{ab} extends smoothly to $\mathcal{I}$ and, in general, is nonvanishing there. Furthermore, since $\delta n_a=0$, we have

$$\delta[\tilde{\nabla}_a n_b]=-\left\{\tilde{\nabla}_{(a}\tilde{\gamma}_{b)c}-\frac{1}{2}\tilde{\nabla}_c\tilde{\gamma}_{ab}\right\}n^c. \tag{51}$$

Substituting from Eqs. (49), (50), and (46) and setting the resulting expression to zero on $\mathcal{I}$ in accordance with Eq. (47), we obtain

$$n_{(a}\tau_{b)c}n^c|_{\mathcal{I}}=0. \tag{52}$$

This, in turn, implies that $\tau_{bc}n^c$ vanishes on $\mathcal{I}$, so we may write

$$\tau_{bc}n^c=\Omega\,\tau_b \tag{53}$$

where τ_b is smooth (and, in general, nonvanishing) at $\mathcal{I}$. This implies that

$$\delta n^a=\delta(\tilde{g}^{ab}n_b)=-\Omega\,\tau^{ab}n_b=-\Omega^2\tau^a. \tag{54}$$

The crucial issue with regard to the applicability of the ideas of the previous section is whether the presymplectic current 3-form[18] $\boldsymbol{\omega}$ extends continuously to $\mathcal{I}$. To investigate this, we express the quantities appearing in Eq. (41) in terms of Ω and variables that extend smoothly to $\mathcal{I}$. Clearly, the unphysical volume element

$$\tilde{\boldsymbol{\epsilon}}=\Omega^4\boldsymbol{\epsilon} \tag{55}$$

and

$$\tilde{P}^{abcdef}\equiv\Omega^{-6}P^{abcdef} \tag{56}$$

[14]The requirement of smoothness could be weakened considerably without affecting our analysis.

[15]For solutions to the vacuum field equations, it follows from the fact that $\Omega=0$ on $\mathcal{I}$ that $\tilde{\nabla}_a\Omega$ is null on $\mathcal{I}$ in the metric $\tilde{g}_{ab}$.

[16]Note that our imposition of this rather rigid structure on $\mathcal{F}$ as a result of our gauge fixing is not done merely for convenience, but is necessary in order that $\boldsymbol{\omega}$ extend to $\mathcal{I}$.

[17]The only difference between our definition and the definition given in [17] concerns the notion of the equivalence of two representatives, ξ^a and ξ'^a. In addition to requiring agreement of ξ^a and ξ'^a at $\mathcal{I}$, we impose the extra requirement that they give rise to the same asymptotic integral (19). However, it is not difficult to show that if ξ^a and ξ'^a agree at $\mathcal{I}$, then they automatically give rise to the same asymptotic integral (19).

[18]As noted in Sec. II, $\boldsymbol{\omega}$ has the ambiguity (5). However, Iyer [20] has shown that if $\mathbf{Y}$ is such that $\boldsymbol{\theta}$ maintains the general form given by Eq. (23) of [13] with the coefficients in that formula being regular, analytic functions of the fields, then $\mathbf{Y}$ must vanish on $\mathcal{I}$. Consequently, in vacuum general relativity, the limit to $\mathcal{I}$ of $\boldsymbol{\omega}$ is, in fact, unique.

extend smoothly to $\mathcal{I}$ and are nonvanishing there. We eliminate the the action of the physical derivative operator, ∇_a, on γ_{ab} in terms of the unphysical derivative operator, $\tilde{\nabla}_a$, via

$$\nabla_a \gamma_{bc} = \tilde{\nabla}_a \gamma_{bc} + 2C^d{}_{a(b}\gamma_{c)d} \tag{57}$$

where (see, e.g., [21])

$$C^c{}_{ab} = 2\Omega^{-1}\delta^c{}_{(a}n_{b)} - \Omega^{-1}n^c\tilde{g}_{ab}. \tag{58}$$

Finally, we substitute

$$\gamma_{ab} = \Omega^{-1}\tau_{ab}. \tag{59}$$

The terms appearing in the resulting expression for $\boldsymbol{\omega}$ may now be classified as follows: (i) Terms in which $\tilde{\nabla}_a$ acts on τ_{1ab} or τ_{2ab}. For these terms, the powers of Ω resulting from Eqs. (55), (56), and (59) cancel, so these terms extend smoothly to $\mathcal{I}$ and are, in general, nonvanishing there. (ii) Terms in which $\tilde{\nabla}_a$ does not act on τ_{1ab} or τ_{2ab} and w^a is proportional to n^a. These terms cancel due to the antisymmetry in τ_{1ab} and τ_{2ab}. (iii) Terms in which $\tilde{\nabla}_a$ does not act on τ_{1ab} or τ_{2ab} but w^a is not proportional to n^a. These terms necessarily contain a contraction of n^a with τ_{1ab} or τ_{2ab}, and Eq. (53) can then be used. The extra power of Ω picked up by the use of this equation ensures that these terms extend smoothly to $\mathcal{I}$, where they are, in general, nonvanishing. The upshot is that $\boldsymbol{\omega}$ extends smoothly to $\mathcal{I}$ and is, in general, nonvanishing there. Thus, with our definition of $\mathcal{F}$, asymptotically flat spacetimes at null infinity in general relativity do indeed fall into the category of "case II" of the previous section.

To apply the proposed prescription of the previous section to define a "conserved quantity," $\mathcal{H}_\xi$, for each BMS generator, ξ^a, and each cross section, $\partial\Sigma$, of $\mathcal{I}$, we need an explicit formula for the pullback, $\bar{\boldsymbol{\omega}}$, of the extension of $\boldsymbol{\omega}$ to $\mathcal{I}$. To do so, we define ${}^{(3)}\boldsymbol{\epsilon}$ by

$$\tilde{\boldsymbol{\epsilon}}_{abcd} = 4\ {}^{(3)}\boldsymbol{\epsilon}_{[abc}n_{d]} \tag{60}$$

so that the pullback, ${}^{(3)}\bar{\boldsymbol{\epsilon}}$, of ${}^{(3)}\boldsymbol{\epsilon}$ to $\mathcal{I}$ defines a positively oriented volume element[19] on $\mathcal{I}$ (see footnote 8). We have

$$\bar{\boldsymbol{\omega}} = -\frac{1}{16\pi}\Omega^{-4}n_a w^a\ {}^{(3)}\bar{\boldsymbol{\epsilon}}. \tag{61}$$

A lengthy but entirely straightforward calculation starting with Eq. (42), making the substitutions (57)–(59), and making heavy use of Eqs. (47), (48), and (53) yields (see also [22,9])

$$\Omega^{-4}n_a w^a|_{\mathcal{I}} = \frac{1}{2}\{-\tau_2^{bc}n^a\tilde{\nabla}_a\tau_{1bc} + \tau_2 n^a\tilde{\nabla}_a\tau_1 + \tau_2 n^a\tau_{1a}\} - [1\leftrightarrow 2] \tag{62}$$

[19]For past null infinity, this volume element would be negatively oriented, resulting in sign changes in some of the formulas below.

where we have written $\tau = \tau^a{}_a$ and "$1\leftrightarrow 2$" denotes the same terms as in the preceding expression with 1 and 2 interchanged.

The above formula can be rewritten in a more useful form as follows. By a direct computation using Eq. (7.5.14) of [21], the variation of the unphysical Ricci tensor at $\mathcal{I}$ is given by

$$\delta\tilde{R}_{ab}|_{\mathcal{I}} = -n_{(a}\tilde{\nabla}_{b)}\tau - n^c\tilde{\nabla}_c\tau_{ab} + n_{(b}\tilde{\nabla}^d\tau_{a)d} + n_{(a}\tau_{b)}. \tag{63}$$

Hence, defining S_{ab} by

$$S_{ab} \equiv \tilde{R}_{ab} - \frac{1}{6}\tilde{R}\tilde{g}_{ab} \tag{64}$$

we obtain

$$\delta S_{ab}|_{\mathcal{I}} = -n_{(a}\tilde{\nabla}_{b)}\tau - n^c\tilde{\nabla}_c\tau_{ab} + n_{(b}\tilde{\nabla}^d\tau_{a)d} + n_{(a}\tau_{b)} - \frac{1}{3}(-n^c\tilde{\nabla}_c\tau + n^c\tau_c)\tilde{g}_{ab}. \tag{65}$$

On the other hand, $\tilde{R}_{ab}$ is related to R_{ab} by the usual conformal transformation formulas (see, e.g., Appendix D of [21]). Setting $R_{ab}=0$ by the vacuum field equations, it follows that [see Eq. (6) of [3]]

$$S_{ab} = -2\Omega^{-1}\tilde{\nabla}_{(a}n_{b)} + \Omega^{-2}n^c n_c\tilde{g}_{ab}. \tag{66}$$

Taking the variation of this equation and evaluating the resulting expression on $\mathcal{I}$ using Eqs. (51), (50), (54) and (53), we obtain

$$\delta S_{ab}|_{\mathcal{I}} = 4n_{(a}\tau_{b)} - n^c\tilde{\nabla}_c\tau_{ab} - n^c\tau_c\tilde{g}_{ab}. \tag{67}$$

Comparing this formula with Eq. (65), we obtain

$$[\tilde{\nabla}^b\tau_{ab} - \tilde{\nabla}_a\tau - 3\tau_a]|_{\mathcal{I}} = 0 \tag{68}$$

as well as

$$[n^b\tilde{\nabla}_b\tau + 2n^b\tau_b]|_{\mathcal{I}} = 0. \tag{69}$$

Using Eq. (69) together with Eq. (65), we see that

$$\Omega^{-4}n_a w^a|_{\mathcal{I}} = \frac{1}{2}[\tau_2^{ab}\delta_1 S_{ab} - \tau_1^{ab}\delta_2 S_{ab}]. \tag{70}$$

Now, the Bondi news tensor, N_{ab}, on $\mathcal{I}$ is defined by [3]

$$N_{ab} = \bar{S}_{ab} - \rho_{ab} \tag{71}$$

where $\bar{S}_{ab}$ denotes the pullback to $\mathcal{I}$ of S_{ab} and ρ_{ab} is the tensor field on $\mathcal{I}$ defined in general by Eq. (33) of [3], which, in our gauge choice, is just $\frac{1}{2}k\bar{\tilde{g}}^0_{ab}$, where $\bar{\tilde{g}}^0_{ab}$ denotes the pullback to $\mathcal{I}$ of $\tilde{g}^0_{ab}$. Since $\delta\rho_{ab}=0$ and since, by Eq. (53), τ^{ab} on $\mathcal{I}$ is tangent to $\mathcal{I}$, we may replace δS_{ab} by δN_{ab} in Eq. (70). Thus, we obtain our desired final formula:

$$\overline{\boldsymbol{\omega}} = -\frac{1}{32\pi}[\tau_2^{ab}\delta_1 N_{ab} - \tau_1^{ab}\delta_2 N_{ab}]\,^{(3)}\overline{\boldsymbol{\epsilon}}. \quad (72)$$

To apply our prescription, we must find a symplectic potential, $\boldsymbol{\Theta}$, for $\overline{\boldsymbol{\omega}}$ on $\mathcal{I}$ which is locally constructed[20] out of the spacetime metric, g_{ab}, and background structure (and depends analytically on the metric), is independent of any arbitrary choices made in specifying the background structure, and is such that $\boldsymbol{\Theta}(g_{ab},\gamma_{ab})$ vanishes for all γ_{ab} whenever g_{ab} is stationary. By inspection, a symplectic potential satisfying all of these properties is given by[21]

$$\boldsymbol{\Theta} = -\frac{1}{32\pi}N_{ab}\tau^{ab}\,^{(3)}\overline{\boldsymbol{\epsilon}}. \quad (73)$$

As discussed in Sec. III, this choice of $\boldsymbol{\Theta}$ will be unique if and only if there does not exist a 3-form $\mathbf{W}$ on $\mathcal{I}$ which is locally constructed (in the sense of footnote 9) out of the physical metric, g_{ab}, and background structure (and depends analytically on the physical metric), is independent of any arbitrary choices made in specifying the background structure, and is such that $\delta\mathbf{W}$ vanishes for all γ_{ab} whenever g_{ab} is stationary. In our case, the only relevant ''background structure'' present is the conformal factor, Ω, since all other background quantities (such as $\tilde{g}^0_{ab}$ and n^a on $\mathcal{I}$) can be reconstructed from Ω and the physical metric. Now, the physical metric, g_{ab}, its curvature, $R_{abc}{}^d$, and (physical) derivatives of the curvature all can be expressed in terms of the unphysical metric, $\tilde{g}_{ab}$, its curvature, $\tilde{R}_{abc}{}^d$, and unphysical derivatives of the unphysical curvature together with Ω and its unphysical derivatives. Therefore, we may view $\mathbf{W}$ as a function of $\tilde{g}_{ab}$, $\tilde{R}_{abc}{}^d$, and unphysical derivatives of $\tilde{R}_{abc}{}^d$, together with Ω and its unphysical derivatives. The requirement that $\mathbf{W}$ vary analytically with g_{ab} (at fixed Ω) then implies that it must depend analytically on $\tilde{g}_{ab}$ at fixed Ω.

In our specification of conditions on the background structure, we required that $\tilde{g}^0_{ab}$ induce a round two-sphere metric of scalar curvature k on all cross sections of $\mathcal{I}$. The choice of k was arbitrary, and could have been fixed at any value. If we keep $\mathcal{F}$ fixed (i.e., consider the same class of physical metrics) but change Ω by $\Omega \to \lambda\Omega$ with λ constant, then $\tilde{g}^0_{ab}$ will induce a round two-sphere metric of scalar curvature $\lambda^{-2}k$ rather than k on all cross sections. We require that under this scaling of Ω (corresponding to modifying an ''arbitrary choice'' in the specification of the background structure), we have $\mathbf{W} \to \mathbf{W}$.

To analyze the implications of this requirement, it is useful to introduce the following notion of the *scaling dimension* [3] of a tensor, $T^{a_1\cdots a_k}{}_{b_1\ldots b_l}$, of type (k,l) which is locally constructed out of the unphysical metric and Ω: If under the scaling $\Omega \to \lambda\Omega$, keeping the physical metric fixed, we have $T^{a_1\cdots a_k}{}_{b_1\ldots b_l} \to \lambda^p T^{a_1\cdots a_k}{}_{b_1\ldots b_l}$, then we define the scaling dimension, s, of $T^{a_1\cdots a_k}{}_{b_1\ldots b_l}$ by

$$s = p + k - l. \quad (74)$$

It follows that the scaling dimension of a tensor does not change under the raising and lowering of indices using the unphysical metric. It is easily seen that the scaling dimension of Ω is $+1$, the scaling dimension of the unphysical metric is 0, and the scaling dimension of the unphysical curvature tensor is -2. Each derivative decreases the scaling dimension by 1, so, for example, the scaling dimension of $n_a = \tilde{\nabla}_a\Omega$ is 0 and the scaling dimension of the jth derivative of the unphysical curvature is $-(j+2)$.

Since the 3-form $\mathbf{W}$ is required to be invariant under scaling of Ω, it must have a scaling dimension of -3. Since $^{(3)}\epsilon_{abc}$ has scaling dimension 0, if we define $w = W_{abc}\,^{(3)}\epsilon^{abc}$, we obtain a scalar with scaling dimension -3. By our assumptions, w must be locally constructed out of Ω and $\tilde{g}_{ab}$ (in the sense of footnote 9) and must vary analytically with $\tilde{g}_{ab}$ at fixed Ω. Presumably, this will imply that we can write w as a convergent sum of terms (with coefficients depending on the conformal factor) of products (with all indices contracted) of the unphysical metric, the unphysical curvature, unphysical derivatives of the unphysical curvature, $n_a = \tilde{\nabla}_a\Omega$ and unphysical derivatives of n^a. (Negative powers of Ω can, of course, occur in the coefficients if they multiply a term which vanishes suitably rapidly at $\mathcal{I}$.) Now, the unphysical metric, the unphysical curvature, and n_a all have have a non-positive scaling dimension and derivatives only further decrease the scaling dimension. Therefore, if any term were composed of more than two factors containing the unphysical curvature tensor, the only way of achieving a scaling dimension of -3 would be to

[20] A major subtlety would have arisen in the meaning of ''locally constructed'' if we had not imposed the rigid background structure given by the Bondi condition (47) together with our fixing of $\tilde{g}^0_{ab}$. If, say, the background structure was specified merely by the ''asymptotic geometry'' as defined on p. 22 of [3], then there would exist diffeomorphisms locally defined in the neighborhood of a point $p \in \mathcal{I}$ which preserve the background structure but cannot be extended to globally defined diffeomorphisms which preserve the background structure. Indeed, a necessary condition for a background-structure-preserving local diffeomorphism to be globally extendible is that it preserve the tensor field ρ_{ab}, defined by Eq. (33) of [3], since ρ_{ab} can be constructed from a global specification of the background structure. Now, locally defined diffeomorphisms that are not globally extendible are not relevant to the definition of ''locally constructed'' given in footnote 9, since that definition requires globally defined diffeomorphisms. Since the allowed (globally defined) diffeomorphisms must locally preserve ρ_{ab}, that quantity would, in effect, count as ''local'' with regard to the definition of ''local construction'' of $\boldsymbol{\Theta}$—even though the construction of ρ_{ab} from the background structure given in [3] involves the global solution to differential equations. Consequently, the Bondi news tensor (which is constructed out of manifestly local quantities and ρ_{ab}) would still be considered as ''locally constructed'' even if the background structure had been specified as in [3]. This subtlety does not arise here, since with our gauge choice, ρ_{ab} and the Bondi news tensor are manifestly local.

[21] That N_{ab} and hence $\boldsymbol{\Theta}$ vanish for all stationary solutions is proved, e.g., on pp. 53–54 of [3].

multiply it by a positive power of Ω, in which case it would vanish at $\mathcal{I}$. Similarly, if the term contained a single factor with two or more derivatives of curvature, it also would have to vanish at $\mathcal{I}$. Similar restrictions occur for terms containing derivatives of n^a. This reduces the possible terms that can occur in w to a small handful, and it is then easily verified that there does not exist an allowed w such that δw is nonzero in general (so that it contributes nontrivially to $\mathbf{\Theta}$) but δw vanishes whenever the physical metric, g_{ab}, is stationary. Therefore, we conclude that $\mathbf{\Theta}$ is unique.

To complete the prescription, we need to specify a stationary ''reference solution'' ϕ_0 satisfying Eq. (36). A natural candidate for ϕ_0 is Minkowski spacetime and, indeed, it should be possible to show that no other stationary solution[22] can satisfy Eq. (36). In Minkowski spacetime, an arbitrary infinitesimal asymptotic symmetry can be written as a sum of a Killing vector field plus a supertranslation. Since Eq. (36) holds automatically whenever either η^a or ξ^a is a Killing vector field, it suffices to check Eq. (36) for the case where both η^a and ξ^a are supertranslations; i.e., on $\mathcal{I}$ they are of the form $\xi^a=\alpha n^a$, $\eta^a=\beta n^a$ where α and β are such that $n^a\tilde{\nabla}_a\alpha=n^a\tilde{\nabla}_a\beta=0$. Since satisfaction of Eq. (36) does not depend upon the choice of representative of the infinitesimal asymptotic symmetry, we may assume that η^a and ξ^a satisfy the Geroch-Winicour gauge condition [17] $\nabla_a\eta^a=\nabla_a\xi^a=0$ (see below). In that case, $\int_{\partial\Sigma}\mathbf{Q}[\mathcal{L}_\eta\xi]$ will vanish and Eq. (36) reduces to

$$0=\int_{\partial\Sigma}\{\eta\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\xi\phi_0)-\xi\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\eta\phi_0)\}. \tag{75}$$

From Eq. (39) we obtain, on $\mathcal{I}$,

$$\eta^c\theta_{cab}(\phi,\delta\phi)=\frac{1}{16\pi}\tilde{\epsilon}_{abcd}V^c\eta^d \tag{76}$$

where

$$V^a\equiv\Omega^{-1}[\tilde{\nabla}_b\tau^{ab}-\tilde{\nabla}^a\tau-3\tau^a] \tag{77}$$

and it should be noted that V^a has a smooth limit to $\mathcal{I}$ on account of Eq. (68). The pullback of $\eta\cdot\boldsymbol{\theta}$ to $\mathcal{I}$ is thus

$$\eta\cdot\overline{\boldsymbol{\theta}}=-\frac{1}{16\pi}\beta n_aV^a n\cdot{}^{(3)}\overline{\boldsymbol{\epsilon}}. \tag{78}$$

In using this equation to evaluate the term $\eta\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\xi\phi_0)$ in Eq. (75), we must substitute χ_{ab} for τ_{ab} where

$$\chi_{ab}\equiv\Omega\mathcal{L}_\xi g_{ab}=\Omega^{-1}[\mathcal{L}_\xi\tilde{g}_{ab}-2K\tilde{g}_{ab}] \tag{79}$$

with

$$K\equiv\Omega^{-1}\xi^a n_a. \tag{80}$$

(Thus, $\chi_{ab}=2X_{ab}$ in the notation of [17]; it follows directly from the definition of infinitesimal asymptotic symmetries that χ_{ab} and K extend smoothly to $\mathcal{I}$.) It may then be seen by inspection of Eq. (19) of [17] that $\boldsymbol{\theta}(\phi_0,\mathcal{L}_\xi\phi_0)$ is proportional to the ''linkage flux'' (see below) associated with ξ^a. However, from the formula for the linkage flux for supertranslations in Minkowski spacetime given in Eq. (10) of [23], it may be verified that that $\int_{\partial\Sigma}\eta\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\xi\phi_0)$ cancels $\int_{\partial\Sigma}\xi\cdot\boldsymbol{\theta}(\phi_0,\mathcal{L}_\eta\phi_0)$, so Eq. (75) is indeed satisfied, as we desired to show.

Thus, for the case of null infinity in general relativity, the general prescription proposed in Sec. IV instructs us to define a ''conserved quantity,'' $\mathcal{H}_\xi$, for each infinitesimal BMS symmetry ξ^a and each cross section, $\partial\Sigma$, of $\mathcal{I}$ by

$$\delta\mathcal{H}_\xi=\int_{\partial\Sigma}[\delta\mathbf{Q}-\xi\cdot\boldsymbol{\theta}]-\frac{1}{32\pi}\int_{\partial\Sigma}N_{ab}\tau^{ab}\xi\cdot{}^{(3)}\overline{\boldsymbol{\epsilon}} \tag{81}$$

together with the requirement that $\mathcal{H}_\xi=0$ for all ξ^a and all cross sections in Minkowski spacetime.

By our above arguments, there exists a unique $\mathcal{H}_\xi$ satisfying the above requirements. How does this prescription compare with the one previously given by Dray and Streubel [5]? From our general analysis of Sec. IV, it follows that our prescription automatically yields the flux formula

$$\mathbf{F}_\xi=\mathbf{\Theta}(g_{ab},\mathcal{L}_\xi g_{ab})=-\frac{1}{32\pi}N_{ab}\chi^{ab}\,{}^{(3)}\overline{\boldsymbol{\epsilon}}. \tag{82}$$

Equation (82) agrees with the flux formula proposed by Ashtekar and Streubel [8] [see Eq. (19) of [23]]. But it was shown by Shaw and Dray [6] that the Dray-Streubel prescription also yields the Ashtekar-Streubel flux formula. Therefore, the difference between our $\mathcal{H}_\xi$ and the ''conserved quantity'' proposed by Dray and Streubel must be a quantity that depends locally on the fields at the cross section $\partial\Sigma$ and yet—since the flux associated with the difference of these quantities vanishes—for a given solution, is independent of the choice of cross section (i.e., this difference, if nonzero, would be a truly conserved quantity). If we restrict our attention to spacetimes that are asymptotically flat at both null and spatial infinity, the equivalence of our prescription to that of Dray and Streubel would follow from the fact that they both yield the ADM conserved quantities in the limit as the cross section approaches spatial infinity. However, it is instructive to show the equivalence of the two prescriptions directly (without assuming asymptotic flatness at spatial infinity), and we now turn our attention to doing so.

Let $\partial\Sigma$ be a cross section of $\mathcal{I}$ and let ξ^a be a representative of an infinitesimal asymptotic symmetry (i.e., an infinitesimal BMS representative). We may uniquely decompose ξ^a into a part that is everywhere tangent to $\partial\Sigma$ on $\partial\Sigma$ plus a supertranslation. Since both our prescription and that of Dray and Streubel are linear in ξ^a, it suffices to consider the equivalence of the prescription for each piece separately, i.e., to consider separately the cases where (a) ξ^a is everywhere tangent to $\partial\Sigma$ and (b) ξ^a is a supertranslation.

[22]If t^a denotes the timelike Killing vector field, then $\int_{\partial\Sigma}\mathbf{Q}[t]$ is proportional to the Komar formula for mass and is nonvanishing for all stationary solutions other than Minkowski spacetime. We expect that Eq. (36) will fail when η^a is an asymptotic boost and ξ^a is an asymptotic spatial translation such that their commutator yields t^a.

Consider, first, case (a), where as discussed in Sec. IV, a true Hamiltonian exists. In case (a), Eq. (81) is simply

$$\delta\mathcal{H}_\xi=\int_{\partial\Sigma}\delta\mathbf{Q}. \tag{83}$$

One might think that the solution to this equation would be simply $\mathcal{H}_\xi=\int_{\partial\Sigma}\mathbf{Q}$, which corresponds to the Komar formula with the correct numerical factor for angular momentum [see Eq. (44) above]. However, although $\int_{\partial\Sigma}\delta\mathbf{Q}$ is well defined and independent of choice of infinitesimal BMS representative ξ^a (as it must be according to the general considerations of Sec. IV), it was shown in [17] that the value of $\int_{\partial\Sigma}\mathbf{Q}$ depends upon the choice of infinitesimal BMS representative and, in this sense, is ill defined unless a representative is specified. It was also shown in [17] that the Geroch-Winicour condition $\nabla_a\xi^a=0$ in M (where ∇_a is the *physical* derivative operator) picks out a class of representatives which makes $\int_{\partial\Sigma}\mathbf{Q}$ well defined. [By Eq. (79), the Geroch-Winicour condition is equivalent to $\chi=0$, where $\chi=\tilde{g}^{ab}\chi_{ab}$.] We write $\mathbf{Q}_{GW}$ to denote $\mathbf{Q}$ when ξ^a has been chosen so as to satisfy the Geroch-Winicour condition. It was shown in [17] that $\int_{\partial\Sigma}\mathbf{Q}_{GW}$ is equivalent to a previously proposed ''linkage formula'' [16] for defining ''conserved quantities.'' Furthermore, this linkage formula has the property that when ξ^a is everywhere tangent to $\partial\Sigma$, it yields zero in Minkowski spacetime[23] as desired. This suggests that the solution to Eq. (83) together with the requirement that $\mathcal{H}_\xi$ vanish in Minkowski spacetime is $\mathcal{H}_\xi=\int_{\partial\Sigma}\mathbf{Q}_{GW}$. However, it is far from obvious that this formula satisfies Eq. (83), since when we vary the metric, we also must, in general, vary ξ^a in order to continue to satisfy the Geroch-Winicour gauge condition, $\chi=0$. Indeed, under a variation of the metric, $\delta\tilde{g}_{ab}=\Omega\tau_{ab}$, keeping ξ^a fixed it follows from Eq. (79) that

$$\begin{aligned}\delta\chi&=\delta(\tilde{g}^{ab}\chi_{ab})=\delta(\Omega^{-1}g^{ab}\mathcal{L}_\xi g_{ab})\\&=\Omega^{-1}\mathcal{L}_\xi\gamma=\Omega^{-1}\mathcal{L}_\xi(\Omega\tau)=\mathcal{L}_\xi\tau+K\tau\end{aligned} \tag{84}$$

where, as previously defined above, $\tau=\tilde{g}^{ab}\tau_{ab}$. Consequently, in order to preserve the Geroch-Winicour condition, it will be necessary to vary the infintesimal BMS representative by $\delta\xi^a=\Omega^2u^a$ (see [17]) where u^a satisfies

$$2\Omega^{-1}\nabla_a(\Omega^2u^a)=-\mathcal{L}_\xi\tau-K\tau. \tag{85}$$

Since $\nabla_a u^a=\tilde{\nabla}_a u^a-4\Omega^{-1}u^a n_a$, this relation can be expressed in terms of unphysical variables as

$$2\Omega\tilde{\nabla}_a u^a-4u^a n_a=-\mathcal{L}_\xi\tau-K\tau. \tag{86}$$

Clearly, we have

$$\delta\int_{\partial\Sigma}\mathbf{Q}_{GW}=\int_{\partial\Sigma}\delta\mathbf{Q}-\frac{1}{16\pi}\int_{\partial\Sigma}\epsilon_{abcd}\nabla^c(\Omega^2u^d) \tag{87}$$

where u^a satisfies Eq. (86). We wish to show that the second term on the right side of Eq. (87) vanishes. To do so, it is convenient to introduce a null vector field l^a as follows. At points of $\partial\Sigma$ we take l^a to be the unique (past-directed) null vector that is orthogonal to $\partial\Sigma$ and satisfies $l^an_a=1$. We extend l^a to all of $\mathcal{I}$ by requiring that $\mathcal{L}_nl^a=0$ on $\mathcal{I}$. Finally, we extend l^a off of $\mathcal{I}$ via the geodesic equation $l^b\tilde{\nabla}_bl^a=0$. A calculation similar to that given in Eq. (17) of [17] shows that the integrand of the second term in Eq. (87) can be written as

$$I_{ab}\equiv\epsilon_{abcd}\nabla^c(\Omega^2u^d)=\left[l^c\tilde{\nabla}_cY+\frac{1}{2}Y\tilde{\nabla}_cl^c+\tilde{D}_cs^c\right]{}^{(2)}\tilde{\epsilon}_{ab} \tag{88}$$

where s^a denotes the projection of u^a to $\partial\Sigma$; $\tilde{D}_a$ and ${}^{(2)}\tilde{\epsilon}_{ab}$ are the derivative operator and volume element on $\partial\Sigma$ associated with the induced unphysical metric, $\tilde{q}_{ab}$, on $\partial\Sigma$, and we have written

$$Y\equiv\frac{1}{2}[\mathcal{L}_\xi\tau+K\tau]. \tag{89}$$

The term $\tilde{D}_cs^c$ is a total divergence and integrates to zero.[24] After a significant amount of algebra, it can be shown that the remaining terms in Eq. (88) can be expressed as

$$\mathbf{I}'=\frac{1}{2}\mathcal{L}_\xi\left[\left(\mathcal{L}_l\tau+\frac{1}{2}\tau\tilde{\nabla}_al^a\right){}^{(2)}\tilde{\boldsymbol{\epsilon}}\right]. \tag{90}$$

These remaining terms integrate to zero since ξ^a is tangent to $\partial\Sigma$. This establishes that

$$\delta\int_{\partial\Sigma}\mathbf{Q}_{GW}=\int_{\partial\Sigma}\delta\mathbf{Q} \tag{91}$$

and thus the unique solution to Eq. (83) which vanishes in Minkowski spacetime is

$$\mathcal{H}_\xi=\int_{\partial\Sigma}\mathbf{Q}_{GW} \tag{92}$$

which is equivalent to the linkage formula. This agrees with the Dray-Streubel expression in case (a).

We turn our attention now to case (b) where ξ^a is a supertranslation and thus takes the form [17]

$$\xi^a=\alpha n^a-\Omega\tilde{\nabla}^a\alpha+O(\Omega^2) \tag{93}$$

where α is such that on $\mathcal{I}$ we have $n^a\tilde{\nabla}_a\alpha=0$. Direct substitution of Eq. (93) into the variation of Eq. (44) yields, on $\mathcal{I}$ [20],

[23]This fact follows immediately from the equivalence of Eqs. (21) and (22) of the first reference of [6].

[24]It is erroneously stated in [17] that $\tilde{q}_{ab}\tilde{\nabla}^au^b$ is an intrinsic divergence. The dropping of that term does not affect any of the results in the body of that paper. However, the formula given in footnote 20 of [17] is valid only when χ ($=2X$ in the notation of [17]) vanishes on $\mathcal{I}$.

$$\delta Q_{ab} = -\frac{1}{16\pi}\tilde{\epsilon}_{abcd}\tilde{\nabla}^c(\alpha\tau^d - \tau^{de}\tilde{\nabla}_e\alpha) \tag{94}$$

from which it follows that the pullback, $\delta\overline{\mathbf{Q}}$, of $\delta\mathbf{Q}$ to $\mathcal{I}$ is given by

$$\delta\overline{\mathbf{Q}} = -\frac{1}{16\pi}U\cdot{}^{(3)}\overline{\boldsymbol{\epsilon}} \tag{95}$$

where

$$U^a = \tilde{\nabla}^a(\alpha\tau^b n_b) - \alpha n^b\tilde{\nabla}_b\tau^a - n^a\tau^b\tilde{\nabla}_b\alpha + n_b\tilde{\nabla}_c\alpha\tilde{\nabla}^b\tau^{ac}. \tag{96}$$

The pullback of $\xi\cdot\boldsymbol{\theta}$ to $\mathcal{I}$ is given by Eq. (78) above (with the substitutions $\eta\to\xi$ and $\beta\to\alpha$).

Thus, our general prescription instructs us to define $\mathcal{H}_\xi$ in case (b) by the requirement that $\mathcal{H}_\xi = 0$ in Minkowski spacetime together with the equation

$$\delta\mathcal{H}_\xi = -\frac{1}{16\pi}\int_{\partial\Sigma}\left[U^a l_a - \alpha V^a n_a + \frac{1}{2}\alpha N_{ab}\tau^{ab}\right]{}^{(2)}\boldsymbol{\epsilon} \tag{97}$$

where l_a is any covector field on $\mathcal{I}$ satisfying $n^a l_a = 1$. A lengthy calculation [20] shows that the solution to this equation is the expression given by Geroch [3], namely

$$\mathcal{H}_\xi = \frac{1}{8\pi}\int_{\partial\Sigma}P^a l_a\,{}^{(2)}\boldsymbol{\epsilon} \tag{98}$$

where

$$P^a = \frac{1}{4}\alpha K^{ab}l_b + (\alpha D_b l_c + l_b D_c\alpha)\overline{g}^{cd}N_{de}\overline{g}^{e[b}n^{a]}. \tag{99}$$

Here D_a is the derivative operator on $\mathcal{I}$ defined on pp. 46–47 of [3]; $\overline{g}^{ab}$ is the (non-unique) tensor field on $\mathcal{I}$ satisfying $\overline{g}_{ac}\overline{g}^{cd}\overline{g}_{db} = \overline{g}_{ab}$ where $\overline{g}_{ab}$ denotes the pullback to $\mathcal{I}$ of $\tilde{g}_{ab}$, and $K^{ab} = {}^{(3)}\overline{\epsilon}^{acd}\,{}^{(3)}\overline{\epsilon}^{bef}\Omega^{-1}\overline{C}_{cdef}$ where $\Omega^{-1}\overline{C}_{cdef}$ denotes the pullback to $\mathcal{I}$ of the limit to $\mathcal{I}$ of $\Omega^{-1}\tilde{C}_{cdef}$, where $\tilde{C}_{cdef}$ denotes the unphysical Weyl tensor. Equation (98) agrees with the Dray-Streubel prescription in case (b). Consequently, our prescription agrees with that given by Dray and Streubel for all infitesimal BMS representatives ξ^a and all cross sections $\partial\Sigma$, as we desired to show.

VI. SUMMARY AND OUTLOOK

In this paper, using ideas arising from the Hamiltonian formulation, we have proposed a general prescription for defining notions of ''conserved quantities'' at asymptotic boundaries in diffeomorphism covariant theories of gravity. The main requirement for the applicability of our ideas is that the symplectic current $(n-1)$-form $\boldsymbol{\omega}$ extend continuously to the boundary. If, in addition, the pullback of $\boldsymbol{\omega}$ vanishes at the boundary (case I), then a Hamiltonian associated with each infinitesimal asymptotic symmetry exists, and the value of the Hamiltonian defines a truly conserved quantity. On the other hand, if the pullback of $\boldsymbol{\omega}$ fails to vanish in general at the boundary (case II), our prescription requires us to find a symplectic potential on the boundary which vanishes for stationary solutions. When such a symplectic potential exists and is unique—and when a ''reference solution'' ϕ_0 can be found satisfying the consistency condition (34)—we have provided a well-defined prescription for defining a ''conserved quantity,'' $\mathcal{H}_\xi$, for each infinitesimal asymptotic symmetry, ξ^a, and cross section $\partial\Sigma$. This ''conserved quantity'' is automatically local in the fields in an arbitrarily small neighborhood of the cross section and has a locally defined flux given by the simple formula (32). For the case of asymptotically flat spacetimes at null infinity in vacuum general relativity, our proposal was shown to yield a unique prescription which, furthermore, was shown to agree with the one previously given by Dray and Streubel [5] based upon entirely different considerations. In this way, we have provided a link between the Dray-Streubel formula and ideas arising from the Hamiltonian formulation of general relativity.

Since our approach does not depend on the details of the field equations—other than that they be derivable from a diffeomorphism covariant Lagrangian—there are many possible generalizations of the results we obtained for vacuum general relativity. We now mention some of these generalizations, all of which are currently under investigation.

Perhaps the most obvious generalization is to consider asymptotically flat spacetimes at null infinity in general relativity with matter fields, ψ, also present. If the asymptotic conditions on ψ are such that the $\boldsymbol{\omega}$ continues to extend continuously to $\mathcal{I}$ and are such that the physical stress-energy tensor, T_{ab}, satisfies the property that $\Omega^{-2}T_{ab}$ extends continuously to $\mathcal{I}$ (so that ''T_{ab} vanishes asymptotically to order 4'' in the terminology of [3]), then an analysis can be carried in close parallel with that given in Sec. V for the vacuum case. For minimally coupled fields (i.e., fields such that the curvature does not explicitly enter the matter terms in the Lagrangian), it follows from the general analysis of [13] that there will be no matter contributions to $\mathbf{Q}$ from the term $\mathbf{X}^{ab}\nabla_{[a}\xi_{b]}$ [see Eq. (11) above]. [Even for non-minimally coupled fields such as the conformally invariant scalar field, the $\mathbf{X}^{ab}\nabla_{[a}\xi_{b]}$ term in $\mathbf{Q}$ will retain the vacuum form (44) in the limit as one approaches $\mathcal{I}$.] However, in general the symplectic potential $\boldsymbol{\theta}$ and symplectic current $\boldsymbol{\omega}$ will pick up additional contributions due to the matter fields and the other terms in $\mathbf{Q}$ in Eq. (11) may also acquire matter contributions. For the massless Klein-Gordon scalar field, ψ, we require $\Omega^{-1}\psi$ to have a smooth limit to $\mathcal{I}$. In that case, $\boldsymbol{\omega}$ extends continuously to $\mathcal{I}$. Although T_{ab} does not actually vanish asymptotically to order 4 in this case (see the Appendix of [24]), it appears that all the essential features of the analysis of Sec. V carry through nonetheless. In Einstein-Klein-Gordon theory no additional matter terms occur in $\mathbf{Q}$, so $\mathbf{Q}$ continues to be given by Eq. (44). Furthermore, the extension to $\mathcal{I}$ of the pullback to surfaces of constant Ω of the matter field contribution to $\boldsymbol{\theta}$ satisfies the property that it vanishes for stationary solutions. Consequently, in this case we can define $\boldsymbol{\Theta}$ on $\mathcal{I}$ by simply adding this additional matter contribution to $\boldsymbol{\theta}$ to the right side of Eq. (73). The upshot is that the explicit matter contributions to formula (81) cancel, so that $\mathcal{H}_\xi$ is again given by the linkage formula (92) when ξ^a is tangent to $\partial\Sigma$ and is given by Eq. (98) when ξ^a is a supertranslation. However, the flux formula (82) will pick up additional terms arising from the additional matter contributions to $\boldsymbol{\theta}$ and hence to $\boldsymbol{\Theta}$. Similar results hold for nonminimally coupled scalar fields, such as the conformally coupled scalar field.[25]

The analysis is similar in the case of higher derivative gravity theories if we *impose*, in addition to the usual asymptotic conditions at null infinity, the requirement that $\Omega^{-2}R_{ab}$ extend continuously to $\mathcal{I}$. (Of course, there is no guarantee that the field equations will admit a reasonable number of solutions satisfying this property.) If we consider a Lagrangian which, in addition to the Einstein-Hilbert term (38), contains terms which are quadratic and/or higher order in the curvature and its derivatives, then additional terms will appear in $\mathbf{Q}$ as well as $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ (see [13]). However, it appears that none of these additional terms will contribute to $\mathcal{H}_\xi$ or its flux when the limit to $\mathcal{I}$ is taken. Thus, it appears that the formulas for both the ''conserved quantities'' and their fluxes will be the same in higher derivative gravity theories as in vacuum general relativity.[26]

Our proposal also can be applied to situations where the asymptotic conditions considered are very different from those arising in vacuum general relativity. Thus, for example, it should be possible to use our approach to define notions of total energy and radiated energy in dilaton gravity theories in 2-dimensional spacetimes. It should also be possible to use our approach for asymptotically anti–de Sitter spacetimes in general relativity with a negative cosmological constant. When suitable asymptotic conditions are imposed, the asymptotically anti–de Sitter spacetimes should lie within case I of Sec. IV, so it should be possible to define truly conserved quantities conjugate to all infinitesimal asymptotic symmetries. It would be of interest to compare the results that would be obtained by our approach with those of previous approaches [25].

Finally, we note that many of the ideas and constructions of Sec. IV would remain applicable if $\mathcal{B}$ were an ordinary timelike or null surface $\mathcal{S}$ in the spacetime, M, rather than an asymptotic boundary of M. Thus, one could attempt to use the ideas presented here to define notions of quasi-local energy contained within $\mathcal{S}$ and/or energy radiated through $\mathcal{S}$. However, it seems unlikely that a unique, natural choice of $\boldsymbol{\Theta}$ will exist in this context, so it seems unlikely that this approach would lead to a unique, natural notion of quasi-local energy. Nevertheless, by considering the case where $\mathcal{S}$ is the event horizon of a black hole, it is possible that the ideas presented in this paper may contain clues as to how to define the entropy of a nonstationary black hole in an arbitrary theory of gravity obtained from a diffeomorphism covariant Lagrangian.

[25]For Maxwell and Yang-Mills fields, a new issue of principle arises as a result of the additional gauge structure of these theories. If we merely require the vector potential A_a to extend smoothly to $\mathcal{I}$, then $\boldsymbol{\omega}$ will extend continuously to $\mathcal{I}$ and, by the general analysis of Sec. IV, the integral defining Ω_Σ will always exist. However, Ω_Σ will not be gauge invariant. (Thus, a Hamiltonian on Σ conjugate to gauge transformations will fail to exist in general in much the same way as a Hamiltonian conjugate to infinitesimal asymptotic symmetries fails to exist in general.) Consequently, in these cases it appears that substantial gauge fixing at $\mathcal{I}$ would be needed in order to obtain gauge invariant expressions for ''conserved quantities.''

[26]The interpretation of this result would be that, although higher derivative gravity theories may have additional degrees of freedom, these extra degrees of freedom are massive and do not propagate to null infinity (and/or they give rise to instabilities and are excluded by our asymptotic assumptions).

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