Self-force on a static scalar test charge outside a Schwarzschild black hole

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The finite part of the self-force on a static scalar test charge outside a Schwarzschild black hole is zero. By direct construction of Hadamard's elementary solution, we obtain a closed-form expression for the minimally coupled scalar field produced by a test-charge held fixed in Schwarzschild spacetime. Using the closed-form expression, we compute the necessary external force required to hold the charge stationary. Although the energy associated with the scalar field contributes to the renormalized mass of the particle (and thereby its weight), *we find there is no additional self-force acting on the charge*. This result is unlike the analogous electrostatic result, where, after a similar mass renormalization, there remains a finite repulsive self-force acting on a static electric test-charge outside a Schwarzschild black hole. We confirm our force calculation using Carter's mass-variation theorem for black holes. The primary motivation for this calculation is to develop techniques and formalism for computing all forces—dissipative and non-dissipative—acting on charges and masses moving in a black-hole spacetime. In the Appendix we recap the derivation of the closed-form electrostatic potential. We also show how the closed-form expressions for the fields are related to the infinite series solutions.

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I. INTRODUCTION

In order to gain deeper quantitative understanding of highly relativistic binary star systems spiraling toward coalescence, a number of authors have used perturbation theory to study the motion of test-particles orbiting black holes. (See, e.g., [1–4].) The basic idea is to solve the linearized field equations with out-going radiation boundary conditions on a Schwarzschild or Kerr black-hole background. The source of the perturbing field is generally chosen to be a test particle (endowed with a small scalar charge, electric charge, or mass charge) moving on a bound geodesic orbit, e.g., a circle. Once the perturbing field (scalar, electromagnetic, or gravitational) is calculated, the energy radiated per orbit (i.e., the time-averaged energy flux) can be computed by a performing a surface integral over a distant sphere surrounding the system. The rate energy carried away by the radiation is then equated to a loss of orbital energy of the particle; thus one can compute the rate of orbital decay.[1] In effect, this energy-balance argument gives the *time-averaged* radiation-reaction force associated with imposing out-going radiation boundary conditions [6]. This method has been used successfully to compute the inspiral rate of coalescing binaries to very high relativistic order, and it has been used to check analogous post-Newtonian calculation of the inspiral [7]. These perturbation calculations also add insight into effects such as wave ''tails.'' (See [1] and [8] for post-Newtonian and perturbation-theory discussions of tails.) However, in spite of the success of the perturbation approach, the method, as it has been applied, has a drawback: it has only been used to compute *time-averaged*, *dissipative* forces on the particle.

The time-averaging is problematic for two reasons: (1) During the late stages of inspiral the orbit will be decaying swiftly, and there will not be many orbits left before the final splat to average over. (2) In the curved spacetime near a black hole, the self-forces experienced by the particle are not instantaneous forces. Rather these forces arise because the fields produced by the particle at one instant travel away from the particle, encounter the curvature of the spacetime, and then scatter back and interact with the particle at a later time. (See DeWitt and DeWitt [9] for an illuminating discussion of this point.) Thus the fields produced when the particle moves through, say, periastron, and suffers the maximum coordinate acceleration, will not come back to interact with the particle until later in the orbit when the coordinate acceleration will be more gentle. In other words, the state of motion that produced the fields will not be the state of motion affected by the fields. Although it is unlikely that taking into account the non-local nature of the self-forces will qualitatively change any of the results found by the ''averaging'' technique, the issue has never been rigorously addressed with perturbation theory.

In addition to the dissipative forces, the perturbing fields will produce other non-dissipative—conservative—self-forces on the particle, and these forces have not been studied by the conventional perturbation-theory approach [1–4]. (As DeWitt and DeWitt [9] have demonstrated, these forces can be computed using a different perturbative approach.) For example, suppose we have an electric charge in a circular orbit around a Schwarzschild black hole. Even if we neglect the dissipative effects of the radiation reaction, which cause a secular decay of the orbit, the charged particle still does not travel on an exact geodesic of the spacetime. The particle will feel an additional conservative force—proportional to square of the charge—that pushes it slightly off the geodesic.

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[1] Finding a gravitational-wave signal in noisy detector data requires an accurate prediction of the orbital decay rate. See Thorne [5].

As this force is not radiative in nature, the portion of the field that produces it is explicitly discarded in the conventional perturbation approach because it falls off faster than $O[1/r]$, and therefore any effect it might have on the motion is missed by a technique that uses a surface integral on a distant sphere to determine the back-reaction effect. (2) Another perturbation calculation by Gal'tsov [10] has used the ''half-retarded–minus–half-advanced'' Green's function to find the radiation reaction forces. But these calculations also miss the conservative (i.e., *time symmetric*) parts of the force, because they simply cancel when the two parts of the Green's function are subtracted. In other words, the Gal'tsov result differs from the DeWitt-DeWitt [9] result in that it misses the conservative part of the force.

We can try (but as we will see, fail) to understand the origin of the conservative forces by considering a static electric test-charge [11] held fixed by some non-conducting mechanical struts outside a Schwarzschild black hole. The charge experiences a self-force (which is clearly not a radiation-reaction force in any conventional sense) proportional to the square of the electric charge. A physical origin of such a force might be naively described as follows: the surface of the uncharged black hole will act as a conductor, and therefore the presence of the external charge will induce a dipolar charge distribution on the horizon [12]. The magnitude of the induced dipole moment will depend on the size of the dipole (in this case, the mass M of the Schwarzschild black hole) and the magnitude e of the external charge. This leads one to believe that the charge will experience an attractive dipolar force scaling as b_s^{-5}, where b_s is the Schwarzschild radial coordinate of the charge. This force will be in addition to the natural attractive force (the weight of the particle) which, according to Newton, should scale as b_s^{-2}. Thus our naive suspicion is that, in addition to the weight, the strut holding the charge fixed will have to counteract an attractive self-force of the form

$$F_{(\mathrm{naive})} \sim \frac{e^2 M}{b_s^5}. \tag{1.1}$$

Smith and Will (SW) [13] have calculated the force required to hold an electric charge fixed outside a Schwarzschild black hole. They find the total force exerted by the mechanical strut holding the charge fixed must be

$$F_{(\mathrm{strut})} = \frac{M \mu_{\mathrm{ren}}}{b_s^2}\left(1 - \frac{2M}{b_s}\right)^{-1/2} - \frac{e^2 M}{b_s^3}. \tag{1.2}$$

Here μ_{ren} is the renormalized mass of the test-charge. We use units in which $G = c = 1$. As expected, the first term scales as $O[b_s^{-2}]$, and shows that the strut must support the weight of the particle; such a term would be present whether or not the particle is electrically charged. The second term shows—contrary to the ''physical intuition'' presented above— that the black hole tries to *repel* the charged particle and the repulsion scales as $O[b_s^{-3}]$.[2] SW show for almost any choice of variables in Eq. (1.2) the first term dominates, and the second term only lightens the load a small amount.

The fact that our naive physical intuition led to both the wrong direction and the wrong scaling of the force suggests such calculations should be carried out in detail before conclusions are drawn. In this paper we carry out the analogous calculation for a scalar charge in the presence of a Schwarzschild black hole, i.e., we compute the force required to hold a *scalar* charge fixed. The primary result of this paper is to show that, although the scalar field does contribute to the renormalized mass, there is no force analogous to the second term in Eq. (1.2) for a scalar charged particle. The primary motivation for this calculation is much broader: to begin to develop techniques and formalism for tackling a sequence of problems, namely, calculating all the forces, linear in the fields (dissipative and non-dissipative), that act on test charges and masses moving in the proximity of a black hole.

As the implicit underlying motivation for this work is calculating the forces on *moving* charges, we can ask: does our no–self-force–on–a–*static*–scalar–charge result carry over to the case of a moving charge? Not directly. However, we can gain some intuition about extending our static results to dynamic results for scalar charges by examining the static and dynamic cases for electric charges. Although the SW force calculation is only valid in the static limit, DeWitt and DeWitt [9] show that when the electric charge is in slow motion around the black hole, in addition to the conventional radiation reaction force, there is a conservative repulsive force acting on the charge. This force is independent of the velocity, and clearly corresponds to the force found in the static calculation, i.e., the second term in Eq. (1.2).[3] Therefore, it seems reasonable to expect a correspondence between the static case and the slow-motion case for scalar charges. For scalar charges, this means the absence of a self-force on a static charge should imply there is no conservative (repulsive or attractive) force on a slowly orbiting charge. This, however, does not rule out the presence of a higher-order, conservative force that scales as, say, the *square* of the orbital velocity. (This issue is under vigorous investigation [16].) As a consequence, although both electric and scalar charges will spiral inward due to radiation reaction forces, scalar charges will not suffer the same persistent, velocity-independent, conservative force trying to hold them off the geodesic.

As a matter of principle, many of the delicate issues of radiation reaction forces in curved spacetime we will be dealing with in this paper have already been addressed in

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[2]Although the second term in Eq. (1.2) appears only to be accurate to leading order in M, the result is exact for static charges; the appearance is only a beautiful artifact of Schwarzschild coordinates. Switching to harmonic or isotropic coordinates [14,15] makes the second term appear as an infinite series in M. See Eq. (4.2b).

[3]DeWitt and DeWitt only compute the force to leading order in the mass of the hole and their result appears to agree exactly with the second term in Eq. (1.2). However, the agreement is really only at leading order in M. See the previous footnote.

quite a general context by DeWitt and Brehme [17] (using a world-tube calculation), by Quinn and Wald [18] (using an axiomatic approach), and Mino, Sasaki and Tanaka [19] (using matched asymptotic expansions). However, in practice, the results presented in these papers have never been applied to give even simple results like Eq. (1.2), nor did they work out the general expressions for forces on scalar charges.

We begin in Sec. II by calculating the scalar field produced by a static scalar charge in Schwarzschild spacetime. The standard method for solving the field equations on a black-hole background is to use separation of variables and decompose the solution into Fourier, radial and angular modes (spherical harmonics). (See Appendix B for an example.) The result is an infinite-series solution for the field. This method works well for computing the far-zone properties of the field (e.g., the energy flux) where the field is weak and can be accurately described by the first few terms in the series. However, when computing a force, such as Eq. (1.2) (or its scalar analogue, as we are doing here), the behavior of the field on a distant sphere will not suffice; we need to know the detailed behavior of the field up close to the particle. For this it will not be sufficient to describe the field by a few multipoles. Describing the singular nature of the field near the charge requires an infinite number of multipoles [20,21]; therefore it is not surprising that SW did not use the standard multipolar expansion of the electrostatic field (e.g., [22,23]) in their force calculation. Rather, when embarking on their electrostatic force calculation SW comment: ''. . . an exact calculation is made possible by the fortuitous existence of a previously discovered analytic solution to the [static] curved-space Maxwell's equations.'' The exact solution to which they refer is an old result due to Copson and modified by Linet [24] which gives a closed-form expression for the electrostatic potential of a fixed charge residing in Schwarzschild spacetime. [See Eq. (B5).] The Copson-Linet formula has the advantage that the singular nature of the field near the electric charge is manifest, thus allowing simple calculations in the close proximity of the particle. We begin by constructing the scalar-field analogue of the Copson-Linet result: a closed-form expression for the field of a fixed scalar charge in Schwarzschild spacetime. Our derivation is similar to Copson's, and is based on constructing the Hadamard ''elementary solution'' [25] of the scalar-static field equation. As in the Copson-Linet electrostatic solution, our solution will clearly show the divergent behavior of the field near the particle. In Appendix A, we outline Copson's derivation of the closed-form expression for the electrostatic field with the Hadamard formalism. In Appendix B, we generate some interesting summation formulas by equating the closed-form solutions with the infinite series solutions [22,23].

As a historical note, we mention that Copson [26] obtained his solution for a static electric charge in a Schwarzschild spacetime in 1928. This is approximately 40 years before the term ''black hole'' was coined [27] and researchers began formulating the *no-hair* theorem. However, Copson was probably the first to note one of the important features of the *no-hair* theorem: '' . . . the potential of an electron on the boundary sphere $R=\alpha$ [on the horizon] is independent of its position on the sphere, a rather curious result.''

Our force calculation in Sec. III closely follows the SW calculation of the electrostatic force. We use the exact expression for the scalar-static field found in Sec. II to compute the stress-energy tensor and the force density in a local, freely-falling frame near the fixed scalar charge. By integrating the force density over a small spherical volume of radius ϵ centered on the charge (spherical in the freely-falling frame) we compute the total force the strut must supply to hold the charge fixed. In the limit where the radius of the spherical region of integration shrinks to zero, there remains a formally divergent piece of the force scaling as $(\text{charge})^2/\epsilon$. This factor multiplies an acceleration that is identical to the acceleration in the first term in Eq. (1.2); therefore the infinite term is simply absorbed into the definition of the mass of the particle, i.e., in Eq. (3.30) we define

$$\mu_{\text{ren}} \equiv \mu_{\text{bare}} + \frac{1}{2}\lim_{\epsilon\to 0}\frac{q^2}{\epsilon}, \tag{1.3}$$

where q is the scalar charge of the particle, and μ_{bare} is the bare mass of the particle. In this way, the scalar charge does contribute to the renormalized mass of the particle in exactly the same way as the electric charge contributes to the renormalized mass in Eq. (1.2). The form of classical mass renormalization depicted in Eq. (1.3) is seen in all calculations of this type (e.g., [13,28,17]). The main conclusion of this paper is that, after the formally infinite piece is absorbed into the renormalized mass, there remains no scalar counterpart to the second term in Eq. (1.2). Also in Sec. III, we verify this answer using conservation of energy.

Neither the charge's contribution to the renormalized-mass term or to the repulsive term in Eq. (1.2) should be confused with the contribution to the gravitational force on the particle due the stress energy of the electric field perturbing the metric of the spacetime. Such a force would scale as μe^2. In this calculation we are explicitly ignoring corrections to the metric. See the discussion following Eq. (4.2e).

In Sec. IV, we discuss a number of alternative methods for solving similar problems, as well as similarities and differences in the scalar-static and electrostatic results. We also collect what is known about the forces on static charges (mass, electric and scalar) in Schwarzschild spacetime. This gives a clear indication of what future research is needed.

II. SOLUTION OF THE SCALAR FIELD FOR A FIXED CHARGE IN SCHWARZSCHILD SPACETIME

In order to find the self-force acting on the scalar charge, we first solve the massless scalar field equation

$$\Box V \equiv (1/\sqrt{-g})(\sqrt{-g}g^{\alpha\beta}V_{,\alpha})_{,\beta} = 4\pi\rho \tag{2.1}$$

in Schwarzschild spacetime. Here commas denote partial differentiation, Greek indices run 0 to 3, and Latin indices will run 1 to 3.[4] The source ρ for our field will be a point-like [29] scalar charge which can be described by

$$\rho(t,\mathbf{x}) = q\int_{-\infty}^{\infty}(1/\sqrt{-g})\,\delta^4(x^\alpha - b^\alpha(\tau))d\tau$$

$$= \frac{q}{u^t(t)}\frac{\delta^3(\mathbf{x}-\mathbf{b}(t))}{\sqrt{-g}}, \tag{2.2}$$

where $b^\alpha(\tau)$ is the spacetime trajectory of the charged body, τ is the proper time measured along the path, and $u^t = db^0/d\tau = dt/d\tau$. [Notice, $\int \rho d(\text{proper volume}) = \int \rho u^t\sqrt{-g}d^3x = q$ in a frame comoving with the charge.] We will later restrict our attention to a static field and a fixed source charge at $\mathbf{b} = b\hat{\mathbf{z}}$, but, for the present, we will leave the time-dependence in the equations.

We use isotropic coordinates [14] to describe the Schwarzschild geometry. The line element is

$$ds^2 = -\frac{(2r-M)^2}{(2r+M)^2}dt^2 + \left(1+\frac{M}{2r}\right)^4(dx^2+dy^2+dz^2). \tag{2.3}$$

In these coordinates Eq. (2.1) can be written

$$C^{ij}\tilde{V}_{,ij}(\omega,\mathbf{x}) + C^j\tilde{V}_{,j}(\omega,\mathbf{x}) + C\tilde{V}(\omega,\mathbf{x})$$

$$= 4\pi(1+M/2r)^4\tilde{\rho}(\omega,\mathbf{x}), \tag{2.4}$$

where

$$C^{ij} = \text{diag}(1,1,1), \tag{2.5a}$$

$$C^j = h(r)\frac{x^j}{r} = \frac{d}{dr}[\ln(1-(M/2r)^2)]\frac{x^j}{r}, \tag{2.5b}$$

$$C = \omega^2\frac{(1+M/2r)^6}{(1-M/2r)^2}, \tag{2.5c}$$

and we have used the Fourier transform

$$V(t,\mathbf{x}) = \int_{-\infty}^{\infty}\tilde{V}(\omega,\mathbf{x})e^{-i\omega t}d\omega \tag{2.6}$$

to eliminate the time variable. We can also write Eq. (2.4) in the more compact form

$$\nabla^2\tilde{V} + h(r)\tilde{V}_{,r} + C(r,\omega)\tilde{V} = 4\pi(1+M/2r)^4\tilde{\rho}(\omega,\mathbf{x}), \tag{2.4$'$}$$

where ∇^2 is the flat-space Laplacian. Our primary attention will be focused on the static—zero frequency—case ($\omega = C = 0$), but the formalism we are developing is valid for the "Helmholtz"-type equation in Eqs. (2.4) and (2.4′). Following *Hadamard* (pp. 92–107), the elementary solution (i.e., the Green's function [30]) for Eq. (2.4) takes the form

$$U_{elem} = \Gamma^{-1/2}(U_0 + U_1\Gamma + U_2\Gamma^2 + \cdots), \tag{2.7}$$

where Γ is the square of the geodesic distance (in the sense of the "metric" $C^{ij} = \delta^{ij}$) from the source point $\mathbf{x}'$ to the field point $\mathbf{x}$, i.e.,

$$\Gamma = (x-x')^2 + (y-y')^2 + (z-z')^2. \tag{2.8}$$

Since we are working in three dimensions (an odd number of dimensions), there is no natural-log term ($\ln\Gamma$) in the elementary solution. The U_n's are non-singular functions everywhere outside the horizon. Recall, in isotropic coordinates [14] the horizon is at $r = M/2$. The simple form of Γ in Eq. (2.8) is a consequence of the isotropic coordinates we are using; therefore the technique we are developing cannot be easily extended to geometries that do not have spatially isotropic coordinates, e.g., Kerr geometry.

The formula for the leading order behavior of the series is given by *Hadamard* (p. 94):

$$U_0(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{\det C^{ij}}}\exp\left\{-\int_0^\lambda[C^{ij}\Gamma_{,ij} + C^j\Gamma_{,j} - 6]\frac{d\lambda}{4\lambda}\right\}, \tag{2.9}$$

where λ is the arc length measured along the geodesic connecting the source and field points. Letting θ denote the angle between the two spatial vectors $\mathbf{x}$ and $\mathbf{x}'$ and using Eqs. (2.5) and (2.8) we have

$$U_0(\mathbf{x},\mathbf{x}') = \exp\left\{-\int_0^\lambda[h(r)(r-r'\cos\theta)]\frac{d\lambda}{2\lambda}\right\}, \tag{2.10a}$$

$$= \exp\left\{-\int_{r'}^{r}h(r'')\frac{dr''}{2}\right\}, \tag{2.10b}$$

$$= \sqrt{\frac{1-(M/2r')^2}{1-(M/2r)^2}}, \tag{2.10c}$$

where $r = |\mathbf{x}|$, $r' = |\mathbf{x}'|$, and r'' is the dummy integration variable over r. We have also used the geometric relationship $(r-r'\cos\theta)d\lambda = \lambda dr$.

Because we are looking for an axially symmetric solution, we may assume the U_n's are functions of only the radial variables [e.g., notice $U_0 = U_0(r,r')$]. We now substitute Eq. (2.7) into Eq. (2.4), use relations such as $\delta^{ij}\Gamma_{,i}\Gamma_{,j} = 4\Gamma$, and collect powers of Γ. The result is

[4]Since the Ricci scalar curvature R vanishes in the Schwarzschild spacetime, it would seem that including coupling to the curvature (e.g., a conformally invariant term $1/6RV$) in Eq. (2.1) would have no effect on our results. However, if we include coupling to the curvature, the stress-energy that enters the force calculation [Eq. (3.9a)] would have to be modified also. Therefore, we make no claim that our results hold for nonminimally coupled fields.

$$-(2U_{0,r}+hU_0)\left(\frac{r^2-r'^2}{2r}\right)\Gamma^{-3/2}+\sum_{n=0}\left[(2n+1)(2U_{n+1,r}+hU_{n+1})\left(\frac{r^2-r'^2}{2r}\right)+2(2n^2+n-1)U_{n+1}+\frac{1}{r}(rU_n)_{,rr}+hU_{n,r}+\frac{2n-1}{2r}(2U_{n,r}+hU_n)+C(r,\omega)U_n\right]\Gamma^{n-1/2}=0. \tag{2.11}$$

Since U_{elem} is a solution to the homogeneous equation everywhere (except at the source point) the coefficients of each power of Γ must vanish independently. The first term gives an equation for U_0

$$2U_{0,r}+hU_0=0. \tag{2.12}$$

Notice that our Eq. (2.9) has already given us a particular solution [Eq. (2.10c)] to this equation. Setting the coefficient of $\Gamma^{n-1/2}$ to zero and using an integrating factor, we obtain a recursion relation for the U_n's

$$\frac{U_{n+1}(r,r')}{U_0}=\frac{-1}{(r^2-r'^2)^{n+1}}\int_{r'}^{r}\frac{(r''^2-r'^2)^n}{2n+1}\left[\frac{(r''U_n)_{,r''r''}}{U_0}-\frac{2r''U_{0,r''}U_{n,r''r''}}{U_0^2}+(2n-1)(U_n/U_0)_{,r''}+r''C(r'',\omega)(U_n/U_0)\right]dr''. \tag{2.13}$$

In the integrand the U_n's are functions of the dummy integration variable r'' and the source point r', i.e., $U_n=U_n(r'',r')$. This recursion relation allows us to construct a series solution for the Green's function for the Schwarzschild-Helmholtz equation Eqs. (2.4) or (2.4′).[5]

We now explicitly assume that the charge and the field are static, i.e., we assume that $\omega=C(r,\omega)=0$ in Eq. (2.13). Beginning with U_0 from Eq. (2.10c) we can construct

$$\frac{U_1}{U_0}=-\frac{1}{2}\,\frac{(2M)^2}{(4r'^2-M^2)(4r^2-M^2)}\equiv-\frac{1}{2}\gamma, \tag{2.14a}$$

$$\frac{U_2}{U_0}=\frac{3}{8}\,\frac{(2M)^4}{(4r'^2-M^2)^2(4r^2-M^2)^2}=\frac{3}{8}\gamma, \tag{2.14b}$$

$$\frac{U_3}{U_0}=-\frac{5}{16}\,\frac{(2M)^6}{(4r'^2-M^2)^3(4r^2-M^2)^3}=-\frac{5}{16}\gamma. \tag{2.14c}$$

Substituting these into Eq. (2.7), the pattern is immediately clear and the summation is elementary

$$U_{elem}(\mathbf{x},\mathbf{x}')=\frac{U_0}{\sqrt{\Gamma}}\left(1-\frac{1}{2}\gamma\Gamma+\frac{3}{8}(\gamma\Gamma)^2-\frac{5}{16}(\gamma\Gamma)^3+\cdots\right) \tag{2.15a}$$

$$=\frac{U_0}{\sqrt{\Gamma}}\,\frac{1}{\sqrt{1+\Gamma\gamma}} \tag{2.15b}$$

$$=\frac{1}{\sqrt{\Gamma}}\,\frac{r(4r'^2-M^2)}{r'\sqrt{(4r'^2-M^2)(4r^2-M^2)+4M^2\Gamma}}. \tag{2.15c}$$

Equation (2.15c) is a closed-form expression for the Green's function for Eq. (2.4) (with C=0). In the limit $M=0$ it reduces to $|\mathbf{x}-\mathbf{x}'|^{-1}$, the Green's function for Poisson's equation in flat space. We use Eq. (2.15c) in precisely the same manner we would use the Green's function for Poisson's equation: we integrate it against the source to obtain a particular solution to the inhomogeneous equation. In our case the source is the static scalar test-charge held fixed at the spatial position $\mathbf{b}$, and we have

$$V_{part}(\mathbf{x},\mathbf{b})=-\int\left[\frac{q}{u^t(b)\sqrt{-g(b)}}\left(1+\frac{M}{2b}\right)^4\delta^3(\mathbf{x}'-\mathbf{b})\right]\times U_{elem}(\mathbf{x},\mathbf{x}')d^3\mathbf{x}' \tag{2.16a}$$

$$=-q\frac{2b-M}{2b+M}\,\frac{1}{\sqrt{\Gamma(\mathbf{x},\mathbf{b})}}\times\frac{4rb}{\sqrt{(4b^2-M^2)(4r^2-M^2)+4M^2\Gamma(\mathbf{x},\mathbf{b})}} \tag{2.16b}$$

$$=-q\sqrt{\frac{b_h-M}{b_h+M}}\times\frac{1}{\sqrt{r_h^2-2r_hb_h\cos\theta+b_h^2-M^2\sin^2\theta}}. \tag{2.16c}$$

In the last two steps we have explicitly included the factor $1/u^t(b)=\sqrt{-g_{00}(b)}=(2b-M)/(2b+M)=\sqrt{(b_h-M)/(b_h+M)}$; in the last step we have converted to

[5]We can gain confidence in the recursion relation Eq. (2.13) by applying it to the true Helmholtz equation in flat spacetime [Eq. (2.4) with $M=0$, but $C\equiv\omega^2\neq0$]. In this case, Eq. (2.9) gives $U_0=1$, and the recursion relation Eq. (2.13) gives $U_1=-\omega^2/2$, $U_2=\omega^4/24,\ldots$, $U_n=(-1)^n\omega^{2n}/(2n!)$. Summing the series gives the Green's function for the Helmholtz equation

$$U_{elem}=\frac{\cos(\omega|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}.$$

Now compute the inverse Fourier transform and we have

$$G(x,x')=\frac{1}{8\pi}\left\{\frac{\delta[t'-(t-|\mathbf{x}-\mathbf{x}'|)]}{|\mathbf{x}-\mathbf{x}'|}+\frac{\delta[t'-(t+|\mathbf{x}-\mathbf{x}'|)]}{|\mathbf{x}-\mathbf{x}'|}\right\}.$$

This is the half-advanced *plus* half-retarded Green's function. See [21], Eq. (6.61).

harmonic coordinates [14]. The leading minus sign in Eq. (2.16) is a consequence of the source term in Eq. (2.1); our source is ''+'' $4\pi\rho$ and not the familiar ''−'' $4\pi\rho$ of electrostatics. (Like scalar charges attract.) As we are now dealing strictly with a static solution, we have also dropped the *twiddle* denoting the Fourier transform in Eq. (2.4).

Equation (2.16) is a *particular* solution to the scalar-static field equation, but is it the desired solution to the equation? In other words, does it satisfy all the boundary conditions? First, the field and its derivatives are well behaved outside—and on—the horizon; thus our solution has no unphysical regions of infinite energy (save, of course, at the location of the charge). For $r \gg b > M/2$ we see from Eq. (2.16b)

$$V_{part}(r\to\infty)\approx -\frac{q}{r}\left(\frac{2b-M}{2b+M}\right). \tag{2.17}$$

As the charge is lowered toward the horizon, the factor $(2b-M)/(2b+M)$ extinguishes the field measured by a distant observer. Notice if the charge is lowered to the horizon $(b=M/2)$ the field completely disappears. (See, e.g., [31] for discussion.) Thus, it is the extinction factor—which had its origin in the factor of $1/u^t(t)$ in Eq. (2.2) and has survived throughout the calculation—that enforces *no scalar hair on the black hole*.[6]

Equation (2.17) is the appropriate asymptotic form of the scalar field; therefore the particular solution Eq. (2.16) is the desired solution which satisfies all the boundary conditions [32,33]

$$V(\mathbf{x},\mathbf{b})=V_{part}(\mathbf{x},\mathbf{b}). \tag{2.18}$$

Before embarking on the force calculation, we note a remarkable feature of the closed-form solution we have found. Despite the fact that every term in the Hadamard series is divergent at the horizon [Eqs. (2.14)], the closed-form expression for the elementary solution is well behaved on the horizon. The easiest way to see this is with the harmonic-coordinate expression Eq. (2.16c) evaluated on the horizon $(r_h=M)$

$$V_{Horizon}=-\frac{q}{b_h-M\cos\theta}\sqrt{\frac{b_h-M}{b_h+M}}. \tag{2.19}$$

It is also interesting to note that the horizon is not a surface of constant ''potential.'' In Sec. I our (flawed) intuition about the self-force on a static electric charge was predicated on the horizon acting as conducting surface; therefore we should not be surprised that the force on the static scalar charge will be different.

[6]The factor $1/u^t$ is present even in a special relativity. Notice it is just the inverse of the Lorentz factor $\gamma=dt/d\tau=1/\sqrt{1-v^2}$. As a mnemonic, the strength of the ''charge'' depends on the spin of the field: for masses (spin 2) we have $m=\gamma^1 m_{\rm rest}$, for electric charges (spin 1) we have $e=\gamma^0 e_{\rm rest}$, and clearly, by induction, for scalar charges (spin 0) we have $q=\gamma^{-1}q_{\rm rest}$.

III. SELF-FORCE ON A STATIC SCALAR CHARGE

A. Local method

We begin our calculation of the self-force with a picturesque description (*Gedankenexperiment*) of how such a measurement could be made. We imagine a test-charge with bare mass μ_{bare} and scalar charge q held fixed by a non-conducting system of mechanical struts outside the horizon of a Schwarzschild black hole. A non-conducting experimenter at the apex of a vertical, ballistic trajectory momentarily comes to rest with respect to the fixed charge. At this moment, she reaches out and measures the force required to hold the charge fixed, i.e., she measures the force needed to just lift the charge off the strut. The spacetime event $\mathcal{B}$ where/when the force is measured will be taken as the origin of the free-falling observer's coordinates $x^{\bar{\alpha}}$. (The over-bar denotes coordinates in the local, freely-falling frame of the observer [14].) Clearly, by symmetry, we can choose our coordinate system such that the particle is located on the z-axis, and the freely-falling coordinate system is aligned so that the $\bar{z}$-axis coincides with the z-axis. Thus the only component of the force for the freely-falling observer to measure will be $F^{\bar{z}}$. Although this is an elaborate scheme to define the force measurement, working in the freely-falling frame where the charge is momentarily at rest is the surest way to establish unambiguously how the scalar field of the particle contributes to the renormalized mass.

As no scalar charges have ever been observed, nor scalar fields measured, our assumption that the scalar field does not interact with the experimental apparatus (the struts) and the experimenter seems to be quite plausible.

In order to compute the force in the free-falling frame of the observer, we need to be able to convert quantities from the isotropic coordinates of Eq. (2.16) to the coordinates of the observer. The defining feature of the free-falling frame is the metric is locally flat: i.e.,

$$(g_{\bar{\alpha}\bar{\beta}})_{\mathcal{B}}=\mathrm{diag}(-1,1,1,1)\equiv\eta_{\bar{\alpha}\bar{\beta}}, \tag{3.1a}$$

$$(g_{\bar{\alpha}\bar{\beta},\bar{\gamma}})_{\mathcal{B}}=0, \tag{3.1b}$$

$$g_{\bar{\alpha}\bar{\beta}}(x^{\bar{\gamma}})=\eta_{\bar{\alpha}\bar{\beta}}+O[(x^{\bar{\gamma}})^2]. \tag{3.1c}$$

In particular, the Christoffel symbols, $\Gamma^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}=(1/2)g^{\bar{\alpha}\bar{\sigma}}[g_{\bar{\sigma}\bar{\gamma},\bar{\beta}}+g_{\bar{\sigma}\bar{\beta},\bar{\gamma}}-g_{\bar{\beta}\bar{\gamma},\bar{\sigma}}]\sim O[x^{\bar{\sigma}}]$ near the event $\mathcal{B}$. Fortunately, SW have done the difficult work in finding the transformations from isotropic coordinates (un-barred) to the free-falling coordinates (barred). They are given by

$$\bar{t}=\frac{1-M/2b}{1+M/2b}t+\frac{M}{b^2}\frac{1}{(1+m/2b)^2}t(z-b)+O[(x^\mu-b^\mu)^3] \tag{3.2a}$$

$$x^{\bar{j}}=(1+M/2b)^2(x^j-b^j)+\frac{M}{2b^2}\frac{1-M/2b}{(1+M/2b)^5}\delta^{\overline{j3}}\bar{t}^2$$
$$-\frac{M}{2b^2}(1+m/2b)[2(x^j-b^j)(z-b)-\delta^{\overline{j3}}|\mathbf{x}-\mathbf{b}|^2]$$
$$+O[(x^\mu-b^\mu)^3]. \tag{3.2b}$$

These can be used to find

$$\frac{1}{\sqrt{\Gamma(\mathbf{x},\mathbf{b})}}=\frac{1}{|\mathbf{x}-\mathbf{b}|}$$
$$=\frac{1}{|\bar{\mathbf{x}}-\bar{\mathbf{X}}(\bar{t})|}(1+M/2b)^2$$
$$\times\left[1-\frac{M}{2b^2}\frac{1}{(1+M/2b)^3}\bar{z}+O[(x^{\bar{\alpha}})^2]\right], \tag{3.2c}$$

where $\bar{\mathbf{X}}(\bar{t})$ is the position of the charge as viewed in the freely-falling frame

$$X^{\bar{j}}=\frac{1}{2}a_g\delta^{\overline{3j}}\bar{t}^2+O[\bar{t}^3], \tag{3.2d}$$

and a_g is the acceleration of the fixed charge as measured in the freely falling frame at the moment the experimenter comes to rest at the apex of her geodesic trajectory

$$a_g\equiv\frac{M}{b^2}\frac{1}{(1+M/2b)^3(1-M/2b)} \tag{3.2e}$$
$$=\frac{M}{b_s^2}\left(1-\frac{2M}{b_s}\right)^{-1/2}. \tag{3.2f}$$

In the second line we have converted to Schwarzschild coordinates. Notice this is the same as the acceleration appearing in Eq. (1.2). We can use Eqs. (3.2) to evaluate the scalar field and its derivatives in the free-falling frame

$$V(\bar{t},\bar{\mathbf{x}})=-\frac{q}{|\bar{\mathbf{x}}-\bar{\mathbf{X}}(\bar{t})|}\left[1-\frac{1}{2}a_g\bar{z}+O[(x^{\bar{\alpha}})^2]\right], \tag{3.3a}$$

$$V(\bar{t}=0,\bar{\mathbf{x}})=-q\left[\frac{1}{\bar{r}}-\frac{a_g}{2}n^{\bar{3}}+O[\bar{r}\times\text{even number of }(n^{\bar{l}})'\text{s}]\right], \tag{3.3b}$$

$$V_{,\bar{t}}(\bar{t}=0,\bar{\mathbf{x}})=-qO[\bar{r}^0\times\text{odd number of }(n^{\bar{l}})'\text{s}], \tag{3.3c}$$

$$V_{,\bar{k}}(\bar{t}=0,\bar{\mathbf{x}})=-q\left[-\frac{n^{\bar{k}}}{\bar{r}^2}+\frac{a_g}{2\bar{r}}(n^{\bar{k}}n^{\bar{3}}-\delta^{\overline{3k}})+n^{\bar{k}}O[\bar{r}^0\times\text{even number of }(n^{\bar{l}})'\text{s}]\right], \tag{3.3d}$$

$$V_{,\overline{tt}}(\bar{t}=0,\bar{\mathbf{x}})=-q\left[\frac{a_g}{\bar{r}^2}-\frac{a_g^2}{2}\frac{n^{\bar{3}}}{\bar{r}}+O[\bar{r}^{-1}\times\text{even number of }(n^{\bar{l}})'\text{s}]\right], \tag{3.3e}$$

$$V_{,\overline{kt}}(\bar{t}=0,\bar{\mathbf{x}})=-\frac{q}{\bar{r}}[n^{\bar{k}}O[\bar{r}^0\times\text{odd number of }(n^{\bar{l}})'\text{s}]], \tag{3.3f}$$

where $\bar{r}=|\bar{\mathbf{x}}|$ and $n^{\bar{k}}=x^{\bar{k}}/\bar{r}$. The spatial indices can be freely raised and lowered with δ^{ij}.

Notice the extinction factor present in Eq. (2.16) does not extinguish the charge in the freely-falling frame Eq. (3.3a), that is, in the freely-falling frame the dominant behavior of the field is simply (charge/distance), independent of how deep the charge is in the Schwarzschild potential.

We will compute the force required to hold the charge fixed by integrating the force density [13,34]

$$f^{\bar{z}}=T^{\bar{z}\bar{\beta}}{}_{;\bar{\beta}} \tag{3.4}$$

over the physical extent of the charged body at the instant of time $(\bar{t}=0)$ when the measurement is made. More precisely, since we are describing the particle as a Dirac δ-function, we will integrate over an infinitesimal sphere of radius $\bar{\epsilon}$ centered on the particle and take the limit as $\bar{\epsilon}\to 0$. The stress-energy tensor $T^{\bar{z}\bar{\beta}}$ will have contributions from the bare mass of the particle and the scalar field; thus we have

$$F^{\bar{z}}_{(\text{strut})}=\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leqslant\bar{\epsilon}}[f^{\bar{z}}]_{\bar{t}=0}d^3\bar{x}$$
$$=\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leqslant\bar{\epsilon}}[T^{\bar{z}\bar{\beta}}{}_{;\bar{\beta}}]_{\bar{t}=0}d^3\bar{x} \tag{3.5a}$$

$$=\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leqslant\bar{\epsilon}}[T^{\bar{z}\bar{\beta}}_{(\text{bare});\bar{\beta}}]_{\bar{t}=0}d^3\bar{x}+\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leqslant\bar{\epsilon}}[T^{\bar{z}\bar{\beta}}_{(\text{SF});\bar{\beta}}]_{\bar{t}=0}d^3\bar{x}. \tag{3.5b}$$

The first term in the Eq. (3.5b) shows that the strut must support the bare weight of the particle. This term is present whether or not the particle is charged. Using the stress-energy tensor for the bare mass of a point particle located at $\mathbf{b}=b\hat{\mathbf{z}}$

$$T^{\alpha\beta}_{(\text{bare})}=\mu_{\text{bare}}\frac{\dot{b}^\alpha(b)\dot{b}^\beta(b)}{\sqrt{-g(b)}u^t(b)}\delta^3(\mathbf{x}-\mathbf{b}), \tag{3.6}$$

SW found the necessary force the strut must supply to support the bare weight of the particle to be

$$\bar{F}^{z}_{(\text{bare})} = \lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}} [T^{z\bar{\beta}}_{(\text{bare});\bar{\beta}}]_{\bar{t}=0} d^3\bar{x}$$

$$= \frac{M\mu_{\text{bare}}}{b^2}\frac{1}{(1+M/2b)^3}\frac{1}{1-M/2b} = \mu_{\text{bare}} a_g . \tag{3.7}$$

The second term in Eq. (3.5b) involves the stress-energy tensor of the scalar field. In evaluating this integral we make use of Eq. (3.1) and note that the connection coefficients in this frame are $O[\bar{x}^{\alpha}]$; thus we can write

$$\bar{F}^{z}_{(\text{SF})} = \lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}} [T^{\bar{z}\bar{\beta}}_{(\text{SF});\bar{\beta}}]_{\bar{t}=0} d^3\bar{x}$$

$$= \lim_{\bar{\epsilon}\to 0}\left\{ \int_{\bar{r}\leq\bar{\epsilon}} [T^{\bar{z}\bar{k}}_{(\text{SF}),\bar{k}} + T^{\bar{z}\bar{t}}_{(\text{SF}),\bar{t}} + O[x^{\bar{\alpha}} T^{\bar{\beta}\bar{\gamma}}]]_{\bar{t}=0} d^3\bar{x}\right\}. \tag{3.8}$$

We will denote the three contributions to Eq. (3.8) as $\bar{F}^{z}_{(\text{SF1})}$, $\bar{F}^{z}_{(\text{SF2})}$ and $\bar{F}^{z}_{(\text{SF3})}$ respectively. We also make use of Eq. (3.1) in writing the stress-energy tensor for the scalar field

$$T^{\bar{\beta}\bar{\gamma}} \equiv \frac{1}{4\pi}\left[g^{\bar{\alpha}\bar{\sigma}} g^{\bar{\beta}\bar{\tau}} V_{,\bar{\sigma}} V_{,\bar{\tau}} - \frac{1}{2} g^{\bar{\alpha}\bar{\beta}} g^{\bar{\sigma}\bar{\tau}} V_{,\bar{\sigma}} V_{,\bar{\tau}}\right] \tag{3.9a}$$

$$= \frac{1}{4\pi}\left[\eta^{\bar{\alpha}\bar{\sigma}} \eta^{\bar{\beta}\bar{\tau}} V_{,\bar{\sigma}} V_{,\bar{\tau}} - \frac{1}{2} \eta^{\bar{\alpha}\bar{\beta}} \eta^{\bar{\sigma}\bar{\tau}} V_{,\bar{\sigma}} V_{,\bar{\tau}} + O[(x^{\bar{\alpha}})^2 V_{,\bar{\sigma}} V_{,\bar{\tau}}]\right]. \tag{3.9b}$$

We now substitute Eq. (3.9b) into Eq. (3.8) and treat the terms in reverse order. The third term in Eq. (3.8) gives a contribution of the form

$$\bar{F}^{z}_{(\text{SF3})} = \frac{1}{4\pi}\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}} [O[x^{\bar{\alpha}} T^{\bar{z}\bar{t}}_{(\text{SF})}]]_{\bar{t}=0} d^3\bar{x}$$

$$= \frac{1}{4\pi}\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}} [O[x^{\bar{\alpha}} V_{,\bar{\beta}} V_{,\bar{\gamma}}] + O[(x^{\bar{\alpha}})^3 V_{,\bar{\beta}} V_{,\bar{\gamma}}]]_{\bar{t}=0} d^3\bar{x}. \tag{3.10}$$

Using Eqs. (3.3), we see the most singular terms come from $(V_{,\bar{k}} V_{,\bar{l}})$; therefore we have

$$\bar{F}^{z}_{(\text{SF3})} \sim \lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}}\left[\frac{1}{\bar{r}}\times[\text{term with odd number of } (n^{\bar{l}})\text{'s}] + \bar{r}^0\times[\text{term with even number of } (n^{\bar{l}})\text{'s}]\right] d\bar{r}\, d\bar{\Omega}. \tag{3.11}$$

The first term contains an odd number of unit vectors, and therefore will vanish when we integrate over the solid angle. The second term vanishes as $\bar{\epsilon}\to 0$, and similarly for higher powers of $\bar{r}$. These two tricks are used repeatedly in evaluating the remaining integrals in Eq. (3.8). Thus we have

$$F_{(\text{SF3})} = 0. \tag{3.12}$$

We now evaluate the second term in Eq. (3.8) using Eq. (3.3) and ruthlessly discarding terms that do not survive the limit or the angular integration:

$$\bar{F}^{z}_{(\text{SF2})} = -\lim_{\bar{\epsilon}\to 0}\frac{1}{4\pi}\int_{\bar{r}\leq\bar{\epsilon}} [V_{,\bar{z}} V_{,\bar{z}\,\bar{t}}]\bar{r}^2 d\bar{r}\, d\bar{\Omega} \tag{3.13a}$$

$$= \frac{1}{4\pi} q^2 a_g \lim_{\bar{\epsilon}\to 0}\int_0^{\bar{\epsilon}} \frac{1}{\bar{r}^2} d\bar{r} \oint n^{\bar{z}} n^{\bar{z}} d\bar{\Omega} \tag{3.13b}$$

$$= \frac{1}{3} q^2 a_g \lim_{\bar{\epsilon}\to 0}\int_0^{\bar{\epsilon}} \frac{1}{\bar{r}^2} d\bar{r}. \tag{3.13c}$$

Using the divergence theorem and Eq. (3.9b), the first term in Eq. (3.8) can be written

$$\bar{F}^{z}_{(\text{SF1})} = \frac{1}{4\pi}\lim_{\bar{\epsilon}\to 0}\int_{\bar{r}\leq\bar{\epsilon}} [T^{\bar{z}\bar{k}}_{(\text{SF}),\bar{k}}]_{\bar{t}=0} d^3\bar{x}$$

$$= \lim_{\bar{\epsilon}\to 0}\oint_{\bar{r}=\bar{\epsilon}} [T^{\bar{z}\bar{k}}_{(\text{SF})}]_{\bar{t}=0} n^{\bar{k}} \bar{r}^2 d\bar{\Omega}$$

$$= \frac{1}{4\pi}\lim_{\bar{\epsilon}\to 0}\oint_{\bar{r}=\bar{\epsilon}}\left[V_{,\bar{z}} V_{,\bar{k}} - \frac{1}{2}\delta^{\bar{z}\bar{k}}(-V_{,\bar{t}} V_{,\bar{t}} + \delta^{\bar{l}\bar{m}} V_{,\bar{l}} V_{,\bar{m}}) + O[(x^{\bar{\alpha}})^2 V_{,\bar{\beta}} V_{,\bar{\gamma}}]\right]_{\bar{t}=0} \bar{r}^2 n^{\bar{k}} d\bar{\Omega}. \tag{3.14}$$

All the terms except the first vanish either by $\oint\cdots d\bar{\Omega}=0$, or $\lim_{\bar{\epsilon}\to 0}\cdots=0$. Using Eq. (3.3) and integrating the first term in Eq. (3.14) over the solid angle, we are left with

$$\bar{F}^{z}_{(\text{SF1})} = \frac{1}{3} a_g \lim_{\bar{\epsilon}\to 0}\frac{q^2}{\bar{\epsilon}}. \tag{3.15}$$

Combining the results in Eqs. (3.5b), (3.7), (3.12), (3.13c), and (3.15) we have

$$\bar{F}^{z}_{(\text{strut})} = \left[\mu_{\text{bare}} + \frac{q^2}{3}\lim_{\bar{\epsilon}\to 0}\left(\frac{1}{\bar{\epsilon}} + \int_0^{\bar{\epsilon}}\frac{1}{\bar{r}^2} d\bar{r}\right)\right] a_g \tag{3.16a}$$

$$= \left[\mu_{\text{bare}} + \frac{q^2}{3}\int_0^{\infty}\frac{1}{\bar{r}^2} d\bar{r}\right] a_g \tag{3.16b}$$

$$= \mu_{\text{ren}} a_g , \tag{3.16c}$$

where we have defined the leading factor in the bracket as the ''renormalized'' mass μ_{ren}. The renormalized mass has the same functional form as the renormalized mass for the electric charge in the SW calculation. Converting to Schwarzschild coordinates [14], we get

$$F_{(\mathrm{strut})}=\frac{M\mu_{\mathrm{ren}}}{b_s^2}\left(1-\frac{2M}{b_s}\right)^{-1/2}+\{\text{nothing depending on } q\}. \tag{3.17}$$

There is no self-force of the form seen in the second term of Eq. (1.2).

B. Global method

We verify our no-self-force result by means of a global, energy-conservation calculation. Suppose, instead of measuring the force on the charge while the charge is in place (as we did in the last subsection), the free-falling observer lowers the charge a small amount $\delta\bar{b}$. The work done on the experimenter will be

$$\delta\bar{W}=-F^{\bar{z}}\delta\bar{b}=-F^{\bar{z}}(1+M/2b)^2\delta b, \tag{3.18}$$

where we have used Eq. (2.3) to convert the free-falling displacement $\delta\bar{b}$ to an isotropic coordinate displacement δb. The experimenter then converts this energy into a photon and fires the photon to asymptotic infinity. The energy received at infinity will be red-shifted

$$\delta E_{\mathrm{received}}=\sqrt{-g_{00}(b)}\,\delta\bar{W}. \tag{3.19}$$

By conservation of energy this change in the system will be manifested by a change in asymptotic mass $\mathcal{M}$ of the system

$$\delta\mathcal{M}=-\delta E_{\mathrm{received}}=[1-(M/2b)^2]F^{\bar{z}}\delta b. \tag{3.20}$$

Thus we have

$$F^{\bar{z}}=\frac{1}{1-(M/2b)^2}\,\frac{\delta\mathcal{M}}{\delta b}\,. \tag{3.21}$$

We now use the *total mass variation law* of Carter [35], which shows how the asymptotic mass will differ between two situations where the gravitational and matter status of the spacetime is slightly altered. In our case we compute the difference in asymptotic mass before and after we make the small displacement of the charge. The relationship is given by

$$\delta\mathcal{M}-\frac{\kappa}{8\pi}\delta\mathcal{A}=\frac{1}{8\pi}\delta\int G^0_0\sqrt{-g}d^3x+\frac{1}{16\pi}\int G^{\mu\nu}h_{\mu\nu}\sqrt{-g}d^3x. \tag{3.22}$$

Here κ is the surface gravity of the black hole, $\mathcal{A}$ is the area of black hole, $G^{\mu\nu}$ is the Einstein tensor, and $h_{\mu\nu}$ is the difference in the metric between the two configurations. In Eq. (3.22) we have neglected terms involving the spin of the black hole. Throughout this paper we have assumed that the metric is unperturbed by the presence of the charge; therefore the last term in Eq. (3.22) vanishes. For a Schwarzschild black hole the *area* term in Eq. (3.22) is just the change in the mass of the black hole. During our slow displacement we will assume that no matter or radiation goes down the hole, and therefore this term will vanish. (We revisit this point at the end of the section.) Using Einstein's equation to write $G^0_0=8\pi T^0_0$ all that remains of Eq. (3.22) is

$$\begin{aligned}\delta\mathcal{M}&=\delta\int T^0_0\sqrt{-g}d^3x\\&=\delta\int T^0_{(\mathrm{bare})0}\sqrt{-g}d^3x\\&\quad+\delta\int T^0_{(\mathrm{SF})0}\sqrt{-g}d^3x.\end{aligned} \tag{3.23}$$

The first integral can be computed by Eq. (3.6). SW give the result

$$\mathcal{E}_{\mathrm{bare}}\equiv\int T^0_{(\mathrm{bare})0}\sqrt{-g}d^3x=\mu_{\mathrm{bare}}\left(\frac{1-M/2b}{1+M/2b}\right). \tag{3.24}$$

Noting that the metric is diagonal, the scalar field is static, and employing the definition of the stress tensor for a scalar field Eq. (3.9a), we can write the second integral in Eq. (3.23) as

$$\begin{aligned}\mathcal{E}_{\mathrm{SF}}&\equiv\int T^0_{(\mathrm{SF})0}\sqrt{-g}d^3x=\frac{1}{8\pi}\int[g^{jk}V_{,j}V_{,k}]\sqrt{-g}d^3x\\&=\frac{1}{8\pi}\int[(g^{jk}V_{,j}V\sqrt{-g})_{,k}-V(g^{jk}V_{,j}\sqrt{-g})_{,k}]d^3x.\end{aligned} \tag{3.25}$$

The first term in Eq. (3.25) can be converted to two surface integrals: one over the horizon, the other over a sphere at $r\to\infty$. The field and its derivatives are well behaved on the horizon, but $g^{jk}\sqrt{-g}$ vanishes there; therefore the integral on the horizon vanishes. The other surface integral vanishes in the limit $r\to\infty$. Using the original field equation Eq. (2.1), we have

$$(\sqrt{-g}g^{jk}V_{,j})_{,k}=4\pi\sqrt{-g}\rho. \tag{3.26}$$

Thus we can write

$$\mathcal{E}_{\mathrm{SF}}=-\frac{1}{2}\int\rho V\sqrt{-g}d^3x=-\frac{q}{2u^t(b)}\int V\delta^3(\mathbf{x}-\mathbf{b})d^3x. \tag{3.27}$$

Unfortunately, the scalar field is divergent at the source point **b**; therefore we must renormalize. We do so by modeling our source as a charged, spherical shell of radius ϵ, i.e.,

$$\frac{\delta^3(\mathbf{x}-\mathbf{b})}{\sqrt{-g}}\to\lim_{\epsilon\to0}\frac{\delta^1(|\mathbf{x}-\mathbf{b}|-\epsilon)}{4\pi\epsilon^2}. \tag{3.28}$$

Performing the integration and using the metric to convert the radius of the ball to free-falling coordinates $\bar{\epsilon}=(1+M/2b)^2\epsilon$, we have

$$\mathcal{E}_{\rm SF}=\frac{1}{2}\left(\frac{1-M/2b}{1+M/2b}\right)\lim_{\bar{\epsilon}\to 0}\frac{q^2}{\bar{\epsilon}}. \tag{3.29}$$

As expected the functional form is the same here as in Eq. (3.24).

Combining Eq. (3.24) and Eq. (3.29) defining the renormalized mass

$$\bar{\mu}_{\rm ren}=\mu_{\rm bare}+\frac{1}{2}\lim_{\bar{\epsilon}\to 0}\frac{q^2}{\bar{\epsilon}}, \tag{3.30}$$

and using Eq. (3.21) and Eq. (3.2e), we have

$$\bar{F^z}=\bar{\mu}_{\rm ren}a_g . \tag{3.31}$$

This agrees exactly with our previous calculation Eq. (3.16c): no finite part of the self-force.

We close this section with a pedagogical comment on the global energy conservation method for computing the force. We note that the calculation was predicated on the assumption that the *area* term in Eq. (3.22) vanished, i.e.,

$$\frac{\kappa}{8\pi}\delta\mathcal{A}=\delta(\text{mass of the hole})=0. \tag{3.32}$$

In terms of modern black hole theory, this is a valid assumption: No particles were dropped into the hole. The charge was displaced slowly, so no transverse fields (i.e., radiation) heated the horizon. Therefore, the area remains unchanged. However, in Sec. III A we gave a primitive derivation of the force which did not appear to explicitly rely on any sophisticated properties of black holes. (Implicitly, we did assume that the mass of the hole remained constant when the observer wiggled the charge to measure the force.) An interesting interpretation of the two force calculations is to accept the primitive derivation in Sec. III A as the correct force. Then, when we evaluate the right hand side of Eq. (3.23) and show that it is the same as our local (primitive) force calculation, we have *verified* that the area of the black hole did not change when we lowered the charge. This tells us, in spite of the fact that energy of the scalar-static field (or the electric field for an electric charge) extends clear down to the horizon, that none of the energy near the horizon is pushed across the horizon when we lower the charge.

IV. CONCLUSIONS AND DISCUSSION

We have developed a formalism for constructing the Hadamard elementary solution for the Schwarzschild-Helmholtz equation (2.4). This formalism was chosen because it expresses the singular nature of field in the near proximity of the charge. In the case of a static charge, we are able to find a closed-form expression for the field. We have used this expression to show (after mass renormalization) there is no self-force on a static scalar charge outside a Schwarzschild black hole

Although we have patterned our discussion after Copson [26] and SW [13], there are alternative methods for computing the scalar field and the self-force. For example, in the static limit, Linet [36] has used *generalized axially symmetric potential* (GASP) theory [37] to derive the Green's function for the scalar field. However, this type of construction has not been extended to the Helmholtz-type equation depicted in Eq. (2.4). Lohiya [38] has demonstrated a concise method for determining the force on a static electric charge. Lohiya's method also uses the Copson-Linet closed-form expression for the electrostatic potential; however it is unclear how to extend Lohiya's method to moving particles. The formalism we have developed can be extended to moving charges. The recursion relation, Eq. (2.13), can be integrated with $\omega\neq 0$. Although it may be hard to find a simple summation of the results as in Eq. (2.15b), it is possible to obtain the Green's function to the first several orders in Γ.

Now that we have computed the (absence of) forces acting on a static scalar charge, let us assemble what is known about all the forces on a static charge outside a Schwarzschild black hole. In order to express this, let us slightly change the thought experiment. We will give the test charge a mass μ, an electric charge e, and a scalar charge q. We will support the charge on a strut as before, but, instead of measuring the force supplied by the strut at some moment $\bar{t}=0$, we kick the strut out from under the charge and find the instantaneous acceleration of the falling charge in harmonic coordinates. After the particle begins to move there will also be a radiation-reaction force, so we must make the measurement at the moment we remove the strut. The metric can be used to convert quantities from free-falling (proper) coordinates to harmonic coordinates [14]. The result is

$$\left[\mu\frac{d^2r_h}{dt_h^2}\right]_{\bar{t}=0}=\left(\frac{r_h-m}{r_h+m}\right)^{3/2}\bar{F}^z_{(\rm strut)} . \tag{4.1}$$

Using this to convert Eqs. (1.2) and (3.17), and expanding in the post-Newtonian quantity M/r_h, we have

$$\left[\mu\frac{d^2r_h}{dt_h^2}\right]_{\bar{t}=0}=\frac{M}{r_h^2}\Bigg\{\mu\left[-1+4\left(\frac{M}{r_h}\right)-9\left(\frac{M}{r_h}\right)^2+16\left(\frac{M}{r_h}\right)^3 + \text{known terms}\right] \tag{4.2a}$$

$$+\frac{e^2}{M}\left[\left(\frac{M}{r_h}\right)-6\left(\frac{M}{r_h}\right)^2+\frac{39}{2}\left(\frac{M}{r_h}\right)^3 + \text{known terms}\right] \tag{4.2b}$$

$$+\frac{q^2}{M}[\text{all terms are known to be zero}] \tag{4.2c}$$

$$+\frac{\mu^2}{M}\left[2\left(\frac{M}{r_h}\right)-\frac{87}{4}\left(\frac{M}{r_h}\right)^2 + \text{unknown terms}\right] \tag{4.2d}$$

$$+O[e^2\mu]+O[q^2\mu]+O[\mu^3]\Bigg\}. \tag{4.2e}$$

Here r_h denotes the radial position of the particle in harmonic coordinates. In line (4.2a) we have recovered the velocity independent terms of the geodesic equation of motion expressed in harmonic coordinates. Since this is just a Taylor expansion of the first term in Eq. (1.2), we know these terms to all orders in M/r_h. Thankfully, the first three terms are in agreement with the second post-Newtonian equations of motions. (See e.g., [39,40].) Line (4.2b) is just the expansion of the second term in Eq. (1.2), and thus we know these terms to all orders in M/r_h. Line (4.2c) is the scalar-charge part, which we have shown to vanish for all orders in M/r_h. For moving charges, there will very likely be non-zero terms. In line (4.2d), only the first two terms are known from second post-Newtonian calculations. (See e.g., [39,40].) The question remains, can the unknown terms in line (4.2d) be obtained by methods similar to those used to find the electric and scalar forces, that is, by looking at the field (metric) perturbations produced by the mass of the test particle? Obviously there are a number of conceptual issues to tackle in answering this question. For example, when the metric itself is the perturbed field, can we define a freely falling observer in the same way as we did in the scalar-charge force calculation? When solving for the metric perturbation, how do the stresses in the strut affect the solution? This is currently under vigorous investigation [41]. Line (4.2e) represents higher order effects, such as additional forces on the particle due to the change in the metric produced by the electric field of the particle. Such terms will also arise from a gauge change, say, $r \rightarrow r + O[\mu]$. These terms are clearly second order in the perturbations.

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APPENDIX A: RECAP OF ELECTROSTATIC CHARGE IN SCHWARZSCHILD SPACETIME

In this section we summarize the results of Copson [26] and obtain the closed-form solution for the electrostatic potential using our Hadamard construction. Using isotropic coordinates and assuming the field is strictly static, Maxwell's equations for the electrostatic potential can be written

$$C^{ij}A_{0,ij}(\mathbf{x}) + C^{j}A_{0,j}(\mathbf{x}) = -16\pi e \frac{b^2(2b-M)}{(2b+M)^3}\delta^3(\mathbf{x}-\mathbf{b}), \tag{A1}$$

where

$$C^j = h(r)\frac{x^j}{r} = \frac{d}{dr}\left\{\ln\left[\frac{(1+M/2r)^3}{1-M/2r}\right]\right\}\frac{x^j}{r}. \tag{A2}$$

Equation (A1) is in the same form as Eq. (2.4); therefore we proceed using Eq. (2.9) and we get

$$U_0^{(\mathrm{el})} = \frac{r}{r'}\left(\frac{2r'+M}{2r+M}\right)^{3/2}\left(\frac{2r-M}{2r'-M}\right)^{1/2}. \tag{A3}$$

The superscript (el) denotes that these are parts of the electrostatic solution. Using the recursion relation Eq. (2.13), we have

$$U_1^{(\mathrm{el})}(r,r') = \frac{3}{2}U_0^{(\mathrm{el})}\gamma, \tag{A4a}$$

$$U_2^{(\mathrm{el})}(r,r') = -\frac{5}{8}U_0^{(\mathrm{el})}\gamma^2, \tag{A4b}$$

$$U_3^{(\mathrm{el})}(r,r') = \frac{7}{16}U_0^{(\mathrm{el})}\gamma^3, \tag{A4c}$$

$$U_4^{(\mathrm{el})}(r,r') = -\frac{45}{128}U_0^{(\mathrm{el})}\gamma^4, \tag{A4d}$$

where γ is the same as in Eq. (2.14a). Once again the summation is elementary:

$$U_{\mathrm{elem}}^{(\mathrm{el})}(r,r') = \frac{U_0^{(\mathrm{el})}}{\sqrt{\Gamma}}\frac{(1+2\gamma\Gamma)}{\sqrt{1+\Gamma\gamma}}. \tag{A5}$$

As in the scalar case, we integrate the elementary solution against the source and obtain a particular solution to electrostatic field Eq. (A1)

$$A_0^{\mathrm{part}}(\mathbf{x},\mathbf{b}) = \frac{e}{(b_h+M)(r_h+M)} \times \frac{b_h r_h - M^2\cos\theta}{\sqrt{r_h^2 - 2r_h b_h\cos\theta + b_h^2 - M^2\sin^2\theta}}, \tag{A6}$$

where we have switched to harmonic coordinates [14]. This is Copson's [26] 1928 solution ''for the potential of an electron in the Schwarzschild field.'' As in the scalar case we must ask: does the particular solution satisfy the boundary conditions? It is well behaved at the horizon, so there is no problem there. However, as $r_h \rightarrow \infty$ the potential does not give the correct value

$$A_0^{\mathrm{part}}(r_h \rightarrow \infty) \sim \frac{e}{r_h}\frac{b_h}{b_h+M} \neq \frac{e}{r_h}. \tag{A7}$$

The fact that the field does not behave as e/r_h for large r_h suggests by Gauss's law that we have found a solution with some additional charge lying around. However, our solution satisfies the homogeneous (source-free) equation everywhere outside the horizon except at the source point where there is a charge e. We must conclude that we found a solution with some charge on the horizon. In order to fix the boundary condition, we need to add a monopolar solution of the homogeneous equation, that is, we need to add an image charge. It is easy to see what is needed,

$$A_0^{\rm homog} = \frac{eM}{(r_h+M)(b_h+M)}, \tag{A8}$$

and check that this satisfies the homogeneous equation outside the horizon. Linet [24] noticed the discrepancy in Eq. (A7) and added this piece to Copson's result. Combining the two pieces gives the final result: the electrostatic potential for a fixed charge outside a Schwarzschild black hole

$$A_0(\mathbf{x},\mathbf{b}) = \frac{e}{(b_h+M)(r_h+M)} \times \left\{ \frac{b_h r_h - M^2\cos\theta}{\sqrt{r_h^2 - 2r_h b_h \cos\theta + b_h^2 - M^2 \sin^2\theta}} + M \right\}. \tag{A9}$$

The potential $A_0^{\rm part}$ in Eq. (A6) is the particular solution constructed directly from the Hadamard elementary solution. If the Hadamard potential $A_0^{\rm part}$ is taken to be the actual potential and the force calculation is carried out (i.e., repeat the SW calculation using the method similar to Sec. III A), the resulting force is zero. This means that although our construction of the Hadamard solution was strictly a local calculation, when we summed the series we found a solution with just enough charge on the horizon to cancel the repulsive force in Eq. (1.2). This also means that the repulsive force that SW found for the electric charge is due solely to the part of the potential $A_0^{\rm homog}$ which is tacked on to satisfy the boundary conditions. In other words, the second term in Eq. (1.2) is simply the force produced by the image charge on the horizon, and the force can be computed from Eq. (A8) directly:

$$F_{\rm self} = e\left[\frac{d}{dr_h} A_0^{\rm homog}(r_h, b_h)\right]_{r_h=b_h} = -\frac{e^2 M}{(b_h+M)^3} = -\frac{e^2 M}{b_s^3}. \tag{A10}$$

This gives a physical interpretation of the repulsive force; the charge outside the black hole is repelled by an image charge inside the horizon.

APPENDIX B: COMPARING CLOSED-FORM SOLUTIONS WITH SERIES SOLUTIONS

Equating our closed-form solutions for the scalar-static and electrostatic fields with the conventional infinite series solutions, we can obtain some interesting summation formulas that do not appear in the standard references [42–44].

For a fixed point source Eq. (2.1) is easily solved by separation of variables. The angular dependence is expressed by Legendre polynomials, and the resulting radial equation is also Legendre's equation. Equating the series solution to our close-form solution Eq. (2.16c) we have

$$V(\mathbf{x_h},\mathbf{b_h}) = -q\sqrt{\frac{b_h-M}{b_h+M}} \frac{1}{\sqrt{r_h^2 - 2r_h b_h\cos\theta + b_h^2 - M^2\sin^2\theta}} \tag{B1a}$$

$$= -\frac{q}{M}\sqrt{\frac{b_h-M}{b_h+M}} \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) \times \begin{Bmatrix} P_l(r_h/M) Q_l(b_h/M), & \text{if } r_h<b_h \\ P_l(b_h/M) Q_l(r_h/M), & \text{if } r_h>b_h \end{Bmatrix}. \tag{B1b}$$

Here the P_l and Q_l are the Legendre functions. This summation is a special case of Eq. 28 in MacRobert [45].

If the field point is located on the horizon ($M=r_h<b$) or on the axis ($\theta=0$) we can verify this formula using the standard summation formula [42–44]

$$\sum_{n=0}^{\infty} (2n+1) Q_n(x) P_n(y) = \frac{1}{x-y} |x|>1 \text{ and } |x|>|y|. \tag{B2}$$

Applying this summation formula to Eq. (B1), we get

$$V(\text{horizon}) = -\frac{q}{b_h - M\cos\theta}\sqrt{\frac{b_h-M}{b_h+M}}. \tag{B3}$$

Clearly the horizon is not a surface of constant "potential." This is in contrast with the electrostatic case where the horizon is a surface of constant potential. [See Eq. (A9).] On the axis of symmetry [i.e., $\theta=0$, so $P_l(\cos\theta)=1$ for all values of l] we can also use Eq. (B2) to sum Eq. (B1)

$$V(\text{axis}) = -\frac{q}{|r_h-b_h|}\sqrt{\frac{b_h-M}{b_h+M}}. \tag{B4}$$

In the electrostatic case, the series solution can be similarly obtained by separation of variables. The radial functions are derivatives of Legendre functions. Equating the analytic expression with the series solution gives an interesting summation formula

$$A_0(\mathbf{x},\mathbf{b}) = \frac{e}{(b_h+M)(r_h+M)} \left\{ \frac{b_h r_h - M^2\cos\theta}{\sqrt{r_h^2 - 2r_h b_h\cos\theta + b_h^2 - M^2\sin^2\theta}} + M \right\} \tag{B5a}$$

$$= -\frac{e}{M^3}\sum_{l=0}^{\infty}\frac{2l+1}{l(l+1)} P_l(\cos\theta) \begin{Bmatrix} (r_h-M)(b_h-M) P_l'(r_h/M) Q_l'(b_h/M), & \text{if } r_h<b_h \\ (r_h-M)(b_h-M) P_l'(b_h/M) Q_l'(r_h/M), & \text{if } r_h>b_h \end{Bmatrix}. \tag{B5b}$$

Here it is understood when $l=0$ we make the replacement

$$\frac{(r_h-M)}{l}P_l'(r_h/M)\rightarrow M \quad (\text{when } l=0). \tag{B6}$$

Notice that the horizon $(r_h=M)$ is a surface of constant potential. The closed-form expression [modulo the homogeneous piece in Eq. (A8)] was computed in 1928 by Copson [26], who expanded the result in terms of radial functions and discovered the summation formula. The series result was rederived by Cohen and Wald [22], and Hanni and Ruffini [23]. The asymptotic form of the solution can be seen immediately from the closed-form result, or from the series by noting that $Q_0(x\rightarrow\infty)\approx 1/x$. We see that

$$A_0(r\rightarrow\infty)\approx\frac{e}{r}, \tag{B7}$$

which is the correct behavior. Equation (B1) can also be obtained from Eq. (B5) by differentiating and using Legendre's equation.

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