Entropy of rotating Misner string spacetimes

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(Received 22 April 1999; published 23 March 2000)

Using a boundary counterterm prescription motivated by the AdS-CFT conjecture, I evaluate the energy, entropy and angular momentum of the class of Kerr-NUT-bolt-AdS spacetimes. As in the non-rotating case, when the NUT charge is nonzero the entropy is no longer equal to one-quarter of the area due to the presence of the Misner string. When the cosmological constant is also non-zero, the entropy is bounded from above.

PACS number(s): 04.70.Dy, 04.20.−q

The thermodynamic properties of gravity have for long appeared to be inextricably connected with the presence of black holes [1,2]. A physical entropy S and temperature β^{-1} can be ascribed to a given black hole configuration, where these quantities are respectively proportional to the area and surface gravity of the event horizon(s).

Recently it has been demonstrated that entropy can be associated with a broader and qualitatively different gravitational system, one containing Misner strings [3]. These objects are the gravitational analogues of Dirac strings, and arise whenever the gravitational field in the Euclidean regime has a U(1) isometry group (generated by a Killing vector ξ which is timelike in the Lorentzian regime) with a fixed point set of co-dimension $d_f<d-2$ (called a ''nut'' [4]). The existence of any fixed point set makes it impossible to everywhere foliate the spacetime with surfaces of constant τ, leading to a difference between the total energy (H_∞) and free energy of the gravitational system, which thermodynamically is proportional to its entropy. In general a spacetime can contain both black holes (for which $d_f=d-2$—called a ''bolt'') and Misner strings, and the total gravitational entropy will receive contributions from both of these objects.

Explicit demonstration of these ideas has been given in a number of cases, for spacetimes with and without cosmological constant. Since the Misner string contribution to the entropy is divergent, at first only the relative entropy (and energy) between a spacetime with a bolt/nut configuration and its asymptotically matched pure nut counterpart was calculated [3,5,6]. However it has more recently been shown that the entropy of Misner strings can be intrinsically defined [7,9], even if no bolts are present. By adding to the action an additional boundary term which is a functional of the intrinsic curvature invariants on the boundary, the equations of motion are unaffected and the gravitational entropy (and total and free energies) is finite whether or not there is a bolt. The inclusion of this boundary term is motivated from recent work [10,11] on the conjectured AdS-CFT duality, which equates the bulk gravitational action of an asymptotically AdS spacetime with the quantum effective action of a conformal field theory (CFT) defined on the AdS boundary. The coefficients in the additional term may be uniquely fixed by demanding that it be finite for Schwarzchild-AdS spacetime. However the spacetime need not be locally AdS asymptotically—locally asymptotically flat cases may also be included [7].

The purpose of this paper is to extend these considerations to include rotation. Specifically I consider the class of Euclidean Kerr and Kerr-AdS solutions with and without NUT charge in four dimensions. This class of spacetimes forms an important test case for the counterterm prescription and has received relatively little attention in the literature [12,13]. Even for Kerr spacetimes with zero nut charge and cosmological constant the problem of computing quasilocal energy is very difficult and has only been carried out in the slow-rotating limit [14]. The counterterm prescription given in Ref. [7] extends to the full Kerr-NUT class, reproducing the values of the mass and angular momentum without any background spacetime subtractions. When the nut charge is non-vanishing, I find that the presence of rotation does not admit the existence of regular spacetime solutions unless a bolt is also present. Furthermore the rotation parameter has no upper bound when the nut charge and cosmological constant are nonzero. The entropies for these spacetimes are also computed and are not proportional to their horizon areas due to the Misner strings. This is the first calculation of the entropy of rotating spacetimes with nut charge.

Consider a Euclidean manifold M with metric $g_{\mu\nu}$, covariant derivative ∇_μ, and time coordinate τ which foliates M into non-singular hypersurfaces Σ_τ with unit normal u_μ. $\Theta^{\mu\nu}$ (whose trace is Θ) denotes the extrinsic curvature of any boundary(ies) ∂M of the manifold M (internal and/or at infinity), with induced metric(s) γ. The path-integral formulation of quantum gravity implies that the Euclidean action $I=-\log Z$ to lowest order in $\hbar$, where Z is the partition function of an ensemble

$$Z=\int [Dg][D\Phi]\exp[-I(g,\Phi)] \tag{1}$$

with the path integral taken over all metrics g and matter fields Φ that are appropriately indentified under the period β of τ. The thermodynamic definition $\log Z=S-\beta H_\infty$ then implies that the entropy of a given spacetime is

$$S=\beta H_\infty-I=\beta(E+\Omega\cdot J)-I \tag{2}$$

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where E and J are respectively the energy and angular momentum of the spacetime at infinity and Ω the angular velocity at the event horizon. The entropy is then the difference between the value the action would have (βH_∞, the total energy) if there were no breakdown of foliation and its actual value (proportional to the free energy).

One could compute this difference by removing small neighborhoods N^i_ϵ of the fixed point sets and strings so that $I = I_{M^i_\epsilon} - \Sigma_i I_{N^i_\epsilon}$. Rewriting the $I_{M^i_\epsilon}$ into Hamiltonian form (taking care to include the additional surface terms due to these new boundaries), one finds that the only non-zero contributions to the Hamiltonian are from the boundaries at infinity and along the strings. When the contributions $I_{N^i_\epsilon}$ from the small neighborhoods of the fixed point sets are reinserted, their surface terms are non-vanishing and yield the one-quarter of the areas of the neighborhoods removed, i.e. of the bolts and the strings.

The action is generally taken to be a linear combination of a volume (or bulk) term

$$I_v = -\frac{1}{16\pi}\int_M d^d x \sqrt{g}(R + 2\Lambda + \mathcal{L}(\Phi)) \tag{3}$$

and a boundary term

$$I_b = -\frac{1}{8\pi}\int_{\partial M} d^{d-1}x \sqrt{\gamma}\Theta(\gamma) \tag{4}$$

(chosen to yield a well-defined variational principle), where $\mathcal{L}(\Phi)$ is the matter Lagrangian and Λ the cosmological constant. When evaluated on solutions both I_v and I_b are typically divergent, yielding divergent values for both the string area and Hamiltonian terms and hence for the entropy. One method of dealing with this difficulty is to compute everything relative to some chosen reference background spacetime (suitably matched in its asymptotic and topological properties) whose boundary(ies) have the same induced metric(s) as those in the original spacetime [15–17]; the reference spacetime is then interpreted as the vacuum for that sector of the quantum theory. Such a choice is not always unique [18], nor is it always possible to embed a boundary with a given induced metric into the reference background. Indeed, for Kerr spacetimes this latter problem forms a serious obstruction towards calculating the subtraction energy, and calcuations have only been performed in the slow-rotating regime [14].

The counterterm proposal involves adding a term I_{ct} to the action, where [7]

$$I_{ct} = \frac{2}{l}\frac{1}{8\pi}\int_{\partial M_\infty} d^3x \sqrt{\gamma}\sqrt{1 + \frac{l^2}{2}R(\gamma)} \tag{5}$$

with $l = \sqrt{3/|\Lambda|}$ (a slight variant of this prescription was given by Lau [8]). The coefficients of the $R(\gamma)$ term and the overall action are determined by demanding that the Schwarzchild-AdS solution have finite total action $I_T = I_v + I_b + I_{ct}$. The prescription (5) has been shown to be sufficient for evaluating the actions, entropies and total energies for the Schwarzchild, Taub-bolt, and Taub-NUT (Newman-Unti-Tamburino) spacetimes, along with their AdS and topological extensions without the use of any background subtractions [7,9]. It is motivated by the conjectured AdS-CFT correspondence: divergences appearing in the stress-energy tensor of the boundary CFT are just the standard ultraviolet divergences of quantum field theory and may be removed by adding counterterms to the action which depend only on the intrinsic geometry of the boundary. Quantities such as energy, entropy and (as will be shown) angular momentum are then intrinsically defined for a given spacetime, rather than with respect to a reference background. Furthermore, Eq. (5) applies even in the $l \to \infty$ limit, thereby including asymptotically locally flat cases, unlike the prescriptions in Refs. [10,11] to which Eq. (5) reduces for large l.

Using Eq. (5) the entropy is

$$S = \beta H_\infty - (I_v + I_b + I_{ct}) \tag{6}$$

where all quantities are evaluated on a given solution, and where $H_\infty = M + \Omega J$, with $M = Q[\partial/\partial\tau]$ and $J = Q[\partial/\partial\phi]$ being the conserved charges associated with the Killing vectors $\partial/\partial\tau$ and $\partial/\partial\phi$ where $\Omega = a/(r_+^2 - a^2 - N^2)$, with r_+ defined below. These conserved charges are given by [16,19]

$$Q[\xi] = \frac{1}{8\pi}\int_{\partial M_\infty \cap \Sigma_\tau}\left[\Theta^{\mu\nu} - \Theta\gamma^{\mu\nu} + \frac{2}{\sqrt{\gamma}}\frac{\delta I_{ct}}{\delta\gamma_{\mu\nu}}\right]u_\mu \xi_\nu \tag{7}$$

which may be shown by taking the variation of the action with respect to the boundary metric $\gamma_{\mu\nu}$ at infinity .

The class of Euclidean Kerr-NUT-AdS spacetimes has the metric form

$$\begin{aligned} ds^2 = {} & \frac{V(r)(d\tau - [2N\cos(\theta) - a\sin^2(\theta)]d\phi)^2 + \mathcal{H}(\theta)\sin(\theta)^2[a\,d\tau - (r^2 - N^2 - a^2)d\phi]^2}{\chi^4(r^2 - [N + a\cos(\theta)]^2)} \\ & + (r^2 - [N + a\cos(\theta)]^2)\left(\frac{dr^2}{V(r)} + \frac{d\theta^2}{\mathcal{H}(\theta)}\right) \end{aligned} \tag{8}$$

where the Einstein field equations imply

TABLE I. Action and entropy for rotating black holes with NUT charge.

Spacetime	Periodicity	Action	Entropy
Kerr	$\frac{4\pi r_+(r_+^2-a^2)}{(r_+^2+a^2)}$	$\frac{\pi(r_+^2-a^2)^2}{(r_+^2+a^2)}$	$\pi(r_+^2-a^2)$
Kerr-bolt	$8\pi N$	$\pi\frac{(r_+^2-a^2)^2-N^4}{r_+^2+a^2-N^2}$	$\pi(r_+^2-a^2+N^2)$
Kerr-AdS	$\frac{4\pi r_+(l^2+a^2)(r_+^2-a^2)}{3r_+^4+(l^2-a^2)r_+^2+a^2l^2}$	$-\frac{\pi(r_+^2-l^2)(r_+^2-a^2)^2}{(3r_+^4+(l^2-a^2)r_+^2+a^2l^2)\chi^2}$	$\pi\frac{r_+^2-a^2}{\chi^2}$
Kerr-bolt-AdS	$8\pi N$	$-\pi\frac{(r_+^2-N^2-a^2)[r_+^4-(a^2+l^2)r_+^2+(N^2-a^2)(3N^2-l^2)]}{(3r_+^4+(l^2-a^2-6N^2)r_+^2+(N^2-a^2)(3N^2-l^2))\chi^2}$	$\pi\frac{3r_+^6+(l^2-4a^2-15N^2)r_+^4+(a^2+3N^2)^2r_+^2+(N^2-a^2)^2(3N^2-l^2)}{(3r_+^4+(l^2-a^2-6N^2)r_+^2+(N^2-a^2)(3N^2-l^2))\chi^2}$

$$\mathcal{H}=1+\frac{qN^2}{l^2}+\frac{[2N+a\cos(\theta)]^2}{l^2}$$
$$V(r)=\frac{r^4}{l^2}+\frac{[(q-2)N^2-a^2+l^2]r^2}{l^2}-2mr-\frac{(a+N)(a-N)(qN^2+l^2+N^2)}{l^2} \tag{9}$$

and where the periodicity in τ and the parameters q and χ are chosen so that conical singularities are avoided. There are string singularities at both the north and the south poles in this metric when $N\neq 0$.

In the (θ,ϕ) section these considerations may be implemented as follows. Writing the metric as

$$ds^2=g_{\tau\tau}\left(d\tau+\frac{g_{\tau\phi}}{g_{\tau\tau}}d\phi\right)^2+g_{rr}dr^2+g_{\theta\theta}d\theta^2+\left(g_{\phi\phi}-\frac{g_{\tau\phi}^2}{g_{\tau\tau}}\right)d\phi^2,$$

conical singularities in the (θ,ϕ) section will be absent provided the metric in this section is conformal to $d\theta^2+\theta^2d\phi^2$ near $\theta=0$, and to $d\theta^2+(\theta-\pi)^2d\phi^2$ near $\theta=\pi$. Expanding the (θ,ϕ) part of the metric about these respective points yields $\chi^2=\mathcal{H}(0)$ and $\chi^2=\mathcal{H}(\pi)$. For $N\neq 0$ these relations cannot be simultaneously satisfied. However for the form of the metric given above there are string singularities for each of these values of θ when $N\neq 0$, so this is a moot point. Transforming $\tau\rightarrow\tau\pm 2N\phi$ respectively removes the string singularities at $\theta=0,\pi$; one can then use two nonsingular coordinate patches at each of the poles and then match them elsewhere via a simple coordinate transformation. Rather than do that I shall take $\chi=\sqrt{1+a^2/l^2}$ so that conical singularities are manifestly removed when $N=0$. Further requiring that the conformal factor be unity when a$=0$ yields $q=-4$.

The periodicity in τ is more subtle. The location of the nut is at $r=\sqrt{a^2+N^2}\equiv r_N$, where the area of surfaces orthogonal to the (r,τ) section vanishes. The Misner string singularity runs along the postive and negative z axes from the nut to infinity. Regularity along the z axes then implies that τ has period $8\pi N$. However regularity in the (r,τ) section implies that τ also has period $2\pi/\kappa$ where $\kappa=\sqrt{-\nabla_\mu\zeta_\nu\nabla^\mu\zeta^\nu/2}$, with $\zeta=\partial/\partial\tau+\Omega(\partial/\partial\phi)$ being the Killing field normal to the horizon, and where $\Omega=a/(r_+^2-(a+N)^2)$ is the angular velocity of the horizon. Explicitly

$$\kappa=\frac{V'(r_+)}{4\chi^2(r_+^2-r_N^2)} \tag{10}$$

where $V(r_+)=0$, r_+ being the location of the foliation breakdown. Interpreting this latter equation as determining m

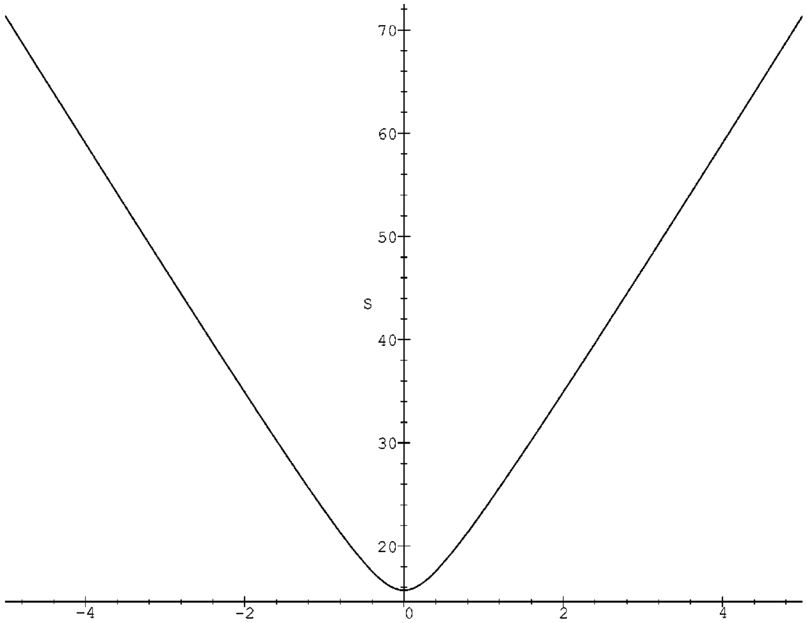

FIG. 1. The entropy as a function of $x = a/N$ in the Kerr-bolt case.

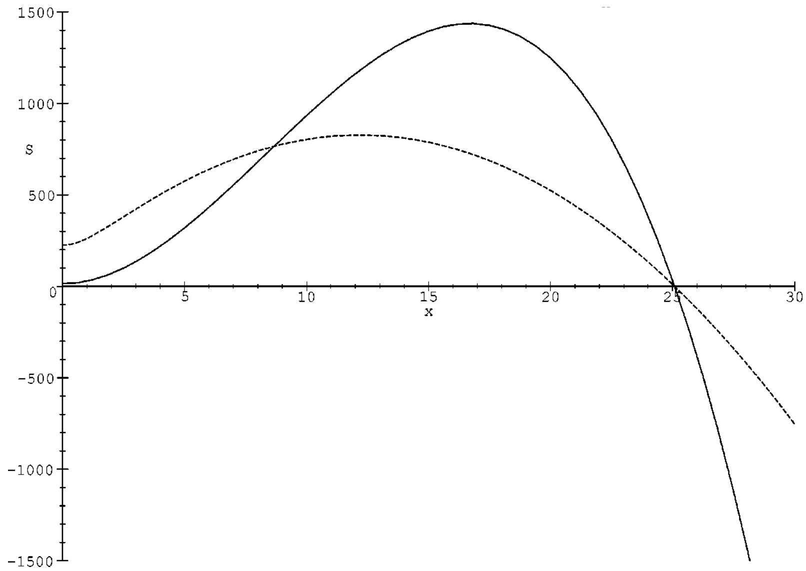

FIG. 2. The entropy (solid line) and l^2/N^2 (dashed line) as a function of $x = a/N$ for $K = 4$ in the Kerr-bolt-AdS case.

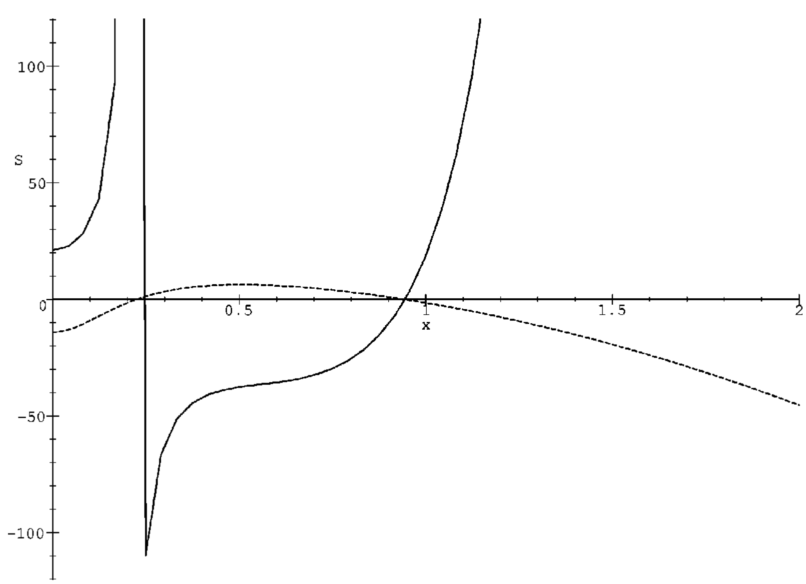

FIG. 3. The same plot as Fig. 2, but for $K = 1.01$.

in terms of r_+, and then equating these two periods yields a quartic constraint (cubic if $l\to\infty$) on r_+ in terms of a, N and l

$$K^4-\frac{1}{6}\frac{(y^2+x^2)K^3}{\sqrt{1+x^2}}-\frac{1}{3}\frac{(x^2+6-y^2)K^2}{1+x^2}+\frac{1}{6}\frac{(y^2+x^2)K}{\sqrt{1+x^2}}+\frac{1}{3}\frac{(y^2-3)(x-1)(x+1)}{(1+x^2)^2}=0 \tag{11}$$

where $K=r_+/r_N$, $x=a/N$ and $y=l/N$. Solutions to this equation for the parameter K constitute a valid Kerr-bolt-AdS metric that is free of singularities. Solutions to the $y\to\infty$ limit of this equation furnishes the allowed parameter space for the Kerr-bolt solutions.

A more careful treatment is required if $r_+=r_N$; in this case regularity of the solutions demands that $V(r)$ have a double root there. However this requirement turns out to be incompatible with the τ-periodicity constraints unless $a=0$. Hence $r_+>r_N$, and there are no regular Kerr-NUT or Kerr-NUT-AdS solutions (the former observation was made in Ref. [12]).

Denoting the unit radial normal to ∂M (with induced metric $\gamma_{\mu\nu}$) by n^ν, the quasilocal mass and angular momenta are given by the expressions

$$M=\int d^2x\sqrt{\sigma}\xi^\nu(ku_\nu+j_\nu) \tag{12}$$

$$J=\int d^2x\sqrt{\sigma}\psi^\nu(ku_\nu+j_\nu) \tag{13}$$

where $\xi=\partial/\partial\tau$, $\psi=\partial/\partial\phi$, and k is the trace of the extrinsic curvature $k_{\mu\nu}=\sigma_\mu{}^\alpha\sigma_\nu{}^\beta\nabla_\alpha n_\beta$ is the extrinsic curvature of the 2-boundary which is the intersection of ∂M and Σ_τ, with metric $\sigma_{\mu\nu}=\gamma_{\mu\nu}-n_\mu n_\nu$. The vector j_ν is

$$j_\nu=\sigma_\nu{}^\beta n^\alpha\nabla_\beta u_\alpha \tag{14}$$

is the angular momentum vector of the 2-boundary. Using the formulas (7), I find after somewhat lengthy and tedious calculation

$$M=\frac{m}{\chi^4},\quad J=\frac{ma}{\chi^4} \tag{15}$$

for each of the Kerr, Kerr-AdS, Kerr-bolt and Kerr-bolt-AdS solutions, the parameter m obeying the constraints mentioned in the previous paragraph in the bolt case. The actions, Hamiltonians, and entropies for each case are finite. The results are given in Table I; omitted is the Hamiltonian for each case, which is simply $H_\infty=(m/\chi^4)(1+a\Omega)$. After substituting the solutions for m and r_+, the action in the Kerr-bolt case is $4\pi Nm$, in agreement with Ref. [12]. Note that for the bolt solutions the entropy is not one-quarter of the area, due to the presence of the Misner string.

One of the more unusual results which follows from Table I is that in the Kerr-bolt-AdS solution the entropy is not positive for all values of the parameters. The entropy in this case is most easily analyzed by using Eq. (11) to write y^2 in terms of K and x. For given values of K one then can numerically search for regions where $y^2>0$ (i.e. a real cosmological constant) as a function of x. The entropy can then be determined for these same values in the allowed regions. It is always positive for any values of a and l provided $19K^6-78K^4+3K^2+8>0$, or $K>K_m=2.02282556\ldots$. However for $1<K<K_m$ there exist ranges of values of a/N for which the entropy is negative, and for certain values of a the entropy diverges to $-\infty$. Similar properties have been noted in the non-rotating AdS NUT and bolt solutions [7,9]. The entropy $s=S/N^2$ as a function of $x=a/N$ is plotted in Fig. 1 for the Kerr-bolt solution – for small x $s\approx 5\pi+\mathcal{O}(x^2)$, and for large x, $s\to 4\pi x$. The parameter m is an everywhere defined increasing function of x, and approaches $2N$ for large x. The behavior is quite different in the the Kerr-bolt-AdS solution. For a given $K>K_m$, s reaches a maximum somewhere between $x=0$ and $x=x_m$; for $x>x_m$, the parameter $l^2<0$ and so there are no allowed solutions. For $K<K_m$, the entropy (and m) will be negative in some allowed region of x (i.e. where $l^2>0$). Figures 2 and 3 show typical cases. Note that the entropy is bounded above in the K=4 case.

To summarize, the prescription (5) has been shown to apply to spacetimes with non-zero angular momentum, and so removes the troublesome aspects of evaluating physical quantities in gravity relative to some chosen background [14,18]. Indeed, the expressions for M, J and S could all be given quasilocally at finite radius R, although I have omitted them here for the sake of brevity. As with the non-rotating case, these results must be carefully interpreted. For $N\neq 0$ the boundary at infinity is not a direct product $S^1\times S^2$ but instead is a squashed S^3. Consequently continuation to the Lorentzian regime is not straightfoward the way it is in the $N=0$ cases. The most promising possibility is that of interpreting the path integral over all metrics as the partition function for an ensemble of spacetimes with fixed NUT charge [3,5]. A more complete understanding of the thermodynamics of these solutions (and how to interpret the negative values of the entropy in the AdS case), as well as the relationship between these results and the behavior of a conformal field theory on the boundary remain interesting questions for further study.

The work was supported by a National Science Foundation grant PHY94-07194 and by the Natural Sciences and Engineering Research Council of Canada. I am grateful to the ITP and to the Physics Department at UCSB for their hospitality.

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