Two-dimensional effective action for matter fields coupled to the dilaton

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(Received 8 November 1999; published 22 March 2000)

We revise the calculation of the one-loop effective action for scalar and spinor fields coupled to the dilaton in two dimensions. Applying the method of covariant perturbation theory for the heat kernel we derive the effective action in an explicitly covariant form that produces both the conformally invariant and the conformally anomalous terms. For scalar fields the conformally invariant part of the action is nonlocal. The obtained effective action is proved to be infrared finite. We also compute the one-loop effective action for scalar fields at finite temperature.

PACS number(s): 04.60.Kz, 04.62.+v, 11.10.Lm, 11.25.Db

I. INTRODUCTION

The effective action and conformal anomaly of quantum fields coupled to the dilaton in two dimensions have been the subject of a number of recent papers. The main motivation for the study of quantum field models in two-dimensional (2D) dilaton-gravity backgrounds comes from the fact that such models naturally arise after the spherical (or dimensional) reduction from higher-dimensional field theories and gravity. For a description of the spherical reduction procedure leading to dilaton gravity in two dimensions, we refer to [1,2].

Two dimensional models seem to be easy to quantize, and in some cases they admit exact solutions at classical and quantum levels. Black hole physics is one of the most interesting applications of such models [1,3]. Seemingly, the two-dimensional results could provide information about higher-dimensional quantum physics. The question of the effective action for 2D dilaton gravity and its relation to Hawking radiation is addressed in many papers ([4–7], to mention a few). For the history and contemporary state of this problem see a recent review by Kummer and Vassilevich [8]. However, the applicability of these two-dimensional considerations to the Hawking effect in four dimensions is hampered by serious problems [6,9,10] (some of these problems are related to the dimensional-reduction anomalies and may be resolved via their thorough analysis [11]).

We study here only two-dimensional models, so, we are not concerned with problems related to the dimensional reduction or higher-dimensional quantum physics. Specifically, in this paper we focus on the one-loop effective action for quantum matter fields interacting with the background dilaton and gravity in two dimensions, and related infrared problems. Surprisingly, there is no consensus in the literature even about this relatively simple problem. We derive the one-loop effective action, which, in our opinion, corrects and supplements other known results on this subject.

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II. HEAT KERNEL FOR SCALAR FIELDS COUPLED TO THE DILATON IN TWO DIMENSIONS

Let us begin with the classical action for the scalar matter field η coupled to the background metric $g_{\mu\nu}$ and the background dilaton field ϕ,

$$S=-\frac{1}{2}\int \mathrm{d}^2x\, g^{1/2}\mathrm{e}^{-2\phi}\nabla^\mu\eta\nabla_\mu\eta. \tag{1}$$

We do not specify here the function ϕ, which can be an arbitrary smooth function. Following the procedure of [4,12] we redefine field variables and rewrite the action (1) in terms of new scalar fields $\tilde{\eta}=\mathrm{e}^{-\phi}\eta$. Then the action takes the form

$$S=-\frac{1}{2}\int \mathrm{d}^2x\, g^{1/2}\{\nabla^\mu\tilde{\eta}\nabla_\mu\tilde{\eta}-\tilde{\eta}^2[\Box\phi-(\nabla_\mu\phi)(\nabla^\mu\phi)]\}. \tag{2}$$

The one-loop effective action for this model is defined as

$$W=\frac{1}{2}\mathrm{Tr}\ln F(\nabla), \tag{3}$$

where the differential operator corresponding to the action (2) reads

$$F(\nabla)=\Box+\Box\phi-(\nabla_\mu\phi)(\nabla^\mu\phi). \tag{4}$$

The widely accepted technique to compute the effective action is to use the trace anomaly of the energy-momentum tensor (Weyl anomaly), $T=2g^{\mu\nu}(\delta W/\delta g^{\mu\nu})$. Combined with the proper boundary conditions it provides enough information to derive unambiguously the one-loop effective action in the absence of the dilaton, the Polyakov action [13]. A similar method was applied to the system of quantum scalar fields coupled to the dilaton [4,12,14–16]. The operator (4) describes a 2D conformal model, and, in this case, W is also restored by the integration of the conformal anomaly. Such an effective action is known as the anomaly-induced action. Because this action is completely defined by T, the main subject of calculations and controversies in the existing literature was the computation of the anomaly itself. Unfortunately, unlike the Polyakov action, the anomaly-induced action is incomplete because it may contain conformally invariant terms that cannot be fixed by knowledge of the anomaly alone. These missing terms are important, for they lead to a non-zero (though traceless) energy-momentum tensor. This ambiguity is an artifact of the method, and its origin is obvious. The integral of the Weyl anomaly is, in fact, the difference between the effective actions in a physical spacetime and in a reference one [17]. Without a dilaton the reference spacetime is implicitly assumed to be flat with the same topology as the physical 2D manifold. The presence of the dilaton leads to a nontrivial conformally invariant effective action in the reference spacetime. As we will show explicitly, this action is generically nonlocal; hence, it can contribute to the Hawking radiation from the 2D dilaton black holes.

In order to obtain the complete effective action we use a method, which is different from the one we just described. Our approach to this problem has two important features: (1) it is manifestly covariant throughout all calculations; (2) it does not make use of the trace anomaly, thus, both anomaly-producing and conformally invariant terms come from the same calculation.

We begin with the heat kernel for the operator (4) and express the one-loop effective action as an integral over the proper time s, [18,19],

$$W=-\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\,\mathrm{Tr}\,K(s). \tag{5}$$

In coordinate representation Tr denotes the functional trace, $\mathrm{Tr}\,K(s)=\int \mathrm{d}^D x\,\mathrm{tr}\hat{K}(s|x,x)$ in arbitrary dimensions D, where tr denotes the matrix trace over any internal degrees of freedom that may be present in a field theory. The heat kernel $\hat{K}(s)$ is defined as a solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{K}(s|x,y)=\hat{F}(\nabla^x)\hat{K}(s|x,y), \quad \hat{K}(0|x,y)=\hat{1}\,\delta(x-y). \tag{6}$$

For the computation of the heat kernel we employ the covariant perturbation theory of Barvinsky and Vilkovisky [20–23]. As a basis for our calculations we use a general expression for the trace of the heat kernel in arbitrary spacetime dimensions obtained in Refs. [23–25] up to the third order in curvatures,

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^D x\, g^{1/2} \times\Bigg\{1+s\hat{P}+s^2\sum_{i=1}^{5} f_i(-s\Box_2)\Re_1\Re_2(i)+s^3\sum_{i=1}^{29} F_i \times(-s\Box_1,-s\Box_2,-s\Box_3)\Re_1\Re_2\Re_3(i)+\mathrm{O}[\Re^4]\Bigg\}, \tag{7}$$

for the generic field operator

$$\hat{F}(\nabla)=\hat{1}\Box-\frac{\hat{1}}{6}R(x)+\hat{P}(x). \tag{8}$$

Here R is the Ricci scalar, and $\hat{P}$ is an arbitrary potential term, which depends on the background fields and curvatures. In Eq. (7) we introduced the collective notation for background field strengths (''curvatures''), $\Re=(R_{\mu\nu},\hat{\mathcal{R}}_{\mu\nu},\hat{P})$, which includes the commutator curvature,

$$[\nabla_\mu,\nabla_\nu]\eta=\hat{\mathcal{R}}_{\mu\nu}\eta. \tag{9}$$

The form factors f_i and F_i in Eq. (7) are analytic functions of the dimensionless argument $s\Box$ that act on tensor invariants constructed from the field strengths. It is assumed that the operator arguments $\Box_i$ in the form factors are acting on the curvatures at the corresponding spacetime points, $\Re_i=\Re(x_i)$, and after that all spacetime points are made coincident, $x_1=x_2=x_3=x$.

For straightforward applications of this result, the differential operator for a field model should be of the form (8). It was already shown [23,24] that the heat kernel (7) correctly reproduces the Polyakov action, where the operator is just $F(\nabla)=\Box$. The operator (4) also belongs to the class of models (8) with the following specifications,

$$\mathrm{tr}\hat{1}=1,\hat{P}=\left(\frac{1}{6}R+\Box\phi-(\nabla_\mu\phi)(\nabla^\mu\phi)\right)\hat{1}. \tag{10}$$

Furthermore, the basis of 29 tensor structures in the third order [23,25] can be considerably reduced using the identities

$$\hat{\mathcal{R}}_{\mu\nu}=0, \quad R_{\mu\nu}=\frac{1}{2}g_{\mu\nu}R. \tag{11}$$

It is useful to express the heat kernel (7) in terms of two background field objects, the Ricci scalar R and the dilaton field ϕ, instead of R and P. Integrating by parts and discarding total derivatives we represent the first local term of Eq. (7) in the form

$$\int \mathrm{d}^2x\, g^{1/2}\,P(x)=\int \mathrm{d}^2x\, g^{1/2}\,\phi\Box\phi. \tag{12}$$

The expression for the trace of the heat kernel for operator (4) reads

$$\mathrm{Tr}\,K(s)=\frac{1}{4\pi s}\int d^2x\, g^{1/2}\Bigg\{1+s\phi\Box\phi+s^2\Bigg[\frac{1}{2}f(-s\Box_2)(\Box\phi_1)(\Box\phi_2)+\Bigg(\frac{1}{32}f(-s\Box_2)-\frac{1}{8}\left(\frac{f(-s\Box_2)-1}{s\Box_2}\right)$$
$$+\frac{3}{8}\left(\frac{f(-s\Box_2)-1-\frac{1}{6}s\Box_2}{(s\Box_2)^2}\right)\Bigg)R_1R_2+\left(\frac{1}{4}f(-s\Box_2)-\frac{1}{2}\,\frac{f(-s\Box_2)-1}{s\Box_2}\right)(\Box\phi_1-(\nabla\phi_1)^2)R_2\Bigg]$$
$$+s^3[M_1(-s\Box_1,-s\Box_2,-s\Box_3)R_1R_2R_3+M_2(-s\Box_1,-s\Box_2,-s\Box_3)R_1R_2(\Box\phi_3)+M_3(-s\Box_1,-s\Box_2,$$
$$-s\Box_3)R_1(\Box\phi_2)(\Box\phi_3)+M_4(-s\Box_1,-s\Box_2,-s\Box_3)(\Box\phi_1)(\Box\phi_2)(\Box\phi_3)]+\mathrm{O}[\mathfrak{R}^4]\Bigg\}. \quad (13)$$

In any expression, which depends on ϕ and R, like Eq. (13), we assume $\mathfrak{R}=(\phi,R)$. All second-order form factors are expressed via the basic one

$$f(-s\Box)=\int_0^1 d\alpha e^{\alpha(1-\alpha)s\Box}. \quad (14)$$

The third-order form factors M_i for $i=1\ldots4$ are functions of the dimensionless arguments $\xi_k=-s\Box_k$, $k=1\ldots3$ and are listed in Appendix A. They are formed with the basic form factors (14) and

$$F(-s\Box_1,-s\Box_2,-s\Box_3)=\int_{\alpha\geqslant0}d\alpha_1\,d\alpha_2\,d\alpha_3\,\delta(1-\alpha_1-\alpha_2-\alpha_3)\times\exp(s(\alpha_2\alpha_3\Box_1+\alpha_1\alpha_3\Box_2+\alpha_1\alpha_2\Box_3)). \quad (15)$$

The form factors in Eqs. (7) and (13) are analytical functions of the proper time, that can be exhibited, for example, by rewriting the following form factor:

$$\frac{1}{s(\Box_1-\Box_2)}(f(-s\Box_1)-f(-s\Box_2))=\int_0^1 d\alpha\,\alpha(1-\alpha)\int_0^1 d\beta\times\exp(s\alpha(1-\alpha)((1-\beta)\Box_1+\beta\Box_2)). \quad (16)$$

Local coefficients of the Schwinger-DeWitt expansion [18,19], which is often used in quantum field theory, can be easily obtained from the nonlocal expression (7) by the simple expansion of all form factors in powers of the proper time [24].

III. ONE-LOOP EFFECTIVE ACTION FOR SCALAR FIELDS COUPLED TO THE DILATON IN TWO DIMENSIONS

The trace of the heat kernel is a classical object, which, nevertheless, contains complete information about all quantum averages. For example, the trace anomaly in two dimensions is completely defined by the first Schwinger-DeWitt coefficient, $a_1(x)=\mathrm{tr}\,\hat{P}(x)$ [the potential term figures here in the form (10), rather than in the integrated form (12)]. This is a local expression, and any derivations of the one-loop effective action based just on the coefficient a_1 ignore complex conformally invariant, nonlocal structures of the heat kernel. Such methods work well in the case of pure 2D gravity and give the Polyakov effective action, but they fail in the case of dilaton gravity models. However, these procedures still might be valid for another field model [5,8] discussed in the closing section of this paper.

In two dimensions, even after subtracting the ultraviolet divergences, the resulting one-loop renormalized effective action is not generally defined because of bad behavior of the heat kernel trace in the large proper time limit (infrared divergence) [21]. However, in our model (2)–(4) we can control the infrared behavior using the asymptotic behavior of the form factors [24],

$$f(-s\Box)=-\frac{1}{s}\,\frac{2}{\Box}+\mathrm{O}\left(\frac{1}{s^2}\right),\quad s\to\infty \quad (17)$$

$$F(-s\Box_1,-s\Box_2,-s\Box_3)=\frac{1}{s^2}\left(\frac{1}{\Box_1\Box_2}+\frac{1}{\Box_1\Box_3}+\frac{1}{\Box_2\Box_3}\right)+\mathrm{O}\left(\frac{1}{s^3}\right),\quad s\to\infty, \quad (18)$$

and prove the infrared finiteness of W. Indeed, we find that

$$\frac{1}{s}\mathrm{Tr}\,K(s)=\mathrm{O}\left(\frac{1}{s^2}\right),\quad s\to\infty \quad (19)$$

and, hence, the proper time integral is convergent at the upper limit.

In order to compute the integral (5) we apply the technique of Ref. [24]. Let us reproduce here some differential equations that basic form factors of the nonlocal heat kernel satisfy

$$-s\frac{\Box_1\Box_2\Box_3}{D}F(-s\Box_1,-s\Box_2,-s\Box_3)=\frac{d}{ds}(sF(-s\Box_1,-s\Box_2,-s\Box_3))+\frac{\Box_1(\Box_3+\Box_2-\Box_1)}{2D}f(-s\Box_1)+\frac{\Box_2(\Box_1+\Box_3-\Box_2)}{2D}f(-s\Box_2)+\frac{\Box_3(\Box_1+\Box_2-\Box_3)}{2D}f(-s\Box_3), \quad (20)$$

$$\frac{f(-s\Box)-1}{s\Box}=\frac{d}{ds}\left(-\frac{2}{\Box}f(-s\Box)\right)+\frac{1}{2}f(-s\Box), \quad (21)$$

$$\frac{f(-s\Box)-1-\frac{1}{6}s\Box}{(s\Box)^2}=\frac{d}{ds}\left(-\frac{2}{3\Box}\frac{f(-s\Box)-1}{s\Box}-\frac{1}{3\Box}f(-s\Box)\right)+\frac{1}{12}f(-s\Box), \quad (22)$$

where D is the expression

$$D=\Box_1{}^2+\Box_2{}^2+\Box_3{}^2-2\Box_1\Box_2-2\Box_1\Box_3-2\Box_2\Box_3. \quad (23)$$

Applying these relations to the heat kernel (13) we can present some part of it in the form of a total derivative over the proper time:

$$\frac{1}{s}\mathrm{Tr}\,K(s)=\frac{1}{4\pi}\int dx\,g^{1/2}\left\{\frac{d}{ds}[f_1(s|\Box_2)R_1R_2+f_2(s|\Box_2)(\Box\phi_1-(\nabla\phi_1)^2)R_2+N_1(s|\Box_1,\Box_2,\Box_3)R_1R_2R_3+N_2(s|\Box_1,\Box_2,\Box_3)R_1R_2(\Box\phi_3)+N_3(s|\Box_1,\Box_2,\Box_3)R_1(\Box\phi_2)(\Box\phi_3)+N_4(s|\Box_1,\Box_2,\Box_3)(\Box\phi_1)(\Box\phi_2)\times(\Box\phi_3)]+\frac{1}{s^2}+(\Box\phi)\frac{1}{s\Box}(\Box\phi)+h(s|\Box_2)(\Box\phi_1)(\Box\phi_2)+H(s|\Box_1,\Box_2,\Box_3)R_1(\Box\phi_2)(\Box\phi_3)+\mathrm{O}[\mathfrak{R}^4]\right\}, \quad (24)$$

where

$$f_1(s|\Box_2)=\frac{1}{\Box_2}\left(\frac{1}{8}f(-s\Box_2)-\frac{1}{4}\frac{f(-s\Box_2)-1}{s\Box_2}\right), \quad (25)$$

$$f_2(s|\Box_2)=\frac{1}{\Box_2}f(-s\Box_2), \quad (26)$$

and

$$h(s|\Box_2)=\frac{1}{2}f(-s\Box_2), \quad (27)$$

$$H(s|\Box_1,\Box_2,\Box_3)=-\frac{1}{2\Box_1}\frac{1}{\Box_2-\Box_3}(\Box_2f(-s\Box_2)-\Box_3f(-s\Box_3)). \quad (28)$$

The third-order form factors $N_i(s|\Box_1,\Box_2,\Box_3)$ may be found in Appendix B. As can be seen from Eq. (24), not all of the terms in Tr $K(s)$ admit the form of a total derivative. As a result, in contrast to the Polyakov action [13,24], the effective action (5) in two dimensions depends on the ultraviolet cutoff parameter μ.

For the sake of convenience we split the total renormalized effective action into two parts, $W_{\rm ren}=W_{\rm fin}+W_\mu$. $W_{\rm fin}$ is a part defined by the total derivative terms of Eq. (24) while W_μ is defined by the rest (including any higher order terms). The calculation of the finite terms becomes trivial as we perform the proper time integration. Using Eqs. (20), (21) again we can check that the form factors f_i and N_i vanish at the upper limit, $s\to\infty$. Thus,

$$W_{\rm fin}=\frac{1}{8\pi}\int d^2x\,g^{1/2}\{f_1(s=0)R_1R_2+f_2(s=0)\times(\Box\phi_1-(\nabla\phi_1)^2)R_2+N_1^{\rm sym}(s=0)R_1R_2R_3+N_2^{\rm sym}(s=0)R_1R_2(\Box\phi_3)+N_3^{\rm sym}(s=0)R_1(\Box\phi_2)(\Box\phi_3)+N_4^{\rm sym}(s=0)(\Box\phi_1)(\Box\phi_2)(\Box\phi_3)\}, \quad (29)$$

where $N_i^{\rm sym}(i=1,\dots 4)$ are symmetrized in their arguments, $\Box_1,\Box_2,\Box_3$, according to the symmetries of the tensor structures they are acting on. All of the third-order ($\mathfrak{R}^3$) contributions to the heat kernel trace vanish, leaving the expression

$$W_{\rm fin}=\frac{1}{96\pi}\int d^2x\,g^{1/2}\left\{R\frac{1}{\Box}R-12(\nabla\phi)^2\frac{1}{\Box}R+12\phi R\right\}. \quad (30)$$

This part coincides with the usual form of the one-loop effective action derived by the integration of the trace anomaly [2,4,12,14].

To treat the remaining terms of the effective action (when the zeroth order term is already subtracted [26]) we apply the proper time cutoff regularization,

$$W_\mu = -\frac{1}{8\pi}\int d^2x\, g^{1/2}\int_{1/\mu^2}^{L} ds \times\left\{\frac{1}{s}\phi\Box\phi + h(s|\Box_2)(\Box\phi_1)(\Box\phi_2) + H(s|\Box_1,\Box_2,\Box_3)R_1(\Box\phi_2)(\Box\phi_3) + O[\mathfrak{R}^4]\right\}, \tag{31}$$

where infrared and ultraviolet cutoff parameters are introduced correspondingly at the upper and lower limits of the proper time integral in order to single out terms of this integral that are apparently divergent. However, the integral as a whole appeared to be infrared finite and independent of the parameter L, as will be demonstrated in a moment. As far as ultraviolet divergences are concerned, only the first, local, term of integral (31) is divergent when $\mu\to\infty$. Then, the key element of our computations is the integral:

$$\int_0^L ds f(-s\Box) = \int_0^1 d\alpha \frac{e^{\alpha(1-\alpha)L\Box}-1}{\alpha(1-\alpha)\Box} = -\frac{2}{\Box}(\ln(-L\Box)+\mathbf{C}), \tag{32}$$

where $\mathbf{C}$ is the Euler constant. The dependence on the infrared cutoff parameter L in first two terms of integral (31) cancels, as does the L dependence in the form factor (28) of the third term. The resulting expression reads

$$W_\mu = \frac{1}{8\pi}\int d^2x\, g^{1/2}\left\{(\Box\phi)\ln\left(-\frac{\Box}{\mu^2}\right) \times\phi - \frac{\ln(\Box_2/\Box_3)}{(\Box_2-\Box_3)}\frac{1}{\Box_1}R_1(\Box\phi_2)(\Box\phi_3) + O[\mathfrak{R}^4]\right\}. \tag{33}$$

It should be emphasized that the nonlocal form factors of Eq. (33) can be used for physical applications only when expressed in terms of the Green function, e.g., after converting them into the mass spectral integrals [21,22],

$$-\ln\left(-\frac{\Box}{\mu^2}\right) = \int_0^\infty dm^2\left(\frac{1}{m^2-\Box} - \frac{1}{m^2-\mu^2}\right), \tag{34}$$

$$-\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2} = \int_0^\infty dm^2 \frac{1}{m^2-\Box_1}\frac{1}{m^2-\Box_2}, \tag{35}$$

where $1/(m^2-\Box)$ is the massive Euclidean Green function with zero boundary conditions at spacetime infinity.

Combining the two pieces, $W_{\rm fin}$ and W_μ, we get the final result for the renormalized effective action

$$W_{\rm ren} = \frac{1}{96\pi}\int d^2x\, g^{1/2}\left\{R\frac{1}{\Box}R - 12(\nabla\phi)^2\frac{1}{\Box}R + 12\phi R + 12(\Box\phi)\ln\left(-\frac{\Box}{\mu^2}\right)\phi - 12\frac{\ln(\Box_2/\Box_3)}{(\Box_2-\Box_3)}\frac{1}{\Box_1} \times R_1(\Box\phi_2)(\Box\phi_3) + O[\mathfrak{R}^4]\right\}. \tag{36}$$

Equation (36) is one of the main results of this paper. This effective action is covariant by construction. It evidently reproduces the conformally anomalous part [4] and unambiguously fixes new conformally invariant terms that were not derived previously in the literature. It can be seen explicitly from Eq. (36) that, with an exception for the anomalous $R\phi$ term, terms of the first and third orders in the dilaton field are absent from the conformally invariant part of this one-loop effective action. This plausibly indicates that all higher-order terms are even in powers of the dilaton. This conjecture can be tested using the higher-order perturbation expansions for the heat kernel in flat space found in Refs. [27,28].

In agreement with Refs. [7,29], we see that the ϕ^2 terms of the effective action are nonlocal, and do not have the local form derived in [5,8]. We should stress that these terms are infrared finite. Only the ultraviolet regularization parameter μ enters the answer, and not the infrared cutoff as was suggested in [7].

The form of the last term in Eq. (36) is different from the one in Ref. [7] because the covariantization procedure of nonlocal terms used there is incorrect. To check our result we can perform the opposite operation, namely, to sum the series of the terms quadratic in the dilaton into a flat space object. To do so we take the expression (33) in a flat space assuming that the original metric is related to the flat space one $\overline{g}_{\mu\nu}$ via a conformal factor: $g_{\mu\nu} = e^{-2\sigma}\overline{g}_{\mu\nu}$. To return to the original metric we have to use the equation for the variation of the effective action form factor [30],

$$\int d^2x\, g^{1/2}\delta_\sigma\left(\ln\left(-\frac{\overline{\Box}_2}{\mu^2}\right)\right)\mathfrak{R}_1\mathfrak{R}_2 = \int d^2x\, g^{1/2}\frac{\ln(\Box_1/\Box_2)}{\Box_1-\Box_2}\delta_\sigma(\Box_2)\mathfrak{R}_1\mathfrak{R}_2 + O[\mathfrak{R}^3], \tag{37}$$

where $\overline{\Box}$ is defined for the flat-space metric $\overline{g}_{\mu\nu}$. Equation (37) follows directly from the rule of variation of the Euclidean Green function [18],

$$\delta_\sigma\frac{1}{m^2-\Box} = \frac{1}{m^2-\Box}\delta_\sigma(\Box)\frac{1}{m^2-\Box}, \tag{38}$$

and Eqs. (34), (35). We know that $\delta_\sigma(\Box) = -2(\delta\sigma)\Box$, where $\delta\sigma = \sigma$, and upon substituting the nonlocal curvature expression for the conformal factor, $\sigma(g) = -\frac{1}{2}(1/\Box)R$, into Eq. (37) one can see that it becomes nothing but the second term of Eq. (33). As a result, it is possible to rewrite the terms quadratic in the dilaton field in a flat spacetime form,

$$W_\mu = \frac{1}{8\pi}\int d^2x \left\{ (\bar{\Box}\phi)\ln\left(-\frac{\bar{\Box}}{\mu^2}\right)\phi + O[\phi^4] \right\}. \quad (39)$$

When expanded in powers of the curvatures, the effective action (39) again becomes an infinite series.

IV. PARTIAL SUMMATION OF THE DILATON EFFECTIVE ACTION

In the previous sections we have derived the one-loop effective action as a perturbation series in powers of the dilaton field ϕ. So far no explicit form of the dilaton field was assumed, and we could rewrite this expansion in terms of the potential $P(\phi)$. But instead of doing a perturbative expansion using the potential as a small parameter, we perform a partial summation of the effective action and obtain a result, which is nonperturbative in terms of P. To simplify calculations we work in a flat spacetime in the present and following sections. In other words, we perform all operations only with the conformal part of $W_{\rm ren}$, Eq. (39), and we can restore the whole covariant result, expressed in terms of R and P, at the end of the derivations. For the sake of convenience, we keep here the covariant notation $\Box$ instead of the flat-space one $\bar{\Box}$.

Let us begin with the equation for the potential P,

$$P = \Box\phi - (\nabla_\mu\phi)(\nabla^\mu\phi). \quad (40)$$

We rewrite this equation as a linear differential equation by substituting the following ansatz for the dilaton field [4]:

$$\phi = -\ln\Omega. \quad (41)$$

The solution of the resulting equation on Ω,

$$(\Box + P)\Omega = 0, \quad (42)$$

reads

$$\Omega = 1 - \frac{1}{\Box + P}P, \quad (43)$$

where the boundary condition $\Omega = 1$ at $|x| \to \infty$ is assumed. In principle, more general solutions containing zero modes, $\Box\Omega_0 = 0$, are allowed, but the requirement of covariant perturbation theory [21] that all background fields including the dilaton field (41) vanish at spacetime infinity puts $\Omega_0 = 1$.

The effective action, which is known up to the third order in the dilaton field, now can be written in a nonperturbative form by inserting Eqs. (41) and (43) into Eq. (39). The result reads

$$W_\mu = \frac{1}{8\pi}\int d^2x\, (\Box\phi)\ln\left(-\frac{\Box}{\mu^2}\right)\phi, \quad (44)$$

where

$$\phi(x) = -\ln\left(1 - \frac{1}{\Box + P}P\right). \quad (45)$$

Thus, we obtain a partially summed form of the one-loop effective action. This summation is partial, because not all higher-order terms are included in Eqs. (44), (45) but only those containing the form factor $\ln(-\Box/\mu^2)$. Such a summation with help of a new auxiliary scalar field, which is expressed in a nonlocal way through the perturbation (curvature) [31], is very similar to the summation of the Ricci scalar terms in the 4D covariant effective action performed in Ref. [32].

To reproduce the perturbation series we first obtain the expansion of Ω,

$$\Omega(x) = 1 - \frac{1}{\Box}P + \frac{1}{\Box}\left(P\frac{1}{\Box}P\right) + O[P^3], \quad (46)$$

which gives us an approximation for the dilaton

$$\phi(x) = \frac{1}{\Box}P + \frac{1}{2}\left(\frac{1}{\Box}P\right)\left(\frac{1}{\Box}P\right) - \frac{1}{\Box}\left(P\frac{1}{\Box}P\right) + O[P^3]. \quad (47)$$

This series obviously coincides with an iterative solution of Eq. (40). The perturbative expansion of Eq. (44) reads

$$\begin{aligned} W_\mu = \frac{1}{8\pi}\int d^2x \Bigg\{ & P\ln\left(-\frac{\Box}{\mu^2}\right)\left(\frac{1}{\Box}P\right) \\ & + \frac{1}{2}P\ln\left(-\frac{\Box}{\mu^2}\right)\left(\left(\frac{1}{\Box}P\right)\left(\frac{1}{\Box}P\right)\right) \\ & + \frac{1}{2}\Box\left(\left(\frac{1}{\Box}P\right)\left(\frac{1}{\Box}P\right)\right)\ln\left(-\frac{\Box}{\mu^2}\right)\left(\frac{1}{\Box}P\right) \\ & - \left(P\frac{1}{\Box}P\right)\ln\left(-\frac{\Box}{\mu^2}\right)\left(\frac{1}{\Box}P\right) \\ & - P\ln\left(-\frac{\Box}{\mu^2}\right)\frac{1}{\Box}\left(P\frac{1}{\Box}P\right) + O[P^4] \Bigg\}. \end{aligned} \quad (48)$$

The leading term of this expansion is apparently similar to the expression obtained in Ref. [7] with the reservation of a different meaning for the regularization parameter μ.

V. DILATON EFFECTIVE ACTION AT FINITE TEMPERATURE

An obvious use for the obtained one-loop effective action is its application to the calculation of the stress tensor. So far we have worked in a Euclidean spacetime. According to rules of covariant perturbation theory, one makes the transition to Minkowski spacetime only after deriving quantum averages and currents from the Euclidean effective action [20]. Similarly, in the calculation of the energy-momentum tensor for quantum fields in a black hole background boundary conditions corresponding to the Unruh, Boulware, or Hartle-Hawking vacua are to be specified after the variation over the metric. Authors of [6,9] completed this procedure by making the effective action local by introducing auxiliary fields and imposing the proper boundary conditions on them.

The other way to introduce boundary conditions corresponding to the Hartle-Hawking vacuum is to consider the field system at some fixed temperature $T=1/\beta$. This is relatively easy to do, because the Killing vector always exists in two dimensions, in contrast to higher dimensions [33–35]. Therefore, without losing generality, we can make a conformal transformation to a flat space where the Euclidean time is periodic, i.e., the flat spacetime has the topology of a cylinder. In our new flat space the anomaly-generating part of the effective action (30) vanishes, so we deal only with the conformally invariant part (33). This is what one would expect, because the anomalous part does not depend on temperature. In our treatment of the finite-temperature effective action for scalar fields coupled to the dilation we will follow the computational scheme of Refs. [34,35]. For general notions of finite temperature field theory we refer to [33], and references on some earlier works can be found in [35].

Let us start with the flat-space limit of the trace of the heat kernel (7),

$$\mathrm{Tr}\, K(s)=\frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^D x \times\left\{1+sP+s^2\frac{1}{2}Pf(-s\Box)P+\mathrm{O}[P^3]\right\}. \tag{49}$$

Here we restrict our consideration to terms of the second order, because it gives the first nonlocal contribution to the finite-temperature effective action W^β. Using the form (10) for the potential term we rewrite this heat kernel in terms of the dilaton field,

$$\mathrm{Tr}\, K(s)=\frac{1}{(4\pi s)^{D/2}}\int \mathrm{d}^D x\left\{1+s\phi\Box\phi +s^2\frac{1}{2}(\Box\phi)f(-s\Box)(\Box\phi)+\mathrm{O}[\phi^3]\right\}. \tag{50}$$

We are calculating W^β in a way similar to the zero temperature case (5),

$$W^\beta=-\frac{1}{2}\int_0^\infty\frac{\mathrm{d}s}{s}(\mathrm{Tr}\, K^\beta(s)-\mathrm{Tr}\, K(s)|_{\phi=0}), \tag{51}$$

where we subtract the zeroth-order term of the zero-temperature $\mathrm{Tr}\, K(s)$ from the heat kernel at some finite temperature $1/\beta$. It is well known [33] that one can express the heat kernel at finite temperature as an infinite sum of the zero temperature heat kernels at separated points x and x',

$$K^\beta(s|\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=\sum_{n=-\infty}^{\infty}K(s|\tau,\boldsymbol{x};\tau'+\beta n,\boldsymbol{x}'), \tag{52}$$

where τ is the Euclidean time and $\boldsymbol{x}$ are the spatial coordinates. Then, the two-dimensional $\mathrm{Tr}\, K^\beta$ can be found via the heat kernel in one dimension,

$$\mathrm{Tr}\, K^\beta(s)=\theta_3(0,\mathrm{e}^{-\beta^2/4s})\frac{\beta}{(4\pi s)^{1/2}}\int \mathrm{d}^1x\ \mathrm{tr}\, K^{(1)}(s|\boldsymbol{x},\boldsymbol{x}), \tag{53}$$

where we have introduced the Jacobi theta function,

$$\theta_3(a,b)\equiv\sum_{n=-\infty}^{n=\infty}\mathrm{e}^{2nai}b^{n^2}. \tag{54}$$

The zeroth-order term of the effective action requires special treatment. First of all, it is removed by the renormalization procedure in the zero temperature quantum field theory (QFT) [see Eq. (51)], which amounts to subtracting the $n=0$ term from the sum (52). Straightforward computation of the proper time integral (51) with the subsequent summation over n gives us a numerical coefficient in front of the $1/\beta$ term, which figures in the final expression (58) below.

The dilaton-dependent part of W_β can be computed after the Poisson re-summation,

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-(\beta^2/4s)n^2}=\frac{\sqrt{4\pi s}}{\beta}\sum_{k=-\infty}^{\infty}\mathrm{e}^{-(4\pi^2 s/\beta^2)k^2}. \tag{55}$$

The term $k=0$ of a new sum over k corresponds to the high temperature limit $T\to\infty$ $(\beta=0)$, and it is ultraviolet finite. The rest of the k-sum corresponds to a single term $n=0$ $(T=0)$ of the original n-sum, thus, it needs to be regularized. All of this follows from the fact that ultraviolet counterterms introduced in a field theory at zero temperature are sufficient for renormalization of the finite temperature field theory. Here we use the zeta function regularization,

$$W^\beta_{k\neq 0}=-\frac{1}{2}\frac{\partial}{\partial\epsilon}\left[\frac{\mu^{2\epsilon}}{\Gamma(\epsilon)}\int_0^\infty\frac{\mathrm{d}s}{s^{1-\epsilon}}\mathrm{Tr}\, K^\beta|_{k\neq 0}(s)\right]_{\epsilon=0}, \tag{56}$$

where ϵ is a small positive parameter and Γ is the gamma function.

On the other hand, the sum over k is infrared finite, with an exception for the $k=0$ term. Infrared divergences appearing in different orders of the perturbation theory in P are, in fact, artificial and disappear in the final result expressed in terms of ϕ, but we need to introduce an auxiliary mass to treat them at intermediate stages,

$$W^{\beta}_{k=0}=-\frac{1}{2}\int_0^{\infty}\frac{ds}{s}\,e^{-sm^2}(\mathrm{Tr}\,K^{\beta}(s)|_{k=0} - \mathrm{Tr}\,K(s)|_{\phi=0})|_{m^2=0}. \tag{57}$$

One can observe that the infrared poles in $W^{\beta}_{k=0}$, which come from the local P and nonlocal P^2 contributions, mutually cancel.

The final result reads

$$W^{\beta}_{\mathrm{ren}}=-\int dx\left\{\frac{\pi}{12\beta}+\frac{\beta}{8\pi}(\Box\phi)\left[2\ln\left(\frac{\beta\mu}{4\pi}\right) -\Psi\left(\frac{i\beta\sqrt{-\Box}}{4\pi}\right)-\Psi\left(-\frac{i\beta\sqrt{-\Box}}{4\pi}\right)\right]\phi+\mathrm{O}[\phi^3]\right\}, \tag{58}$$

where Ψ is the psi function. The obtained finite temperature effective action is infrared finite but depends on the ultraviolet regularization parameter μ. Even though arguments of the psi functions are imaginary, the combination of Ψ's in Eq. (58) is equivalent to the real part of Ψ. The thermal form factor found is a smooth function of the inverse temperature β, and its different asymptotics can be easily analyzed.

We emphasize that expression (58) is valid for arbitrary temperature, and high and low temperature asymptotics can be derived from it. The most popular expansion in thermal field theory is the high temperature one, $\beta\to 0$. We found that this expansion admits a local form,

$$\frac{W^{\beta}_{\mathrm{ren}}}{\beta}=-\int dx\left\{\frac{\pi}{12\beta^2}+\frac{1}{4\pi}\left[\ln\left(\frac{\beta\mu}{4\pi}\right)+\mathbf{C}\right]\phi\Box\phi +\frac{\zeta(3)}{64\pi^3}\beta(\Box\phi)(\Box\phi)+\frac{\zeta(5)}{1024\pi^5}\beta^3(\Box\phi)(\Box^2\phi) +\frac{\zeta(7)}{16384\pi^5}\beta^5(\Box\phi)(\Box^3\phi)+\mathrm{O}[\phi^3]+\mathrm{O}[\beta^8]\right\},$$

$$\beta\to 0. \tag{59}$$

This series has a form similar to high temperature expansions in four dimensions [33,35].

The other important limit is, of course, the low temperature asymptotic,

$$\frac{W^{\beta}_{\mathrm{ren}}}{\beta}=\int dx\left\{\frac{1}{8\pi}(\Box\phi)\ln\left(-\frac{\Box}{\mu^2}\right)\phi-\frac{1}{\beta^2}\frac{\pi}{12}(1+4\phi^2) -\frac{1}{\beta^4}\frac{8\pi^3}{15}\phi\left(\frac{1}{\Box}\phi\right)+\mathrm{O}[\phi^3]+\mathrm{O}[1/\beta^6]\right\},\quad \beta\to\infty. \tag{60}$$

Restoring the effective action to the original two-dimensional spacetime form one can see that the leading, temperature independent, term of this expansion is just W_{ren} at zero temperature (33).

In curved spacetime the total result will be the sum of the anomalous part of the effective action (30), and Eq. (58) with all flat quantities, metric and derivatives, being expressed in terms of physical (curved spacetime) ones. We are not aware of similar results in the literature to which one could make any comparisons.

VI. DISCUSSION

We have calculated the one-loop effective action for scalar fields interacting with a background dilaton field in curved spacetime. The main results are Eqs. (36) and (44) for the zero-temperature case and Eq. (58) for the finite-temperature one. Strictly speaking, these results are obtained for a generic dilaton field as long as the corresponding potential P has the form (10), and decreases sufficiently rapidly at infinity. These results are applicable to arbitrary two-dimensional spacetimes with the topology of either a disk or a cylinder, because in two dimensions there is always a Killing vector, and using nonsingular conformal transformations the problem can be reduced to one for a flat spacetime with the corresponding topology.

The difference between our approach and a similar attempt to use perturbation theory made in the interesting paper [7] is that we are able to control the infrared divergences that appear in two-dimensional calculations. We computed the one-loop effective action as an expansion in powers of the dilaton field rather than in the potential term of a differential operator, and proved the infrared finiteness of the effective action order by order. The resulting covariant nonlocal effective action does not depend on the infrared cutoff, and is proved to be valid up to the third order in the spacetime curvature and the dilaton. For one particular field model in a two-dimensional flat spacetime infrared finiteness of the one-loop two-point functions was demonstrated in Ref. [36]. Lombardo, Mazzitelli and Russo [7] found, using results of Ref. [36], the nonlocal form factor entering the conformally invariant part of the 2D effective action. Unfortunately, their generalization of this result to curved spacetime (covariantization) was not correct. The correct form of the covariantization procedure makes use of Eq. (37). By applying it to the terms quadratic in the dilaton we summed them up into a single flat-space term (39). Then, the dilaton field could be expressed in terms of the potential term, Eq. (44), and, if necessary, be expanded in powers of the potential and/or curvatures.

Equation (36) demonstrates that the effective action for scalars interacting with the dilaton does not admit the exact form proposed by Kummer and Vassilevich in Refs. [5,8], but contains additional conformally invariant terms that are nonlocal. Here we would like to explain the origin of this discrepancy. In order to calculate the one-loop effective action corresponding to the bosonic operator (4) they substituted the fermionic representation of the determinant of the operator

$$\hat{F}(\nabla)=\overline{g}^{\mu\nu}(\mathcal{D}_{\mu}\mathcal{D}_{\nu}+\hat{1}\,\partial_{\mu}\partial_{\nu}\phi), \tag{61}$$

in a flat spacetime for an original bosonic representation. Note that we work in the Euclidean signature and the derivative $\mathcal{D}_{\mu}$ is defined (for the case $\varphi=\psi=\phi$ in the notation of [5]) as

$$\mathcal{D}_{\mu}=\partial_{\mu}-\gamma^{5}\epsilon^{\nu}_{\mu}\partial_{\nu}\phi. \tag{62}$$

Here γ^5 belongs to the algebra of gamma matrices and has the property $(\gamma^5)^2=\hat{1}$. This is an absolutely legitimate procedure as long as one deals with formal definitions of the operator determinants in flat spacetime (3). But the renormalization procedure generically breaks the identity of these two representations, hence, the corresponding renormalized effective actions do not coincide. To check this fact by an explicit calculation, one can note that the operator (61) also belongs to the class of operators of the type (8). Therefore, one can directly apply the covariant perturbation theory to this model. Let us write down the corresponding potential term,

$$\hat{P}=\hat{1}\,\overline{\Box}\,\phi, \tag{63}$$

and the commutator curvature,

$$\hat{\mathcal{R}}_{\mu\nu}=\gamma^{5}(\epsilon_{\mu}{}^{\alpha}\partial_{\nu}\partial_{\alpha}\phi-\epsilon_{\nu}{}^{\alpha}\partial_{\mu}\partial_{\alpha}\phi). \tag{64}$$

We performed the computation of the one-loop effective action along the lines of the calculational scheme of Secs. II and III and using the tables of form factors of Ref. [23]. We found that the terms of third order in ϕ, which in this case is equivalent to order $\mathfrak{R}^3$, vanishes, and the ϕ^2 terms have no infrared singularity. The final answer for the effective action corresponding to the operator (61) is local and finite:

$$W_{\rm ren}=-\frac{1}{8\pi}\int d^{2}x\,\phi\overline{\Box}\phi+O[\phi^{4}]. \tag{65}$$

Indeed, this expression is in good agreement with Eq. 42 of Ref. [5] (the opposite sign is attributed to the Euclidean spacetime signature). However, as one can see, it is completely different from the effective action for the original bosonic model found in Sec. III above. Technically, this stems from the fact that in order to remove the ultraviolet divergences one should subtract the first two terms of the Schwinger-DeWitt expansion of the heat kernels. But the Schwinger-DeWitt coefficients for the operators in question are absolutely different, because their heat kernels are different. Nevertheless, the remarkable result of Ref. [5] is the exact expression for the effective action for the spinor model (61). This two-dimensional one-loop effective action is local, infrared finite, and does not depend on a regularization parameter. In principle, we could start with a covariant version of the differential operator (61) and derive the covariant effective action, but it not necessary because all conformally invariant terms in that effective action can be obtained from Eq. (65) by covariantization, while the anomalous part is proved to be exact [5].

A few remarks are in order about the applicability of the perturbative expansion to physically interesting cases. The covariant perturbation theory works well when derivatives of the background fields are much bigger than powers of these fields [21],

$$\nabla\nabla\mathfrak{R}\gg\mathfrak{R}^{2}. \tag{66}$$

Since the anomalous part (30) is exact, we are concerned only about the conformal terms, (33) or (39). For instance, if one wants to use the results above to study black hole physics and consider the 2D dilaton gravity inspired by the spherical reduction from four dimensions, then the dilaton field is defined as $\phi=-\ln r$ (where $4\pi r^2$ is the area of a surface of the constant radius r). At infinity the 2D Schwarzschild metric is asymptotically flat, and the 2D curvature and potential obviously satisfy the condition (66). As for the vicinity of the horizon, note that W_{μ} in Eq. (39) is expressed in terms of the flat spacetime operator $\overline{\Box}$. This means that all quantities in Eq. (66) should also be defined in the flat metric $\overline{g}_{\mu\nu}$. One can check that the condition $\overline{\nabla\nabla}P \gg P^2$ (in the ''tortoise'' coordinates) is also satisfied near the horizon. This means that our results should be valid both at infinity and at the horizon of the 2D dilaton black hole.

Kummer and Vassilevich [2,5,8] considered the case of a more general differential operator, with the dilaton coupling being defined in terms of two arbitrary functions of the dilaton, $\varphi(\phi)$ and $\psi(\phi)$. The results of this paper are obtained for the simplest case $\varphi(\phi)=\psi(\phi)=\phi$. Strong reasons to study more general models are discussed in Ref. [8]; here, we just would like to note that such nontrivial dilaton couplings can be incorporated into our computational method. This is possible to do, because such models correspond to the minimal second-order operator when expressed in terms of new redefined metric and covariant derivative [2], but we leave this generalization for some other publication.

In conclusion, we would like to make some general comments on the validity of the anomaly-induced effective actions. Such effective actions are very attractive, because they are relatively easy to derive, and they have rather simple structures. However, as was shown above, the anomaly-induced effective actions are incomplete even for simple two-dimensional field models. This is also true for four dimensions, and an interesting recent work [37] would be a good illustration. The authors applied the nonlocal effective action obtained by integration of the four-dimensional trace anomaly (the generalized Riegert action) to the study of the Hawking radiation. Indeed, this action is nonlocal, therefore, it captures some essential features of the energy-momentum tensor in curved spacetime. Nevertheless, as correctly noted in [37], further applications of the Riegert action encounter serious obstacles, because it is ill-defined at infinity [38]. The four-dimensional nonlocal effective action of Barvinsky and Vilkovisky [21,23] is the only known action with the right properties [38]. There are many ways to split this effective action into anomalous and conformal parts [39], and only one of them gives the Riegert action. In order to obtain physically consistent results we should always use the entire effective action, not just its anomaly-generating part.

ACKNOWLEDGMENTS

We thank V. Frolov and S. Solodukhin for stimulating discussions, and P. Sutton for reading the manuscript. This work was supported by Natural Sciences and Engineering Research Council of Canada. A. Z. is grateful to the Killam Trust for its financial support.

APPENDIX A: THIRD-ORDER FORM FACTORS FOR THE TRACE OF THE HEAT KERNEL

Below we display form factors of the third order in Eq. (13) of Sec. II. They are expressed in terms of dimensionless arguments $\xi_i = -s\Box_i$, $i=1,\dots,3$, and the denominator,

$$\Delta = \xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3. \tag{A1}$$

$$\begin{aligned} M_1(\xi_1,\xi_2,\xi_3) = & -F(\xi_1,\xi_2,\xi_3)\frac{\xi_1^2\xi_2^2\xi_3^2}{3\Delta^3} - f(\xi_1)\frac{1}{32\Delta^3\xi_2}(\xi_1{}^6 - 4\xi_1{}^5\xi_2 - 4\xi_1{}^5\xi_3 + 3\xi_1{}^4\xi_2\xi_3 + 24\xi_1{}^3\xi_2{}^2\xi_3 + 5\xi_1{}^4\xi_3{}^2 \\ & + 24\xi_1{}^3\xi_2\xi_3{}^2 - 2\xi_1{}^2\xi_2{}^2\xi_3{}^2 + 32\xi_1\xi_2{}^3\xi_3{}^2 - 25\xi_1{}^2\xi_2\xi_3{}^3 - 36\xi_1\xi_2{}^2\xi_3{}^3 + 5\xi_2{}^3\xi_3{}^3 - 5\xi_1{}^2\xi_3{}^4 - 9\xi_2{}^2\xi_3{}^4 + 4\xi_1\xi_3{}^5 \\ & + 5\xi_2\xi_3{}^5 - \xi_3{}^6) - \left(\frac{f(\xi_1)-1}{\xi_1}\right)\frac{1}{8\Delta^2\xi_2}(\xi_1{}^4 - 2\xi_1{}^3\xi_3 - 12\xi_1{}^2\xi_2\xi_3 - 10\xi_1\xi_2{}^2\xi_3 + 8\xi_1\xi_2\xi_3{}^2 - 2\xi_2{}^2\xi_3{}^2 + 2\xi_1\xi_3{}^3 \\ & + 3\xi_2\xi_3{}^3 - \xi_3{}^4) - \left(\frac{f(\xi_1)-1-\frac{1}{6}\xi_1}{(\xi_1)^2}\right)\frac{3}{8D\xi_2}(\xi_1{}^2 + 4\xi_1\xi_2 + \xi_2\xi_3 - \xi_3{}^2) + \frac{1}{\xi_2-\xi_3}\,\frac{\xi_2}{32\xi_1}(f(\xi_2)-f(\xi_3)) \\ & + \frac{1}{\xi_2-\xi_3}\,\frac{\xi_2}{8\xi_1}\left(\frac{f(\xi_2)-1}{\xi_2} - \frac{f(\xi_3)-1}{\xi_3}\right) + \frac{1}{\xi_2-\xi_3}\,\frac{3\xi_2}{8\xi_1}\left(\frac{f(\xi_2)-1-\frac{1}{6}\xi_2}{(\xi_2)^2} - \frac{f(\xi_3)-1-\frac{1}{6}\xi_3}{(\xi_3)^2}\right), \end{aligned} \tag{A2}$$

$$\begin{aligned} M_2(\xi_1,\xi_2,\xi_3) = & F(\xi_1,\xi_2,\xi_3)\frac{\xi_1\xi_2\xi_3{}^2}{D^2} - f(\xi_1)\frac{1}{12D^2\xi_2}(2\xi_1{}^4 - 5\xi_1{}^3\xi_3 + 3\xi_1{}^2\xi_2{}^2 + 3\xi_1{}^2\xi_3{}^2 + \xi_1\xi_3{}^3 + 11\xi_1\xi_2\xi_3{}^2 + 11\xi_1\xi_2{}^2\xi_3 \\ & + \xi_1\xi_2{}^3 - 6\xi_2{}^2\xi_3{}^2 + 4\xi_2{}^3\xi_3 + 4\xi_2\xi_3{}^3 - \xi_2{}^4 - \xi_3{}^4) + f(\xi_3)\frac{1}{12D^2\xi_2}(-\xi_3{}^4 + \xi_1\xi_3{}^3 + 3\xi_1{}^2\xi_3{}^2 - 5\xi_1{}^3\xi_3 \\ & + 2\xi 1^4 + \xi_2\xi_3{}^3 + 17\xi_1\xi_2\xi_3{}^2 - 12\xi_1{}^2\xi_2\xi_3 - 6\xi_1{}^3\xi_2 + 17\xi_1\xi_2{}^2\xi_3 + 4\xi_1{}^2\xi_2{}^2) \\ & - \frac{f(\xi_1)-1}{\xi_1}\,\frac{\xi_1}{2D\xi_2}(\xi_1 - \xi_2 - \xi_3) + \frac{f(\xi_3)-1}{\xi_3}\,\frac{1}{2D\xi_2}(\xi_1{}^2 - \xi_1\xi_3 - 3\xi_2\xi_3 - \xi_1\xi_2) \\ & + \frac{1}{\xi_2-\xi_3}\,\frac{2\xi_2+\xi_3}{12\xi_1}(f(\xi_2)-f(\xi_3)) + \frac{1}{\xi_2-\xi_3}\,\frac{\xi_2}{2\xi_1}\left(\frac{f(\xi_2)-1}{\xi_2} - \frac{f(\xi_3)-1}{\xi_3}\right), \end{aligned} \tag{A3}$$

$$\begin{aligned} M_3(\xi_1,\xi_2,\xi_3) = & -F(\xi_1,\xi_2,\xi_3)\frac{\xi_2\xi_3}{\Delta} + f(\xi_1)\frac{1}{2D}(2\xi_3 - \xi_1) - f(\xi_2)\frac{1}{2\xi_1 D}(2\xi_1\xi_3 + \xi_2{}^2 - \xi_3{}^2 - \xi_1{}^2) \\ & + \frac{1}{\xi_2-\xi_3}\,\frac{\xi_2}{2\xi_1}(f(\xi_2)-f(\xi_3)), \end{aligned} \tag{A4}$$

$$M_4(\xi_1,\xi_2,\xi_3) = \frac{1}{3}F(\xi_1,\xi_2,\xi_3) - f(\xi_1)\left(\frac{1}{\xi_3} - \frac{\xi_1}{2\xi_2\xi_3}\right). \tag{A5}$$

APPENDIX B: THIRD-ORDER FORM FACTORS FOR COMPUTATION OF THE EFFECTIVE ACTION

Here is the list of third-order form factors in Eq. (24) of Sec. III with D defined by Eq. (23):

$$\begin{aligned} N_1(s|\Box_1,\Box_2,\Box_3) &= sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{\Box_1\Box_2\Box_3}{3D^2} + f(-s\Box_1)\frac{1}{8D^2\Box_1\Box_2}(\Box_1{}^4 - 2\Box_1{}^3\Box_3 + 2\Box_1\Box_3{}^3 - \Box_3{}^4 \\ &\quad - 2\Box_1{}^3\Box_2 + 3\Box_2\Box_3{}^3 - 8\Box_1{}^2\Box_2\Box_3 + 8\Box_1\Box_2\Box_3{}^2 - 10\Box_1\Box_2{}^2\Box_3 - 2\Box_2{}^2\Box_3{}^2) \\ &\quad - \left(\frac{f(-s\Box_1)-1}{s\Box_1}\right)\frac{1}{4D\Box_1\Box_2}(\Box_1{}^2 + 4\Box_1\Box_2 + \Box_2\Box_3 - \Box_3{}^2) + \frac{1}{\Box_2 - \Box_3}\frac{\Box_2}{\Box_1} \\ &\quad \times\left[-\frac{1}{8}\left(\frac{1}{\Box_2}f(-s\Box_2) - \frac{1}{\Box_3}f(-s\Box_3)\right) + \frac{1}{4}\left(\frac{1}{\Box_2}\frac{f(-s\Box_2)-1}{s\Box_2} - \frac{1}{\Box_3}\frac{f(-s\Box_3)-1}{s\Box_3}\right)\right], \end{aligned} \tag{B1}$$

$$\begin{aligned} N_2(s|\Box_1,\Box_2,\Box_3) &= -sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{\Box_3}{D} + \frac{\Box_1 - \Box_2 - \Box_3}{D\Box_2}f(-s\Box_1) \\ &\quad + \frac{\Box_1\Box_3 - \Box_1{}^2 + 3\Box_2\Box_3 + \Box_1\Box_2}{D\Box_2\Box_3}f(-s\Box_3) - \frac{1}{\Box_1 - \Box_3}\frac{\Box_1}{\Box_2}\left(\frac{1}{\Box_1}f(-s\Box_1) - \frac{1}{\Box_3}f(-s\Box_3)\right), \end{aligned} \tag{B2}$$

$$N_3(s|\Box_1,\Box_2,\Box_3) = sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{1}{\Box_1}, \tag{B3}$$

$$N_4(s|\Box_1,\Box_2,\Box_3) = -sF(-s\Box_1,-s\Box_2,-s\Box_3)\frac{D}{\Box_1\Box_2\Box_3}. \tag{B4}$$

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[1] J. G. Russo, L. Susskind, and L. Thorlacius, Phys. Lett. B **292**, 13 (1992).
[2] W. Kummer, H. Liebl, and D. V. Vassilevich, Mod. Phys. Lett. A **12**, 2683 (1997).
[3] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, R1005 (1992).
[4] V. Mukhanov, A. Wipf, and A. Zelnikov, Phys. Lett. B **332**, 283 (1994).
[5] W. Kummer and D. V. Vassilevich, Phys. Rev. D **60**, 084021 (1999).
[6] R. Balbinot and A. Fabbri, Phys. Rev. D **59**, 044031 (1999).
[7] F. C. Lombardo, F. D. Mazzitelli, and J. S. Russo, Phys. Rev. D **59**, 064007 (1999).
[8] W. Kummer and D. V. Vassilevich, Ann. Phys. (Leipzig) **8**, 801 (1999).
[9] R. Balbinot and A. Fabbri, Phys. Lett. B **458**, 112 (1999).
[10] S. Nojiri and S. D. Odintsov, Phys. Lett. B **463**, 57 (1999).
[11] V. P. Frolov, P. Sutton, and A. I. Zelnikov, Phys. Rev. D **61**, 024021 (2000).
[12] J. S. Dowker, Class. Quantum Grav. **15**, 1881 (1998).
[13] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981).
[14] T. Chiba and M. Siino, Mod. Phys. Lett. A **12**, 709 (1997).
[15] R. Bousso and S. Hawking, Phys. Rev. D **56**, 7788 (1997).
[16] S. Nojiri and S. D. Odintsov, Mod. Phys. Lett. A **12**, 2083 (1997).
[17] J. S. Dowker, Class. Quantum Grav. **11**, L7 (1994).
[18] B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[19] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. **119**, 1 (1985).
[20] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B282**, 163 (1987).
[21] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B333**, 471 (1990).
[22] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B333**, 512 (1990).
[23] A. O. Barvinsky, Yu. V. Gusev, G. A. Vilkovisky, and V. V. Zhytnikov, Report of the University of Manitoba (1993).
[24] A. O. Barvinsky, Yu. V. Gusev, G. A. Vilkovisky, and V. V. Zhytnikov, J. Math. Phys. **35**, 3543 (1994).
[25] A. O. Barvinsky, Yu. V. Gusev, G. A. Vilkovisky, and V. V. Zhytnikov, J. Math. Phys. **35**, 3525 (1994).
[26] E. S. Fradkin and G. A. Vilkovisky, Lett. Nuovo Cimento Soc. Ital. Fis. **19**, 47 (1977).
[27] S. F. Wilk, Y. Fujiwara, and T. A. Osborn, Phys. Rev. A **24**, 2187 (1981).
[28] P. Haberl, D. Fliegner, M. G. Schmidt, and C. Schubert, in *Heat Kernel Techniques and Quantum Gravity*, Discourses in Mathematics and Its Applications, No. 4, edited by S. A. Fulling (Texas A&M University, College Station, TX, 1995), p. 87.

[29] S. Nojiri and S. D. Odintsov, Phys. Rev. D **59**, 044003 (1999).
[30] A. O. Barvinsky, Yu. V. Gusev, G. A. Vilkovisky, and V. V. Zhytnikov, Nucl. Phys. **B439**, 561 (1995).
[31] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. **73B**, 209 (1978).
[32] A. G. Mirzabekian, G. A. Vilkovisky, and V. V. Zhytnikov, Phys. Lett. B **369**, 215 (1996).
[33] J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).
[34] Yu. V. Gusev and A. I. Zelnikov, Class. Quantum Grav. **15**, L13 (1998).
[35] Yu. V. Gusev and A. I. Zelnikov, Phys. Rev. D **59**, 024002 (1999).
[36] H. Balasin, W. Kummer, O. Piguet, and M. Schweda, Phys. Lett. B **287**, 138 (1992).
[37] R. Balbinot, A. Fabbri, and I. Shapiro, Phys. Rev. Lett. **83**, 1494 (1999).
[38] J. Erdmenger and H. Osborn, Nucl. Phys. **B483**, 431 (1997).
[39] A. O. Barvinsky, A. G. Mirzabekian, and V. V. Zhytnikov, in *Quantum Gravity: Proceedings of the Sixth Moscow Quantum Gravity Seminar*, edited by V. Berezin (World Scientific, Singapore, 1997), gr-qc/9510037.