Holographic formulation of quantum general relativity

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We show that there is a sector of quantum general relativity, in the Lorentzian signature case, which may be expressed in a completely holographic formulation in terms of states and operators defined on a finite boundary. The space of boundary states is built out of the conformal blocks of $SU(2)_L \oplus SU(2)_R$, WZW field theory on the n-punctured sphere, where n is related to the area of the boundary. The Bekenstein bound is explicitly satisfied. These results are based on a new Lagrangian and Hamiltonian formulation of general relativity based on a constrained $Sp(4)$ topological field theory. The Hamiltonian formalism is polynomial, and also left-right symmetric. The quantization uses balanced $SU(2)_L \oplus SU(2)_R$ spin networks and so justifies the state sum model of Barrett and Crane. By extending the formalism to $Osp(4/N)$ a holographic formulation of extended supergravity is obtained, as will be described in detail in a subsequent paper.

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I. INTRODUCTION

There has been recently much interest in holographic formulations of theories of quantum gravity.[1] In addition to the original argument based on the Bekenstein bound of 't Hooft [1] and Susskind [2], there is also a very interesting argument based on results of topological quantum field theory advocated by Crane [3] and others [4–6] that suggests that quantum cosmological theories should be based on states and observables living on boundaries inside the universe. These two arguments reinforce each other in an interesting way: the Bekenstein bound [13] tells us that there should be a finite amount of information per unit area of the boundary while topological quantum field theories provides a large class of quantum field theories with finite dimensional state spaces associated to boundaries.

For these reasons, several years ago a holographic formulation of quantum general relativity was presented [7]. The theory was holographic in that the physical state space had the explicit form

$$\mathcal{H}_{\mathcal{B}} = \sum_a \mathcal{H}_a, \tag{1}$$

where a are the eigenvalues of the area operator $\hat{A}$ (which is known by both construction [8–10] and general theorems [11] to have a discrete spectra). The eigenspaces of definite area were constructed explicitly in terms of the conformal blocks of $SU(2)_q$ Wess-Zumino-Witten (WZW) conformal field theory on the punctured two sphere. More explicitly, the areas are expressed in terms of a sums of total quantum spins j_i associated with the punctures, so that in the large k limit [12]

$$a(j_i) = \sum_i l_{Pl}^2 \sqrt{j_i(j_i+1)}, \tag{2}$$

$$\mathcal{H}_{a(j_i)} = \mathcal{V}_{j_i}, \tag{3}$$

where $\mathcal{V}_{j_i}$ is the space of conformal blocks (or intertwiners) on the punctured two sphere.

It then follows from the formula for the dimension of these spaces that the Bekenstein bound [13] is satisfied, so that [7]

$$\dim(\mathcal{H}_A) \leqslant e^{c/4G_B \hbar}, \tag{4}$$

where $c = \sqrt{3}/\ln(2)$ in quantum general relativity and G_B is the "bare" Newton's constant. Thus, this result implies that the macroscopic Newton's constant, which is not so far predicted by the theory, should be $G = G_B/c$.

Finally a complete set of boundary observables based on the gravitational fields at the boundary exists that is both sufficient to make complete measurements of the physical state and expressed explicitly in terms of operators in the conformal field theory [7].

Another property of this formulation is that the bulk state which describes the physics in the interior of the boundary is the Chern-Simons state of Kodama [14], which is known to have a semiclassical interpretation in terms of de Sitter or anti–de Sitter spacetime [14,15].

These results show that, at least for quantum general relativity, completely holographic formulations exist.

Given the recent interest in holographic formulations of $\mathcal{M}$ theory [16–20], it is then very natural to try to extend these results to $\mathcal{N}=8$ supersymmetry, to provide a candidate for a completely background independent formulation of $\mathcal{M}$ theory. This goal was the impetus of the present work. However, in order to accomplish the supersymmetric extension, certain issues had to be addressed, which led to a new formulation of general relativity at both the classical and quantum level. As these may be of independent interest, they are presented here. A subsequent paper will presents an extension of the present results to theories with extended supersymmetry, some of which may be candidate for such a formulation of $\mathcal{M}$ theory, in a (3+1)-dimensional compactification [21].

The new formulation presented here is related to the Ashtekar formulation [22–24], but differs from it in that it is

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[1] Thanks are due to Yi Ling for discussions during the course of joint work on the supersymmetric extension, to appear [38].

entirely left-right symmetric. Both self-dual and antiself-dual fields are kept in the theory, although they are in the end related to each other through constraints that play the role of the reality conditions. At the same time, the formulation is entirely polynomial.[2]

This formulation has several features that are of interest for holographic formulations of the theory. First, because the reality conditions are part of the algebra of constraints, the Lorentzian form of the theory is more easily studied. Second, the extension to the supersymmetric case is somewhat easier, as will be seen in the subsequent paper. Third, it allows a more transparent treatment of the splitting between kinematical and dynamical constraints, in both the bulk and boundary theories.

This last point is the most important and is worth elaborating on. The basic idea of the formalism is that general relativity is expressed as a constrained topological field theory, for the group $G=Sp(4)$. This group, which double covers the anti–de Sitter group contains $H=SU(2)_L \oplus SU(2)_R$ as a subgroup. What is meant by a constrained topological field theory is that all derivative terms, and hence the structure of the canonical theory, is the same as a topological field theory with no local degrees of freedom. The local degrees of freedom arise because of the imposition of local, nonderivative constraints, which reduce the explicit gauge symmetry from G to the subgroup H. The fields in the coset G/H become the gravitational degrees of freedom, further, the constraints in the coset become nonlinear and in fact become the generators of spacetime diffeomorphisms.

What is interesting is that to extend to the case of $\mathcal{N}$-supersymmetry, all that is needed is to extend the structure just described so that $G=Osp(4/\mathcal{N})$ and the subgroup H is some supersymmetric extension of $SU(2)_L \oplus SU(2)_R$, with at most half the supersymmetry generators of G. This, and several related ideas, are discussed in [21]. In this paper we describe the classical and quantum physics of the nonsupersymmetric theory.

II. GENERAL RELATIVITY AS A CONSTRAINED TQFT

In this section we introduce new way of writing general relativity as a constrained topological quantum field theory, (TQFT), which we call the ambidextrous formalism.[3] For the nonsupersymmetric case we study here, the theory is based on a connection valued in the Lie algebra $G=Sp(4)$ [which double covers $SO(3,2)$ the anti–de Sitter group]. Thus, this approach is similar to that of MacDowell and Mansouri, in which general relativity is found as a consequence of breaking the $SO(3,2)$ symmetry of a topological quantum field theory down to $SO(3,1)$ [29]. However it differs from that approach in that the beginning point is a $\int B\wedge F$ theory.

The $Sp(4)$ connection is written $A_{\alpha\beta}$ where the four dimensional indices $\alpha,\beta,=1,\ldots,4$ will be often broken down into a pair $\alpha=(A,A')=(0,1,0',1')$ of $SU(2)$ indices expressing the fact that $SU(2)_L \oplus SU(2)_R \subset Sp(4)$. Thus, the connection is

$$A_{\alpha\beta}=\{A_{AB},A_{A'B'},A_{AA'}\}. \tag{5}$$

The components of the connection $A_{AA'}$ which parametrize the coset $Sp(4)/SU(2)_L \oplus SU(2)_R$ will be taken to represent the frame fields $e_{AA'}$ so we will take

$$A_{AA'}=\frac{1}{l}e_{AA'}, \tag{6}$$

where l has dimensions of length.

We take for our starting point a modification of the $Sp(4)$ $B\wedge F$ theory. This is given by

$$I^0=\imath\int_{\mathcal{M}}\frac{1}{g^2}(B_\alpha^{\ \beta}\wedge F_\beta^{\ \gamma}\gamma_{5\gamma}^{\ \alpha})-\frac{e^2}{2}(B_\alpha^{\ \beta}\wedge B_\beta^{\ \gamma}\gamma_{5\gamma}^{\ \alpha})$$
$$+\frac{-\imath k}{4\pi}\int_{\partial\mathcal{M}}(Y_{CS}(A_{AB})-Y_{CS}(A_{A'B'})), \tag{7}$$

where $B_\alpha^{\ \beta}$ is a two form valued in the adjoint representation of $Sp(4)$, Y_{CS} is the $SU(2)$ Chern-Simons action, g and e are dimensionless coupling constants and k is as usual an integer. The variational principle given by Eq. (7) is well defined only in the presence of certain boundary conditions, which are the subject of the next section.

γ_5 is given by

$$\gamma_{5\alpha}^{\beta}=\begin{pmatrix}\delta_A^B & 0\\ 0 & -\delta_{A'}^{B'}\end{pmatrix}. \tag{8}$$

The inclusion of the γ_5 is necessary if we want the action to be parity invariant.[4] However, its presence breaks the $SP(4)$ invariance down to $SU(2)_L \oplus SU(2)_R$. To see this we expand to find

[2]Another way of modifying the Ashtekar formalism that uses two connections is given in [25].

[3]We may note that there is more than one way to represent general relativity with a cosmological constant as a constrained topological quantum field theory. The earliest such approach to the authors knowledge is that of Plebanski [26], studied also in [27]. Alternatively, one can deform a topological field theory of the form of $\int \mathrm{Tr}\, F\wedge F$, as described in [28] (see also [8]). What is new in the present presentation is the representation of general relativity as a constrained topological field theory for the de Sitter group $SO(3,2)$. For reasons that will be apparent soon, the present formulation is more suited both to the Lorentzian regime and to the theory with vanishing cosmological constant.

[4]Note that this is not required to reproduce classical general relativity, as this can be done with parity asymmetric actions [23]. However, we insist on it here as we want to develop a form of the quantum theory which is explicitly parity invariant. It is interesting to note that Eq. (7) remains an action for general relativity if the γ_5 is replaced by δ_α^β. In this case the action is chiral but the $SP(4)$ gauge symmetry is broken only by the constraints.

$$I^0 = -\imath \int_{\mathcal{M}} \frac{1}{g^2}(B_{AB} \wedge F^{AB} - B_{A'B'} \wedge F^{A'B'}) - \frac{e^2}{2}(B_{AB} \wedge B^{AB} - B_{A'B'} \wedge B^{A'B'}) + \frac{\imath k}{4\pi}\int_{\partial \mathcal{M}} (Y_{CS}(A_{AB}) - Y_{CS}(A_{A'B'})). \tag{9}$$

Thus we see that we have $-\imath$ times the difference between the actions for the $\int B \wedge F$ theories for $SU(2)_L$ and $SU(2)_R$. The mixed components $B^{AA'}$ have disappeared from the theory. The reason for preferring this choice will be clear shortly.

We now impose two constraints that set the $SU(2)_L \oplus SU(2)_R$ components of $B^{\alpha\beta}$ to be equal to the self-dual and anti-self-dual two forms constructed from $e^{AA'}$. With constraints that do this the action has the form

$$I^1 = -\imath \int_{\mathcal{M}} \frac{1}{g^2}(B^{AB} \wedge F_{AB} - B^{A'B'} \wedge F_{A'B'}) - \frac{e^2}{2}(B_{AB} \wedge B^{AB} - B_{A'B'} \wedge B^{A'B'}) + \lambda_{AB} \wedge \left(\frac{1}{l^2} e^{AA'} \wedge e^{B}_{\ A'} - B^{AB}\right) + \lambda_{A'B'} \wedge \left(\frac{1}{l^2} e^{A'A} \wedge e^{B'}_{\ A} - B^{A'B'}\right) + \frac{\imath k}{4\pi}\int_{\partial \mathcal{M}} (Y_{CS}(A_{AB}) - Y_{CS}(A_{A'B'})). \tag{10}$$

It is not hard to show that the equations of motion of this action reproduce those of general relativity with a cosmological constant. To see this we note the forms of the $Sp(4)$ curvatures,

$$F^{AB} = f^{AB} + \frac{1}{l^2} e^{AA'} \wedge e^{B}_{\ A'}, \tag{11}$$

$$F^{AA'} = \mathcal{D} e^{AA'}, \tag{12}$$

where f^{AB} is the $SU(2)_L$ curvature of the connection A_{AB} and $\mathcal{D}$ is the $SU(2)_L \oplus SU(2)_R$ covariant derivative. $F^{AA'}$ is the torsion. [The definition of $F^{A'B'}$ is the same as Eq. (11) with primed indices.] The λ_{AB} and $\lambda_{A'B'}$ field equations set

$$B^{AB} = \frac{1}{l^2} e^{AA'} \wedge e^{B}_{\ A'} \equiv \frac{1}{l^2} \Sigma^{AB}, \tag{13}$$

$$B^{A'B'} = \frac{1}{l^2} e^{A'A} \wedge e^{B'}_{\ A} \equiv \frac{1}{l^2} \Sigma^{A'B'}. \tag{14}$$

Putting the solutions to these field equations back into the action, we find

$$I^1 = -\imath \int_{\mathcal{M}} \frac{1}{G}(e^{AA'} \wedge e^{B}_{\ A'} \wedge f_{AB} - e^{A'A} \wedge e^{B'}_{\ A} \wedge f_{A'B'}) + \Lambda e^{AA'} \wedge e^{B}_{\ A'} \wedge e_{AB'} \wedge e^{B'}_{B} + \frac{\imath k}{4\pi}\int_{\partial \mathcal{M}} (Y_{CS}(A_{AB}) - Y_{CS}(A_{A'B'})), \tag{15}$$

where

$$G = g^2 l^2 \tag{16}$$

and

$$\Lambda = \frac{2}{l^4}\left(\frac{1}{g^2} - \frac{e^2}{2}\right). \tag{17}$$

This is an action for general relativity in first order form. The reason for the funny signs and factors of $\imath$ is that

$$e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd} = (-\imath)(\Sigma^{AB}\Sigma_{AB} - \Sigma^{A'B'}\Sigma_{A'B'}), \tag{18}$$

with a similar identity holding for the curvature term.

To show the complete correspondence with general relativity we may consider the A_{AB} and $A_{A'B'}$ field equations which (ignoring the boundary terms) are

$$\frac{\delta I^1}{\delta A_{AB}} : D \wedge B^{AB} = 0, \tag{19}$$

$$\frac{\delta I^1}{\delta A_{A'B'}} : \overline{D} \wedge B^{A'B'} = 0. \tag{20}$$

Together with Eqs. (13) and (14) these give

$$\mathcal{D} \wedge \Sigma^{AB} = \mathcal{D} \wedge \Sigma^{A'B'} = 0, \tag{21}$$

which implies that the $SU(2)_L \oplus SU(2)_R$ connections A_{AB} $A_{A'B'}$ are the metric compatible torsion free connections associated with the frame fields $e^{AA'}$. This in turn implies that the torsion

$$F_{AA'} = \nabla \wedge e_{AA'} = 0. \tag{22}$$

The action is then

$$I^1 = -\imath \int_{\mathcal{M}} \frac{1}{G}(e^{AA'} \wedge e^{B}_{\ A'} \wedge f_{AB}[A(e)] - e^{A'A} \wedge e^{B'}_{\ A} \wedge f_{A'B'}[A(e)]) + \Lambda e^{AA'} \wedge e^{B}_{\ A'} \wedge e_{AB'} \wedge e^{B'}_{B} + \frac{\imath k}{4\pi}\int_{\partial \mathcal{M}} (Y_{CS}(A_{AB}) - Y_{CS}(A_{A'B'})), \tag{23}$$

which is Einstein's action with a cosmological constant.

Equivalently, we may plug the solutions to the constraints into the remaining field equations to find the field equations for general relativity. The complete set of field equations are Eqs. (13), (14), (19), and (20) together with

$$\frac{\delta I^1}{B^{AB}} = \frac{1}{g^2}F_{AB} - e^2 B_{AB} - \lambda_{AB} = 0, \tag{24}$$

its double with primed indices everywhere and

$$\frac{\delta I^1}{e^{AA'}} : e^B{}_{A'} \wedge \left(\frac{1}{l^2} B_{AB} + \lambda_{AB} \right) + e^A{}_{B'} \wedge \left(\frac{1}{l^2} B_{A'B'} + \lambda_{A'B'} \right) = 0. \tag{25}$$

Plugging Eq. (24) and its primed double into Eq. (25) we then find the Einstein equation

$$\frac{1}{G}(f^A_B \wedge e^{BA'} + f^{A'}_{B'} \wedge e^{AB'}) - \Lambda e^{AB'} \wedge e^B_{B'} \wedge e^{A'}_B = 0. \tag{26}$$

Thus, we have shown how general relativity with a cosmological constant may be derived as a constrained $Sp(4)$ $B \wedge F$ theory.

We may note that because the action contributes two terms to the cosmological constant, there is the possibility of canceling the cosmological constant, while preserving the structure which derives from an $SP(4)$ connection. From Eq. (17) we see that $\Lambda = 0$ for

$$e^2 = \frac{2}{g^2}. \tag{27}$$

This is interesting as it implies that the cosmological constant vanishes at a kind of self-dual point.

This derivation has also shown that there is some redundancy in the field equations that follow from Eq. (10). As is known from [23] the left and right handed field equations decouple so the right handed part of the connection can be gotten instead from the left handed part of the connection by imposing reality conditions, $A_{A'B'} = \overline{A_{AB}}$.

Thus, *as far as the bulk equations of motion are concerned* we can further constrain the $Sp(4)$ symmetry of the $B \wedge F$ theory down to only $SU(2)_L$ by setting $B^{A'B'} = B^{AA'} = 0$ to find the bulk action

$$I^2 = \frac{1}{g^2}\int_{\mathcal{M}} (B^{AB} \wedge F_{AB} + \lambda_{AB}(e^{AA'} \wedge e^B{}_{A'} - B_{AB})). \tag{28}$$

In the Euclidean case this self-dual action suffices, while in the Lorentzian case it must be supplemented by the reality condition, $A_{A'B'} = \overline{A_{AB}}$. It will be important to keep this in mind when we turn to the study of the boundary theory.

III. THE ROLE OF THE BOUNDARY TERMS IN THE FIELD EQUATIONS

When we take the boundary conditions into account we must impose some boundary conditions to insure that the action (10) is functionally differentiable. The equations we need to worry about are the A_{AB} and $A_{A'B'}$ field equations. When we make a variation the boundary contributes a term of the form

$$\delta I^1_{boundary} = \imath \int_{\partial\mathcal{M}} \left[\delta A_{AB} \wedge \left(\frac{k}{4\pi} f^{AB} - \frac{1}{g^2 l^2}\Sigma^{AB} \right) - \delta A_{A'B'} \wedge \left(\frac{k}{4\pi} f^{A'B'} - \frac{1}{g^2 l^2}\Sigma^{A'B'} \right) \right]. \tag{29}$$

In order for the action to be functionally differentiable we must then impose a boundary condition that makes Eq. (29) vanish.

There are several different boundary conditions that might be imposed. We will be interested here in a set of boundary conditions that extend the ''self-dual boundary conditions'' studied, in the case of the Euclidean theory, in [7]. These were motivated by the fact that they allow de Sitter of anti–de Sitter spacetime to be solutions. In the Lorentzian theory we can impose similar conditions, but the details are different, as we now describe.[5]

In the Euclidean case we imposed in [7] the condition that the pullbacks of the fields into the boundary satisfied the pull back of the self-dual equations, expressed on two forms as

$$\vec{f}^{AB} = \frac{4\pi}{l^2 g^2 k}\vec{\Sigma}^{AB}, \tag{30}$$

where $\vec{f}$ indicates the pull back of the two forms into the boundary. These are of course satisfied by de Sitter or anti–de Sitter spacetime, as the full two forms satisfy these conditions. However, in the Euclidean case there are an infinite number of other spacetimes whose two forms pulled back to the boundary satisfy Eq. (30). This is because the left and right handed parts of the curvature are independent in the Euclidean case. As a result, the Euclidean theory with Eq. (30) imposed on the boundary has a solution space given by one degree of freedom for each point on the boundary. This may be verified explicitly by linearized analysis [7].

In the Lorentzian case the left and right handed parts of the Weyl curvature are not independent, they are complex conjugates of each other. Hence we cannot impose Eq. (30) and have $f^{A'B'}$ vary independently on the boundary, as in the Euclidean case. In fact, in that case it can be verified that the result of the reality conditions is to limit the freedom in the solutions to Eq. (30) to oscillations of the boundary in de Sitter or anti–de Sitter spacetime. To have an infinite dimensional space of classical solutions in the Lorentzian case we must relax the boundary conditions.

To see how this may be done, we note first that Eq. (29) can be made to vanish in two ways. We can either fix the connection on the boundary and require that $\delta\vec{A}$ vanishes or we can require that the self-dual conditions (30) be satisfied. However it is also possible to study mixed conditions in which we choose the first solution for some components of $\vec{A}$ and the second for the other components.

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[5]I am grateful to Abhay Ashtekar, Yi Ling and Roger Penrose for discussions on these conditions.

One thing we would like to retain in the Lorentzian case is the relationship between the dimension of the state space of the boundary theory and the area of the boundary, discovered in [7] as this provides a realization of the Bekenstein bound [13]. However, as we will see when we study the canonical quantization, this only requires that the self-dual boundary conditions (30) be imposed on the pull back of the two forms into the intersections of the boundary and the spatial slices. For the other components we can relax the boundary conditions. One natural way to do this is the following.[6]

First, we fix a time slicing of the boundary $\partial\mathcal{M}$. We choose a time coordinate, t, such that these are $t=$const slices. t is then fixed up to a one parameter time reparametrization group $t\rightarrow t'=f(t)$. We will then impose the self-dual condition (30) on the $t=$const spatial slices of $\partial\mathcal{M}$.

We want to weaken the boundary conditions by imposing Eq. (30) on only some of the mixed space-time components of the boundary. We can do this locally by fixing coordinates σ^1,σ^2 on the $t=$const slices of $\partial\mathcal{M}$. We then fix the self dual boundary conditions for the following components:

$$\vec{f}^{AB}_{\sigma^1\sigma^2}=\frac{4\pi}{g^2kl^2}\vec{\Sigma}^{AB}_{\sigma^1\sigma^2}, \tag{31}$$

$$\vec{f}^{A'B'}_{\sigma^1\sigma^2}=\frac{4\pi}{g^2kl^2}\vec{\Sigma}^{A'B'}_{\sigma^1\sigma^2} \tag{32}$$

and

$$\vec{f}^{AB}_{t\sigma^1}=\frac{4\pi}{g^2kl^2}\vec{\Sigma}^{AB}_{t\sigma^1}, \tag{33}$$

$$\vec{f}^{A'B'}_{t\sigma^1}=\frac{4\pi}{g^2kl^2}\vec{\Sigma}^{A'B'}_{t\sigma^1}. \tag{34}$$

However, we make the remaining terms in Eq. (29) vanish by putting

$$\delta A^{AB}_{\sigma^1}=\delta A^{A'B'}_{\sigma^1}=0. \tag{35}$$

Clearly these conditions are compatible with the reality conditions, and they result in a functionally differentiable action. At the same time the fact that the self-dual conditions (30) are not imposed on all pull-backs of the two forms on the boundary means that the solution space is larger. While A_1 is fixed, there is now no condition on $\dot{A}_{\sigma^2}$. Consequently that component of the connection is allowed to evolve, so long as Eqs. (31),(32) are satisfied.

The rationale for these conditions comes from the quantum theory, more particularly from the form of the boundary Hilbert space, which is constructed as in [7] from the spaces of intertwiners of the quantum deformed gauge group. In the quantum theory we define the boundary conditions by a condition that the spin networks intersect the boundary at a discrete set of points, called punctures, whose labels do not evolve. This does constrain certain components of the connection (up to local gauge transformations). The reason is that the traces of the holonomies around a loop γ that surrounds a single puncture are fixed by the quantization of the conditions (31),(32). Thus, fixing certain components of the connection on the boundary is a consequence of fixing the boundary conditions in the quantum theory in such a way that the labels on the punctures that determine the Hilbert space of the boundary theory are fixed and do not evolve. But by doing so we weaken the boundary conditions on other components of the connection. This gives the boundary state more freedom to evolve within those fixed Hilbert spaces. We will see how this works when we come to the quantum theory in Sec. VI.

To complete the specification of the boundary conditions we will then anticipate the role of the punctures in the quantum theory and fix a discrete set of preferred points on the spatial boundary. Each such puncture is surrounded by a local region and in each of these we may introduce local coordinates (r,θ) which are angular coordinates with the puncture at the origin. These can then be joined yielding a single coordinate patch on the whole punctured sphere, which reduces to an angular coordinate system in the neighborhood of each puncture. Bringing back t we then have a coordinate system (r,θ,t) on the whole of ∂M minus the world lines of the punctures. We then apply the above conditions with $\theta=\sigma^1$ and $r=\sigma^2$. The boundary conditions (35) then imply that the holonomies of the $SU(2)_L\oplus SU(2)_R$ connections around loops in the spatial boundary that surround single punctures are fixed.

Finally note that compatibility of Eqs. (31),(32) with the field equations requires that

$$G\Lambda=\frac{8\pi}{l^2g^2k}, \tag{36}$$

which gives us a relation

$$k=\frac{4\pi}{1-\frac{g^2e^2}{2}}. \tag{37}$$

This is an interesting relation, as k must be an integer. We see that at the self dual point where $G^2\Lambda=0$, $k\rightarrow\infty$.

IV. THE CANONICAL FORMALISM

To understand the relationship between the $Sp(4)$ gauge invariance and diffeomorphism invariance, as well as to prepare to discuss the quantization we study in this section the canonical formulation of the theory we have just introduced. We do this by making a 3+1 decomposition of the action (10) in the usual way [23], with the spacetime manifold decomposed as $\mathcal{M}=\Sigma\times R$, with Σ a three manifold. Here we ignore boundary terms; their effects are included in the following section.

[6]For more details concerning the application of these boundary conditions, see [38].

Before beginning, we must fix a point of view concerning the relationship between the complex quantities such as the self-dual connections and the real metric of spacetime. We will take here the approach in which all fields are assumed to be complex and then set up the canonical formalism for this case. We will then consider the reality conditions to be a restriction on the space of solutions which is imposed after the canonical formalism is set up. This is natural for considerations of the quantization, because it parallels the situation of the quantum theory in which the operator algebra is defined over the complexes, while the reality conditions are imposed by the choice of an inner product. At the level of the abstract algebra, before the inner product is imposed, it makes no sense to restrict to the real sector, as this is done by restricting certain operators to be Hermitian, but this is not defined in the absence of the inner product.

We now proceed to the $3+1$ decomposition. We write the action in terms of space and time coordinates separately, with spacetime index $\mu=(0,a)$, with $a=1,2,3$, we have

$$I=-\imath\int dt\int_{\Sigma}\epsilon^{abc}\Big\{\frac{1}{g^2}B^{AB}_{ab}\dot{A}_{cAB}-\frac{1}{g^2}B^{A'B'}_{ab}\dot{A}_{cA'B'}+\frac{1}{g^2}A^{AB}_0[\mathcal{D}_aB_{bcAB}]-\frac{1}{g^2}A^{A'B'}_0[\mathcal{D}_aB_{bcA'B'}]+e_{AA'0}\Big[\frac{2}{g^2l}B^{AB}_{ab}e_{cB}{}^{A'}$$
$$+\frac{2}{l^2}\lambda^{AB}_{ab}e_{cB}{}^{A'}-\frac{2}{g^2l^2}B^{A'B'}_{ab}e^{A}_{cB'}-\frac{2}{l^2}\lambda^{A'B'}_{ab}e_{cB'}{}^{A}\Big]+B^{AB}_{0a}\Big[\frac{1}{g^2}f_{bcAB}+\frac{1}{g^2l^2}e_{bA}{}^{A'}e_{cBA'}-e^2B_{bcAB}-\lambda_{bcAB}\Big]-B^{A'B'}_{0a}\Big[\frac{1}{g^2}f_{bcA'B'}$$
$$+\frac{1}{g^2l^2}e_{bA'}{}^{A}e_{cB'A}-e^2B_{bcA'B'}-\lambda_{bcA'B'}\Big]+\lambda^{AB}_{0a}\Big[\frac{1}{l^2}e_{bA}{}^{A'}e_{cBA'}-B_{bcAB}\Big]+\lambda^{A'B'}_{0a}\Big[\frac{1}{l^2}e_{bA'}{}^{A'}e_{cB'A}-B_{bcA'B'}\Big]\Big\}. \tag{38}$$

The canonical momenta for the forms B, λ, and $e_{0AA'}$ all vanish, as do the canonical momenta of the time components $A_{0AB}, A_{0A'B'}$. This gives the primary constraints. The nonvanishing canonical momenta are for A_{aAB} and $A_{aA'B'}$ are, respectively,

$$\pi^a_{AB}=\frac{-\imath}{g^2}\epsilon^{abc}B_{bcAB},$$

$$\pi^a_{A'B'}=\frac{\imath}{g^2}\epsilon^{abc}B_{bcA'B'}. \tag{39}$$

The π^a's are, as usual, vector densities.

We now come to the secondary constraints. First there are the $SU(2)_L\oplus SU(2)_R$ gauge constraints, which are

$$G^{AB}=\mathcal{D}_a\pi^{aAB}=0, \tag{40}$$

$$G^{A'B'}=\mathcal{D}_a\pi^{aA'B'}=0. \tag{41}$$

These preserve the vanishing of the canonical momenta of A_{0AB} and $A_{0A'B'}$. The vanishing of the canonical momenta for $e_{0AA'}$ is more complicated, and gives the four secondary constraints,

$$G^{AA'}=\frac{1}{l^2}[\pi^{cAB}-\imath\lambda^{*cAB}]e_{cB}{}^{A'}+\frac{1}{l^2}[\pi^{cA'B'}-\imath\lambda^{*cA'B'}]e_{cA}{}^{B'}$$
$$=0. \tag{42}$$

One might expect that as the $Sp(4)/(SU(2)_L\oplus SU(2)_R)$ gauge symmetry seems to be explicitly broken by the constraints in the action, these would become second class constraints. Instead, as we shall see, these four equations become the Hamiltonian and diffeomorphism constraints of the theory.

From the vanishing of the canonical momenta for the mixed space-time components of the two forms B^{AB}_{0a} and $B^{A'B'}_{0a}$ we get two more sets of constraints,

$$I^{aAB}=\epsilon^{abc}\Big[\frac{1}{g^2}f^{AB}_{bc}+\frac{1}{g^2l^2}e^{AA'}_{b}e_{cA'}{}^{B}-e^2B^{AB}_{bc}-\lambda^{AB}_{bc}\Big], \tag{43}$$

$$I^{aA'B'}=\epsilon^{abc}\Big[\frac{1}{g^2}f^{A'B'}_{bc}+\frac{1}{l^2}e^{A'A}_{b}e_{cA}{}^{B'}-e^2B^{A'B'}_{bc}+\lambda^{A'B'}_{bc}\Big]. \tag{44}$$

The first pair, I^{aAB} and $I^{aA'B'}$, may be solved to express the λ^{AB}_{ab} and $\lambda^{A'B'}_{ab}$ in terms of the other fields. These constraints are then eliminated with the primary constraints which are the vanishing of the λ's momenta.

The preservation of the vanishing of the canonical momenta for the mixed components λ^{AB}_{0a} and $\lambda^{A'B'}_{0a}$ results in six more constraints that show that the π^{aAB} and $\pi^{aA'B'}$ are fixed to be the duals of the self-dual two forms constructed from the frame fields:

$$J^{aAB}=\pi^{aAB}+\frac{\imath}{g^2l^2}\epsilon^{abc}e^{AA'}_{b}e_{cA'}{}^{B}=0, \tag{45}$$

$$J^{aA'B'}=\pi^{aA'B'}-\frac{\imath}{g^2l^2}\epsilon^{abc}e^{A'A}_{b}e_{cA}{}^{B'}=0. \tag{46}$$

They can be solved to eliminate the $e_a^{AA'}$ in terms of the π^{aAB} and the quantities $N_{AA'}$. These are four quantities defined by

$$N_{AA'}=t^{\mu}e_{\mu AA'}, \tag{47}$$

where t^{μ} is the timelike unit normal [23]. They are subject to the one constraint $N^{AA'}N_{AA'}=2$, which follows from $t^{\mu}t_{\mu}=-1$. They therefore represent three independent quantities, which together with the nine π^{aAB} allow us to express the twelve $e_a^{AA'}$ as

$$e_a^{AA'}=e_{aA}^{\ \ B}N_B^{\ A'}=\frac{1}{\sqrt{q}}\epsilon_{abc}\pi^{bBC}\pi^{cA}_{\ \ C}N_B^{\ A'}. \tag{48}$$

We also have the complex conjugate of these relations

$$e_a^{A'A}=e_{aA'}^{\ \ B'}N_{B'}^{\ A}=\frac{1}{\sqrt{\bar{q}}}\epsilon_{abc}\pi^{bB'C'}\pi^{cA'}_{\ \ C}N_{B'}^{\ A}, \tag{49}$$

where q and $\bar{q}$ are made from the determinants of π^{AAB} and $\pi^{aA'B'}$ in the usual way.

We may note that even for the complex case, the π^{aAB} and $\pi^{aA'B'}$ are not independent quantities. This is because the pullback of the self-dual and anti-self-dual three forms of a metric define the same metric. As a result there is an additional second class constraint, which is

$$R^{ab}=\pi^{aAB}\pi^{b}_{AB}-\pi^{aA'B'}\pi^{b}_{A'B'}=0. \tag{50}$$

We will come back to the role this plays after we have isolated the Hamiltonian constraint of the theory.

For completeness we mention also two more sets of constraints, which express the Lagrange multiplier fields in terms of other quantities. They they play no role in what follows as the Lagrange multipliers are in any case eliminated but we give them for completeness.

The preservation of the vanishing of the momenta for the λ_{ab}'s result in constraints

$$J_a^{AB}=B_{0a}^{AB}-\frac{2}{l^2}e_{0A'}^{\ \ A}e_a^{\ A'B}=0, \tag{51}$$

$$J_a^{A'B'}=B_{0a}^{A'B'}-\frac{2}{l^2}e_{0A}^{\ \ A'}e_{aA'A}=0. \tag{52}$$

Similarly, the preservation of the vanishing of the momenta for the B_{ab}'s result in constraints

$$I_{aAB}=-\frac{1}{g^2}\mathcal{D}_a A_{0AB}+\frac{1}{g^2}\dot{A}_{aAB}+\frac{1}{g^2l^2}e_{0AA'}e_{aB}^{\ \ A'}-2e^2B_{0a}^{AB}+\lambda_{0a}^{AB}=0, \tag{53}$$

$$I_{aA'B'}=-\frac{1}{g^2}\mathcal{D}_a A_{0A'B'}+\frac{1}{g^2}\dot{A}_{aA'B'}+\frac{1}{g^2l^2}e_{0A'A}e_{aB'}^{\ \ A}-2e^2B_{0a}^{A'B'}+\lambda_{0a}^{A'B'}=0. \tag{54}$$

Using all the I and J constraints, we find that G^{AB} and $G^{A'B'}$ are unchanged, and indeed are first class constraints that generate $SU(2)_L\oplus SU(2)_R$ internal gauge transformations. However, the components of the Gauss law in the coset $Sp(4)/SU(2)_L\oplus SU(2)_R$ now become

$$G^{AA'}=\frac{1}{l^2}e_{cB}^{\ \ A'}\left[\frac{\Lambda g^2}{l^2}\pi^{cAB}-\frac{2\imath}{g^2}\epsilon^{abc}f_{bc}^{AB}\right]+\frac{1}{l^2}e_{cB'}^{\ \ A}\left[\frac{\Lambda g^2}{l^2}\pi^{cA'B'}+\frac{2\imath}{g^2}\epsilon^{abc}f_{bc}^{A'B'}\right], \tag{55}$$

where the cosmological constant is defined by Eq. (17).

We now use Eqs. (48) and (49) to write these in terms of six new constraints,

$$G^{AA'}=\frac{9G}{N^2E^3}\left(\frac{N_D^{A'}}{\sqrt{q}}C^{AD}+\frac{N_{D'}^{A}}{\sqrt{q}}C^{A'D'}\right), \tag{56}$$

where

$$C^{AD}=-\imath\pi^{aC}_{\ \ B}\pi^{bD}_{\ \ C}f_{ab}^{AB}+\frac{\Lambda g^4}{l^2}\epsilon_{abc}\pi^{aC}_{\ \ B}\pi^{bD}_{\ \ C}\pi^{cAB}, \tag{57}$$

$$C^{A'D'}=\imath\pi^{aC'}_{\ \ B'}\pi^{bD'}_{\ \ C'}f_{ab}^{A'B'}+\frac{\Lambda g^4}{l^2}\epsilon_{abc}\pi^{aC'}_{\ \ B'}\pi^{bD'}_{\ \ C'}\pi^{cA'B'}. \tag{58}$$

These constraints must vanish independently because they transform separately under $SU(2)_L$ and $SU(2)_R$ transformations. Thus, from the closure of the constraint algebra we have

$$\{G^{AB},G^{CC'}\}\approx C^{AB}\approx 0, \tag{59}$$

$$\{G^{A'B'},G^{CC'}\}\approx C^{A'B'}\approx 0. \tag{60}$$

Now we must recall that the two conjugate pairs (A_{aAB},π^{aAB}) and $(A_{aA'B'},\pi^{aA'B'})$ are mutually commuting, so that

$$\{C^{AB},C^{A'B'}\}=0. \tag{61}$$

Furthermore, we know from work of Jacobson in [30] that the four C^{AB} contain the standard Hamiltonian and diffeomorphism constraints of the Ashtekar formalism, and thus make a first class algebra. The same is then true for the $C^{A'B'}$. It follows that the algebra of the four $G^{AA'}$ is first class and contains the Hamiltonian and diffeomorphism constraints of the theory. To see this in more detail, consider the four vector $V^{AA'}=V^{\mu}e_{\mu}^{AA'}$ as a parameter of the constraints

$$G(V)=\int V^{AA'}G_{AA'}=\int W^{AB}C_{AB}+W^{A'B'}C_{A'B'}, \tag{62}$$

where for simplicity we have set

$$W^{AB}=\frac{9G}{N^2E^3}\frac{N_B^{A'}}{\sqrt{q}}V^{AA'} \tag{63}$$

and similarly for $W^{A'B'}$. We have thus expressed $G(V)$ in terms of Lagrange multipliers and two copies of the Ashtekar constraints. Thus, their algebra is first class. It also follows that the algebra of $G^{AA'}$ with both G^{AB} and $G^{A'B'}$ is first class.

FInally, we must deal with the is the remaining constraint R^{ab} given by Eq. (50). Its Poisson bracket with $G(V)$ gives a remaining set of constraints, which are

$$\begin{aligned} S^{ab}(x)&\equiv\{R^{ab}(x),G(V)\}\\ &=\mathcal{D}_c(W_A^D\pi_B^{cE}\pi_{D)E}^{(b})\pi^{aAB}\\ &\quad+\mathcal{D}_c(W_{A'}^{D'}\pi_{B'}^{cE'}\pi_{D')E'}^{(b})\pi^{aA'B'}=0. \end{aligned} \tag{64}$$

This is actually a well known condition, it is the reality condition for the Ashtekar formalism, which guarantees that $\dot{q}^{ab}$ is real. Here it is recovered as a constraint, even in the complexified case. It implies a relationship between the real parts of A_{aAB} and $A_{aA'B'}$.

In fact we can now give a simple interpretation of the resulting formalism. With all fields complex, what we have are two copies of the Ashtekar formalism, one with positive chirality and one with negative chirality. However, the left and right sectors are related by the constraints R^{ab} and S^{ab} that require that all metric quantities constructed from the left and right handed sectors agree. Given that the constraints come in the combination $G^{AA'}$ given by Eq. (56) we have only four spacetime constraints, so the two copies of the Ashtekar formalism evolve together with common lapses and shifts. Thus, as in the Ashtekar formalism, once one sets the constraints R^{ab} and S^{ab} to be zero, they are preserved in time, so that the metric quantities continue to agree, whether computed from the left or right sector. Finally, even in the presence of the constraints, the internal gauge constraints are independent, so that the local gauge symmetry is $SU(2)_L\oplus SU(2)_R$.

Finally, so far we have not made a restriction to real metrics. To do so is simple, we restrict to the subspace of the solution space for which q^{ab} and its time derivative are real. Given the relations just found the equivalence to the Ashtekar formalism guarantees that real initial data will evolve to a real spacetime.

V. THE BOUNDARY THEORY IN THE CANONICAL FORMALISM

We now include in the canonical analysis the effects of the boundary term in the action, proportional to the Chern-Simons invariant of the pull back of the connection on the boundary. This analysis was first done in the chiral formulation in [7], here we extend it to the ambidextrous formulation.

With the boundary terms included, the primary constraints that define the nonvanishing momenta are

$$S^{aAB}(x)\equiv\pi^{aAB}(x)+\frac{\imath}{g^2}B^{*aAB}(x)-\frac{\imath k}{4\pi}\int d^2S^{ab}(\sigma)A_b^{AB}\delta^3(x,S(\sigma))=0, \tag{65}$$

$$S^{aA'B'}(x)\equiv\pi^{aAB}(x)-\frac{\imath}{g^2}B^{*aA'B'}(x)+\frac{\imath k}{4\pi}\int d^2S^{ab}(\sigma)A_b^{A'B'}\delta^3(x,S(\sigma))=0. \tag{66}$$

What is important for the construction of the boundary theory is the interaction of the boundary term in the definition of the momenta (65) and the generalized Gauss's law constraints that come from the $A_0^{\alpha\beta}$ field equations. Recall that the Gauss's law for $SU(2)_L\oplus SU(2)_R$ has the form

$$G^{AB}\equiv\frac{\imath}{g^2}\mathcal{D}_aB^{*aAB}, \tag{67}$$

$$G^{A'B'}\equiv\frac{\imath}{g^2}\mathcal{D}_aB^{*aA'B'}. \tag{68}$$

If we use the definition of the momenta from (65), (66) in the Gauss's law we find, after integrating by parts, that

$$\begin{aligned} G(\Lambda)&\equiv\int_\Sigma\Lambda_{AB}G^{AB}=\int_\Sigma\Lambda_{AB}\frac{\imath}{g^2}\mathcal{D}_aB^{*aAB}\\ &=-\int_\Sigma\mathcal{D}_a(\Lambda_{AB})\pi^{aAB}+\int_{\partial\Sigma}d^2S^{ab}\Lambda_{AB}\\ &\quad\times\left(\frac{\imath k}{4\pi}f_{ab}^{AB}-\pi_{ab}^{*AB}\right), \end{aligned} \tag{69}$$

with an identical expression for $G^{A'B'}$. Thus, in addition to the bulk constraints we found in the previous section, there are two boundary constraints given by

$$G^B(\lambda)=\int_{\partial\Sigma}d^2S^{ab}\lambda_{AB}\left(\pi_{ab}^{*AB}-\frac{\imath k}{4\pi}f_{ab}^{AB}\right), \tag{70}$$

$$\overline{G}^B(\overline{\lambda})=\int_{\partial\Sigma}d^2S^{ab}\overline{\lambda}_{A'B'}\left(\pi_{ab}^{*A'B'}-\frac{\imath k}{4\pi}f_{ab}^{A'B'}\right). \tag{71}$$

These implement Eqs. (31),(32), which were the spatial parts of the boundary conditions we imposed to make the action functionally differentiable.

The next thing to notice is that the boundary term in the primary constraints (65),(66) have the effect of modifying the Poisson brackets for fields pulled back into the boundary. We can see this by computing their algebra. Defining $S(f) \equiv \int_\Sigma f_{aAB} S^{aAB}$, we find

$$\{S(f),S(g)\} = \frac{\imath k}{2\pi}\int_{\partial\Sigma} d^2 S^{ab} f_{aAB} g_b^{AB}, \qquad (72)$$

with similar relations holding for the equations with the primed indices.

We can now characterize the kinematics of the boundary theory classically. The phase space of the boundary theory, which we will call $\Gamma^{\partial\Sigma}$ can be characterized by fields pulled back to the spatial boundary, which are written $\vec{A}_a^{AB}, \vec{A}_a^{A'B'}$, $\vec{\pi}_{ab}^{*AB}$ and $\vec{\pi}_{ab}^{*A'B'}$. (Note that for these pullback fields the abstract indices $a,b,c,\ldots$ are two dimensional.) The latter commute with all other boundary fields and hence label sectors of the boundary phase space. (They fail to commute with connection variables normal to the boundary, which are not part of the phase space of the boundary theory.)

By the constraints π_{ab}^{*AB} and $\pi_{ab}^{*A'B'}$ are determined [up to the $SU(2)_L \oplus SU(2)_R$ gauge invariance] in terms of the two metric on the boundary.

The actual degrees of freedom of the boundary phase space are given by the $SU(2)_L \oplus SU(2)_R$ connection A_a, pulled back into the boundary. To find their Poisson brackets one must construct the Dirac brackets by inverting the second class constraints (72). This is done in detail in [7], the result is

$$\{\vec{A}_a^{AB}(\sigma), \vec{A}_{bCD}(\sigma')\} = \frac{2\pi}{k}\epsilon_{ab}\delta^2(\sigma\sigma')\,\delta_{CD}^{(AB)}. \qquad (73)$$

These are in fact the Poisson brackets of two dimensional Chern-Simons theory.

However, the curvatures of the boundary connection are determined by the boundary terms in the Gauss's law (69). These require

$$\vec{f}^{AB} = \frac{4\pi}{kl^2}\vec{e}^{AC'} \wedge \vec{e}^{B}{}_{C'},$$

$$\vec{f}^{A'B'} = \frac{4\pi}{kl^2}\vec{e}^{A'C} \wedge \vec{e}_C^{B'}. \qquad (74)$$

There are relations between $\vec{f}^{AB}$ and $\vec{f}^{A'B'}$. These follow from the constraints which express the fact that the pull backs of the self-dual and anti-self-dual two forms into the spatial boundaries Σ define the same two geometry. These require that the invariants constructed from $\vec{A}^{AB}$ and $\vec{A}^{A'B'}$ must be equal.

Thus, the phase space of the boundary theory is that of $SU(2)_L \oplus SU(2)_R$ Chern-Simons theory, with an external field constraining the curvatures.

The Hamiltonian of the theory may be constructed, following the standard procedure, by extending the Hamiltonian constraint by a boundary term so that the expression is functionally differentiable even when the lapse function is nonvanishing on the boundary. To extract the Hamiltonian we may choose $W^{AB} = \tau\epsilon^{AB}$ and $W^{A'B'} = \tau\epsilon^{A'B'}$. The Hamiltonian then must have the form

$$H(\tau) = \int_\Sigma \tau[\epsilon^{AB}C_{AB} + \epsilon^{A'B'}C_{A'B'}] + \int_{\partial\Sigma}\tau h, \qquad (75)$$

where we require that the time coordinate τ match the slicing of the boundary given by the preferred $t=$const surfaces that go into the definition of the boundary conditions. This means that continued to the boundary τ must be a function of t which is constant on the $t=$const surfaces.

The condition that H be functionally differentiable requires that h be a functional defined on the boundary, of the form

$$\int_{\partial\Sigma}\tau h = 4\imath\int_{\partial\Sigma} d^2 S_a \tau[\pi_A^{aB}\pi_B^{bC}A_{bC}^{A} - \pi_{A'}^{aB'}\pi_{B'}^{bC'}A_{bC'}^{A'}]. \qquad (76)$$

When the constraints are satisfied this last expression, Eq. (76), is the Hamiltonian of the theory. We see that it is a functional on the boundary, which is both as required by diffeomorphism invariance and consistent with the holographic hypothesis.

VI. QUANTIZATION

We may now sketch the quantization of the ambidextrous theory. We only emphasize those aspects which differ from the treatment given for the Euclidean signature theory in [7], to which the reader may refer for more details. We begin with the bulk theory and then add the boundary degrees of freedom.

We work first in the connection representation. Initially the configuration space is defined to be the space of complexified $SU(2)_L \oplus SU(2)_R$ connections, mod internal gauge transformations:

$$\mathcal{C}^{gauge} = \frac{(A^{AB}, A^{A'B'})}{G^{AB} \times G^{A'B'}}. \qquad (77)$$

Functionals on $\mathcal{C}^{gauge}$ will live in a Hilbert space, subject to a suitable norm such as that given in [31,11] called $\mathcal{H}^{gauge}$.

We must now discuss a subtle but important issue, having to do with the use of the spin network states to describe the Lorentzian signature theory. For the case that the gauge group is real $SU(2)_L \oplus SU(2)_R$ the resulting space of states has a basis given by the spin networks, as discussed in [10,11]. In these states the edges of the spin networks are labeled by pairs of integers j_L, j_R corresponding to the finite dimensional representations of $SU(2)_L \oplus SU(2)_R$. In the present case, where the spacetime is Lorentzian the connections actually live in the complexification of $SU(2)_L \oplus SU(2)_R$. This means that there is additional freedom in the choice of states, arising from the fact that the gauge group is noncompact. One might choose, for example, to label the spin networks with continuous as well as discrete labels, corresponding to the full set of representations of the gauge group.

The strategy guiding the present approach is to set up a quantization of the complexification of general relativity at the kinematical level, and then impose the reality conditions as operator equations, by realizing Eqs. (50) and (64) on a suitable space of states. Thus, in principle we do have the freedom to work within a kinematical state space which is considerably enlarged from that defined in [10,11] by extending the labels on the spin networks to all representations of the complexifications of $SU(2)_L \oplus SU(2)_R$. Given that there are continuous families of representations this greatly expands the state space. This poses a very important issue, which is that it may no longer be possible to choose an inner product for the space of diffeomorphism invariant states that renders it separable.[7] This would be a disaster, which must be avoided if possible.

In fact, it is possible to avoid this disaster. To do this we work within the Hilbert space whose basis is labeled by spin networks whose edges are labeled only by pairs of ordinary spins (j_L, j_R). The reason is that we will be implementing the Lorentzian signature theory as long as we work in a space of states in which it is possible to express, and solve, the operator forms of the reality conditions, Eqs. (50) and (64). In this theory the kinematical theory differs from that of the Euclidean theory in that every measure of three geometry, such as areas and volumes, has a right value and a left value, which come from the corresponding labels on the states. This extension is the way that the spin network theory can express the fact that it gives a kinematical description of the complexification of geometry, in essence the complex part of any function of the three metric is the difference between its left and right value. The reality conditions will, as we will see shortly, be expressed by conditions that require the left and right geometries to be equal.

Following the methods developed in [32,10,11] we are then free to impose the condition that the states are invariant under spatial diffeomorphisms. Given the choice of kinematical inner product on $\mathcal{H}^{gauge}$ defined by the $SU(2)_L \oplus SU(2)_R$ spin networks we construct in the usual way a unitary representation of the spatial diffeomorphism group Diff(Σ) on $\mathcal{H}^{gauge}$. The gauge invariant states live in a subspace which is called $\mathcal{H}^{diffeo}$. These are by now standard constructions which were done at the heuristic level in [32,33,8,10] and then treated rigorously in [31,34,11].

Once $\mathcal{H}^{diffeo}$ is constructed there remain three more sets of constraints to impose which are the Hamiltonian constraint $\mathcal{C}(N)=0$ and the constraints $R^{ab}=0$ and $S^{ab}=0$ that determines that the left and right handed fields define the same metric geometry.

There are two ways we could handle the constraints $R^{ab}=0$ and $S^{ab}=0$ that tie the left and right sectors to each other. The orthodox Dirac method would require that, as these together make a second class set, that they be solved explicitly and eliminated before the quantization. One way to do this is to eliminate the right handed quantities $(A_a^{A'B'}, \pi^{aA'B'})$ together with the $SU(2)_R$ gauge freedom in favor of the left handed quantities. This would result in the Ashtekar formalism. However, as the Hamiltonian is a constraint, there is a second option which can be tried, which may preserve the chiral symmetry of the theory. This is to realize R^{ab} as an operator equation on physical states, so that we try to define and solve simultaneously

$$\hat{R}^{ab}|\Psi\rangle = 0 \tag{78}$$

and

$$\mathcal{C}(N)|\Psi\rangle = 0 \tag{79}$$

on states in $\mathcal{H}^{diffeo}$.

We then *define* the quantization of S^{ab} by

$$\hat{S}^{ab} \equiv [\hat{\mathcal{C}}[N], \hat{R}^{ab}]. \tag{80}$$

This is, of course, a formal expression that requires a regularization procedure to specify completely. It then follows that physical states that satisfy Eqs. (79) and (78) also satisfy

$$\hat{S}^{ab}|\Psi\rangle = 0. \tag{81}$$

To see how this works, recall that standard constructions give a normalizable basis for $\mathcal{H}^{gauge}$ in terms of spin networks for the algebra $SU(2)_L \oplus SU(2)_R$. The edges are labeled by pairs of spins (j_L, j_R) and the nodes are labeled by pairs of intertwiners (μ_L, μ_R).

Using this basis it is easy to impose the condition (78) on states. The reason is that $R^{ab}=0$ is equivalent to the requirement that all area and volume observables constructed from π^{aAB} and $\pi^{aA'B'}$ are equal. For general states in the spin network basis, the areas and volumes constructed from π^{aAB}, may be called the ''left quantum geometry.'' These will differ from those constructed from $\pi^{aA'B'}$, which we may call the ''right handed geometry.'' Classically $R^{ab}=0$ is equivalent to the statement that the right handed areas and volumes are equal to the left handed ones, for every region of the three manifold.

The states in the spin network basis which are spanned by eigenstates of left and right handed area and volume operators, such that the eigenvalues of the left handed areas always equal the eigenvalues of the right-handed areas, live in a subspace $\mathcal{H}^{sym} \subset \mathcal{H}^{gauge}$ which is spanned by the subset of spin networks whose labels satisfy $j_L = j_R$ and $\mu_L = \mu_R$. Representations of $SU(2)_L \oplus SU(2)_R$ which satisfy $j_L = j_R$ are called *balanced*.

Such representations have been employed by Barrett and Crane in a proposal for a state sum model to represent quan-

[7]There are delicate issues concerning the treatment of the norm on states with high valence nodes, but these may be resolved leading to separable Hilbert space.

tum general relativity [35] and have been studied recently in [36,37]. It is quite interesting to find it arising also within the Hamiltonian framework.[8]

The restriction to balanced spin networks implements half the reality conditions. The other half are, as we argued above, automatically satisfied on states which are solutions to the Hamiltonian constraints. For the purposes of studying the boundary theory, we need be concerned with one class of solutions to the bulk Hamiltonian constraint, which are those that are derived from the Chern-Simons state [14]. We thus now show that the state can be extended to the present case.

To do this we show that the loop transform of the $SU(2)_L \oplus SU(2)_R$ Chern-Simons state induces, as in [7] a finite dimensional space of boundary states, all of which satisfy the bulk Hamiltonian constraints. These are expressed in terms of arbitrary (but quantum deformed) $SU(2)_L \oplus SU(2)_R$ spin networks. The reality condition (50) is then implemented on this solution space by restricting the spin networks in the transform to the balanced spin networks. This restriction commutes with the imposition of the constraints, so that the result also provides, by the formal argument above, a solution to Eq. (64).

The Chern-Simons state for $SU(2)_L \oplus SU(2)_R$ is given by

$$\Psi_{CS}(A) = e^{(k'/4\pi)(S_{CS}[A_{AB}] - S_{CS}[A_{A'B'}])}. \qquad (82)$$

It is straightforward to show using the usual methods [14,39] that this solves independently both the left and right parts of the Hamiltonian constraint (58). In the spin network basis, suitably quantum deformed [12], the corresponding solutions space is given in the bulk by

$$\Psi(\Gamma) = \int DA e^{(k'/4\pi)(S_{CS}[A_{AB}] - S_{CS}[A_{A'B'}])} T[\Gamma]. \qquad (83)$$

Here $k' = 6\pi/G^2\Lambda$. Γ then refer to $SU_q(2)_L \oplus SU_q(2)_R$ quantum spin networks, q deformed with $q = e^{2\pi i/(k'+2)}$ and $T[\Gamma]$ is a suitably framed product of traces of Wilson loops associated to Γ. This defines a finite dimensional state space, parameterized by boundary states we will discuss shortly. We may note that the fact that the cosmological constant is common to the left and right sector means that they have the same quantum deformation parameter.

The transform (83) defines a space of physical states, which may be called $\mathcal{H}^{physical}$. The restriction that defines this may be stated as follows: a functional of quantum deformed spin networks $\phi(\Gamma)$ is in $\mathcal{H}^{physical}$ if it is invariant under the quantum recoupling rules given in [40]. The boundary theory then consists of equivalence classes of quantum spin networks, under these recoupling relations, which meet the boundary at a fixed set of punctures.

The reality condition (64) can be imposed on the space of states by requiring that the action of the left handed area operator, defined from π^{aAB} is equal to the action of the right handed area operator $\pi^{aA'B'}$ for every two surface in the bulk. These actions are defined on the quantum deformed spin network states in [12]. The condition is solved for every surface when the quantum deformed spin network states are restricted to balanced spin networks.

To show that the Hamiltonian constraint has been solved in a way that is consistent with the imposition of the reality conditions in this form one must check that the restriction to balanced spin networks commutes with the quantum recoupling relations, applied separately to the left and right labels of the spin networks. This is straightforward, as one may use the recoupling relations to express the equivalence classes in terms of trivalent spin networks, after which the imposition of the balanced conditions amount to the trivial requirement that $j_L = j_R$ on all edges. This means that, at least formally, the second reality condition (64) is also solved on this space of states.

We may now take into account the details of the construction of the boundary theory. The Chern-Simons state state becomes a finite dimensional space of states, as described in [7], for each set of punctures on the boundary. These are given by the quantum deformed intertwiners on the punctured boundary The restriction to balanced representations extends to the boundary, we also require that $k = k'$ so that there is only one contribution to the cosmological constant. One then constructs a space of physical states that has the form [7]

$$\mathcal{H}^{phys} = \sum_n \sum_{j_1,\ldots,j_n} \mathcal{H}_{j_1,\ldots,j_n}, \qquad (84)$$

where

$$\mathcal{H}_{j_1,\ldots,j_n} = \mathcal{V}^{balanced}_{j_1,\ldots,j_n} \subset \mathcal{V}^L_{j_1,\ldots,j_n} \otimes \mathcal{V}^R_{j_1,\ldots,j_n}. \qquad (85)$$

Here $\mathcal{V}^{balanced}_{j_1,\ldots,j_n}$ is the linear space of balanced intertwiners in $\mathcal{V}^L_{j_1,\ldots,j_n} \otimes \mathcal{V}^R_{j_1,\ldots,j_n}$, which is the space of conformal blocks for the punctured sphere, for the $SU(2)_L \oplus SU(2)_R$ WZW model. By the balanced condition, the common j_i's label the punctures. The sum extends up to spins $j = k'$, because of the quantum deformation [7].

The full set of physical observables for the theory can be described in terms of operators on $\mathcal{H}^{phys}$ [7]. Among them is the area operator [8,9], which is diagonal in the punctures and whose eigenvalues are given, in the limit of large k' [12] by

$$a[j_i] = \sum_i G\hbar \sqrt{j_i(j_i+1)}. \qquad (86)$$

One then finds that the Bekenstein bound [13] is satisfied, as

$$\ln \mathrm{Dim}(\mathcal{V}^L_{j_1,\ldots,j_n} \otimes \mathcal{V}^R_{j_1,\ldots,j_n}) \leq c a[j_i], \qquad (87)$$

[8]An alternative approach to deriving the Barrett-Crane balanced states as the quantization of an action similar to Eq. (7) is given by [27]. In this approach one proceeds from the classical action to the path integral directly by defining a natural discretization of the $So(4)$ Plebanski action.

with $c=2\sqrt{3}/\ln(2)$.

Finally, to realize the dynamics we must implement the Hamiltonian (76), which we recall is a boundary term. The evolution of the states according to a time defined on the boundary by the function field $\tau\in\partial\mathcal{M}$ is then given by the Schrödinger equation

$$\imath\hbar\frac{\delta}{\delta\tau}\Psi=\int_{\partial\Sigma}\tau\hat{h}\Psi, \qquad (88)$$

where the Hamiltonian is given by an operator representing Eq. (76). The implementation of the Hamiltonian as a quantum operator on the space $\mathcal{H}^{phys}$ is a nontrivial problem, which will represent another step in this program.

Thus, as in the classical theory, the gauge invariance splits up into a kinematical, linear part and a dynamical, nonlinear part, and the splitting affects both the bulk and the boundary theory. The linear kinematical part has to do with the gauge invariance in the subgroup $H=SU(2)_L\oplus SU(2)_R$, while the nonlinear part has to do with the coset $SP(4)/H$. In the bulk the nonlinear part of the gauge invariance turns out to be expressed precisely as the Hamiltonian and diffeomorphism constraints of the theory. In the boundary theory the linear part tells us that the theory has a complete holographic formulation given in terms of states and operators constructed from an ordinary conformal field theory on the two dimensional spatial boundary. The nonlinear part of the gauge invariance, when extended to the boundary, gives rise to the Hamiltonian, that generates physical time evolution. This Hamiltonian respects the preferred time slicing of the boundary that was used for the construction of the boundary conditions on the finite boundary.

Thus, we have found that general relativity with a cosmological constant has, in the Lorentzian case, as well as the Euclidean case studied in [7], a holographic formulation when expressed in terms of finite boundaries. The hope in subsequent work will be to extended this to the $N=8$ supersymmetric case and by doing so obtain results relevant for a holographic formulation of $\mathcal{M}$ theory.

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