Regular (2+1)-dimensional black holes within nonlinear electrodynamics

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(2+1)-regular static black hole solutions with a nonlinear electric field are derived. The source to the Einstein equations is an energy momentum tensor of nonlinear electrodynamics, which satisfies the weak energy conditions and in the weak field limit becomes the (2+1)-Maxwell field tensor. The derived class of solutions is regular: the metric, curvature invariants, and electric field are regular everywhere. The metric becomes, for a vanishing parameter, the (2+1)-static charged BTZ solution. A general procedure to derive solutions for the static BTZ (2+1)-spacetime for any nonlinear Lagrangian depending on the electric field is formulated; for relevant electric fields one requires the fulfillment of the weak energy conditions.

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In general relativity the literature on regular black hole solutions is rather scarce [1,2]. In (3+1)-gravity it is well known that electrovacuum asymptotically flat metrics endowed with timelike and spacelike symmetries do not allow for the existence of regular black hole solutions. Nevertheless, in the vacuum plus the cosmological constant Λ case, the de Sitter solution [3] with a positive cosmological constant is known to be a regular nonasymptotically flat solution (the scalar curvature is equal to 4Λ and all the invariants of the conformal Weyl tensor are zero.) In order to be able to derive regular (black hole) gravitational-nonlinear electromagnetic fields one has to enlarge the class of electrodynamics to nonlinear ones [2]. On the other hand in (2+1)-gravity, which is being intensively studied in these last years [4–7], in the vacuum case all solutions are locally Minkowski (the Riemann tensor is zero); the extension to the vacuum plus cosmological constant allows for the existence of the rotating anti-de Sitter regular black hole [4] (the scalar curvature and the Ricci square invariants are constants proportional to Λ and Λ^2.) The static (2+1)-charged black hole with cosmological constant [the static charged Banados-Teitelboim-Zenelli (BTZ) solution] is singular (when r goes to zero the curvature and the Ricci square invariants blow up). Similarly as in the (3+1)-gravity, one may search for regular solutions in (2+1)-gravity incorporating nonlinear electromagnetic fields to which one imposes the weak energy conditions in order to have physically plausible matter fields. One can look for regular solutions with nonlinear electromagnetic fields of the Born-Infeld type [8–13] or electrodynamics of wider spectra.

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In this work, we are using electromagnetic Lagrangian $L(F)$ depending upon a single invariant $F=1/4F^{ab}F_{ab}$, which we demand in the weak field limit to be equal to the Maxwell Lagrangian $L(F)\rightarrow -F/4\pi$, the corresponding energy momentum tensor has to fulfill the weak energy conditions: for any timelike vector u^a, $u^a u_a=-1$ (we are using signature $-++$) one requires $T_{ab}u^a u^b \geqslant 0$ and $q_a q^a \leqslant 0$, where $q^a=T^a_b u^b$.

The action of the (2+1)-Einstein theory coupled with nonlinear electrodynamics is given by

$$S=\int \sqrt{-g}\left(\frac{1}{16\pi}(R-2\Lambda)+L(F)\right)d^3x, \tag{1}$$

with the electromagnetic Lagrangian $L(F)$ unspecified explicitly at this stage. We are using units in which $c=G=1$. The ambiguity in the definition of the gravitational constant [there is not Newtonian gravitational limit in (2+1)-dimensions] allows us to maintain the factor $1/16\pi$ in the action to keep the parallelism with (3+1)-gravity. Varying this action with respect to gravitational field gives the Einstein equations

$$G_{ab}+\Lambda g_{ab}=8\pi T_{ab}, \tag{2}$$

$$T_{ab}=g_{ab}L(F)-F_{ac}F_b{}^c L_{,F}, \tag{3}$$

while the variation with respect to the electromagnetic potential A_a entering in $F_{ab}=A_{b,a}-A_{a,b}$, yields the electromagnetic field equations

$$\nabla_a(F^{ab}L_{,F})=0, \tag{4}$$

where $L_{,F}$ stands for the derivative of $L(F)$ with respect to F.

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Concrete solutions to the dynamical equations above we present for the static metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2, \tag{5}$$

where $f(r)$ is an unknown function of the variable r. We restrict the electric field to be

$$F_{ab} = E(r)(\delta_a^t \delta_b^r - \delta_a^r \delta_b^t). \tag{6}$$

The invariant F then is given by

$$2F = -E^2(r), \tag{7}$$

thus the electric field can be expressed in terms of the invariant F. Substituting Eqs. (6) and (7) into the electromagnetic field equations (4) we arrive at

$$E(r)L_{,F} = \frac{e}{r}, \tag{8}$$

where e is an integration constant. We choose $e = -q/4\pi$ in order to obtain the Maxwell limit. Then we have

$$E(r)L_{,F} = -\frac{q}{4\pi r}. \tag{9}$$

Using now Eq. (7) we express the derivative L_F as function of r, as follows:

$$L_{,r} = \frac{q}{4\pi r} E_{,r}. \tag{10}$$

We rewrite the Einstein's equations equivalently as

$$R_{ab} = 8\pi(T_{ab} - Tg_{ab}) + 2\Lambda g_{ab}. \tag{11}$$

From Eq. (3) using Eqs. (6) and (7) the trace becomes

$$T = 3L(F) + 2E^2(r)L_{,F}. \tag{12}$$

As it was above pointed out, the Lagrangian $L(F)$ must satisfy (i) correspondence to Maxwell theory, i.e. $L(F) \to -L/4\pi$, and (ii) the weak energy conditions: $T_{ab}u^a u^b \geqslant 0$ and $q_a q^a \leqslant 0$, where $q^a = T^a_b u^b$ for any timelike vector u^a; in our case the first inequality requires

$$-(L + E^2 L_{,F}) \geqslant 0, \tag{13}$$

which can be stated equivalently as

$$L \leqslant EL_{,E} \to L \leqslant \frac{q}{4\pi r}E. \tag{14}$$

The norm of the energy flux q_a, occurs to be always less or equal to zero; for u^a along the time coordinate, $u^a = \delta_t^a/\sqrt{-g_{tt}}$, one has the inequality $q_a q^a = -(L + L_{,F}E^2)^2 \leqslant 0$.

Assume now that one were taking into account additionally the scalar magnetic field $B := F_{\phi r}$, then the Maxwell equations would be

$$\frac{d}{dr}[rEL_{,F}] = 0, \quad \frac{d}{dr}\frac{f}{r}BL_{,F} = 0. \tag{15}$$

On the other hand, the Ricci tensor components, evaluated for the BTZ metric (5), would yield the following relation:

$$A := R_{tt} + f^2 R_{rr} = 0, \tag{16}$$

while the evaluation of the same relation using the electromagnetic energy-momentum would give

$$A = -8\pi L_{,F}\left(\frac{f}{r}B\right)^2. \tag{17}$$

Therefore, the scalar magnetic field should be equated to zero, $B = 0$, thus the only case to be treated is just the one with the electric field E.

As far as the Einstein equations are concerned, the $R_{tt}(= -f^2 R_{rr})$ and $R_{\Omega\Omega}$ components yield respectively the equations

$$f_{,rr} + \frac{f_{,r}}{r} = -4\Lambda + 16\pi(2L(F) + E^2 L_{,F}), \tag{18}$$

$$f_{,r} = -2\Lambda r + 16\pi r(L(F) + E^2 L_{,F}). \tag{19}$$

If one replaces $f_{,r}$ from Eq. (19) and its derivative $f_{,rr}$ into Eq. (18) one arrives, taking into account Eq. (10), at an identity. Therefore one can forget Eq. (18) and integrate the relevant Einstein equation (19):

$$f(r) = -M - \Lambda r^2 + 16\pi \int r[L(F(r)) + E^2 L_{,F}]dr. \tag{20}$$

Summarizing we have obtained a wide class of solutions, depending on a Lagrangian $L(E)$, given by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\, d\Omega^2, \tag{21}$$

the structural function

$$f(r) = -M - (\Lambda - 2C)r^2 + 4q\int \left[r\int \frac{E_{,r}}{r}dr - E\right]dr, \tag{22}$$

which is obtained from Eq. (20) by using Eqs. (10) and (7), where C is a constant of integration, and the Lagrangian $L(E)$ is constrained to

$$L_{,r} = \frac{q}{4\pi r}E_{,r}. \tag{23}$$

We recall that the Lagrangian and the energy momentum tensor have to fulfill the conditions quoted above.

We present now various particular examples.

The static charged BTZ solution [4] is characterized by the function

$$-g_{tt}=f=-M+\frac{r^2}{l^2}-2q^2\ln r, \tag{24}$$

the Lagrangian and the electric field

$$L(E)=\frac{1}{8\pi}E^2=\frac{1}{8\pi}\frac{q^2}{r^2}, \quad E(r)=\frac{q}{r}, \tag{25}$$

where C has been equated to zero and $\Lambda=-1/l^2$. It is worthwhile to point out that the static charged BTZ black hole is singular at $r=0$.

Other interesting examples arise in the Born-Infeld electrodynamics—nonlinear charged (2+1)–black-hole [14]. In this case the structural function is

$$-g_{tt}=f=-M-(\Lambda-2b^2)r^2-2b^2r\sqrt{r^2+q^2/b^2} -2q^2\ln(r+\sqrt{r^2+q^2/b^2}), \tag{26}$$

and the Lagrangian and the electric field are given by

$$L(F)=-\frac{b^2}{4\pi}\left(\sqrt{1+2\frac{F}{b^2}}-1\right)=-\frac{b^2}{4\pi}\left(\frac{r}{\sqrt{r^2+q^2/b^2}}-1\right),$$

$$E(r)=\frac{q}{\sqrt{r^2+q^2/b^2}}, \tag{27}$$

where b is the Born-Infeld parameter, and $C=b^2$. This solution fulfills the weak energy conditions and it is singular at $r=0$. From the Ricci and Kretschmann scalars it follows that in this case there is a curvature singularity at $r=0$ [14].

A new class of solution, which is regular everywhere, is given by the structural function of the form

$$f(r)=-M-\Lambda r^2-q^2\ln(r^2+a^2) \tag{28}$$

where M, a, q, and Λ are free parameters. The Lagrangian and the electric field are given by

$$L(r)=\frac{q^2}{8\pi}\frac{(r^2-a^2)}{(r^2+a^2)^2},$$

$$E(r)=\frac{qr^3}{(r^2+a^2)^2}. \tag{29}$$

This Lagrangian requires to set $C=0$. The Lagrangian and the electric field satisfy the weak energy conditions (14). To express the Lagrangian in terms of F or equivalently E, one has to write r in terms of E by solving the quartic equation for $r(E)$, this will give rise to an explicit r containing radicals of E, which introduced in $L(r)$, finally will bring L as function of E. The expression $L(E)$ is not quite illuminating, thus we omit it here.

To establish that this solution is regular one has to evaluate the curvature invariants [17]. The nonvanishing curvature components, which occur to be regular at $r=0$, are given by

$$R_{0110}=\frac{q^2(a^2-r^2)}{(r^2+a^2)^2}+\Lambda, \tag{30}$$

$$R_{0202}=-f(r)\left(\frac{q^2r^2}{r^2+a^2}+\Lambda r^2\right), \tag{31}$$

$$R_{1212}=f(r)^{-1}\left(\frac{q^2r^2}{r^2+a^2}+\Lambda r^2\right), \tag{32}$$

where 0,1,2 stand respectively for t, r, and Ω.

Evaluating the invariants R, and $R_{ab}R^{ab}$ one has

$$R=\frac{2q^2(r^2+3a^2)}{(r^2+a^2)^2}+6\Lambda, \tag{33}$$

$$R_{ab}R^{ab}=12\Lambda^2+4q^4\frac{r^4+2r^2a^2+3a^4}{(r^2+a^2)^4}+\frac{8\Lambda q^2(3a^2+r^2)}{(r^2+a^2)^2}. \tag{34}$$

Since the metric, the electric field and these invariants behave regularly for all values of r, we conclude that this solution is curvature regular everywhere. Nevertheless, for solutions without any horizon or black hole solutions with an inner and outer horizon, at $r=0$ a conical singularity may arise.

At $r=0$ the function $f(r)$ becomes $f(0)=-M-q^2\ln(a^2)$.

Thus for M positive, $M>0$, and a in the range $0<a<1$, the value of $f(0)$ will be $f(0)=-M+q^2\ln(1/a)^2$, which will be positive, say $f(0):=\beta^2$, if $\ln(1/a)^2>M/q^2$. In such a case, for $0<\beta<1$ the solutions will show angular deficit since the angular variable Ω, which originally runs $0\leqslant\Omega<2\pi$ will now run $0\leqslant\Omega<2\beta\pi$; the parameter a can be expressed in terms of β, q, and M as $a^2=\exp[-(\beta^2+M)/q^2]$. For $\beta=1$, there will be no angular deficit, the ratio of the perimeter of a small circle around $r=0$ to its radius, as this last tends to zero, will be 2π.

If one allows M to be negative, $M<0$, and a to take values in the interval $0<a<1$, then $f(0)$ will be always positive, in this case one can adopt the following parametrization: $-M=\beta^2\cos^2\alpha$, $q^2\ln(1/a)^2=\beta^2\sin^2\alpha$, therefore $f(0)=\beta^2$. One will have angular deficit if $0<\beta<1$, and for $\beta=1$ the resulting $(2+1)$ space-time will be free of singularities. Another possibility with positive $f(0)=\beta^2$ arises for $M<0$, and $a>1$, $f(0)$ can be parametrized as $-M=\beta^2\cosh^2\alpha$, $q^2\ln(1/a)^2=\beta^2\sinh^2\alpha$. Again the values taken by β will govern the existence of angular deficit, for $\beta=1$ the solutions will be regular.

If $f(0)$ is negative, $f(0)=:-\beta^2$, the character of the coordinates t and r changes, the coordinate t becomes spacelike, while r is now timelike and one could think of the singularities, if any, as causal structure singularities because they could arise at the ''time'' $r=0$.

In what follows we shall treat the parameter a as a free one, having in mind the above restrictions to have solutions free of conical singularities.

To establish that this solution represents a black hole, one has to demonstrate the existence of horizons, which require the vanishing of the g_{tt} component, i.e., $f(r)=0$. The roots of this equation give the location of the horizons (inner and outer in our case). The roots—at most four—of the equation $f(r)=0$ can be expressed in terms of the Lambert $W(r)$ function

$$r_{1,2,3,4}=\pm\left[\exp\left(\frac{\Lambda a^2-M}{q^2}\right.\right.\\ \left.\left.-LW\left[\frac{\Lambda}{q^2}\exp\left(\frac{\Lambda a^2-M}{q^2}\right)\right]\right)-a^2\right]^{1/2}. \quad (35)$$

There arise various cases which depend upon the values of the parameters: four real roots (two positive and two negative roots, the negative roots have to be ignored), two complex and two real roots, two complex and one real positive root (the extreme case), and four complex roots (no black holes solutions). Although this analytical expression for the Lambert function can be used in all calculations, [we recall that Lambert function fulfills the following equation: $\ln(LW(x))+LW(x)=\ln(x)$], it occurs also useful to extract information from the graphical behavior of the our $f(r)$ (see figures).

Analytically one can completely treat the extreme black hole case; for it, the derivative of $f(r)$ has to be zero, $\partial_r(f(r))=0$, at the r_{extr}, this gives

$$r_{extr}=\sqrt{-a^2-\frac{q^2}{\Lambda}}>0 \quad (36)$$

for $\Lambda<0$. From this expression one concludes that the following inequality holds: $a^2<-q^2/\Lambda$. Entering now r_{extr} into $f(r)=0$ one obtains a relation between the parameters involved, which can be solved explicitly for the mass—the extreme one—

$$M_{extr}=a^2\Lambda+q^2\left(1+\ln\left[\frac{-\Lambda}{q^2}\right]\right), \quad (37)$$

this M_{extr} varies its values depending on the values given to the parameters a, q, and Λ. We have an extreme black hole characterized by negative cosmological constant, $\Lambda<0$, and positive extreme mass, $M_{extr}>0$, if the parameter a is restricted by the inequality $a^2<-(q^2[1+\ln(-\Lambda/q^2)])/\Lambda$.

For other values of the mass M, one distinguishes the following branches: if $M>M_{extr}$ one has a black hole solution, and if $M<M_{extr}$ there are no horizons.

In Fig. 1 we draw the graph of $f(r)$ which corresponds to regular solutions for fixed values of M and changing the values of the parameters Λ, q. In Fig. 2 we draw the graph of $f(r)$ corresponding to solutions which exhibit a conical singularity at $r=0$, for $f(0)=1/2$, keeping M fixed while Λ and q change.

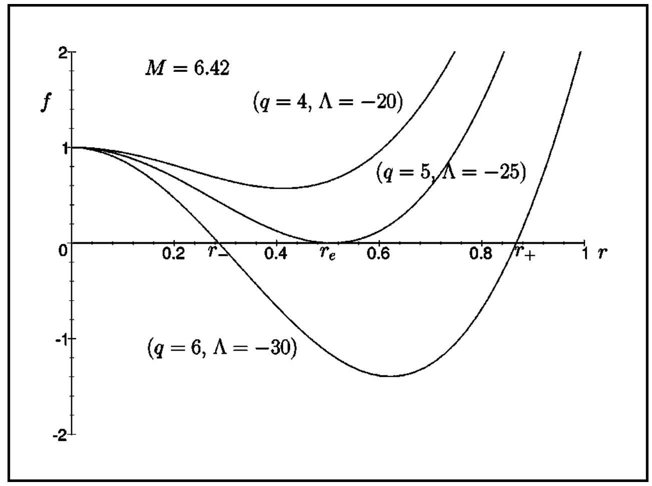

FIG. 1. Behavior of $-g_{tt}$ for $M=6.421$ and for different values of q and Λ corresponding to regular black hole ($q=6$, $\Lambda=-30$), regular extreme black hole ($q=5$, $\Lambda=-25$), horizon-free regular solution ($q=4$, $\Lambda=-20$), where $r_-=0.28$ is the inner horizon, $r_e=0.50$ is the extreme horizon, and $r_+=0.86$ is the event horizon.

If one were interested in the thermodynamics of the obtained solution one would evaluate the temperature of the black hole, which is given in terms of its surface gravity by [15,16]

$$k_B T_H=\frac{\hbar}{2\pi}k. \quad (38)$$

In general, for a spherically symmetric [and for circularly symmetric in $(2+1)$-dimensions] system the surface gravity can be computed via (for our signature)

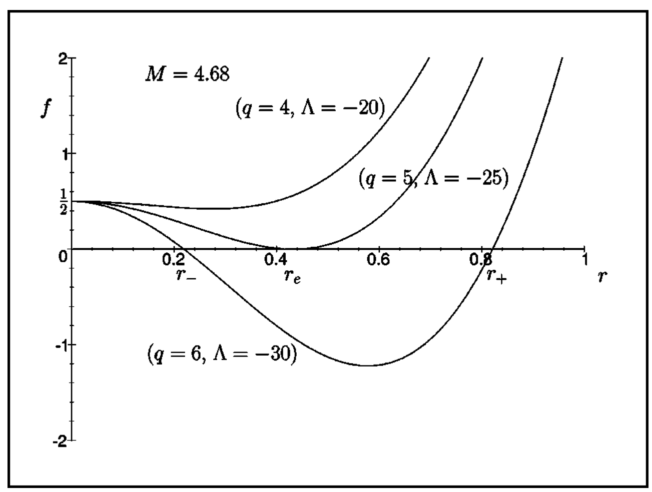

FIG. 2. Behavior of $-g_{tt}$ for $M=4.68$ and for different values of q and Λ for solutions with angular deficit $0\leqslant\Omega<\pi/\sqrt{2}$ corresponding to conical singular black hole ($q=6$, $\Lambda=-30$), conical singular extreme black hole ($q=5$, $\Lambda=-25$), and conical naked singular solution ($q=4$, $\Lambda=-20$), where $r_-=0.21$ is the inner horizon, $r_e=0.43$ is the extreme horizon, and $r_+=0.82$ is the event horizon.

$$k = -\lim_{r \to r_+} \left[\frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt} g_{rr}}} \right], \tag{39}$$

where r_+ is the outermost horizon. For our solution we have from Eqs. (22), (38), and (39) that

$$k_B T = \frac{\hbar}{2\pi} \left(-\Lambda r_+ - \frac{q^2 r_+}{r_+^2 + a^2} \right). \tag{40}$$

Since in our case there is no an analytical expression of r_+ in terms of elementary functions, one cannot give a parameter dependent expression of Eq. (40). It is easy to check that when $q=0$, T in Eq. (40) reduces to the BTZ temperature. In the extreme case (36), the temperature vanishes in Eq. (40). The entropy can be trivially obtained using the entropy formula $S = 4\pi r_+$. Other thermodynamic quantities such as heat capacity and chemical potential can be computed as in [16].

To achieve the maximal extension of our regular black solutions one has to follow step by step the procedure presented in [17] determining first the Kruskal-Szekeres coordinates, and to proceed further to draw the Penrose diagrams.

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