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How much energy do closed timelike curves in 2+1 spacetimes need?

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By noticing that, in open $2+1$ gravity, polarized surfaces cannot converge in the presence of timelike total energy momentum (except for a rotation of 2π), we give a simple argument which shows that, quite generally, closed timelike curves cannot exist in the presence of such an energy condition.

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I. ENERGY CONDITIONS AND POLARIZED SURFACES

There exist different types of causality violations in general relativity (GR). One of them corresponds to spacetimes such as Gödel's universe, where there are closed timelike curves (CTC) passing through each point of spacetime. The causality violation set is not a result of the evolution of certain initial data, but rather it has existed ''forever.'' There is certain evidence, provided by the fact that ''we are not being invaded by hordes of tourists coming from the future,'' that our universe is not of this kind.

Nevertheless, GR allows for causality violations that ''have not existed forever,'' but, instead, are generated through spacetime evolution. In these cases there exists a Cauchy horizon $\mathcal{H}$ (we shall always refer to, say, future Cauchy horizons; the case of past Cauchy horizons is, of course, identical), that can be compactly generated or not. $\mathcal{H}$ is said to be compactly generated (CGCH) if its generators, when directed to the past, always enter a compact region and remain there forever. A spacetime with a CGCH is a possible characterization of time machines for the following two reasons: If an otherwise causally well behaved spacetime is changed in a compact region such that a Cauchy horizon $\mathcal{H}$ appears as a result, then $\mathcal{H}$ is compactly generated [1]; conversely, a CGCH violates strong causality [2].

For obvious reasons, it is interesting to know under what conditions $\mathcal{H}$ can be nonempty. It can be seen that the weak energy condition (WEC) must be violated in an open spacetime with a CGCH [1]. In this sense, the construction of this kind of time machine needs ''quantum matter'' (or the simultaneous creation of a singularity). There is a large amount of semiclassical work in this direction, which we do not intend to review here.

Spacetimes with noncompactly generated Cauchy horizons are allowed by classical GR (as opposed to compactly generated ones), but it is not clear under which conditions they should or should not exist.

We can make progress along these lines working in symmetric models, such that we can reduce the problem to one in $2+1$ gravity. Thus, in what follows we shall restrict ourselves to these low dimensional models, moreover to open ones, i.e., with noncompact (and simply connected) spatial sections (typically, $\mathcal{R}^2$). A possible definition of energy momentum (EM) in these spacetimes is via holonomies. In this way, the total EM is timelike, spacelike, or null, according to whether parallel transport of vectors around loops that enclose all the matter is defined by a rotation, a boost, or a null rotation, respectively. The following results can be obtained in $2+1$ [3]: Under quite general conditions, a CGCH not only violates strong causality, but also stable causality, since there exists at least one closed null geodesic (this does not necessarily occur in $3+1$, as emphasized in Ref. [4]). Even if one allows for WEC violations, under certain conditions on the relationship between positive and negative masses, a CGCH cannot exist if the total EM is timelike (except when it is a rotation of 2π).

The aim of this paper is to give a result similar to the last one mentioned above, but for positive masses and noncompactly generated horizons. Namely, the original calculations of Gott [5] showed that in the spacetime of two particles that gravitationally scatter each other, certain inequality that involves the masses and velocities of the particles is sufficient for the existence of closed timelike curves (CTCs). That this inequality is also a necessary condition can be seen from Cutler's analysis on the global structure of these spacetimes [6]. This inequality, in turn, can be reexpressed as spacelike total EM , and, in summary, the spacetime of two particles does not have CTCs if the EM is not spacelike (this has been first noticed by Deser, Jackiw, and t' Hooft [1]). Kabat [7] has further analyzed systems with more particles, and conjectured that as a general property, CTCs cannot exist in the absence of spacelike EM. Menotti and Seminara [8] have given a proof of this conjecture for systems with rotational symmetry, but, unfortunately, this assumption does not hold either in solutions such as Gott's or in others with different particles. Headrick and Gott [9] have also shown a result related to Kabat's conjecture: if a CTC is deformable to infinity, then its holonomy cannot be timelike, except for a rotation of 2π.

Below we give an argument which shows that, quite generally, this conjecture is true. Basically, the argument is the following: if a Cauchy horizon exists, it can be obtained as a limit of polarized surfaces; on the other hand, these surfaces cannot converge if the total EM is timelike (except when it is a rotation of 2π), leading in this way to a contradiction.

The rest of this paper is devoted to a more detailed description of this simple idea, and heavily relies on the works of Cutler [6] and of Carroll *et al.* [10], to which the reader can refer for further details. Also, some of the tools here used are of the kind of those used in Ref. [3], but in that reference the exposition is somewhat more detailed. Along this work

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we implicitly use some basic properties of curves, CTCs, and causality that can be seen in, e.g., Refs. [11] or [12]. Finally, a comprehensive review of CTCs in $2+1$ can be found in Ref. [9].

The notion of polarized surfaces was originally introduced by Kim and Thorne in their analysis of vacuum fluctuations and wormholes [13], and it is widely used in works that study the stability/instability of Cauchy horizons under quantum test fields.

The nth polarized surface $\Sigma(n)$ is defined as the set of points through which passes a self-intersecting null geodesic (SNG), i.e., a null geodesic that returns to the same point of spacetime, but possibly with a different tangent vector, that circles the system n times (this is made explicit below). Its utility as a ''Cauchy horizon finder'' is a consequence of the following property:

$$\lim_{n\to\infty} \Sigma(n) = \mathcal{H}. \tag{1}$$

Cutler has used this criterion to obtain the global structure of Gott's spacetime, and it has survived a nontrivial check of self-consistency, since Cutler finds that the region where there are CTCs disappears if the total EM is not spacelike, a fact that is known from other sources (e.g., a time function can be globally defined).

For simplicity, let us start discussing the case of two particles; the generalization will be straightforward. Let us suppose that there are CTCs in this spacetime, restricted to a region delimited by a Cauchy horizon $\mathcal{H}$. It is easy to see that the CTCs must circle both particles. Thus, the CTCs can be characterized by the number of times they encircle them, the winding number n. The same holds for the SNGs, and the nth polarized surface $\Sigma(n)$ is, thus, defined as the set of points through which passes a SNG with winding number n.

We first choose a point $q\in\mathcal{H}$ and a curve γ which starts at q and ends at some point p_1, and is completely contained in the region which contains CTCs (except for q, which is not in the region of CTCs but, rather, in its boundary) . That is, $\gamma:[0,1]\to\mathcal{M}$, with $\mathcal{M}$ the spacetime manifold, such that $\gamma(0)=q$ and $\gamma(1)=p_1$. Since p_1 is in the region where there are CTCs, there exists a CTC $\mathcal{C}_1$ that passes through p_1; this CTC circles, say, n times the pair of particles. We now approach p_1 to q along γ, while smoothly deforming the entire curve $\mathcal{C}_1$, keeping n fixed. At a certain point, this deformation will no longer be possible, and the curve that we were deforming will result in a SNG $\mathcal{G}_n$ that starts and ends at a point $q_1\in\Sigma(n)$ (it is not clear when this procedure will converge to a closed curve, but if it does, one can see that it must converge to a SNG). We now take $\mathcal{C}_1$ and we move along it twice, obtaining a curve $\mathcal{C}_2$ that passes through $p_2(=p_1)$. Repeating the whole procedure, we obtain $\mathcal{G}_{2n}$ and a point $q_2\in\Sigma(2n)$ that is closer to $\mathcal{H}$, i.e., there exists a neighborhood $\mathcal{O}$ of q, such that $q_2\in\mathcal{O}$ but $q_1\notin\mathcal{O}$. Thus, a point $q_n\in\Sigma(n)$ will be closer (in the topological sense just mentioned) to $\mathcal{H}$ than another one $q_m\in\Sigma(m)$ with $m<n$. Thus, the succession $\{q_n\}$ converges to q, and, in this way, one expects that Eq. (1) holds. So we find that as a necessary condition for the existence of CTCs, the polarized surfaces should converge.

In the process $\Sigma(n)\to\mathcal{H}$, the initial tangent to the SNG $\mathcal{G}_n$, $k_n^{(i)}$, and the final one, $k_n^{(f)}$, must approach the tangent to the horizon, k (which is a null vector, since $\mathcal{H}$ is a null hypersurface). That is, $k_n^{(f)}\to k$ and $k_n^{(i)}\to k$. Since $\mathcal{G}_n$ is a geodesic, its tangent is parallel transported, i.e., $k_n^{(f)}=Ak_n^{(i)}$, with $A\in\mathcal{SO}(2,1)$. The crucial point is that $A=\mathcal{L}^n$, with $\mathcal{L}$ the holonomic operator that defines the total EM. Now, k must be a fixed null direction, i.e., a null eigenvector of $\mathcal{L}$. So $\mathcal{L}$ must have at least one null eigenvector. It is easy to see that if $\mathcal{L}$ is spacelike or null, it has two and one null eigenvectors, respectively; and if $\mathcal{L}$ is timelike it has no null eigenvector, except when it is the identity (which must correspond to a rotation of 2π, because for a rotation of angle zero the spacetime would be the well behaved vacuum flat metric). Thus, if the total EM is timelike and it is not the identity, there cannot be any fixed null directions and we have reached a contradiction and arrived at our main result.

For more general situations, e.g., if there are an arbitrary number of particles, one must first recall that every subsystem has timelike EM if the total EM is timelike [10]. With this property in hand, one can then repeat the whole construction and show that the polarized surfaces cannot converge if the total EM is timelike.

The property that the evolution of data with timelike total EM is free of singularities and/or Cauchy horizons seems to be a general feature that is not even restricted to particlelike solutions, but that, instead, also holds for fields coupled to gravity (the simplest case of this statement being Einstein-Rosen waves, or a massless scalar field coupled to gravity, if seen as a $2+1$ system). A rigorous proof for vacuum and electrovacuum with a G_2 group of symmetries is contained in the work of Berger *et al.* [14] (the condition of timelike total EM is not made explicit in Ref. [14], but it follows from the boundary conditions there imposed). It can be seen that if one has a universe with timelike total EM, that one needs to add some matter in order to make the total EM spacelike [10], and thus, this ''quite general'' property that CTCs need spacelike total EM gives a precise notion of how much energy is needed for causality violation. A similar result in $3+1$ would, of course, be of the greatest interest.

II. SOME FINAL COMMENTS

The argument that we gave as supporting the property of the polarized surfaces as ''finders'' of Cauchy horizons is essentially the original one of Kim and Thorne. Though it is widely used and it is usually expected to hold under very general conditions, to our knowledge there is no rigorous proof of it. Some parts of the analysis of the previous section are implicit in Cutler's work, so we now make contact with it. Cutler takes a point p through which a SNG passes and chooses two charts of inertial-like coordinates (one chart for each particle). He then explicitly calculates the map $k^i\to k^f$ as a nonlinear map $g(\phi)$ from the circle of null directions at p to itself, and uses the fact that g has two fixed points to obtain the tangent to the horizon (one fixed point corresponds to the tangent to the future horizon, and the other one to the past horizon) and reconstruct it using some symmetries of the spacetime. That is, his map g corresponds, essentially, to our map $\mathcal{L}$. We have here taken advantage of the fact that $\mathcal{L}$ is linear, to see under which conditions there are fixed null directions (the fixed points of g correspond to the null eigenvectors of $\mathcal{L}$); and we have noted that $\mathcal{L}$ defines the total EM, so that the two fixed points that Cutler finds do not depend on the details of the geometry of Gott's spacetime, but rather on the property that its total EM is spacelike.

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