PHYSICAL REVIEW D, VOLUME 61, 081502(R)

Critical phenomena and a new class of self-similar spherically symmetric perfect-fluid solutions

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(Received 13 January 1999; published 20 March 2000)

We consider the self-similar solutions associated with the critical behavior observed in the gravitational collapse of spherically symmetric perfect fluids with the equation of state $p=\alpha\mu$. We identify for the first time the global nature of these solutions and show that it is sensitive to the value of α. In particular, for $\alpha > 0.28$, we show that the critical solution is associated with a new class of asymptotically Minkowski self-similar spacetimes. We discuss some of the implications of this for critical phenomena.

PACS number(s): 04.20.Jb, 04.20.Cv, 04.40.Nr, 98.80.Hw

I. INTRODUCTION

One of the most exciting developments in general relativity in recent years has been the discovery of critical phenomena in gravitational collapse. For a variety of spherically symmetric imploding matter fields, there is a self-similar critical solution containing a naked singularity which separates models which collapse to black holes from those which disperse [1–3]. Sometimes a discrete similarity is involved but, in other circumstances, the critical solution seems to be represented by a continuously self-similar model. This is one which has a homothetic Killing vector and contains no dimensional constants. Perfect-fluid models of this kind necessarily have an equation of state of the form $p=\alpha\mu$ and so only in this case could the critical solution be homothetic.

Self-similar spherically symmetric perfect-fluid solutions have been much studied in general relativity (see [4] and references therein) and the attempt to understand critical phenomena has led to several further studies [5–9]. However, their precise relationship with the critical solution has remained obscure. This is mainly because the full family of such solutions had not been identified when critical phenomena were first discovered. However, recently Carr and Coley [10] have presented a complete asymptotic classification of such solutions. Furthermore, by reformulating the field equations for these models in terms of dynamical systems, Goliath et al. [11,12] have obtained a compact three-dimensional state space representation of the solutions and this leads to another complete picture of the solution space. These investigations have resulted in the discovery of a new class of ‘‘asymptotically Minkowski’’ self-similar spacetimes.

In this paper we shall discuss these new solutions and show why they are intimately related to critical phenomena [5]. We thereby demonstrate for the first time the *global* nature of the critical solution. Although the detailed derivation of these new solutions is given elsewhere, this is the first published announcement of their existence and the first attempt to link them to critical phenomena. The purpose of this paper is therefore to highlight this result in advance of the more extensive analyses (since these are in a much broader context). In particular, we will show how the global features relate to the equation of state parameter α and explain why there is only one such solution for each α. It should be emphasized that numerical studies of the critical solution are always restricted to some finite range of the self-similar variable z. However, as the critical index is approached, the extent of the self-similar region grows and one could in principle go to arbitrarily large values of z. This paper can therefore be regarded as predicting the characteristics of these solutions.

The discussion will mainly be in terms of the compact state space [11,12] but, to extract important physical features, some of the quantities used by Carr and Coley [10] will be plotted. No equations will be used because the discussion is intended to be purely qualitative and thereby accessible to the general reader. However, some technical terms will be used in the next section, so here we give some background references. For an introduction to dynamical systems theory in general relativity, see [13]; for its particular application in the spherically symmetric context, see [11,12]; for more details of the other types of self-similar solutions, see Carr *et al.* [14]. At the end of the paper, we emphasize our key predictions, so that ‘‘critical’’ workers can investigate these.

II. THE SOLUTION SPACE

We shall focus on spherically symmetric self-similar solutions in which the spacetime admits a homothetic Killing vector. This means that all dimensionless variables depend only on the self-similar variable $z\equiv r/t$, where r is the comoving radial coordinate and t is the time coordinate. In a dynamical systems approach these solutions correspond to

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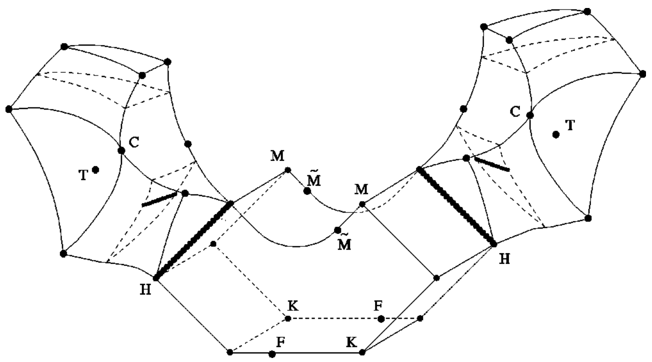

FIG. 1. The three-dimensional state space of self-similar spherically symmetric perfect-fluid models for $\alpha > 1/5$, obtained by matching a spatially self-similar region with two timelike self-similar regions along the lines H [14]. Labelled equilibrium points act as asymptotic states for orbits. The triangular surfaces indicated by dashed lines are sonic surfaces. Two of these surfaces contain a sonic line, indicated by a line of equilibrium points.

orbits in a three-dimensional compact state space. The state space of self-similar spherically symmetric perfect-fluid models (for $\alpha > 1/5$) is presented in Fig. 1. A point in this space corresponds to a certain geometry and matter field configuration on a homothetic (constant z) slice, while an orbit in the state space represents an entire spacetime. All continuous orbits are future and past asymptotic to one of a few solutions with higher symmetry. These appear as equilibrium points on the boundary of the state space and these points are labeled in Fig. 1. A physical description of the solutions asymptotic to them is given in Table I.

The state space is divided into two halves, one corresponding to positive z, the other to negative z. This means that the solutions in one half are the time reverse of solutions in the other half, so all equilibrium points appear twice in Fig. 1. The *sonic surfaces* are also depicted and solutions generally develop a shock wave here [15]. However, in two of the sonic surfaces there is a *sonic line* and solutions which pass through this line can be extended continuously through the sonic surface. Only these solutions will be considered to be physical and the number of such solutions is strongly restricted.

The state space has the advantage of giving a pictorial representation of the relationship between different solutions and the connection between the initial and final states, thereby yielding insights into the global nature of the solutions. However, it has the disadvantage that it is rather abstract. To better understand the physical aspects of the solutions, it is useful to consider some of the physically interesting quantities which arise in the comoving approach. The dependence of these quantities on z corresponds to two-dimensional projections of orbits in the full state space. Following [10], we use: (1) the scale factor S, which fixes the relation between the comoving radial coordinate r and the Schwarzschild radial coordinate $R=Sr$, thus indicating when a solution expands infinitely ($S\rightarrow\infty$) or encounters a singularity ($S\rightarrow 0$) for finite values of z; (2) the velocity V of the spheres of constant z relative to the fluid, which is important for the identification of event horizons ($|V|=1$) and

TABLE I. Interpretation of solutions asymptotic to the given equilibrium points.

Label	Interpretation
C	Solutions with a regular center or infinitely dispersed solutions
M, M̃	Asymptotically Minkowski solutions
K	Nonisotropic singularity solutions
F	Asymptotically Friedmann solutions
T	Exact static solution

naked singularities, see [16]; (3) the density profile μt^2, which gives the matter distribution at a given comoving time t; and (4) the mass function $2m/R$, where $m(r)$ is the mass within radial coordinate r, thus indicating the presence of an apparent horizon ($2m/R=1$), see [17].

We shall first briefly review the previously known families of solutions. All of these solutions are discussed in more detail in [14], where the dependence of the above functions on z are shown explicitly. We shall then consider the new asymptotically Minkowski solutions. This is the first discussion of their physical features and the first detailed analysis of their relevance to critical phenomena.

Asymptotically Friedmann solutions

There are two one-parameter sets of solutions that are asymptotic to the flat Friedmann solution, all of which are connected with one of the Friedmann points F. One has positive z and the other has negative z. Two qualitatively different families can be distinguished: (1) expanding-recollapsing solutions (F–K orbits); and (2) ever-expanding (or ever-contracting) solutions (F–C orbits), where C is to be interpreted as an infinitely dispersed state. The latter family contains the flat Friedmann solution itself. Thus the flat Friedmann solution appears both as an equilibrium point F and as an orbit in state space, corresponding to different slicings.

Asymptotically quasi-static solutions

For each value of α, there is a unique static solution, originally found by Tolman [18]. The corresponding (T–T) orbit traverses the entire state space and spans both positive and negative z. Furthermore, there is a two-parameter set of solutions with behavior resembling the static solution at large $|z|$ (i.e., near $t=0$). They are all associated with K points, corresponding to nonisotropic singularities. As with the asymptotically Friedmann solutions, there are two different families within this class: (1) expanding-recollapsing solutions (K–K orbits); and (2) ever-expanding (or ever-contracting) solutions (K–C orbits). The latter contain the naked-singularity solutions discussed by Ori and Piran [16] and Foglizzo and Henriksen [19]. Unlike the asymptotically Friedmann solutions, the asymptotically quasi-static solutions necessarily span both positive and negative z.

Asymptotically Minkowski solutions

When $\alpha > 1/5$, solutions exist that are ''asymptotically Minkowski,'' in the sense that the state-space orbits asymptote to equilibrium points that correspond to Minkowski space. There are actually two subclasses of such solutions,

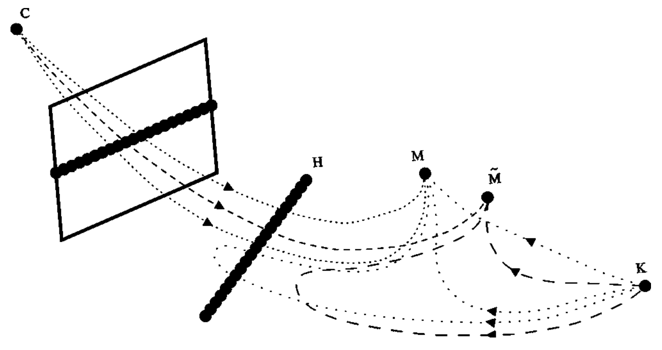

FIG. 2. Asymptotically Minkowski solutions of class A (dotted) and class B (dashed) as orbits in state space. Arrows go from C or K towards the infinitely dispersed state. Densely dotted and short-dashed curves correspond to regular solutions, while sparsely dotted and long-dashed curves correspond to singular solutions.

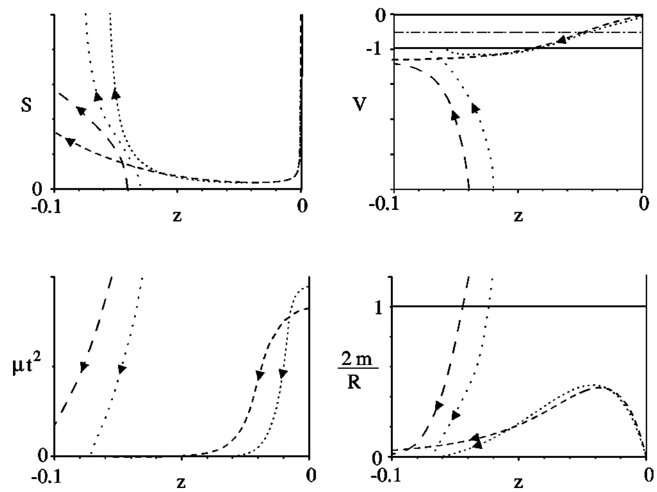

FIG. 3. Physical quantities for the asymptotically Minkowski solutions. The dash-dotted line in the $V(z)$ diagram corresponds to the sonic surface. Other designations are given in the caption of Fig. 2.

associated with different equilibrium points in state space: class A solutions are connected with the M points and are described by two parameters; class B solutions are connected with the $\tilde{\mathrm{M}}$ points and are described by one parameter. Both of these subclasses contain two different families of solutions: (1) singular solutions (K–M, K–$\tilde{\mathrm{M}}$ orbits); and (2) regular solutions (C–M, C–$\tilde{\mathrm{M}}$ orbits). Only the latter contain a sonic point. All of these types of solutions are illustrated in Figs. 2 and 3, where the arrows indicate whether solutions are future asymptotic to M or $\tilde{\mathrm{M}}$.

Solutions in class A have $V \to 1$, $S \to \infty$, $\mu t^2 \to 0$, and $2m/R \to 0$ at some *finite* value $z = z_*$. Although this limit is reached at finite z, it should be pointed out that most investigations of the critical solution use the physical distance R, which is infinite. Examples of such solutions are illustrated by the dotted curves in Figs. 2 and 3. Solutions in class B have $V \to V_* > 1$, $S \to \infty$, $\mu t^2 \to 0$, and $2m/R \to 0$ as $z \to \infty$. Examples of these solutions are represented by the dashed curves in Figs. 2 and 3. Both classes are asymptotically dispersive and solutions asymptotic to K points also have $S \to 0$ at a finite value of z, indicating the formation of a singularity in the past (assuming the time direction indicated in the figures). Even though $2m/R \to 0$ for both classes, the mass m need not vanish.

It should be emphasized that these solutions are not asymptotically flat in the usual global sense, in which there is a certain radial decay of the curvature towards spatial infinity, see, e.g., [20]. We are not considering an isolated system here but rather a fluid spacetime in which the Minkowski geometry is obtained asymptotically along certain coordinate lines. It can be shown that the curvature vanishes asymptotically as the M ($\tilde{\mathrm{M}}$) point is approached. Because the fluid becomes infinitely diluted, the situation is analogous to that of the open Friedmann solution, in which the Milne solution is approached asymptotically along certain timelines at late times (see, e.g., [13]).

III. THE CRITICAL SOLUTION

Critical phenomena in gravitational collapse were first studied by Choptuik [1] and remain an active field of research, see, e.g., [21] and references therein. The solution at the threshold of black-hole formation in spherically symmetric radiation fluid collapse, corresponding to $\alpha = \frac{1}{3}$, was studied by Evans and Coleman [5]. They found it to be a self-similar solution distinguished by the following criteria:

(1) It is everywhere analytic, or at least C^∞. In particular it has a regular center, and also crosses the sonic surface in an analytic way and;

(2) It has a collapsing interior surrounded by an expanding exterior. This means that the radial fluid three-velocity V_R associated with a Schwarzschild foliation (which is different from the function V) has exactly one zero.

Subsequently, other authors [7,8] have used these criteria to investigate the critical solution for other values of α. For a recent review, see [21].

The uniqueness of the critical solution can be understood as follows: For each value of the equation-of-state parameter α, there exists a one-parameter set of solutions with a regular center and a one-parameter set of solutions analytic at the sonic line ([4]). Thus it is not surprising that the first condition leads to a discrete set of solutions. There is only one solution in this set that satisfies the second condition and this is the critical solution.

We now examine the critical solution in terms of both the state space of the self-similar spherically symmetric perfect-fluid solutions and the behavior of the various physical quantities. The results are summarized in Figs. 4 and 5.

Starting from the regular center C, a numerical investigation shows that for all equations of state, the orbit of the critical solution passes through the sonic line and enters the spatially self-similar region (with $|V| > 1$). It turns out that for α in the range $0 < \alpha \lesssim 0.28$, it is of the asymptotically quasi-static kind; it passes through the spatially self-similar region and enters a second timelike self-similar region, finally reaching another sonic point (indicated by 'x' in the figures), which is generally irregular. However, this does not invalidate the solution as being the critical one, since the

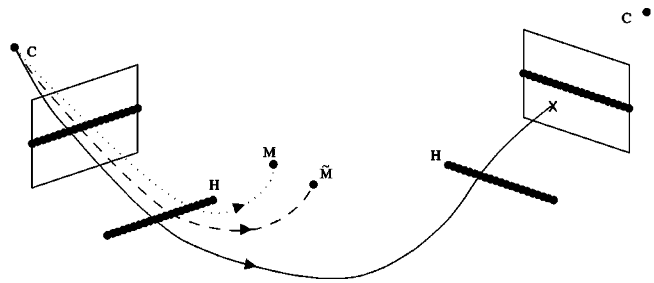

FIG. 4. Critical orbits in the state space. Arrows go from the regular center towards the infinitely dispersed state. Note that the orbits correspond to different equations of state, i.e., different values of α, and thus belong to different state spaces. The three orbits exemplify the cases $0<\alpha\lesssim 0.28$ (full curve), $\alpha\approx 0.28$ (dashed), and $0.28\lesssim\alpha<1$ (dotted).

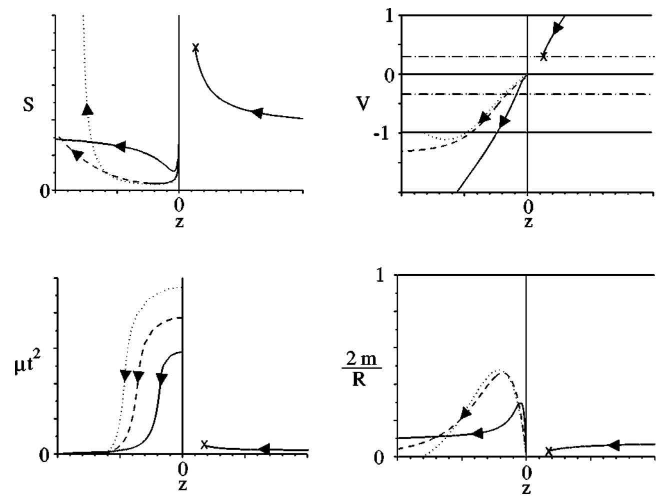

FIG. 5. Physical quantities for the critical solution. The dash-dotted lines in the $V(z)$ diagram correspond to the sonic surfaces. Other designations are given in the caption of Fig. 4. Note that for $\alpha\gtrsim 0.28$ (dotted), the critical solution tends to a finite value of z.

solution describing the inner collapsing region is usually matched to an asymptotically flat exterior geometry sufficiently far from the center. An example of a solution belonging to this class is given by the full curves in Figs. 4 and 5.

We find that for the limiting case $\alpha\approx 0.28$, the critical solution is an asymptotically Minkowski solution of class B, whose orbit ends at an equilibrium point $\tilde{\mathrm{M}}$. In Figs. 4 and 5, this solution corresponds to the dashed lines. For $0.28\lesssim\alpha<1$, we find that the critical solution belongs to the asymptotically Minkowski solutions of class A, whose orbit ends at an equilibrium point M. A solution representing this class is indicated by the dotted curves in Figs. 4 and 5.

For $\alpha\geqslant\alpha_*\approx 0.89$, the investigations in [12,7,8] indicate that the critical solution is already irregular at the first sonic point. As the matching must be performed outside the sonic point [16,19], the solution would then be unphysical. However, Neilsen and Choptuik [22] have recently demonstrated the existence of a regular critical solution for $\alpha\geqslant\alpha_*$ as well. Our present investigation supports their analysis.

To understand what happens for $\alpha=\alpha_*$, we consider equations of state near this value. It turns out that the behavior can be understood in terms of the stability near the sonic line. In order for a solution to be regular at the sonic surface, the corresponding orbit must approach the sonic line along one of (at most) two possible directions. Each of these directions is associated with an eigenvalue—the direction corresponding to the smaller eigenvalue is called *dominant* and is associated with a one-parameter family of solutions (containing just one C^∞ solution), the other is called *secondary* and is associated with an isolated solution. For $\alpha<\alpha_*$, the critical solution corresponds to the secondary direction. However, when α gets close to α_*, the eigenvalue associated with the critical solution approaches that of the other direction. For $\alpha=\alpha_*$, the eigenvalues (and directions) are equal, corresponding to a *degenerate node*, and for $\alpha>\alpha_*$, the critical solution is associated with the dominant direction. Thus a transition from the secondary direction to the dominant direction occurs at $\alpha=\alpha_*$. This transition results in severe numerical difficulties, so very high numerical precision is required to investigate such solutions, as pointed out in [22].

IV. CONCLUSIONS

Our key predictions can be summarized as follows:

(1) When one gets sufficiently close to the critical solution that the large-z behavior can be studied, then this solution should have the various asymptotic features we predict for different values of α.

(2) There should be a sudden transition in the nature of the critical solution as α passes through 0.28, with the solution going from the asymptotically quasi-static form to asymptotically Minkowski form. However, it should be emphasized that the solution is only flat null towards infinity; so one still needs to match to a non-self-similar region on a spacelike surface.

(3) Although the asymptotically flat limit is reached at finite z, most critical workers use the physical distance, which is infinite. However, it should be pointed out that in the stiff case $(\alpha=1)$, the asymptotically flat state is reached at *finite* physical distance, which should lead to some anomalies, see [10]).

(4) Although we have not explained why the critical solution is analytic at the sonic point (this presumably relates to the usual stability criterion), we have used the asymptotic features to explain why the analytic solution is unique for given α. The relationship to solutions which are regular but not analytic at the sonic point is discussed in more detail by Carr and Henriksen [23].

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