Exact solution for the exterior field of a rotating neutron star

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A four-parameter class of exact asymptotically flat solutions of the Einstein-Maxwell equations involving only rational functions is presented. It is able to describe the exterior field of a slowly or rapidly rotating neutron star with a poloidal magnetic field.

I. INTRODUCTION

Observations of binary pulsars [1] indicate that the individual *neutron stars* (NS) in such systems have masses very close to the Chandrasekhar limit of $1.4M_\odot$ of white dwarfs. Theoretically, the issue of the maximum mass of a NS hinges strongly on the equation of state (EOS) and the particle interactions at the high density of the center. Models of NS with strong kaon condensation [2] or even a quark nucleus ("strange star" [3]) can have at most 1.5–$2M_\odot$, which would leave a range for lower mass black holes. However, even modest *differential rotation* [4] may easily increase the maximum mass $2M_\odot$ of a nonrotating NS to above $3M_\odot$ for a nascent NS in a transient phase of a supernova. Moreover, a mass-quadrupole moment Q is also important [5] for achieving correspondence with numerical results [6,7].

In order to analytically model the exterior field of a NS in the framework of *general relativity* (GR), one needs an *exact* asymptotically flat solution of the *Einstein-Maxwell equations* (electrovac spacetimes) possessing at least four arbitrary physical parameters which are the mass M, angular momentum J, magnetic dipole μ, and mass-quadrupole moment Q. The simplest solution, besides, can be envisaged as axially symmetric, the magnetic field sharing the symmetry of the mass and angular momentum distributions; an additional reflection symmetry with respect to the equatorial plane which is expected of the self-gravitating objects (see, e.g., [8] and references therein) must also be imposed.

Our paper aims to present an exact solution which does satisfy the above requirements and, most importantly, one that is a mathematically very *simple* solution admitting a representation exclusively in terms of the rational functions of spheroidal coordinates (previous effort in this direction only led either to the solution which had no reflection symmetry [9] or to solutions [10] which did not permit the ratio-

nal function representation, and consequently could not be written in a concise form).

II. FOUR-PARAMETER EXACT SOLUTION

The reported solution has been constructed with the aid of Sibgatullin's method [11] according to which the complex potentials $\mathcal{E}$ and Φ satisfying Ernst's equations [12] are defined, for specified axis data $e(z) := \mathcal{E}(z, \rho=0)$ and $f(z) := \Phi(z, \rho=0)$, by the integrals

$$\mathcal{E}(z,\rho) = \frac{1}{\pi} \int_{-1}^{1} \frac{e(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}},$$

$$\Phi(z,\rho) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}}. \tag{1}$$

The unknown function $\mu(\sigma)$ is to be found from the singular integral equation

$$\fint_{-1}^{1} \frac{\mu(\sigma)[e(\xi)+\tilde{e}(\eta)+2f(\xi)\tilde{f}(\eta)]d\sigma}{(\sigma-\tau)\sqrt{1-\sigma^2}} = 0 \tag{2}$$

with the normalizing condition

$$\int_{-1}^{1} \frac{\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}} = \pi, \tag{3}$$

where $\xi = z + i\rho\sigma$, $\eta = z + i\rho\tau$, with ρ and z as the Weyl-Papapetrou cylindrical coordinates and $\sigma, \tau \in [-1,1]$; $\tilde{e}(\eta) := e(\bar{\eta})$, $\tilde{f}(\eta) := f(\bar{\eta})$, with the overbar signifying complex conjugation.

In what follows, the axis data $e(z)$ and $f(z)$ are chosen in the form

$$e(z) = \frac{(z-M-ia)(z+ib)+d-\delta-ab}{(z+M-ia)(z+ib)+d-\delta-ab},$$

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$$f(z) = \frac{i\mu}{(z+M-ia)(z+ib)+d-\delta-ab},$$

$$\delta := \frac{\mu^2 - M^2 b^2}{M^2 - (a-b)^2}, \quad d := \frac{1}{4}[M^2 - (a-b)^2], \quad (4)$$

such that the algebraic equation

$$e(z) + \bar{e}(z) + 2f(z)\bar{f}(z) = 0 \quad (5)$$

will have a pair of distinct roots of multiplicity two. This is a key point for having the rational form of the final expressions for $\mathcal{E}(\rho,z)$ and $\Phi(\rho,z)$ after performing the Riemann-Hilbert procedure of the analytic continuation of the functions $e(z)$, $f(z)$ into the complex plane (ρ,z). The resulting expressions for the potentials $\mathcal{E}(\rho,z)$ and $\Phi(\rho,z)$ obtained from Eqs. (1)–(5) are of the following *polynomial* form:[1]

$$\mathcal{E} = (A - 2MB)/(A + 2MB), \quad \Phi = 2i\mu C/(A + 2MB),$$

$$A = 4[(k^2x^2 - \delta y^2)^2 - d^2 - ik^3 xy(a-b)(x^2-1)] - (1-y^2)$$
$$\times [(a-b)(d-\delta) - M^2 b][(a-b)(y^2+1) + 4ikxy],$$

$$B = kx\{2k^2(x^2-1) + [b(a-b)+2\delta](1-y^2)\}$$
$$+ iy\{2k^2 b(x^2-1) - [k^2(a-b) - M^2 b - 2a\delta](1-y^2)\},$$

$$C = 2k^2 y(x^2-1) + [2\delta y - ikx(a-b)](1-y^2), \quad (6)$$

where we have introduced the generalized spheroidal coordinates

$$x = \frac{1}{2k}(r_+ + r_-) \quad \text{and} \quad y = \frac{1}{2k}(r_+ - r_-),$$

$$r_\pm := \sqrt{\rho^2 + (z \pm k)^2}, \quad k := \sqrt{d+\delta}. \quad (7)$$

The four arbitrary real parameters entering the solution are the total mass M, total angular momentum per unit mass $a := J/M$, magnetic dipole moment μ, and mass-quadrupole moment

$$Q = -\frac{M}{4[M^2 - (a-b)^2]}[M^4 + 2M^2(a^2+b^2)$$
$$- (3a+b)(a-b)^3 - 4\mu^2] \quad (8)$$

of the source.

The corresponding complete metric is given by the axisymmetric line element[2]

$$ds^2 = -f(dt - \omega d\varphi)^2 + k^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\right.$$

$$\left.\times\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right], \quad (9)$$

in which the metric coefficients f, γ, and ω are the following rational functions of the coordinates x and y (see, e.g., [14] for details of Sibgatullin's method):

$$f = E/D, \quad e^{2\gamma} = E/16k^8(x^2-y^2)^4, \quad \omega = (y^2-1)L/E,$$

$$E = \{4[k^2(x^2-1) + \delta(1-y^2)]^2 + (a-b)[(a-b)(d-\delta)$$
$$- M^2 b](1-y^2)^2\}^2 - 16k^2(x^2-1)(1-y^2)\{(a-b)$$
$$\times [k^2(x^2-y^2) + 2\delta y^2] + M^2 by^2\}^2,$$

$$D = \{4(k^2 x^2 - \delta y^2)^2 + 2kMx[2k^2(x^2-1) + (2\delta+ab-b^2)$$
$$\times(1-y^2)] + (a-b)[(a-b)(d-\delta) - M^2 b]$$
$$\times(y^4-1) - 4d^2\}^2 + 4y^2\{2k^2(x^2-1)[kx(a-b) - Mb]$$
$$- 2Mb\delta(1-y^2) + [(a-b)(k^2-2\delta) - M^2 b]$$
$$\times(2kx+M)(1-y^2)\}^2,$$

$$L = 8k^2(x^2-1)\{(a-b)[k^2(x^2-y^2) + 2\delta y^2] + M^2 by^2\}$$
$$\times\{kMx[(2kx+M)^2 - a^2 + b^2 - 2y^2(2\delta+ab-b^2)]$$
$$- 2y^2(4\delta d - M^2 b^2)\} - \{4[k^2(x^2-1) + \delta(1-y^2)]^2$$
$$+ (a-b)[(a-b)(d-\delta) - M^2 b](1-y^2)^2\}((1-y^2)$$
$$\times\{2M(2kx+M)[(a-b)(d-\delta) - b(M^2+2\delta)]$$
$$- 4M^2 b\delta + (a-b)(4\delta d - M^2 b^2)\} - 8k^2 Mb(kx+M)$$
$$\times(x^2-1)). \quad (10)$$

Special cases

Equations (6) and (10) admit several well-known classical limits.

(1) In the absence of the magnetic field and vanishing arbitrary quadrupole deformation, i.e., $\mu = b = 0$, only the Ernst potential $\mathcal{E}$ survives which is readily recognizable as that of the Tomimatsu-Sato $\delta = 2$ solution [15] with the mass quadrupole $Q = -\frac{1}{4}(M^3 + 3J^2/M)$.

(2)The stationary pure vacuum limit with a nonvanishing quadrupole parameter b is a particular three-parameter specialization of the Kinnersley-Chitre solution [16].

(3) The magnetostatic limit ($a = b = 0$) is represented by Bonnor's solution [17] for a massive magnetic dipole. For this solution, the quadrupole moment is $Q = \mu^2/M - \frac{1}{4}M^3$.

(4) Reduction to the Kerr metric [18] with total mass M and total angular momentum per unit mass $a := J/M$ is achieved by setting $\mu = 0$ and then formally choosing $b^2 = a^2 - M^2$. The values of M and a remain independent, in particular, $a^2 < M^2$ can be imposed since in this special limit the complex continuation $b \to ib$ is admitted. It should be stressed, however, that there exist general arguments [19] for

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[1]All of the formulas of this paper have been checked with the aid of the MATHEMATICA 3.0 computer program [13].

[2]Throughout the paper, natural units are used in which the gravitational constant and the velocity of light are equal to unity.
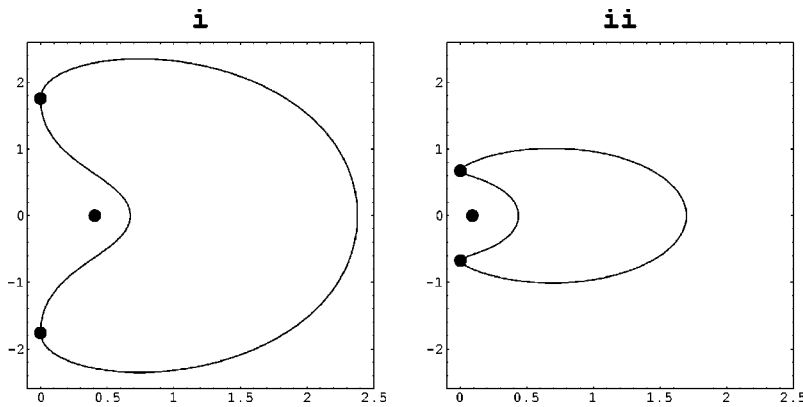
FIG. 1. Ergosphere and singularities: (i) M=4, a=2, b=0.9, μ=2; (ii) M=2, a=1.6, b=−0.2, μ=0.6 (m or m^2, in the natural units).

the interior of the Kerr metric consisting of a perfect fluid according to which a large rotational flattening of the body necessarily implies a large absolute value of the quadrupole moment.

(5) It is remarkable that the hyperextreme part of our solution corresponding to pure imaginary values of k belongs to the Chen-Guo-Ernst family of hyperextreme spacetimes [20]. This branch might represent exterior fields of *relativistic disks*. Their importance for astrophysics was shown by Bardeen and Wagoner [21], cf. also [22]. In the absence of the electromagnetic field an exact global solution for an infinitesimally thin disk of dust has been constructed by Neugebauer and Meinel [23], cf. [24].

Since neutron stars are known to be ''slowly'' rotating astrophysical objects [25] (even at the Kepler frequency $\omega_K = \sqrt{GM/R^3} \simeq 0.5$ ms of a millisecond pulsar the equator would only rotate with a speed of $v_K \simeq c/4$), it is the subextreme part of the metric (9) which should be used for their description. At the same time, we still need to know the location of singularities in our solution to support the physical interpretation we are attributing to it. In Fig. 1 we have plotted in coordinates ρ and z the typical shapes of the infinite redshift surface which one has for the real-valued k, the dots indicating the position of singularities. The two point singularities on the symmetry axis (the poles x=1, y = ±1) belong to the stationary limit surface, while the ring singularity in each case lies at the equatorial plane between the symmetry axis and ergosphere; no singularity outside the infinite red shift surface arises, and hence the metric (9) is describing the exterior fields of compact objects such as neutron stars.

We shall conclude the presentation of our solution by writing out Kinnersley's potential $\mathcal{K}$ [26] the real part of which gives the magnetic component A_ϕ of the electromagnetic four potential:

$$\mathcal{K} := A_\phi + iA_t' = \mu(1-y^2)K/(A+2MB),$$

$$K = 2k^2(x^2-1)[2kx+3M+iy(a-b)]$$
$$-(a-b)[2Ma-Mb(1-y^2)-4i\delta y]+2(2kx+M)$$
$$\times[\delta(1-y^2)+M(M-iby)]+4M\delta, \tag{11}$$

where A_t' is the electric component of a vector potential associated with the dual electromagnetic field tensor. The knowledge of A_ϕ is a necessary basis for the investigation of plasma-dynamical effects around neutron stars. In Fig. 2 the magnetic lines of force are plotted in cylindrical coordinates for two characteristic cases.

III. MATCHING TO NUMERICAL MODELS OF NEUTRON STARS

Our exact axisymmetric solution needs to be *matched* to interior solutions of neutron stars, in order to be realistic. Since the junction conditions on the surface of the NS depend very much on the material details such as equation of state, conductivity, etc., we will restrict ourselves here only to the identification of *asymptotically conserved quantities*. The identification of mass M and angular momentum J of our solution agrees already with the *standard* parameters of
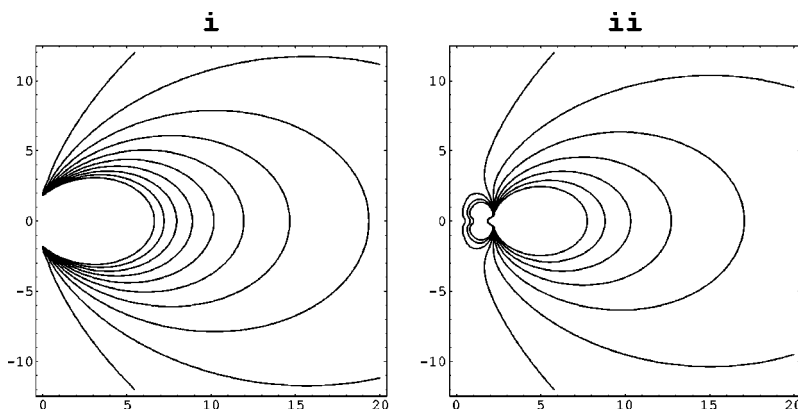

FIG. 2. Magnetic lines of force: (i) M=4, a=2, b=0.4, μ=1.2; (ii) M=2, a=4, b=−1, μ=3 (m or m^2, in the natural units).

asymptotically flat spacetimes in GR [27] and in the astrophysics [28] of NSs.

A. Magnetic field

Normally, the NS' magnetic field $\vec{B}$, predicted already in 1964 by Hoyle, Narlikar, and Wheeler [29] and reaching high values below the upper limit of $\vec{B} \leqslant 3 \times 10^9$ Tesla, is ignored in numerical studies [28] of rapidly rotating NS. More recently, however, axisymmetric solutions of the Einstein-Maxwell equations have been studied [30] using a pseudo-spectral method [31] involving Chebyshev-Legendre polynomials in terms of maximal slicing quasi-isotropic coordinates. Since the magnetic axis is aligned with the rotation axis and only poloidal fields are permitted, this numerical work is particularly suited for a comparison of the electrovac spacetime outside the NS with our exact solution:

For a star close to a sphere and small polar fields $\vec{B} \sim 10^6$ Tesla, these numerical results are within an error of 10^{-3} in agreement with Ferraro's solution [32]

$$A_\phi = 4\pi\rho_0 f_0 \frac{R_*^5}{15r} \sin^2\theta, \quad r > R_*, \tag{12}$$

where R_* is the radius of the star, ρ_0 is the mass density, and f_0 is the constant value taken by the electric current function [30].

The equation above enables us to see how the parameter μ of our solution may depend on the parameters of the corresponding interior metric for small values of Q. Indeed, introducing the Boyer-Lindquist-like coordinates R and θ via the formulas

$$kx = R - M, \quad y = \cos\theta, \tag{13}$$

we easily find from Eq. (11), in the limit $R \to \infty$, that

$$A_\phi = \frac{\mu\sin^2\theta}{R} + O\left(\frac{1}{R^2}\right) = \frac{\mu\sin^2\theta}{r} + O\left(\frac{1}{r^2}\right), \tag{14}$$

since R has a representation $R = r[1 + O(r^{-1})]$ in terms of the isotropic coordinate r [28]. From Eqs. (12) and (14) the desired relation of μ to the parameters determining the interior of a NS follows immediately.

B. Quadrupole moment

An operational way of defining the quadrupole moment of an axially symmetric body in GR has been developed by Ryan [33]. For NS one finds quite generally [7] that the numerical simulations are rather well accounted by the simple quadratic relation

$$Q = -c(M,\text{EOS})\frac{J^2}{M}, \tag{15}$$

where the constant $c = c(M,\text{EOS})$ depends only on the mass M and the equation of state (EOS) for the interior of the NS.

For NSs of 1.4 $M_\odot$ the range of this constant is $c = 2$ to 7.4. It is intriguing that this simple relation quadratic in J holds also for fast rotating NS.

If we reparametrize the arbitrary quadrupole parameter b of our exact solution by the dimensionless parameter Δ via $b = \pm\sqrt{a^2 + 2aM\Delta - M^2}$, we obtain from Eq. (8) for the quadrupole moment ($\mu = 0$):

$$Q = -\left(1 + \frac{1}{2}\Delta^2 - \frac{a\Delta^2(a\Delta^2 - M\Delta \pm \sqrt{a^2 + 2aM\Delta - M^2})}{2(M^2 - a^2 + a^2\Delta^2 - 2aM\Delta)}\right)$$
$$\times \frac{J^2}{M}. \tag{16}$$

Our reparametrization is such that for $\Delta = 0$ we recover the mass-quadrupole moment $Q = -J^2/M$ of the Kerr metric. Thus it is worth pointing out that the mass-quadrupole parameter in our solution is also intimately related with the angular momentum dipole and octupole moments, and this means that the *deformations* of the source are mainly due to *rotation*. In a particular case, for instance, when $M = 1.4M_\odot$ and $a = 0.625M_\odot$, the values of Δ covering the NS range $2 < c < 7.4$ are given by the interval $0.986 < \Delta < 2.068$.

In comparison with Eq. (15), the constant c for our solution depends not only on the mass M, but also on $\Delta > (M^2 - a^2)/2aM$ which can be adjusted to different EOS. The additional piece depending on angular momentum per unit mass, however, arises only in higher order of Δ. Thus the quadrupole moment of our exact solution accounts rather well to the simple quadratic law (15) of NSs. Moreover, in the ''extreme'' limit $a \to M$ we find

$$Q = -\left(1 - \frac{\Delta(\Delta \pm \sqrt{2\Delta})}{2(\Delta - 2)}\right)M^3, \tag{17}$$

from which it can be seen that Q can assume arbitrary values for any given value of M, unlike in the case of the extreme Kerr metric for which $Q = -M^3$. In this particular limit, the ''NS interval'' corresponds to $0.931 < \Delta < 1.584$.

Further study is needed to exhibit the astrophysical significance of our solution in more detail. In view of a simple analytical form of the new electrovacuum metric and clear physical interpretation of the characteristic parameters it possesses, it is anticipated that the solution will prove itself suitable for use in concrete astrophysical applications involving neutron stars.

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