Hard-thermal-loop resummation of the free energy of a quark-gluon plasma

Jens O. Andersen, Eric Braaten, and Michael Strickland*
Physics Department, Ohio State University, Columbus, Ohio 43210
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The quark contribution to the free energy of a hot quark-gluon plasma is calculated to leading order in hard-thermal-loop (HTL) perturbation theory. This method selectively resums higher order corrections associated with plasma effects, such as screening, quasiparticles, and Landau damping. Comparing to the weak-coupling expansion of QCD, the error in the one-loop HTL free energy is of order α_s, but the large $\alpha_s^{3/2}$ correction from QCD plasma effects is included exactly.

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I. INTRODUCTION

Experimental data from the BNL Relativistic Heavy Ion Collider (RHIC) will soon become available. In order to determine if a quark-gluon plasma has been created, a careful comparison of the predictions of hadronic models and QCD has to be made. It is therefore desirable to find a systematic way to calculate the thermodynamic properties and signatures of a quark-gluon plasma within QCD. Asymptotic freedom suggests that at sufficiently high temperatures a straightforward perturbative expansion should suffice. However, at experimentally accessible temperatures, perturbative QCD does not seem to be of any quantitative use [1–3]. The problem is evident in the free energy of the quark-gluon plasma. The weak-coupling expansion has been calculated through order $\alpha_s^{5/2}$ [1–3]. The successive approximations to the free energy show no sign of converging at temperatures that are relevant for heavy-ion collisions.

One possibility for improving the convergence of the perturbative predictions is to apply Padé approximants to the series in $\alpha_s^{1/2}$ [4]; however, this technique can only be applied if several terms in the perturbation series are known. Another possibility is to use hard-thermal-loop (HTL) approximations within a self-consistent Φ-derivable framework [5]. A third approach is to apply HTL perturbation theory [6], which is an extension of the resummation method of Braaten and Pisarski [7] into a systematic perturbative expansion. In two previous papers [6], we calculated the one-loop free energy of pure-glue QCD using HTL perturbation theory. Here we extend that calculation to include quarks, thus completing the calculation of the free energy of a hot quark-gluon plasma to leading order in HTL perturbation theory.

The paper is organized as follows. In the next section, we calculate the quark contribution to the free energy of a hot quark-gluon plasma at one loop in HTL perturbation theory. In Sec. III, we carry out the high-temperature expansion of the free energy, and in Sec. IV we take the low-temperature limit. In Sec. V, we compare the one-loop HTL free energy with the weak-coupling expansion of QCD. We conclude in Sec. VI. In the Appendix, we have collected the integrals that are required in the calculations.

*Currently at: Physics Department, University of Washington, Seattle, WA 98195-1560.

II. QUARK CONTRIBUTION TO HTL FREE ENERGY

The one-loop HTL free energy for an $SU(N_c)$ gauge theory with N_f massless quarks is

$$\mathcal{F}_{\rm HTL}=(N_c^2-1)[(d-1)\mathcal{F}_T+\mathcal{F}_L]+N_cN_f\mathcal{F}_q+\Delta\mathcal{F}, \quad (1)$$

where $\mathcal{F}_T$ and $\mathcal{F}_L$ are the contributions to the free energy from transverse and longitudinal gluons, respectively, $\mathcal{F}_q$ is the contribution to the free energy from each flavor and color of the quarks, and $\Delta\mathcal{F}$ is a counterterm. The quark contribution is given by

$$\mathcal{F}_q=-\sum\!\!\!\!\!\!\int_{\{K\}} \log\det[\not{K}-\Sigma(K)]. \quad (2)$$

The sum-integral in Eq. (2) represents a dimensionally regularized integral over the momentum $\mathbf{k}$ and a sum over the Matsubara frequencies $\omega_n=(2n+1)\pi T$:

$$\sum\!\!\!\!\!\!\int_{\{K\}} \equiv T\sum_{n=-\infty}^{\infty}\mu^{3-d}\int\frac{d^dk}{(2\pi)^d}. \quad (3)$$

The factor of μ^{3-d}, where μ is a renormalization scale, ensures that the regularized free energy has the correct dimensions even for $d\neq 3$. The HTL quark self-energy in Eq. (2) is

$$\Sigma(K)=\frac{m_q^2}{2k}\log\frac{i\omega_n+k}{i\omega_n-k}\gamma_0+\frac{m_q^2}{k}\left(1-\frac{i\omega_n}{2k}\log\frac{i\omega_n+k}{i\omega_n-k}\right)\hat{\mathbf{k}}\cdot\boldsymbol{\gamma}, \quad (4)$$

where $\hat{\mathbf{k}}=\mathbf{k}/k$ and m_q is the thermal quark mass parameter [8].

We proceed to calculate the quark contribution to the HTL free energy. The inverse quark propagator can be written as

$$\not{K}-\Sigma(K)=A_0(K)\gamma_0-A_S(K)\hat{\mathbf{k}}\cdot\boldsymbol{\gamma}, \quad (5)$$

where

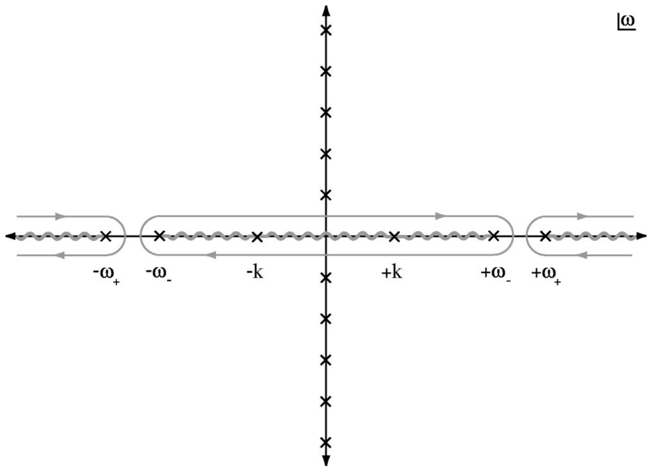

FIG. 1. The quark contribution to the HTL free energy can be expressed as an integral over a contour C that wraps around the branch cuts of $\log[(A_0^2-A_S^2)/(\omega^2-k^2)]$.

$$A_0(K)=i\omega_n-\frac{m_q^2}{2k}\log\frac{i\omega_n+k}{i\omega_n-k},\qquad(6)$$

$$A_S(K)=k+\frac{m_q^2}{k}\left(1-\frac{i\omega_n}{2k}\log\frac{i\omega_n+k}{i\omega_n-k}\right).\qquad(7)$$

We can write the quark contribution (2) as

$$\mathcal{F}_q=-2\sum\!\!\!\!\!\!\int_K \log(k^2+\omega_n^2)-2\sum\!\!\!\!\!\!\int_K \log\left[\frac{A_S^2-A_0^2}{k^2+\omega_n^2}\right],\qquad(8)$$

where we have separated out the free energy of an ideal gas of massless fermions. The first sum-integral in Eq. (8) can be evaluated analytically. In the second sum-integral, the sum over Matsubara frequencies can be expressed as a contour integral:

$$\mathcal{F}_q=-\frac{7\pi^2}{180}T^4+2\int_{\mathbf{k}}\int_C\frac{d\omega}{2\pi i}\log\left[\frac{(A_0-A_S)(A_0+A_S)}{\omega^2-k^2}\right]\frac{1}{e^{\beta\omega}+1},\qquad(9)$$

where the contour C encloses the points $\omega=i\omega_n$ along the imaginary axis in Fig. 1. We have introduced a condensed notation for the dimensionally regularized momentum integral:

$$\int_{\mathbf{k}}=\mu^{3-d}\int\frac{d^dk}{(2\pi)^d}.\qquad(10)$$

The integrand in Eq. (9) has logarithmic branch cuts that run from $-\infty$ to $-\omega_+$, from $-\omega_-$ to ω_-, from ω_+ to $+\infty$, and from $-k$ to k, where $\omega_\pm$ are the quasiparticle dispersion relations that satisfy $A_0\mp A_S=0$, or

$$0=\omega_\pm\mp k-\frac{m_q^2}{2k}\left[\left(1\mp\frac{\omega_\pm}{k}\right)\log\frac{\omega_\pm+k}{\omega_\pm-k}\pm 2\right].\qquad(11)$$

ω_+ is the dispersion relation for the standard quark mode whose helicity equals its chirality. ω_- is the dispersion relation for the *plasmino*, a collective mode whose helicity is opposite to its chirality [8]. The integrand in Eq. (9) also has branch cuts running from $-k$ to k due to the logarithms in Eqs. (6) and (7). The contour C can be deformed to wrap around the branch cuts as shown in Fig. 1. We identify the contributions from the branch cuts that end at $\pm\omega_+$ as the quasiparticle contribution to $\mathcal{F}_q$ from the quark mode. We identify the contribution from $k<|\omega|<\omega_-$ as the quasiparticle contribution from the plasmino. The sum of these contributions is denoted by $\mathcal{F}_{q,\mathrm{qp}}$:

$$\mathcal{F}_{q,\mathrm{qp}}=-4\int_{\mathbf{k}}\left[T\log(1+e^{-\beta\omega_+})+\frac{1}{2}\omega_+ +T\log\frac{1+e^{-\beta\omega_-}}{1+e^{-\beta k}}+\frac{1}{2}(\omega_--k)\right].\qquad(12)$$

We identify the remaining contribution from $|\omega|<k$ as the Landau-damping term and denote it by $\mathcal{F}_{q,\mathrm{Ld}}$:

$$\mathcal{F}_{q,\mathrm{Ld}}=\frac{4}{\pi}\int_{\mathbf{k}}\int_0^k d\omega\,\theta_q\left[\frac{1}{e^{\beta\omega}+1}-\frac{1}{2}\right].\qquad(13)$$

The angle θ_q is

$$\theta_q(\omega,k)=\arctan\frac{\dfrac{\pi m_q^4}{k^2}\left[\dfrac{\omega}{k}+\dfrac{K^2}{2k^2}L\right]}{K^2+2m_q^2+\dfrac{m_q^4}{k^2}\left[1-\dfrac{\omega}{k}L-\dfrac{K^2}{4k^2}(L^2-\pi^2)\right]},\qquad(14)$$

where $L=\log[(k+\omega)/(k-\omega)]$ and $K^2=k^2-\omega^2$. The complete quark contribution to the free energy is the sum of the quasiparticle term (12) and the Landau-damping term (13).

We first simplify the quasiparticle term. The integral of ω_+ is ultraviolet divergent since the asymptotic behavior of the dispersion relation ω_+ is [9]

$$\omega_+(k)\to k+\frac{m_q^2}{k}-\frac{m_q^4}{2k^3}\log\left(\frac{2k^2}{m_q^2}\right).\qquad(15)$$

The integral of (ω_--k) is convergent because the dispersion relation $\omega_-(k)$ approaches the light cone exponentially fast as $k\to\infty$. In order to extract the divergence analytically, we make a subtraction that renders the integral finite in $d=3$ dimensions. The subtraction is then evaluated analytically using dimensional regularization. Our choice of subtraction integral for the quasiparticle term is

$$\mathcal{F}_{q,\mathrm{qp}}^{(\mathrm{sub})}=-2\int_{\mathbf{k}}\left[\sqrt{k^2+2m_q^2}-\frac{m_q^4}{2(k^2+2m_q^2)^{3/2}}\left(\log\frac{2(k^2+2m_q^2)}{m_q^2}-1\right)\right]. \tag{16}$$

After subtracting this term from Eq. (12), we can take the limit $d\rightarrow 3$:

$$\begin{aligned}\mathcal{F}_{\mathrm{qp}}-\mathcal{F}_{\mathrm{qp}}^{(\mathrm{sub})}=&-\frac{2T}{\pi^2}\int_0^\infty dk k^2\left[\log(1+e^{-\beta\omega_+})+\log\frac{1+e^{-\beta\omega_-}}{1+e^{-\beta k}}\right]\\&-\frac{1}{\pi^2}\int_0^\infty dk k^2\left[\omega_+-\sqrt{k^2+2m_q^2}+\frac{m_q^4}{2(k^2+2m_q^2)^{3/2}}\left(\log\frac{2(k^2+2m_q^2)}{m_q^2}-1\right)\right]\\&-\frac{1}{\pi^2}\int_0^\infty dk k^2(\omega_--k).\end{aligned} \tag{17}$$

If we impose a momentum cutoff $k<\Lambda$, our subtraction integral (16) has power divergences proportional to Λ^4 and $m_q^2\Lambda^2$ and logarithmic divergences proportional to $m_q^4\log\Lambda$ and $m_q^4\log^2\Lambda$. The quartic divergence is cancelled by the usual renormalization of the vacuum energy density at zero temperature. Dimensional regularization throws away the power divergences and replaces the logarithmic divergences by poles in $d-3$. In the limit $d\rightarrow 3$, the individual integrals in Eq. (16) are given by Eqs. (A1)–(A3) in the Appendix. The result is

$$\mathcal{F}_{q,\mathrm{qp}}^{(\mathrm{sub})}=\frac{1}{2}m_q^4\left(\frac{2m_q^2}{\mu^2}\right)^{-\epsilon}\frac{\Omega_d}{(2\pi)^d}\left[\frac{1}{\epsilon^2}+\frac{2\log 2}{\epsilon}-\frac{5}{2}+2\log^2 2+\frac{\pi^2}{6}\right], \tag{18}$$

where $d=3-2\epsilon$ and $\Omega_d=2\pi^{d/2}/\Gamma(d/2)$. The last two integrals in Eq. (17) are functions of m_q only and must therefore be proportional to m_q^4. Calculating the integrals numerically, their contributions to Eq. (17) are $(2.166\times 10^{-2})m_q^4$ and $(-1.267\times 10^{-2})m_q^4$, respectively.

We next simplify the Landau-damping term (13). The temperature-independent integral has ultraviolet divergences from the region $k\rightarrow\infty$ with $\omega\sim k$. We must again isolate the divergences by making subtractions. Our choice for the subtraction integral is

$$\mathcal{F}_{q,\mathrm{Ld}}^{(\mathrm{sub})}=-2m_q^4\int_{\mathbf{k}}\int_0^k d\omega\left[\frac{\omega}{k^3(K^2+2m_q^2)}+\frac{K^2L}{2k^4(K^2+2m_q^2)}\right]. \tag{19}$$

Subtracting this from Eq. (13), we can take the limit $d\rightarrow 3$:

$$\begin{aligned}\mathcal{F}_{q,\mathrm{Ld}}-\mathcal{F}_{q,\mathrm{Ld}}^{(\mathrm{sub})}=&\frac{2}{\pi^3}\int_0^\infty d\omega\frac{1}{e^{\beta\omega}+1}\int_\omega^\infty dk k^2\theta_q\\&-\frac{1}{\pi^3}\int_0^\infty d\omega\int_\omega^\infty dk k^2\left[\theta_q-\frac{\pi m_q^4\omega}{k^3(K^2+2m_q^2)}-\frac{\pi m_q^4K^2L}{2k^4(K^2+2m_q^2)}\right].\end{aligned} \tag{20}$$

If we impose ultraviolet cutoffs $\omega<\Lambda$ and $k<\Lambda$ on the energy and momentum, the subtraction integral (19) has logarithmic divergences proportional to $m_q^4\log\Lambda$ and $m_q^4\log^2\Lambda$. They cancel against the corresponding divergences in the quasiparticle subtraction integral (16). The subtraction integral in Eq. (19) is evaluated in the limit $d\rightarrow 3$ using Eqs. (A4),(A5):

$$\mathcal{F}_{q,\mathrm{Ld}}^{(\mathrm{sub})}=-\frac{1}{2}m_q^4\left(\frac{2m_q^2}{\mu^2}\right)^{-\epsilon}\frac{\Omega_d}{(2\pi)^d}\left[\frac{1}{\epsilon^2}+\frac{2\log 2}{\epsilon}-4\log 2+2\log^2 2+\frac{\pi^2}{6}\right]. \tag{21}$$

The last integral in Eq. (20) is a function of m_q only and is therefore proportional to m_q^4. Its contribution to Eq. (20) is $(7.525\times 10^{-3})m_q^4$.

Adding Eqs. (17), (18), (20) and (21), our final result for the quark contribution to the HTL free energy is

$$\begin{aligned}\mathcal{F}_q=&-\frac{2T}{\pi^2}\int_0^\infty dk k^2\left[\log(1+e^{-\beta\omega_+})+\log\frac{1+e^{-\beta\omega_-}}{1+e^{-\beta k}}\right]\\&+\frac{2}{\pi^3}\int_0^\infty d\omega\frac{1}{e^{\beta\omega}+1}\int_\omega^\infty dk k^2\theta_q+(2.342\times 10^{-2})m_q^4.\end{aligned} \tag{22}$$

Since the quark contribution has no logarithmic ultraviolet divergences, the counterterm $\Delta\mathcal{F}$ in Eq. (1) is the same as in the pure-glue case [6]. If we had used a momentum cutoff Λ instead of dimensional regularization, we would need a counterterm proportional to $m_q^2\Lambda^2$ to cancel the quadratic divergence from the quasiparticle term (12).

III. HIGH-TEMPERATURE EXPANSION

If the temperature is much larger than the quark mass parameter m_q, the quark contribution to the free energy can be expanded in powers of m_q/T. The integral in Eq. (9) involves two energy and momentum scales: the ''hard'' scale T and the ''soft'' scale m_q. The terms in the high-temperature expansion can receive contributions from both scales. Dimensional regularization makes it easy to separate these contributions. The soft contribution is obtained by expanding the statistical factor $1/(e^{\beta\omega}+1)$ in Eq. (9) in powers of ω/T. Using the methods in [6], one can show that the soft contribution to $\mathcal{F}_q$ vanishes with dimensional regularization.

The hard contribution is obtained by expanding the logarithm in Eq. (9) in powers of m_q^2. The first term in this expansion is

$$\mathcal{F}_q^{(1)} = -4 m_q^2 \int_{\mathbf{k}} \int_C \frac{d\omega}{2\pi i} \frac{1}{\omega^2 - k^2} \frac{1}{e^{\beta\omega}+1}. \tag{23}$$

The integrand has single poles at $\omega = \pm k$ and can be evaluated using the residue theorem. The momentum integrals can be evaluated analytically and in the limit $d \to 3$ we obtain

$$\mathcal{F}_q^{(1)} = \frac{1}{6} m_q^2 T^2. \tag{24}$$

The second term in the expansion is

$$\mathcal{F}_q^{(2)} = -2 m_q^4 \int_{\mathbf{k}} \int_C \frac{d\omega}{2\pi i} \left[\frac{2}{(\omega^2-k^2)^2} + \frac{1}{k^2(\omega^2-k^2)} - \frac{\omega}{k^3(\omega^2-k^2)} \log\frac{\omega+k}{\omega-k} + \frac{1}{4k^4} \log^2\frac{\omega+k}{\omega-k} \right] \frac{1}{e^{\beta\omega}+1}. \tag{25}$$

Using the residue theorem and collapsing the contour onto the branch cuts from the logarithms, Eq. (25) reduces to

$$\mathcal{F}^{(2)} = 2 m_q^4 \int_{\mathbf{k}} \left\{ \frac{1}{k^2} \frac{\partial}{\partial k} \left(\frac{1}{e^{\beta k}+1} \right) - \frac{1}{k^4} \int_0^k d\omega\, \omega \frac{\partial}{\partial \omega} \left(\frac{1}{e^{\beta\omega}+1} \right) \log\frac{k+\omega}{k-\omega} \right\}. \tag{26}$$

The double integral can be evaluated by first integrating over k and then over ω. Expanding around $\epsilon = 0$ and keeping terms only through ϵ^0, we obtain the finite result

$$\mathcal{F}_q^{(2)} = \frac{2\log 2 - 1}{2\pi^2} m_q^4. \tag{27}$$

The final result for the high-temperature expansion through order m_q^4 is the sum of the first term in Eq. (9) and the terms (24) and (27):

$$\mathcal{F}_q \to -\frac{7\pi^2}{180} T^4 + \frac{1}{6} m_q^2 T^2 + \frac{2\log 2 - 1}{2\pi^2} m_q^4. \tag{28}$$

IV. LOW-TEMPERATURE LIMIT

It is useful to understand the behavior of the HTL free energy in the low-temperature limit where $T \to 0$ with m_q fixed. In this limit, $\mathcal{F}_q$ is proportional to m_q^4. The coefficient could be extracted directly from the final expression (22) for $\mathcal{F}_q$, but it is simpler to compute it from our original expression (8) for the quark contribution to the free energy. As $T \to 0$, the sum over the discrete Matsubara frequencies $\omega_n = (2n+1)\pi T$ becomes an integral over the continuous Euclidean energy ω:

$$\mathcal{F}_q \to -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \int_{\mathbf{k}} \log\left[\frac{A_S^2 - A_0^2}{k^2+\omega^2} \right]. \tag{29}$$

After rescaling the energy $\omega \to k\omega$, we obtain

$$\mathcal{F}_q = -\frac{2}{\pi} \int_0^{\infty} d\omega \int_{\mathbf{k}} k \log\left[\left(1 + \frac{m_q^2}{k^2} f(\omega) \right) \left(1 + \frac{m_q^2}{k^2} \overline{f(\omega)} \right) \right], \tag{30}$$

where

$$f(\omega) = \frac{1}{1+i\omega} + i\left(\frac{\pi}{2} - \arctan\omega \right) \tag{31}$$

and $\overline{f}$ is the complex conjugate of f. Integrating over $\mathbf{k}$, we obtain

$$\mathcal{F}_q = -\frac{2}{\pi} m_q^4 \left(\frac{m_q^2}{\mu^2} \right)^{d-3} \frac{\Omega_d}{(2\pi)^d} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{1-d}{2}\right)}{d+1} \times \int_0^{\infty} d\omega \{ [f(\omega)]^{(d+1/2)} + [\overline{f}(\omega)]^{(d+1/2)} \}. \tag{32}$$

Expanding around $d=3$, we get

$$\mathcal{F}_q = \frac{1}{2\pi} m_q^4 \left(\frac{m_q^2}{\mu^2} \right)^{-\epsilon} \frac{\Omega_d}{(2\pi)^d} \left\{ \left(\frac{1}{\epsilon} + \frac{1}{2} \right) \int_0^{\infty} d\omega [f^2(\omega) + \overline{f}^2(\omega)] - \int_0^{\infty} d\omega [f^2(\omega)\log f(\omega) + \overline{f}^2(\omega)\log \overline{f}(\omega)] \right\}. \tag{33}$$

The integral of f^2 can be evaluated analytically and is purely imaginary. It is cancelled exactly by the integral of $\overline{f}^2$. This cancellation is in accord with the observation that the quark contribution has no logarithmic ultraviolet divergences. The last integral in Eq. (33) must be evaluated numerically. The result is

$$\mathcal{F}_q \to (2.342 \times 10^{-2}) m_q^4, \tag{34}$$

which is identical to the m_q^4 term in our complete expression (22) for the quark contribution to the free energy.

V. COMPARISON WITH WEAK-COUPLING EXPANSION

In this section we present the numerical results for the one-loop HTL free energy (1) with $N_c = 3$ and $N_f = 3$. The quark term is given in Eq. (22). The gluon term is given in Ref. [6]. It depends on the gluon mass parameter m_g and on a renormalization scale μ_3 associated with a logarithmically divergent integral over the three-momentum. We use the weak-coupling limits of the gluon and quark mass parameters [8]:

$$m_g^2 = \frac{2\pi(6+N_f)}{9} \alpha_s(\mu_4) T^2, \tag{35}$$

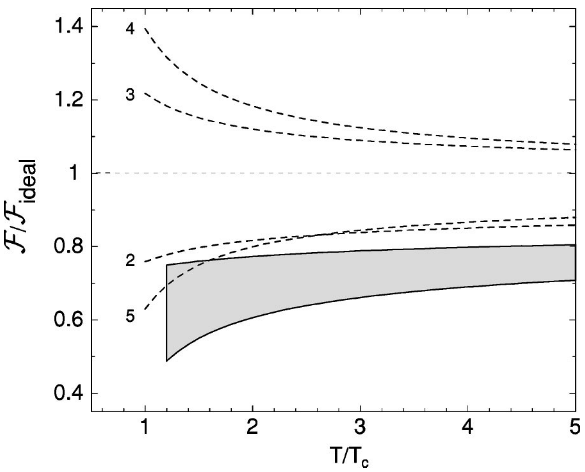

FIG. 2. The free energy for QCD with $N_f=3$ quarks as a function of T/T_c. The HTL free energy is shown as a shaded band that corresponds to varying μ_3 and μ_4 by a factor of 2 around their central values. The weak-coupling expansion through orders α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ are shown as dashed lines labeled by 2, 3, 4 and 5.

$$m_q^2=\frac{2\pi}{3}\alpha_s(\mu_4)T^2, \tag{36}$$

where μ_4 is the renormalization scale for the running coupling constant. We use a parametrization of $\alpha_s(\mu_4)$ that includes the effects of two-loop running:

$$\alpha_s(\mu_4)=\frac{4\pi}{\beta_0\bar{L}}\left(1-\frac{2\beta_1}{\beta_0^2}\frac{\log\bar{L}}{\bar{L}}\right), \tag{37}$$

where $\beta_0=11-\frac{2}{3}N_f$, $\beta_1=51-\frac{19}{3}N_f$, and $\bar{L}=\log(\mu_4^2/\Lambda_{\overline{\rm MS}}^2)$. For the relation between $\Lambda_{\overline{\rm MS}}$ and the critical temperature T_c for the deconfinement phase transition, we use the result $T_c=1.05\,\Lambda_{\overline{\rm MS}}$ calculated for $N_f=4$ flavors of dynamical quarks [10].

The leading-order HTL free energy with $N_f=3$ is shown in Fig 2. It is scaled by the free energy of an ideal gas of quarks and gluons:

$$\mathcal{F}_{\rm ideal}=-\frac{8\pi^2}{45}\left(1+\frac{21}{32}N_f\right)T^4. \tag{38}$$

To illustrate the sensitivity to the choices of the renormalization scales, we take their central values to be $\mu_3=0.717\,m_g$ and $\mu_4=2\pi T$ and we allow variations by a factor of 2. The shaded band indicates the resulting range in predictions. The range of $\mathcal{F}_{\rm HTL}$ comes predominantly from variation in μ_4 at the highest temperatures shown and from variations in μ_3 at the lowest temperatures shown. With our expressions from m_g and α_s, $\mathcal{F}_{\rm HTL}$ diverges either to $+\infty$ or $-\infty$ at $\mu_4=\Lambda_{\overline{\rm MS}}$, depending on whether μ_3 is greater than or less than our central value of $0.717\,m_g$. The free energy of an $SU(3)$ gauge theory with N_f massless quarks has been calculated in the weak-coupling expansion through order $\alpha_s^{5/2}$ [1–3]:

$$\mathcal{F}=-\frac{8\pi^2}{45}T^4\left[\mathcal{F}_0+\mathcal{F}_2\frac{\alpha_s}{\pi}+\mathcal{F}_3\left(\frac{\alpha_s}{\pi}\right)^{3/2}+\mathcal{F}_4\left(\frac{\alpha_s}{\pi}\right)^2+\mathcal{F}_5\left(\frac{\alpha_s}{\pi}\right)^{5/2}+O(\alpha_s^3\log\alpha_s)\right]. \tag{39}$$

The coefficients in this expansion with $\mu_4=2\pi T$ are

$$\mathcal{F}_0=1+\frac{21}{32}N_f, \tag{40}$$

$$\mathcal{F}_2=-\frac{15}{4}\left(1+\frac{5}{12}N_f\right), \tag{41}$$

$$\mathcal{F}_3=30\left(1+\frac{1}{6}N_f\right)^{3/2}, \tag{42}$$

$$\mathcal{F}_4=237.2+15.97N_f-0.413N_f^2+\frac{135}{2}\left(1+\frac{1}{6}N_f\right)\log\left[\frac{\alpha_s}{\pi}\left(1+\frac{1}{6}N_f\right)\right], \tag{43}$$

$$\mathcal{F}_5=-\left(1+\frac{1}{6}N_f\right)^{1/2}[799.2+21.96N_f+1.926N_f^2]. \tag{44}$$

The predictions from the weak-coupling expansion with $\mu_4=2\pi T$ are compared to the HTL free energy in Fig. 2. The expansions of the pressure truncated after orders α_s, $\alpha_s^{3/2}$, α_s^2, and $\alpha_s^{5/2}$ are shown as the dashed lines labeled 2, 3, 4, and 5. As successive terms in the weak-coupling expansion are added, the predictions fluctuate wildly. In addition, the sensitivity to the renormalization scale μ_4 increases at each successive order. Of course, because of asymptotic freedom, the first few terms in the weak-coupling expansion will appear to converge at sufficiently high temperature. However, this occurs only at enormously high temperatures, where all the corrections to the ideal gas are tiny. For example, for $N_f=3$, the $\alpha_s^{3/2}$ correction is smaller than the α_s correction only if $\alpha_s<3\pi/128$. If we use Eq. (37) to extrapolate to high temperature while ignoring the effects of heavier quark flavors, this corresponds to a temperature $T>500\,T_c$.

We now compare the high-temperature expansion of the HTL free energy in Eq. (28) with the weak-coupling expansion. Using the values (35) and (36) for the thermal mass parameters, we find that the α_s correction is overincluded by a factor of $12(3+N_f)/(12+5N_f)$. The $\alpha_s^{3/2}$ correction is included exactly in the leading HTL result. The overincluded α_s correction and the large positive $\alpha_s^{3/2}$ correction combine with higher order corrections in the HTL free energy to give a negative correction that rises slowly with T as shown in Fig. 2.

Lattice gauge theory has been used to calculate the equation of state of a quark-gluon plasma with $N_f=2$ [11,12] and $N_f=4$ [13] flavors of dynamical quarks. These calculations indicate that the pressure, which is the negative of the free energy, approaches that of an ideal gas from below. The approach to the ideal gas is more rapid than the leading order

HTL result. For $N_f=4$, it reaches 80% of the value for an ideal gas already at $T=2.5\,T_c$. For higher values of T, the leading order HTL result lies significantly below the lattice results. This is not of great concern, because the difference can be accounted for by the next-to-leading order correction in HTL perturbation theory. At next-to-leading order, there are two-loop diagrams and one-loop diagrams with HTL counterterms. The contributions of order α_s coming from the hard momentum regions of the two-loop diagrams will reproduce the order-α_s term in the conventional perturbative series (39). The contribution from the HTL counterterm diagram will precisely cancel the order-α_s term in the one-loop HTL free energy. Thus the next-to-leading order correction to $\mathcal{F}_{\rm HTL}/\mathcal{F}_{\rm ideal}$ must approach $[10(24+7N_f)/(32+21N_f)]\alpha_s/\pi$ in the limit $\alpha_s\to 0$. This has the correct sign and roughly the right magnitude to account for the discrepancy with the lattice results.

VI. CONCLUSIONS

We have completed the calculation of the free energy of a quark-gluon plasma to leading order in HTL perturbation theory by calculating the quark contribution. The quark term has a quadratic ultraviolet divergence that vanishes with dimensional regularization, but it has no logarithmic ultraviolet divergences. Comparing our result to the weak-coupling expansions for the free energy, we find that the error is of order α_s but the large correction proportional to $\alpha_s^{3/2}$ is included exactly. It is therefore possible that the HTL perturbative expansion for the free energy will have much better convergence properties than the conventional weak-coupling expansion. To verify this, it will be necessary to extend the calculations of the free energy to next-to-leading order in HTL perturbation theory.

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APPENDIX: INTEGRALS

In this appendix, we collect the results for the integrals that are required to calculate the contribution from quarks to the one-loop HTL free energy. We use dimensional regularization, so that power ultraviolet divergences are set to zero and logarithmic ultraviolet divergences appear as poles in ϵ. In the HTL free energy, the ultraviolet divergences are isolated in subtraction terms that must be expanded around $\epsilon=0$ through order ϵ^0. The integrals required to evaluate the subtractions in the quasiparticle terms are

$$\int_0^\infty dk\,k^{2-2\epsilon}\sqrt{k^2+m^2}=-\frac{1}{16}m^{4-2\epsilon}\left[\frac{1}{\epsilon}+2\log 2-\frac{1}{2}\right], \tag{A1}$$

$$\int_0^\infty dk\,k^{2-2\epsilon}\frac{1}{(k^2+m^2)^{3/2}}=\frac{1}{2}m^{-2\epsilon}\left[\frac{1}{\epsilon}+2\log 2-2\right], \tag{A2}$$

$$\int_0^\infty dk\,k^{2-2\epsilon}\frac{1}{(k^2+m^2)^{3/2}}\log\frac{k^2+m^2}{m^2}=\frac{1}{2}m^{-2\epsilon}\left[\frac{1}{\epsilon^2}-4+\frac{\pi^2}{6}-2\log^2 2+4\log 2\right]. \tag{A3}$$

The integrals required to evaluate the subtractions in the Landau-damping terms are

$$\int_0^\infty d\omega\,\omega\int_\omega^\infty dk\,k^{-1-2\epsilon}\frac{1}{k^2-\omega^2+m^2}=\frac{1}{4}m^{-2\epsilon}\left[\frac{1}{\epsilon^2}+\frac{\pi^2}{6}\right], \tag{A4}$$

$$\int_0^\infty d\omega\int_\omega^\infty dk\;\; k^{-2-2\epsilon}\frac{k^2-\omega^2}{k^2-\omega^2+m^2}\log\frac{k+\omega}{k-\omega}=m^{-2\epsilon}\left[\frac{\log 2}{\epsilon}-2\log 2+\log^2 2\right]. \tag{A5}$$

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