New form of the Kerr solution

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A new form of the Kerr solution is presented. The solution involves a time coordinate which represents the local proper time for free-falling observers on a set of simple trajectories. Many physical phenomena are particularly clear when related to this time coordinate. The chosen coordinates also ensure that the solution is well behaved at the horizon. The solution is well suited to the tetrad formalism and a convenient null tetrad is presented. The Dirac Hamiltonian in a Kerr background is also given and, for one choice of tetrad, it takes on a simple, Hermitian form.

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I. INTRODUCTION

The Kerr solution has been of central importance in astrophysics ever since it was realized that accretion processes would tend to spin up a black hole to near its critical rotation rate [1]. A number of forms of the Kerr solution currently exist in the literature. Most of these are contained in Chandrasekhar's work [2], and useful summaries are contained in the books by Kramer *et al.* [3] and d'Inverno [4]. The purpose of this paper is to present a new form of the solution which has already proved to be useful in numerical simulations of accretion processes. The form is a direct extension of the Schwarzschild solution when written as

$$ds^2=dt^2-\left(dr+\left(\frac{2M}{r}\right)^{1/2}dt\right)^2-r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{1}$$

(Natural units have been employed.) This is obtained from the Eddington-Finkelstein form

$$ds^2=\left(1-\frac{2M}{r}\right)d\bar{t}^2-\frac{4M}{r}d\bar{t}\,dr-\left(1+\frac{2M}{r}\right)\times dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2). \tag{2}$$

by the coordinate transformation

$$t=\bar{t}+2(2Mr)^{1/2}-4M\ln\left(1+\left(\frac{r}{2M}\right)^{1/2}\right). \tag{3}$$

In both metrics r lies in the range $0<r<\infty$, and θ and ϕ take their usual meaning.

The metric (1) has a number of nice features [5], many of which extend to the Kerr case. The solution is well-behaved at the horizon, so can be employed safely to analyze physical processes near the horizon, and indeed inside it [5]. Another useful feature is that the time t coincides with the proper time of observers free-falling along radial trajectories starting from rest at infinity. This is possible because the velocity vector

$$\dot{x}^i=(1,-(2M/r)^{1/2},0,0), \quad \dot{x}_i=(1,0,0,0) \tag{4}$$

defines a radial geodesic with constant θ and ϕ. The proper time along these paths coincides with t, and the geodesic equation is simply

$$\ddot{r}=-M/r^2. \tag{5}$$

Physics as seen by these observers is almost entirely Newtonian, making this gauge a very useful one for introducing some of the more difficult concepts of black hole physics. The various gauge choices leading to this form of the Schwarzschild solution also carry through in the presence of matter and provide a simple system for the study of the formation of spherically symmetric clusters [6] and black holes [5].

A further useful feature of the time coordinate in Eq. (1) is that it enables the Dirac equation in a Schwarzschild background to be cast in a simple Hamiltonian form [5]. Indeed, the full Dirac equation is obtained by adding a single term $\hat{H}_I$ to the free-particle Hamiltonian in Minkowski spacetime. This additional term is

$$\begin{aligned}\hat{H}_I\psi&=i(2M/r)^{1/2}(\partial_r\psi+3/(4r)\psi)\\&=i(2M/r)^{1/2}r^{-3/4}\partial_r(r^{3/4}\psi).\end{aligned} \tag{6}$$

A useful feature of this gauge is that the measure on surfaces of constant t is the same as that of Minkowski spacetime, so one can employ standard techniques from quantum theory with little modification. One subtlety is that the Hamiltonian is not self-adjoint due to the presence of the singularity. This manifests itself as a decay in the wavefunction as current density is sucked onto the singularity [5].

The time coordinate t in the metric of Eq. (1) has many of the properties of a global, Newtonian time. This suggests that an attempt to find an analogue for the Kerr solution might fail due to its angular momentum. The key to understanding how to achieve a suitable generalization is the realization that it is only the local properties of t that make it so convenient for describing the physics of the solution. The natural extension for the Kerr solution is therefore to look for a convenient set of reference observers which generalizes the idea of a family of observers on radial trajectories. In Secs. II

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and III we present a new form of the Kerr solution and show that it has many of the desired properties. In Sec. IV we give various tetrad forms of the solution, and present a Hermitian form of the Dirac Hamiltonian in a Kerr background. Throughout we use Latin letters for spacetime indices and Greek letters for tetrad indices, and use the signature $\eta_{\alpha\beta} = \mathrm{diag}(+---)$. Natural units $c=G=\hbar=1$ are employed throughout.

II. THE KERR SOLUTION

The new form of the Kerr solution can be written in Cartesian-type coordinates (t,x,y,z) in a manner analogous to the Kerr-Schild form [2,4]. In this coordinate system our new form of the solution is

$$ds^2 = \eta_{ij}dx^i dx^j - \left(\frac{2\alpha}{\rho} a_i v_j + \alpha^2 v_i v_j\right) dx^i dx^j, \qquad (7)$$

where η_{ij} is the Minkowski metric,

$$\alpha = \frac{(2Mr)^{1/2}}{\rho} \qquad (8)$$

$$\rho^2 = r^2 + \frac{a^2 z^2}{r^2}, \qquad (9)$$

and a and M constants. The function r is given implicitly by

$$r^4 - r^2(x^2+y^2+z^2-a^2) - a^2 z^2 = 0, \qquad (10)$$

and we restrict r to $0<r<\infty$, with $r=0$ describing the disk $z=0$, $x^2+y^2 \leqslant a^2$. The maximally extended Kerr solution (where r is allowed to take negative values) will not be considered here.

The two vectors in the metric (7) are

$$v_i = \left(1, \frac{ay}{a^2+r^2}, \frac{-ax}{a^2+r^2}, 0\right) \qquad (11)$$

and

$$a_i = (r^2+a^2)^{1/2}\left(0, \frac{rx}{a^2+r^2}, \frac{ry}{a^2+r^2}, \frac{z}{r}\right). \qquad (12)$$

These two vectors play an important role in studying physics in a Kerr background. They are related to the two principal null directions $n_\pm$ by

$$n_\pm = (r^2+a^2)^{1/2} v_i \pm (\alpha\rho v_i + a_i). \qquad (13)$$

For computations it is useful to note that the contravariant components of the spacelike vector in brackets are the same as those of $-a_i$,

$$\alpha\rho v^i + a^i = -(r^2+a^2)^{1/2}\left(0, \frac{rx}{a^2+r^2}, \frac{ry}{a^2+r^2}, \frac{z}{r}\right). \qquad (14)$$

The vector v_i also plays a crucial role in separating the Dirac equation in a Kerr background, and is the timelike eigenvector of the electromagnetic stress-energy tensor for the Kerr-Newman analogue of our form.

III. SPHEROIDAL COORDINATES

The nature of the metric (7) is more clearly revealed if we introduce oblate spheroidal coordinates (r,θ,ϕ), where

$$\cos\theta = \frac{z}{r} \quad 0 \leqslant \theta \leqslant \pi \qquad (15)$$

$$\tan\phi = \frac{y}{x} \quad 0 \leqslant \phi < 2\pi, \qquad (16)$$

so that ρ recovers its standard definition

$$\rho^2 = r^2 + a^2\cos^2\theta. \qquad (17)$$

The use of the symbols r and θ here are standard, though one must be aware that when $M=0$ (flat space) these reduce to *oblate spheroidal* coordinates, and not spherical polar coordinates. This is clear from the fact that r does not equal $\surd(x^2+y^2+z^2)$.

In terms of (t,r,θ,ϕ) coordinates our new form of the Kerr solution is

$$ds^2 = dt^2 - \left(\frac{\rho}{(r^2+a^2)^{1/2}} dr + \alpha(dt - a\sin^2\theta\, d\phi)\right)^2 - \rho^2 d\theta^2 - (r^2+a^2)\sin^2\theta d\phi^2. \qquad (18)$$

This neatly generalizes the Schwarzschild form of Eq. (1), replacing $\surd(2M/r)$ with $\surd(2Mr)/\rho$, and introducing a rotational component. The line element can be simplified further by introducing the hyperbolic coordinate η via $a\sinh\eta = r$, though this can make some equations harder to interpret and will not be employed here. The metric (18) is obtained from the advanced Eddington-Finkelstein form of the Kerr solution,

$$\begin{aligned} ds^2 = &\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 - 2\, dv\, dr \\ &+ \frac{2Mr}{\rho^2}(2a\sin^2\theta) dv\, d\bar{\phi} + 2a\sin^2\theta\, dr\, d\bar{\phi} - \rho^2 d\theta^2 \\ &- \left((r^2+a^2)\sin^2\theta + \frac{2Mr}{\rho^2}(a^2\sin^4\theta)\right) d\bar{\phi}^2, \end{aligned} \qquad (19)$$

via the coordinate transformation

$$dt = dv - \frac{dr}{1 + (2Mr/(r^2+a^2))^{1/2}} \qquad (20)$$

$$d\phi = d\bar{\phi} - \frac{a\, dr}{r^2+a^2+(2Mr(r^2+a^2))^{1/2}}. \qquad (21)$$

This transformation is well-defined for all r, though the integrals involved do not appear to have a simple closed form.

The velocity vector

$$\dot{x}^i=(1,-\alpha(r^2+a^2)^{1/2}/\rho,0,0), \quad \dot{x}_i=(1,0,0,0) \tag{22}$$

defines an infalling geodesic with constant θ and ϕ, and zero velocity at infinity. The existence of these geodesics is a key property of the solution. The time coordinate t now has the simple interpretation of recording the local proper time for observers in free-fall along trajectories of constant θ and ϕ. As in the spherical case, many physical phenomena are simplest to interpret when expressed in terms of this time coordinate. An example of this is provided in the following section, where we show that the time coordinate produces a Dirac Hamiltonian which is Hermitian in form. The difference between this free-fall velocity and the velocity v_i (defined by the gravitational fields) also provides a local definition of the angular velocity contained in the gravitational field.

IV. TETRADS AND THE DIRAC EQUATION

The metric (18) lends itself very naturally to the tetrad formalism. From the principal null directions of Eq. (13) one can construct the following null tetrad, expressed in (t,r,θ,ϕ) coordinates:

$$l^i=\frac{1}{r^2+a^2}(r^2+a^2,r^2+a^2-[2Mr(r^2+a^2)]^{1/2},0,a) \tag{23}$$

$$n^i=\frac{1}{2\rho^2}(r^2+a^2,-(r^2+a^2)-[2Mr(r^2+a^2)]^{1/2},0,a) \tag{24}$$

$$m^i=\frac{1}{\sqrt{2}(r+ia\cos\theta)}(ia\sin\theta,0,1,i\csc\theta). \tag{25}$$

In this frame the Weyl scalars Ψ_0, Ψ_1, Ψ_3 and Ψ_4 all vanish, and

$$\Psi_2=-\frac{M}{(r-ia\cos\theta)^3}. \tag{26}$$

A second tetrad, better suited to computations of matter geodesics, is given by

$$\begin{aligned} e^0{}_i&=(1,0,0,0)\\ e^1{}_i&=(\alpha,\rho/(r^2+a^2)^{1/2},0,-\alpha a\sin^2\theta)\\ e^2{}_i&=(0,0,\rho,0)\\ e^3{}_i&=(0,0,0,(r^2+a^2)^{1/2}\sin\theta). \end{aligned} \tag{27}$$

This defines a frame for all values of the coordinate r, so is valid inside and outside the horizon. Combined with the techniques described in [5] this tetrad provides a very powerful way of analyzing and visualizing motion in a Kerr background.

A further tetrad is provided by reverting to the original Cartesian-type coordinates of Eq. (7) and writing

$$e^\mu{}_i=\delta_i^\mu-\frac{\alpha}{\rho}v_i a_j\eta^{j\mu}, \tag{28}$$

where v_i and a_i are as defined at Eqs. (11) and (12). The inverse is found to be

$$e^i_\mu=\delta^i_\mu+\frac{\alpha}{\rho}\eta^{ij}a_j\delta^k_\mu v_k. \tag{29}$$

This final form of tetrad is the simplest to use when constructing the Dirac equation in a Kerr background. We will not go through the details here but will just present the final form of the equation in a Hamiltonian form. Following the conventions of Itzykson and Zuber [7] we denote the Dirac-Pauli matrix representation of the Dirac algebra by $\{\gamma^\mu\}$ and write $\alpha^i=\gamma^0\gamma^i$, $i=1\ldots3$. Since $e^0_\mu=\delta^0_\mu$, premultiplying the Dirac equation by γ_0 is all that is required to bring it into Hamiltonian form. When this is done, the Dirac equation in a Kerr background becomes

$$i\partial_t\psi=-i\alpha^i\partial_i\psi+m\gamma_0\psi+\hat{H}_K\psi \tag{30}$$

where

$$\hat{H}_K\psi=\frac{\sqrt{2M}}{\rho^2}\bigg((r^3+a^2r)^{1/4}i\partial_r[(r^3+a^2r)^{1/4}\psi] -a\cos\theta r^{1/4}\alpha_\phi i\partial_r(r^{1/4}\psi)-\frac{a\cos\theta}{2}(r^2+a^2)^{1/2}\gamma_5\psi\bigg) \tag{31}$$

and

$$\alpha_\phi=-\sin\phi\alpha_1+\cos\phi\alpha_2. \tag{32}$$

The measure on hypersurfaces of constant t is again the same as that of Minkowski spacetime, since the covariant volume element is simply

$$dx\,dy\,dx=\rho^2\sin\theta\,dr\,d\theta\,d\phi. \tag{33}$$

As with the Schwarzschild case the interaction Hamiltonian $\hat{H}_K$ is not self-adjoint when integrated over these hypersurfaces. This is because the singularity causes a boundary term to be present when the Hamiltonian is integrated.

V. CONCLUSIONS

The Kerr solution is of central importance in astrophysics as ever more compelling evidence points to the existence of black holes rotating at near their critical rate [8]. Any form of the solution which aids physical understanding of rotating black holes is clearly beneficial. The form of the solution presented here has a number of features which achieve this aim. The solution is well suited for studying processes near the horizon, and the compact form of the spin connection for the tetrad of Eq. (28) makes it particularly good for numerical computation. It should also be noted that this gauge admits a simple generalization to a time-dependent form which looks well-suited to the study of accretion and the formation of rotating black holes.

A more complete exposition of the features of this gauge, including the derivation of the Dirac Hamiltonian will be presented elsewhere. One reason for not highlighting more of the advantages here is that many of the theoretical manipulations which exploit these properties have been performed utilizing Hestenes' spacetime algebra [5,9]. This language fully exposes much of the intricate algebraic structure of the Kerr solution and brings with it a number of insights. These are hard to describe without employing spacetime algebra and so will be presented unadulterated in a separate paper.

The fact that the time coordinate measured by a family of free-falling observers brings the Dirac equation into Hamiltonian form is suggestive of a deeper principle. This form of the equations also permits many techniques from quantum field theory to be carried over to a gravitational background with little modification. The lack of self-adjointness due to the source itself is also natural in this framework, as the singularity is a natural sink for the current. In the non-rotating case the physical processes resulting from the presence of this sink are quite simple to analyze [5]. The Kerr case is considerably more complicated, due both to the nature of the fields inside the inner horizon, and to the structure of the singularity. One interesting point to note is that the sink region is described by $r=0$, and so represents a disk, rather than just a ring of matter. This in part supports the results of earlier calculations described in [8], though much work remains on this issue.

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