

# Nonsingular string cosmologies from exact conformal field theories

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Nonsingular two and three dimensional string cosmologies are constructed using the exact conformal field theories corresponding to  $SO(2,1)/SO(1,1)$  and  $SO(2,2)/SO(2,1)$ . All semiclassical curvature singularities are canceled in the exact theories for both of these cosets, but some new quantum curvature singularities emerge. However, considering different patches of the global manifolds allows the construction of nonsingular spacetimes with a cosmological interpretation. In both two and three dimensions, we construct nonsingular oscillating cosmologies, nonsingular expanding and inflationary cosmologies including a de Sitter (exponential) stage with a positive scalar curvature as well as nonsingular contracting and deflationary cosmologies. We analyze these cosmologies in detail with respect to the behavior of the scale factors, the scalar curvature, and the string coupling. The sign of the scalar curvature is changed by the quantum corrections in oscillating cosmologies and evolves with time in the nonoscillating cases. Similarities between the two and three dimensional cases suggest a general picture for higher dimensional coset cosmologies: anisotropy seems to be a generic unavoidable feature, cosmological singularities are generically avoided; and it is possible to construct nonsingular cosmologies where some spatial dimensions are experiencing inflation while the others experience deflation.

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## I. INTRODUCTION AND RESULTS

Fundamental string effects are most important near the Planck scale; thus, it seems that early cosmology will be the most likely area to test string theory. The classical and quantum string dynamics and their associated effects in a wide class of string backgrounds (conformally and nonconformally invariant) have been widely investigated by the present authors [1,2]. New insights and new physical phenomena with respect to string propagation in flat spacetime (and with respect to quantum fields in curved spacetime) have been uncovered [2]. Conformal invariance simplifies the mathematics of the problem but the physics remains mainly unchanged. For low and high mass the string mass spectrum in conformal and nonconformal backgrounds are the same [3].

Fundamental quantum strings demand a conformally invariant background for quantum consistency (conformal invariance is a necessary although not sufficient condition for consistency). However, most curved spacetimes that were historically of physical interest in general relativity and cosmology are not conformally invariant. For instance, black hole spacetimes and Friedmann-Robertson-Walker universes are not of this type. In string theory, a large number of generalizations of these general relativity solutions have been obtained as solutions of the renormalization group  $\beta$ -function equations (low energy effective string equations). These solutions involve, in addition to the metric, a number of massless fields including the dilaton and antisymmetric tensor fields. The  $\beta$ -function equations are the basis for most

investigations in string cosmology. However, they are perturbative and only known to the lowest orders in  $\alpha'$ , and, therefore, the corresponding solutions are only ensured to be conformally invariant to the lowest orders in  $\alpha'$ . It is not generally clear how the higher order corrections will change these solutions, and possible nonperturbative solutions seem to be completely missed in this framework.

A different approach to string cosmology is based on group manifolds and coset spaces and the corresponding Wess-Zumino-Witten (WZW) [4] and gauged WZW models [5]. These models provide new curved backgrounds that are conformally invariant to all orders in  $1/k$  (where  $k \sim 1/\alpha'$  is the level of the WZW model). In the case of group manifolds, these models are generally too simple to describe physically interesting and realistic cosmologies. This is exemplified by the well-studied  $SL(2,R)$  WZW model, which describes anti-de Sitter space. For the coset spaces, on the other hand, the models are generally so complicated that it is difficult even to extract what they describe from a manifest spacetime point of view. In fact, the exact (all orders in  $1/k$ ) metric, dilaton, etc., have only been obtained in a limited number of low-dimensional cases (for a review of such solutions, see Refs. [6,7]). And in most of these cases, the physical spacetime properties have not really been extracted so far. Most studies of coset space gauged WZW models still restrict themselves to lowest order in  $1/k$ , and are thus essentially equivalent to the studies based on the  $\beta$ -function equations.

The purpose of the present paper is to show that interesting nontrivial and nonsingular cosmologies can be obtained

from the cosets  $SO(2,1)/SO(1,1)$  and  $SO(2,2)/SO(2,1)$ . Various aspects of string theory on these cosets have been investigated in the literature, see for instance Refs. [4]–[14]. Cosmological interpretations have been considered in Refs. [12,13,15], but mostly to lowest order in  $1/k$ .

The two dimensional coset  $SO(2,1)/SO(1,1)$ , to all orders in  $1/k$ , has been given a cosmological interpretation in [13], but only in one of the coordinate patches of the global manifold. In this paper, we consider all coordinate patches and we shall show that the more interesting cosmologies actually appear in the coordinate patches considered here.

As for the three dimensional coset  $SO(2,2)/SO(2,1)$ , a cosmological interpretation was attempted in [12], but only in certain coordinate patches and only to the first order in  $1/k$ . The cosmologies obtained in [12], however, are completely changed in the exact theory. Moreover, we show that the more interesting cosmologies are obtained in coordinate patches other than those considered in [12].

Interestingly enough, quantum corrections not only cancel the semiclassical singularities but also change the sign of the scalar curvature for oscillatory cosmologies, which is positive in the semiclassical regime ( $k \rightarrow \infty$ ) and becomes negative for the value of  $k$  dictated by conformal invariance. In nonoscillating cosmologies the scalar curvature evolves from positive for large and negatives times  $t$  to negative near  $t = 0$ .

The paper is organized as follows. In Sec. II, we consider the two-dimensional  $SO(2,1)/SO(1,1)$  coset. We first re-analyze, in Sec. II A, the oscillating cosmology already considered in [13]. We point out the quantum effects on the scalar curvature concerning the singularities and the sign. We also compare with standard 1 + 1 anti-de Sitter space. Sections II B and II C are devoted to the construction of nonsingular nonoscillating cosmologies, obtained from other patches of the global manifold. Both deflationary and inflationary cosmologies are constructed. The corresponding scale factors, Hubble functions, string couplings, etc., are analyzed in detail. In particular, we obtain a nonsingular cosmology going through a de Sitter-like phase of inflation.

In Sec. III, we consider the three-dimensional  $SO(2,2)/SO(2,1)$  coset. Starting from the highly nontrivial exact metric and dilaton [11], we first derive a relatively simple expression for the scalar curvature, showing that *all* semiclassical curvature singularities have disappeared (but some new quantum curvature singularities have appeared). This closely resembles the situation in the two-dimensional case. In Sec. III A, we construct a nonsingular anisotropic oscillating cosmology. We analyze in some detail the scale factors and the scalar curvature, but the string coupling is unfortunately not very well behaved in this case. More interestingly, in Sec. III B, we construct a nonsingular nonoscillating anisotropic cosmology by considering a different coordinate patch [which actually belongs to the  $SO(3,1)/SO(2,1)$  coset]. One scale factor, in this case, experiences a phase of inflationary expansion while the other scale factor experiences deflationary contraction. Moreover, during this phase the string coupling is completely well behaved. Comparison of the two and three dimensional cases strongly indicates that this latter cosmology is a model for the higher

dimensional  $SO(D-1,2)/SO(D-1,1)$  or  $SO(D,1)/SO(D-1,1)$  cosets.

Finally, in Sec. IV, we give our concluding remarks.

## II. TWO-DIMENSIONAL COSET

The  $SO(2,1)/SO(1,1)$  metric and dilaton, to all orders in  $1/k$ , are given by [9]

$$ds^2 = 2(k-2) \left[ \frac{db^2}{4(b^2-1)} - \beta(b) \frac{b-1}{b+1} dx^2 \right], \quad (2.1)$$

$$\Phi(b) = \ln \frac{b+1}{\sqrt{\beta(b)}} + \text{const}, \quad (2.2)$$

where

$$\beta(b) \equiv \left( 1 - \frac{2}{k} \frac{b-1}{b+1} \right)^{-1}. \quad (2.3)$$

In these  $(b,x)$  coordinates, the global manifold is

$$-\infty < b < \infty, \quad -\infty < x < \infty.$$

The scalar curvature is given by

$$R(b) = \frac{4}{k-2} \frac{k(k-4) + k(k-2)b}{[k+2 + (k-2)b]^2}. \quad (2.4)$$

We see here the presence of new quantum curvature singularities at

$$b = -\frac{k+2}{k-2} \quad (2.5)$$

plus coordinate singularities (horizons) at  $b = \pm 1$ .

In the semiclassical limit ( $k \rightarrow \infty$ ), the scalar curvature reduces to

$$R(b) \stackrel{k \rightarrow \infty}{=} \frac{4}{k} \frac{1}{1+b}. \quad (2.6)$$

Comparison of Eqs. (2.4) and (2.6) shows that in the semiclassical limit the quantum curvature singularities coalesce with the coordinate singularities (horizons) at  $b = \pm 1$ .

The fact that the semiclassical curvature singularities become reduced to coordinate singularities, and that new curvature singularities appear in the exact (all orders in  $1/k$ ) theory, will turn out to hold also for the three-dimensional cosets to be discussed in the next section.

For conformal invariance we demand that

$$C \left( \frac{SO(2,1)}{SO(1,1)} \right) = 26 \quad (2.7)$$

leading to  $k = 9/4$ . Thus the curvature singularity (2.6) is located at  $b = -17$ , that is, in the coordinate patch to the left of the left horizon in the  $(b-x)$  diagram. Clearly, it is then possible to construct nonsingular spacetimes by considering

the other coordinate patches. In the present case, we are interested in constructing simple cosmological spacetimes, by which we mean spacetimes with signature  $(-+)$  and line element of the form

$$ds^2 = -dt^2 + A^2(t) dx^2, \quad (2.8)$$

where  $t$  plays the role of cosmic time,  $x$  is the spatial coordinate, and  $A(t)$  is the scale factor.

### A. Oscillating cosmology

Considering the coordinate patch between the two horizons,  $|b| \leq 1$ , and using the parametrization  $b = \cos 2t$ , we get the cosmology [13]

$$ds^2 = \frac{1}{2} \left( -dt^2 + \frac{\tan^2 t}{1 + \frac{8}{9} \tan^2 t} dx^2 \right) \quad (2.9)$$

$$\Phi(t) = \ln(\cos^2 t \sqrt{1 + \frac{8}{9} \tan^2 t}) + \text{const.} \quad (2.10)$$

The scalar curvature

$$R(t) = -72 \frac{4 - \cos^2 t}{(8 + \cos^2 t)^2} \quad (2.11)$$

is oscillating, finite, and always negative

$$R(t) \in \left[ -\frac{9}{2}, -\frac{8}{3} \right], \quad \langle R(t) \rangle = -\frac{5}{\sqrt{2}} \approx -3.53. \quad (2.12)$$

It is an interesting observation that the quantum corrections not only cancel the semiclassical curvature singularities, but also change the sign of the scalar curvature, since semiclassically ( $k \rightarrow \infty$ )

$$R(t) \rightarrow \frac{4}{k} \frac{1}{1 + \cos 2t} \quad (2.13)$$

is always positive (and sometimes positive infinity). We shall see in the next section that this feature also holds in the three-dimensional case.

There is a remarkable similarity between the cosmology (2.9) and the ordinary two-dimensional anti-de Sitter spacetime

$$ds_{AdS}^2 = \frac{1}{2} (-dt^2 + \sin^2 t dx^2), \quad (2.14)$$

where the overall factor  $1/2$  is included for convenience. A direct comparison of the scale factors, Hubble functions, etc., reveals that there is almost no difference between the two cosmologies (2.9) and (2.14). The main difference is found in the scalar curvature, which is constant and negative for the anti-de Sitter cosmology (2.14), while it is oscillating and negative for the coset cosmology (2.9) as we see from Eq. (2.11).

Finally we make a few comments about the dilaton (2.10). In the present notation and conventions, the low-energy ( $k \rightarrow \infty$ ) effective action is

$$S = \int dx \int dt \sqrt{-g} e^\Phi \left[ R + (\nabla \Phi)^2 - \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} + \frac{2}{k} + \dots \right] \quad (2.15)$$

such that the string-coupling  $g$  is given by

$$g_s = e^{-\Phi/2} = (\cos^2 t \sqrt{1 + \frac{8}{9} \tan^2 t})^{-1/2} \quad (2.16)$$

up to an arbitrary positive multiplicative constant. It follows that we should strictly speaking only trust the cosmological solution in the regions near  $t=0$ ,  $t = \pm \pi$ ,  $t = \pm 2\pi$ , etc. That is to say, in the regions where the scale factor is small. In the other regions where the scale factor is large, we should expect that string-loop corrections will be important and possibly change the solution dramatically. Thus, concentrating on the region around  $t=0$ , the line-element equation (2.9) describes a universe experiencing deflationary contraction (negative  $t$ ) followed by deflationary expansion (positive  $t$ ) and always with negative scalar curvature.

### B. Deflationary cosmology

The oscillating cosmology (2.9) was obtained for the value  $k=9/4$ , corresponding to conformal invariance. However, by using other values of  $k$ , it is possible to construct other types of nonsingular cosmologies. In that case, conformal invariance must be ensured by adding other conformal field theories. Moreover, problems with unitarity will possibly appear in such cases. In this subsection, we shall not deal with these problems, but instead concentrate on the possible cosmologies that can be obtained by keeping  $k$  arbitrary.

Taking  $k = -|k| < 0$ , we observe that the curvature singularity (2.6) is located between the two horizons  $b = \pm 1$ . It is then possible to construct nonsingular spacetimes by specializing to the two patches outside the horizons.

Consider first the patch  $b \geq 1$  using the parametrization  $b = \cosh 2t$ . Then Eqs. (2.1)–(2.3) become

$$ds^2 = 2(2 + |k|) \left( -dt^2 + \frac{\tanh^2 t}{1 + (2/|k|)\tanh^2 t} dx^2 \right), \quad (2.17)$$

$$\Phi(t) = \ln \left( \cosh^2 t \sqrt{1 + \frac{2}{|k|} \tanh^2 t} \right) + \text{const.} \quad (2.18)$$

Notice that it is crucial to include the overall prefactor  $2(k-2)$  in the line element to get the desired signature.

In this case the scalar curvature is given by

$$R(t) = -\frac{2|k|}{2+|k|} \frac{(2+|k|)\cosh^2 t + 1}{[(2+|k|)\cosh^2 t - 2]^2}, \quad (2.19)$$

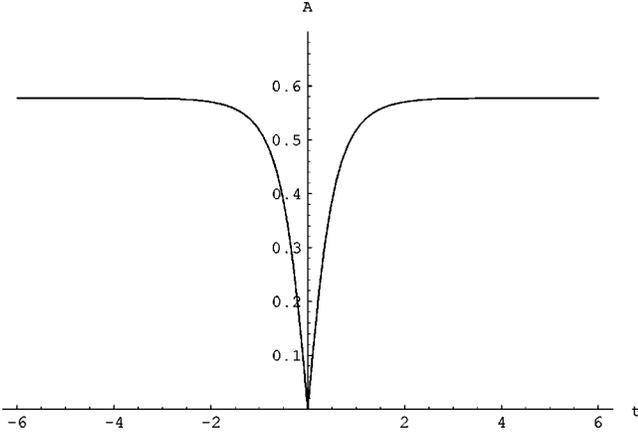


FIG. 1. The scale factor (2.17) as a function of time. Here shown for  $k = -1$ . This cosmology is always deflationary.

while the Hubble function and its derivative become

$$H(t) = \frac{\dot{A}(t)}{A(t)} = \frac{\cosh t}{(\cosh^2 t + (2/|k|)\sinh^2 t)\sinh t}, \quad (2.20)$$

$$\dot{H}(t) = -|k| \frac{(2+|k|)(\cosh^2 t + \sinh^2 t)\cosh^2 t - 2}{[(2+|k|)\cosh^2 t - 2]^2 \sinh^2 t}. \quad (2.21)$$

It follows that the universe has negative scalar curvature and it is always deflationary ( $\dot{H} < 0$ ). More precisely, starting from flat Minkowski space ( $t = -\infty$ ), the universe experiences deflationary contraction until  $t = 0$ . Still deflationary, it then expands and approaches flat Minkowski space again for  $t \rightarrow \infty$  (see Fig. 1).

The string coupling in this case is given by

$$g_s = e^{-\Phi/2} = [\cosh^2 t \sqrt{1 + (2/|k|)\tanh^2 t}]^{-1/2} \quad (2.22)$$

which is always finite and in fact goes to zero for  $t \rightarrow \pm\infty$  (see Fig. 2). Thus, this nonsingular cosmology should be trusted everywhere.

### C. Inflationary cosmology

A more interesting cosmology can be constructed by considering the patch  $b \leq -1$  (and still  $k < 0$ ) instead of  $b \geq 1$ . Using the parametrization  $b = -\cosh 2t$ , Eqs. (2.1)–(2.3) become

$$ds^2 = 2(2+|k|) \left( -dt^2 + \frac{\coth^2 t}{1 + (2/|k|)\coth^2 t} dx^2 \right), \quad (2.23)$$

$$\Phi(t) = \ln(\sinh^2 t \sqrt{1 + (2/|k|)\coth^2 t}) + \text{const.} \quad (2.24)$$

The scalar curvature is now given by

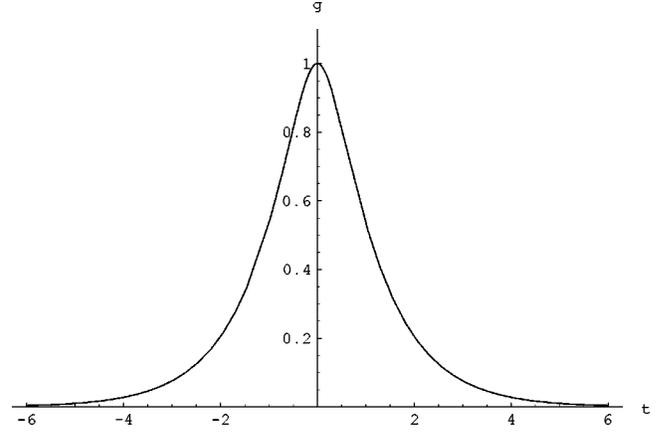


FIG. 2. The string coupling (2.22) corresponding to Fig. 1. The string coupling is finite for all  $t$ .

$$R(t) = \frac{2|k|}{2+|k|} \frac{(2+|k|)\sinh^2 t - 1}{[(2+|k|)\sinh^2 t + 2]^2}, \quad (2.25)$$

while the Hubble function and its derivative become

$$H(t) = -\frac{\sinh t}{(\sinh^2 t + (2/|k|)\cosh^2 t)\cosh t}, \quad (2.26)$$

$$\dot{H}(t) = |k| \frac{(2+|k|)(\cosh^2 t + \sinh^2 t)\sinh^2 t - 2}{[(2+|k|)\sinh^2 t + 2]^2 \cosh^2 t}. \quad (2.27)$$

Now the situation is a little more complicated than in the  $b \geq 1$  patch. However, notice that

$$R(t) > 0 \Leftrightarrow \cosh^2 t > \frac{3+|k|}{2+|k|} \quad (2.28)$$

as well as

$$\dot{H}(t) > 0 \Leftrightarrow \cosh^2 t > \frac{1}{4}(3 + \sqrt{1 + 16/(2+|k|)}). \quad (2.29)$$

Thus the universe starts out as flat Minkowski space at  $t = -\infty$ . It then experiences inflationary ( $\dot{H} > 0$ ) expansion with positive scalar curvature. Just before the scale factor reaches its maximal value at  $t = 0$ , the scalar curvature becomes negative and the expansion becomes deflationary. For  $t > 0$ , the evolution is simply time-reversed (see Fig. 3).

It is particularly interesting that the universe is inflationary ( $\dot{H} > 0$ ) with positive scalar curvature for (large) negative  $t$ . This suggests that the universe goes through some kind of de Sitter phase. This is confirmed by expanding the scale factor for large negative  $t$

$$A(t) = \sqrt{\frac{\coth^2 t}{1 + (2/|k|)\coth^2 t}} \approx \sqrt{\frac{|k|}{2+|k|}} \left( 1 + \frac{2|k|}{2+|k|} e^{2t} \right), \quad t \ll 0; \quad (2.30)$$

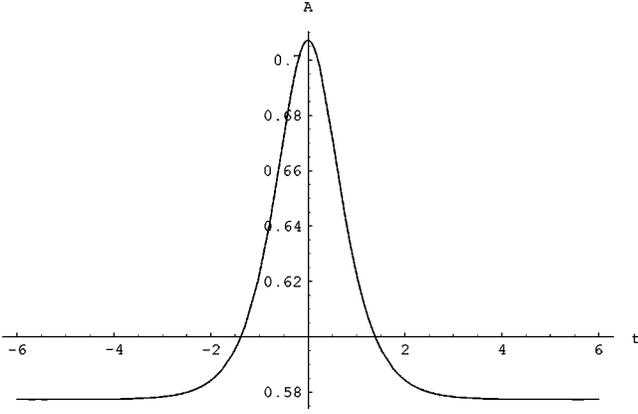


FIG. 3. The scale factor (2.23) as a function of time. Here shown for  $k=-1$ . This cosmology is expanding and inflationary for large negative  $t$ .

thus there is in fact an element of exponential expansion.

Finally, let us consider the string coupling also in this case. It is given by

$$g_s = e^{-\Phi/2} = \left( \sinh^2 t \sqrt{1 + \frac{2}{|k|} \coth^2 t} \right)^{-1/2} \quad (2.31)$$

which is finite everywhere except near  $t=0$  (see Fig. 4). In particular, the string coupling is finite during the phase of inflationary expansion.

### III. THREE-DIMENSIONAL COSET

The  $SO(2,2)/SO(2,1)$  metric and dilaton, to all orders in  $1/k$ , are given by [11]

$$ds^2 = 2(k-2) \left[ \frac{db^2}{4(b^2-1)} + g_{uu} du^2 + g_{vv} dv^2 + 2 g_{uv} du dv \right], \quad (3.1)$$

$$\Phi(b, u, v) = \ln \frac{(b^2-1)(v-u-2)}{\sqrt{\beta(b, u, v)}} + \text{const}, \quad (3.2)$$

where

$$g_{uu}(b, u, v) = \frac{\beta(b, u, v)}{4u(v-u-2)^2} \left[ \frac{b-1}{b+1}(v-u-2) - \frac{v-2}{k-1} \right],$$

$$g_{vv}(b, u, v) = -\frac{\beta(b, u, v)}{4v(v-u-2)^2} \left[ \frac{b+1}{b-1}(v-u-2) + \frac{u+2}{k-1} \right],$$

$$g_{uv}(b, u, v) = \frac{\beta(b, u, v)}{4(v-u-2)^2} \frac{1}{k-1}, \quad (3.3)$$

and the function  $\beta(b, u, v)$  is given by

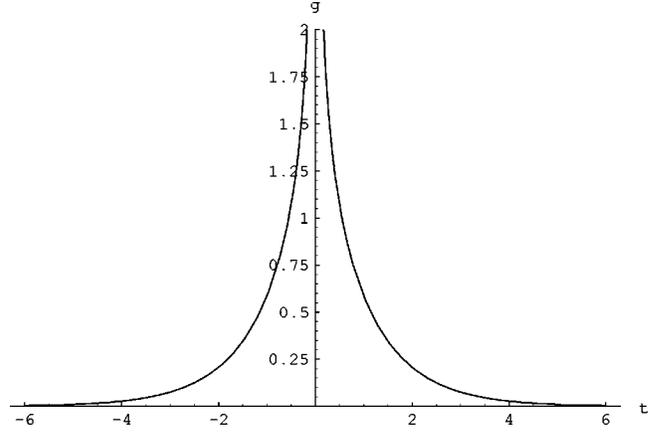


FIG. 4. The string coupling (2.31) corresponding to Fig. 3. The string coupling is finite everywhere except near  $t=0$ .

$$\beta^{-1}(b, u, v) = 1 + \frac{1}{(k-1)(v-u-2)} \times \left[ \frac{b-1}{b+1}(u+2) - \frac{b+1}{b-1}(v-2) - \frac{2}{k-1} \right]. \quad (3.4)$$

In the  $(b, u, v)$  coordinates, the global manifold for  $SO(2,2)/SO(2,1)$  is [10,11]

$$b^2 > 1 \text{ and } uv > 0,$$

$$b^2 < 1 \text{ and } uv < 0, \text{ excluding } 0 < v < u+2 < 2.$$

However, other regions in the  $(b, u, v)$  space can be reached by going for instance to the  $SO(3,1)/SO(2,1)$  coset, for which the metric and dilaton are formally the same as above. For more details, see Refs. [10,11].

The general expression for the scalar curvature, corresponding to the metric (3.3), is quite complicated. It is most conveniently written as a fourth order polynomial in  $b$  divided by the square of a second order polynomial in  $b$ :

$$R(b, u, v) = \frac{4}{k-2} \frac{c_0 + c_1 b + c_2 b^2 + c_3 b^3 + c_4 b^4}{(d_0 + d_1 b + d_2 b^2)^2}, \quad (3.5)$$

where the coefficients  $c_i$  and  $d_j$  are functions of the remaining coordinates  $(u, v)$  and the parameter  $k$

$$c_i = c_i(u, v; k), \quad i=0,1,2,3,4,$$

$$d_j = d_j(u, v; k), \quad j=0,1,2.$$

Their explicit expressions are given in the Appendix.

In the semiclassical limit ( $k \rightarrow \infty$ ), the scalar curvature reduces to

$$R(b, u, v) \stackrel{k \rightarrow \infty}{=} \frac{4}{k} \frac{3 + b^2 - v(b+1) - u(b-1)}{(v-u-2)(b^2-1)} \quad (3.6)$$

in agreement with Ref. [10].

Comparison of the semiclassical expression (3.6) and the exact expression (3.5) for the scalar curvature shows that *all* semiclassical curvature singularities ( $b=1$ ,  $b=-1$ , and  $v=u+2$ ) have disappeared, but that new quantum curvature singularities have appeared. The situation is thus completely analogous to the two-dimensional case: the semiclassical curvature singularities become coordinate singularities (horizons), but new quantum curvature singularities have appeared elsewhere. The curvature singularities in the exact geometry correspond to the zeroes of the denominator in Eq. (3.5), which describe a surface in the  $(b,u,v)$  space

$$\begin{aligned} & -\left[\frac{(1-b)^2}{k-1} + 1 - b^2\right]u + \left[\frac{(1+b)^2}{k-1} + 1 - b^2\right]v \\ & = 2 \left[1 + \frac{1}{(k-1)^2}\right](1-b^2) + 4 \frac{1+b^2}{k-1}. \end{aligned} \quad (3.7)$$

The solutions of this equation precisely correspond to the singularities of the function  $\beta(b,u,v)$ .

By considering different values of  $k$ , one can move the singularity surface around in the global manifold. It is then possible, as in the two-dimensional case, to construct nonsingular spacetimes by considering a single coordinate patch. Notice that for conformal invariance we should demand that

$$C\left(\frac{\text{SO}(2,2)}{\text{SO}(2,1)}\right) = 26, \quad (3.8)$$

leading to  $k = (39 \pm 5\sqrt{13})/23$ .

### A. Oscillating cosmology

Consider first the case where  $k = (39 + 5\sqrt{13})/23 \approx 2.48 \dots$ . It is then possible to construct a cosmology in the coordinate patch

$$|b| \leq 1, \quad u \geq 0, \quad v \leq 0 \quad (3.9)$$

using the parametrization

$$b = \cos 2t, \quad u = 2x^2, \quad v = -2y^2. \quad (3.10)$$

The metric and dilaton are given by

$$ds^2 = 2(k-2)[-dt^2 + g_{xx} dx^2 + g_{yy} dy^2 + 2 g_{xy} dx dy] \quad (3.11)$$

$$\Phi(t,x,y) = \ln \frac{(x^2 + y^2 + 1) \cos^2 t \sin^2 t}{\sqrt{\beta(t,x,y)}} + \text{const}, \quad (3.12)$$

where  $(g_{xx}, g_{yy}, g_{xy})$  and  $\beta(t,x,y)$  are obtained from Eqs. (3.3) and (3.4)

$$\begin{aligned} g_{xx}(t,x,y) &= \frac{\beta(t,x,y)}{(x^2 + y^2 + 1)^2} \left[ (x^2 + y^2 + 1) \tan^2 t + \frac{y^2 + 1}{k-1} \right], \\ g_{yy}(t,x,y) &= \frac{\beta(t,x,y)}{(x^2 + y^2 + 1)^2} \left[ (x^2 + y^2 + 1) \cot^2 t + \frac{x^2 + 1}{k-1} \right], \\ g_{xy}(t,x,y) &= -\frac{\beta(t,x,y)}{(x^2 + y^2 + 1)^2} \frac{xy}{k-1}, \end{aligned} \quad (3.13)$$

as well as

$$\beta^{-1}(t,x,y) = 1 + \frac{(1+x^2)\tan^2 t + (1+y^2)\cot^2 t + (k-1)^{-1}}{(k-1)(x^2 + y^2 + 1)}. \quad (3.14)$$

It is easy to see that this is a nonsingular cosmology with  $t$  playing the role of cosmic time,  $(x,y)$  as the spatial coordinates, and the signature is  $(-++)$ , as it should be. The fact that the cosmology is nonsingular follows since, in the coordinate patch (3.9) considered here and for  $k \approx 2.48$ , the Eq. (3.7) has no real solutions since the left hand side is obviously negative while the right hand side is positive. The cosmology is however somewhat complicated, it is periodic in time, but nonhomogeneous and highly nonisotropic. In fact, the ‘‘scale factors’’ for the two spatial directions are oscillating with a phase difference of  $\pi/2$ .

To gain a little more insight into this cosmology, we consider the region near the spatial ‘‘origin’’  $(x,y) = (0,0)$ . The scalar curvature reduces to

$$R(t,0,0) = \frac{1}{k-2} \frac{\tilde{c}_0 + \tilde{c}_2 \cos^2 2t + \tilde{c}_4 \cos^4 2t}{[k^2 - (k-2)^2 \cos^2 2t]^2}, \quad (3.15)$$

where the coefficients  $(\tilde{c}_0, \tilde{c}_2, \tilde{c}_4)$  are given by

$$\begin{aligned} \tilde{c}_0 &= k^2[-(k-1)^{-1} - 13(k-1) + 6(k-1)^2], \\ \tilde{c}_2 &= (k-2)^2[2(k-1)^{-1} + 4 - 30(k-1) \\ & \quad - 4(k-1)^2], \\ \tilde{c}_4 &= -(k-2)^4[(k-1)^{-1} + 2]. \end{aligned} \quad (3.16)$$

It follows that

$$R(t,0,0) \in \left[ \frac{-(k-1) + 2 - 3(k-1)^{-1}}{k-2}, \frac{-(k-1)^{-1} - 13(k-1) + 6(k-1)^2}{k^2(k-2)} \right] \approx [-3.1435 \dots, -2.2988 \dots] \quad (3.17)$$

as well as

$$\langle R(t,0,0) \rangle = \frac{1}{k-2} \left( \frac{1-2k}{k-1} + \frac{k^2-2}{k\sqrt{k-1}} \right) \approx -2.7126 \dots, \quad (3.18)$$

where the numerical values were obtained for  $k=(39+5\sqrt{13})/23$ . As in the two-dimensional case, we notice therefore that the quantum corrections changed the sign of the curvature since semiclassically ( $k \rightarrow \infty$ )

$$R(t,0,0) \stackrel{k \rightarrow \infty}{=} \frac{4}{k} \frac{3 + \cos^2 2t}{2 \sin^2 2t} \quad (3.19)$$

is always positive (and sometimes positive infinity).

In the region near  $(x,y)=(0,0)$ , one can further define scale factors:

$$A(t) = \sqrt{g_{xx}(t,0,0)} = \sqrt{\beta(t,0,0) \left( \tan^2 t + \frac{1}{k-1} \right)}, \quad (3.20)$$

$$B(t) = \sqrt{g_{yy}(t,0,0)} = \sqrt{\beta(t,0,0) \left( \cot^2 t + \frac{1}{k-1} \right)}. \quad (3.21)$$

That is,

$$A(t) = |\sin t| \sqrt{\frac{(k-1)\sin^2 t + \cos^2 t}{[(k-1) + (k-1)^{-1} - 2]\cos^2 t \sin^2 t + 1}}, \quad (3.22)$$

$$B(t) = |\cos t| \sqrt{\frac{(k-1)\cos^2 t + \sin^2 t}{[(k-1) + (k-1)^{-1} - 2]\cos^2 t \sin^2 t + 1}}, \quad (3.23)$$

both of which oscillate between 0 and  $\sqrt{k-1} \approx 1.216 \dots$  with a phase difference of  $\pi/2$ .

Finally, let us return to the dilaton [for generic  $(t,x,y)$ ]. The string coupling is given by

$$g_s = e^{-\Phi/2} = \left[ \frac{(x^2 + y^2 + 1)\cos^2 t \sin^2 t}{\sqrt{\beta(t,x,y)}} \right]^{-1/2} = [(x^2 + y^2 + 1)|\cos t \sin t|]^{-1/2} \cdot \left[ \cos^2 t \sin^2 t + \frac{(1+x^2)\sin^4 t + (1+y^2)\cos^4 t + (k-1)^{-1}\cos^2 t \sin^2 t}{(k-1)(x^2 + y^2 + 1)} \right]^{-1/4}. \quad (3.24)$$

Thus the string coupling blows up at  $t=0$ ,  $t=\pm\pi/2$ ,  $t=\pm\pi$ , etc. It means that we should only trust the solution in the intermediate regions where  $g_s$  is not large.

### B. Nonoscillating cosmologies

In the previous subsection we obtained an oscillating cosmology using the value  $k=(39+5\sqrt{13})/23 \approx 2.48 \dots$ , corresponding to conformal invariance. Most formulas were, however, presented keeping  $k$  arbitrary, and the oscillating cosmologies in fact exist for arbitrary  $k > 2$ . In this subsection we shall show that it is possible to construct nonsingular three-dimensional nonoscillating cosmologies when  $k < 1$ . Interestingly enough, the condition (3.8) of conformal invariance gives rise to the possibility  $k=(39-5\sqrt{13})/23 \approx 0.912 \dots$ , but in the following we just keep  $k$  arbitrary but less than 1. Possible problems with unitarity will not be dealt with here.

Thus we take  $k < 1$  and consider first the patch

$$b \geq 1, \quad u \geq 0, \quad v \leq 0 \quad (3.25)$$

and use the parametrization

$$b = \cosh 2t, \quad u = 2x^2, \quad v = -2y^2, \quad (3.26)$$

Actually the patch (3.25) is not part of the global manifold for  $SO(2,2)/SO(2,1)$ , so one has to go to the de Sitter coset  $SO(3,1)/SO(2,1)$  instead [10,11]. The general expressions for the metric and dilaton Eqs. (3.1)–(3.4) are, however, unchanged.

Using the parametrization (3.26), the new metric and dilaton are

$$ds^2 = 2(2-k)(-dt^2 + g_{xx} dx^2 + g_{yy} dy^2 + 2 g_{xy} dx dy), \quad (3.27)$$

$$\Phi(t,x,y) = \ln \frac{(x^2 + y^2 + 1)\cosh^2 t \sinh^2 t}{\sqrt{\beta(t,x,y)}} + \text{const}, \quad (3.28)$$

where now

$$\begin{aligned}
 g_{xx}(t,x,y) &= \frac{\beta(t,x,y)}{(x^2+y^2+1)^2} \left[ (x^2+y^2+1)\tanh^2 t + \frac{y^2+1}{1-k} \right], \\
 g_{yy}(t,x,y) &= \frac{\beta(t,x,y)}{(x^2+y^2+1)^2} \left[ (x^2+y^2+1)\coth^2 t + \frac{x^2+1}{1-k} \right], \\
 g_{xy}(t,x,y) &= -\frac{\beta(t,x,y)}{(x^2+y^2+1)^2} \frac{xy}{1-k},
 \end{aligned} \tag{3.29}$$

and the function  $\beta(t,x,y)$  is given by

$$\begin{aligned}
 \beta^{-1}(t,x,y) \\
 &= 1 + \frac{(1+x^2)\tanh^2 t + (1+y^2)\coth^2 t + (1-k)^{-1}}{(1-k)(x^2+y^2+1)}.
 \end{aligned} \tag{3.30}$$

Notice the similarity with Eqs. (3.11)–(3.14): trigonometric functions became hyperbolic functions and  $k-1$  became  $1-k$ . Since we now consider the case where  $k < 1$ , it is then clear that Eqs. (3.27)–(3.30) describes a cosmology with the correct signature  $(-++)$ . Furthermore, it is easily seen from Eq. (3.7) that the cosmology is nonsingular. The change from trigonometric functions to hyperbolic functions obviously has dramatic consequences. The cosmology is however still nonisotropic and nonhomogeneous, but it is no longer oscillating. In fact, the time-dependence of the metric actually disappears for  $t \rightarrow \pm\infty$ , i.e., the universe becomes static in these limits (however, the dilaton stays time-dependent). More precisely, both “scale factors” start out with constant values at  $t = -\infty$ . Then one of them increases monotonically towards a maximal value, while the other one decreases monotonically to zero for  $t \rightarrow 0_-$ . For  $t > 0$ , their behavior is simply time-reversed.

As in the previous subsection, it is useful to consider in more detail the region near the spatial origin  $(x,y) = (0,0)$ . The scalar curvature then reduces to

$$R(t,0,0) = -\frac{1}{2-k} \frac{\tilde{c}_0 + \tilde{c}_2 \cosh^2 2t + \tilde{c}_4 \cosh^4 2t}{[k^2 - (2-k)^2 \cosh^2 2t]^2}, \tag{3.31}$$

with the coefficients  $(\tilde{c}_0, \tilde{c}_2, \tilde{c}_4)$  are still given by Eq. (3.16). It follows that

$$R(\pm\infty, 0, 0) = \frac{-(1-k)^{-1} + 2}{2-k} \leq (\sqrt{3}-1)^2 \approx 0.536 \dots, \tag{3.32}$$

$$R(0,0,0) = \frac{-3(1-k)^{-1} - 2 - (1-k)}{2-k} < -1, \tag{3.33}$$

where the numerical values were obtained using the fact that  $k < 1$ . Thus the scalar curvature is either positive or negative (depending on the precise value of  $k < 1$ ) for large  $|t|$ , but always becomes negative for  $|t|$  close to zero.

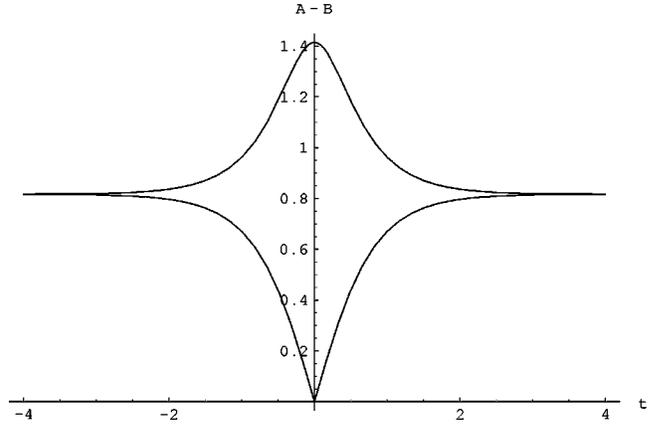


FIG. 5. The two scale factors (3.34) and (3.35) as a function of time. Here shown for  $k = -1$ . For large negative  $t$ , one scale factor is expanding and inflationary while the other scale factor is contracting and deflationary.

As for the oscillating cosmology in the previous subsection, let us define scale factors analogous to Eqs. (3.20) and (3.21). In the present case, one obtains

$$\begin{aligned}
 A(t) \\
 &= |\sinh t| \sqrt{\frac{(1-k)\sinh^2 t + \cosh^2 t}{[(1-k) + (1-k)^{-1} + 2]\cosh^2 t \sinh^2 t + 1}},
 \end{aligned} \tag{3.34}$$

$$\begin{aligned}
 B(t) &= \cosh t \sqrt{\frac{(1-k)\cosh^2 t + \sinh^2 t}{[(1-k) + (1-k)^{-1} + 2]\cosh^2 t \sinh^2 t + 1}}
 \end{aligned} \tag{3.35}$$

(see Fig. 5). It follows that

$$A(0) = 0, \quad B(0) = \sqrt{1-k}, \quad A(\pm\infty) = B(\pm\infty) = \sqrt{\frac{1-k}{2-k}}. \tag{3.36}$$

A careful analysis of the corresponding Hubble functions and their derivatives shows that for  $t \ll 0$ , one scale factor is contracting and deflationary while the other scale factor is expanding and inflationary. This is most easily seen from the expansions

$$A(t) \approx \sqrt{\frac{1-k}{2-k}} \left( 1 - 2 \frac{1-k}{2-k} e^{2t} \right), \quad t \ll 0, \tag{3.37}$$

$$B(t) \approx \sqrt{\frac{1-k}{2-k}} \left( 1 + 2 \frac{1-k}{2-k} e^{2t} \right), \quad t \ll 0, \tag{3.38}$$

and still taking into account that  $k < 1$ .

It is interesting that the scale factor  $A(t)$  is very similar to the scale factor of the deflationary two-dimensional cosmology of Sec. II B, while the scale factor  $B(t)$  is very similar to the scale factor of the inflationary two-dimensional cosmology of Sec. II C. In particular, the scale factor  $B(t)$  has an element of exponential expansion as seen from Eq. (3.38). In

that sense, the two-dimensional coset comprises all the main features of the three-dimensional coset.

The string coupling in this case [generic  $(t,x,y)$ ] is given by

$$g_s = e^{-\Phi/2} = \left[ \frac{(x^2 + y^2 + 1) \cosh^2 t \sinh^2 t}{\sqrt{\beta(t,x,y)}} \right]^{-1/2} = [(x^2 + y^2 + 1) \cosh t |\sinh t|]^{-1/2} \cdot \left[ \cosh^2 t \sinh^2 t + \frac{(1 + x^2) \sinh^4 t + (1 + y^2) \cosh^4 t + (1 - k)^{-1} \cosh^2 t \sinh^2 t}{(1 - k)(x^2 + y^2 + 1)} \right]^{-1/4} \quad (3.39)$$

which is finite everywhere except near  $t=0$ . Asymptotically ( $t \rightarrow \pm\infty$ ) it goes to zero (see Fig. 6). Thus the solution should be trusted everywhere except near  $t=0$ .

It is also possible to construct a nonsingular nonoscillating cosmology in the patch

$$b \leq -1, \quad u \geq 0, \quad v \leq 0, \quad (3.40)$$

using the parametrization

$$b = -\cosh 2t, \quad u = 2x^2, \quad v = -2y^2, \quad (3.41)$$

and still taking  $k < 1$  [also in this case one has to go to the de Sitter coset  $SO(3,1)/SO(2,2)$ ; see Ref. [10,11]]. However, the resulting cosmology is identical to the previous one, up to an interchange of  $x$  and  $y$ .

#### IV. CONCLUSIONS

The cosmological geometries found in this paper are exact conformal field theories. That is, they describe string vacua. Physically, these are spacetimes where the (massless) dilaton and gravitons fields are present. We find that such manifolds provide nonsingular spacetimes with cosmological interpretation. These are of two types: oscillating and nonoscillating cosmologies. The nonoscillating metrics start as Minkowski spacetimes for early times, evolve through inflationary expansion (with *positive scalar curvature*), then pass through a deflationary contraction ending as a Minkowski spacetime for late times.

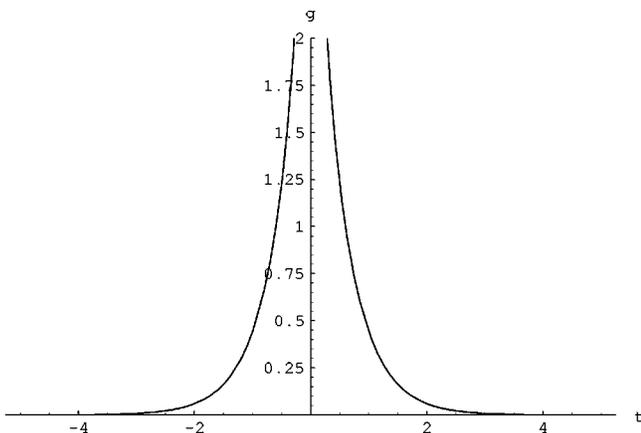


FIG. 6. The string coupling (3.39) corresponding to Fig. 5. The string coupling is finite everywhere except near  $t=0$ .

All cosmological spacetimes from these cosets string vacua turn to be anisotropic. This seems to be a general and unavoidable feature of string vacua cosmologies. Physically, this must be related to the masslessness of the matter sources of these geometries. Notice that de Sitter spacetime in general relativity is exactly isotropic and has a nonzero cosmological constant as matter source.

The coset string cosmologies studied in this paper correspond to analogies of de Sitter spacetime in string vacua. In general relativity, the global de Sitter manifold describes a contracting phase (for  $t \leq 0$ ) and then an expanding universe for  $t \geq 0$ . Only the expanding patch describes the physical space. Similarly, in string cosmologies one should not consider the physical space as the whole global manifold but only a part of it. (The situation is somehow analogous to the Schwarzschild black hole in general relativity. The global Kruskal manifold describes a black hole and its ‘mirror’ disconnected space; only half of the global manifold describes the physical space [the exterior plus the interior of the black hole].) It is a generic feature of coset string cosmologies that the inflationary expanding phase, when it appears it does so for  $t < 0$ .

In both two and three dimensions, the coset cosmologies considered in this paper are related to each other by combinations of simple translations  $t \rightarrow t + \pi/2$ , analytic continuation  $t \rightarrow it$ , and changes of the level  $k$ . This can be seen directly from the explicit expressions for the metric, dilaton, and string coupling, and follows more generally from the fact that the different cosmologies correspond to different coordinate patches of the same global manifold. It was shown in Refs. [8,10] that the transformations relating the different patches are generalized T-duality transformations, in the sense that different patches arise from different gauge fixings in the gauged WZW construction. The dualities in question are thus generalizations of the well-known dualities relating vector and axial gaugings in the two-dimensional case [9].

The similarities between the two and three dimensional cases studied here suggest a general picture for the four dimensional cosmologies; anisotropy seems to be a generically unavoidable feature and cosmological singularities are generically avoided. For coset dimensions higher than four, it is possible to construct nonsingular cosmologies with four dimensions inflating while the others (internal ones) deflate.

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## APPENDIX A

In this Appendix we list the explicit expressions for the coefficients appearing in the scalar curvature (3.5)

$$c_0(u, v; k) = (u^2 + v^2)[1 - (k-1) + (k-1)^2] - 2uv[-5 - (k-1) + (k-1)^2] + (v-u)[-(k-1)^{-2} + 8 + 2(k-1) - 5(k-1)^2] - (k-1)^{-3} - 2(k-1)^{-2} - 14(k-1)^{-1} - 20 - (k-1) + 6(k-1)^2, \quad (\text{A1})$$

$$c_1(u, v; k) = (v^2 - u^2)[3 - 2(k-1) + (k-1)^2] + (v+u)[-2(k-1)^{-2} + 10(k-1)^{-1} - 8 + 6(k-1) - 2(k-1)^2], \quad (\text{A2})$$

$$c_2(u, v; k) = (v^2 + u^2)[3 - (k-1) - (k-1)^2] - 2uv[5 + (k-1) - (k-1)^2] + (v-u)[16(k-1)^{-1} - 30 + 10(k-1) + 4(k-1)^2] + 2(k-1)^{-3} - 36(k-1)^{-1} + 60 - 22(k-1) - 4(k-1)^2, \quad (\text{A3})$$

$$c_3(u, v; k) = (v^2 - u^2)[1 - (k-1)^2] - (v+u)[-2(k-1)^{-2} - 2(k-1)^{-1} + 8 - 2(k-1) - 2(k-1)^2], \quad (\text{A4})$$

$$c_4(u, v; k) = (v-u)[(k-1)^{-2} - 4(k-1)^{-1} + 6 - 4(k-1) + (k-1)^2] - (k-1)^{-3} + 2(k-1)^{-2} + 2(k-1)^{-1} - 8 + 7(k-1) - 2(k-1)^2, \quad (\text{A5})$$

as well as

$$\begin{aligned} d_0(u, v; k) &= (v-u)k - 2k [1 + (k-1)^{-1}], \\ d_1(u, v; k) &= 2(v+u), \\ d_2(u, v; k) &= (v-u)(2-k) - 2(2-k) [1 - (k-1)^{-1}]. \end{aligned} \quad (\text{A6})$$

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