Spectrum of type IIB supergravity on $AdS_5\times T^{11}$: Predictions on $\mathcal{N}=1$ SCFT's

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We derive the full Kaluza-Klein spectrum of type IIB supergravity compactified on $AdS_5\times T^{11}$ with $T^{11}=SU(2)\times SU(2)/U(1)$. From the knowledge of the spectrum and general multiplet shortening conditions, we make a refined test of the AdS-CFT correspondence, by comparison between various shortenings of $SU(2,2|1)$ supermultiplets on AdS_5 and different families of boundary operators with protected dimensions. Additional towers of long multiplets with rational dimensions, that are not protected by supersymmetry, are also predicted from the supergravity analysis to occur in the SCFT at leading order in N and g_sN.

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I. INTRODUCTION

One of the most fascinating properties of the AdS-conformal field theory (CFT) correspondence [1,2,3] is the deep relation between supergravity and gauge theory dynamics, at least in the regime where the supergravity approximation (small space time curvature) is a reliable description of a more fundamental theory such as string or M theory [4,5]. This occurs in the regime where g_sN (g_s being the string coupling) and/or N are large.

Although many tests have been performed in the case of maximal supersymmetry, relating for instance, the dynamics of N coincident $D3$ branes (for large N) and type IIB supergravity compactified on $AdS_5\times S^5$ [4], much less is known on the dual theories for a lower number of supersymmetries [6], where the candidate models exhibit a far richer structure since they contain a variety of matter multiplets with additional symmetries other than the original R symmetry dictated by the supersymmetry algebra [7].

A particularly interesting class of models is obtained by assuming that S^5 is replaced by a five-dimensional coset manifold $X_5=G/H$ with some Killing spinors. As shown in [8] there is a unique such manifold $X_5=T^{pq}=SU(2)\times SU(2)/U(1)$ with $p=q=1$, where p and q define the embedding of the $H=U(1)$ group into the two $SU(2)$ groups. The supergravity theory on $AdS_5\times T^{11}$ is an $\mathcal{N}=2$ supergravity theory with a matter gauge group $G=SU(2)\times SU(2)$. The corresponding four dimensional conformal field theory must then be [9] an $\mathcal{N}=1$ Yang Mills theory with a flavor symmetry G such that an accurate test of the AdS-CFT correspondence could be made using the knowledge of the entire spectrum of the supergravity side of this theory.

The conformal field theory description of type IIB on $AdS_5\times T^{11}$ was constructed by Klebanov and Witten [9] and it was the first example of a conformal theory describing branes at conifold singularities. The same theory was later reobtained by Morrison and Plesser [10] by adopting a general method of studying branes at singularities [11]. In fact, under certain conditions, a conical singularity in a Calabi-Yau space of complex dimension n can be described by a cone over an Einstein manifold X_{2n-1}. In the case of $X_5=T^{11}$ such construction gives rise to a conformal field theory with "singleton" [12] degrees of freedom A and B each a doublet of the factor groups $SU(2)\times SU(2)$ and with conformal anomalous dimension $\Delta_{A,B}=3/4$. Moreover the gauge group $\mathcal{G}$ is $SU(N)\times SU(N)$ and the two singleton (chiral) multiplets are respectively in the $(N,\overline{N})$ and $(\overline{N},N)$ of $\mathcal{G}$.

A set of chiral operators of this theory which are the analogue of the Kaluza-Klein (KK) excitations of the $\mathcal{N}=4$ Yang-Mills theory with $SU(N)$ gauge group is given by $\mathrm{Tr}(AB)^k$ with R-charge k and in the $(k/2,k/2)$ representation of $SU(2)\times SU(2)$. The existence of this (infinite in the large N, g_sN limit) family of chiral operators (massive $\mathcal{N}=2$ hypermultiplets in the supergravity language) has been

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confirmed by Gubser [13] by a study of the eigenvalues of the scalar Laplacian when performing harmonic analysis of type IIB supergravity on $AdS_5 \times T^{11}$.

Moreover, the matching of gravitational and R-symmetry anomalies in the two theories has been also proved in Ref. [13].

This paper analyzes the complete spectrum of the KK states on $AdS_5 \times T^{11}$ and infers its multiplet structure as done in previous investigations for maximal supersymmetry. In that case the KK spectrum, analyzed in terms of AdS representations in [14,15], was interpreted in terms of $\mathcal{N}=1$ conformal superfields in [3] and in terms of the $\mathcal{N}=4$ ones in [16] and [17]. The multiplet shortening conditions [18] can be inferred from the knowledge of all the mass matrices in the KK spectrum [19,20]. In the case of the $SU(2,2|1)$ superalgebra, the shortening is proven to correspond to three types of shortening of the appropriate representations, as discussed in [21] and [22]: massless AdS multiplets, short AdS multiplets, and semilong AdS multiplets. These multiplets, in the conformal field theory language, correspond to respectively conserved, chiral, and semiconserved superfields which have all protected dimensions and which therefore correspond to very particular shortening conditions in the KK context.

We show a full and detailed correspondence between all the CFT operators and the KK modes for the conformal operators of preserved scaling dimension. We also show that there exist other operators related to long multiplets but having nonrenormalized conformal dimension in the supergravity limit. Interestingly enough, these operators seem to be the lowest dimensional ones for a given structure appearing in the supersymmetric Born-Infeld action of the $D3$-brane on $AdS_5 \times T^{11}$ [23,24,25,26].

The paper is organized as follows. In Sec. II the harmonic analysis type of IIB supergravity on $AdS_5 \times T^{11}$ is performed and the complete mass spectrum of the theory is exhibited. In Sec. III properties of $\mathcal{N}=1$ four-dimensional supersymmetric field theories are recalled, in particular the superfield realization of different short and long superconformal multiplets of the $SU(2,2|1)$ superalgebra. In Sec. IV a comparison of superfields of protected dimensions and states in the KK spectrum is made using the formulas giving the mass-conformal dimension relations as predicted by the AdS/CFT correspondence.

II. HARMONIC ANALYSIS ON T^{11}

In this section we give a summary of the derivation of the full mass spectrum of type IIB supergravity compactified on $AdS_5 \times T^{11}$ obtained by KK harmonic expansion on T^{11}. Since our main goal here is the comparison of the mass spectrum with the composite operators of the CFT at the boundary of AdS_5, we just sketch the general procedure and postpone a detailed derivation of our results to a forthcoming publication [27]. Partial results were obtained in [13,28] using different methods.

A. Harmonic expansion

Let us start with a short discussion of the T^{11} geometry.[1] We consider two copies of $SU(2)$ with generators $T_A, \hat{T}_A$, $(A=1\dots3)$: $[T_A, T_B]=\epsilon_{AB}{}^{C}T_C$.

We decompose the Lie algebra $\mathbb{G}$ of $SU(2)\times SU(2)$ with respect to the diagonal generator

$$T_H \equiv T_3 + \hat{T}_3, \tag{2.1}$$

as $\mathbb{G}=\mathbb{H}+\mathbb{K}$, where the subalgebra $\mathbb{H}$ is made of the single generator T_H and the coset algebra $\mathbb{K}$ contains the generators T_i $(i=1,2)$, $\hat{T}_s$ $(s=1,2)$, and

$$T_5 = T_3 - \hat{T}_3. \tag{2.2}$$

In terms of this new basis the commutation relations are

$$[T_i, T_j] = \frac{1}{2}\epsilon_{ij}(T_H + T_5), \quad [\hat{T}_s, \hat{T}_t] = \frac{1}{2}\epsilon_{st}(T_H - T_5),$$

$$[T_5, T_i] = [T_H, T_i] = \epsilon_i{}^{j} T_j, \quad [T_5 \hat{T}_s] = [T_H, \hat{T}_s] = \epsilon_s{}^{t}\hat{T}_t,$$

$$[T_i, \hat{T}_s] = [T_5, T_H] = 0. \tag{2.3}$$

We introduce the coset representative L of $SU(2)\times SU(2)/U_H(1)$, $U_H(1)$ being the diagonal subgroup of G generated by T_H,

$$L(y^i, y^s, y^5) = \exp(T_i y^i)\exp(\hat{T}_s y^s)\exp(T_5 y^5), \tag{2.4}$$

and constructs the left invariant form on the coset

$$L^{-1}dL = \omega^i T_i + \omega^s \hat{T}_s + \omega^5 T_5 + \omega^H T_H, \tag{2.5}$$

where the one-forms $\{\omega^i, \omega^s, \omega^5, \omega^H\}$ satisfy the Maurer-Cartan equations (MCE's)

$$d\omega^\Lambda + \frac{1}{2} C^{\Lambda}_{\Sigma\Pi}\omega^\Sigma \omega^\Pi = 0, \quad \Lambda,\Pi,\Sigma \equiv \{i,s,5,H\}. \tag{2.6}$$

The one-forms $\omega^K \equiv \{\omega^i, \omega^s, \omega^5\}$ are $\mathbb{K}$ valued and can be identified with the five vielbeins of $G/H = T^{11}$, while ω^H is $\mathbb{H}$ valued and is called the H connection of the coset manifold. It is convenient to rescale the ω^K and define as vielbeins $V^a \equiv (V^i, V^s, V^5)$:

$$V^i = a\,\omega^i, \quad V^s = b\,\omega^s, \quad V^5 = c\,\omega^5, \tag{2.7}$$

where a,b,c are real rescaling factors which will be determined by requiring that T^{11} is an Einstein space [29,30].

Once we have the vielbeins, we may construct the Riemann connection one-form $\mathcal{B}^{ab} \equiv -\mathcal{B}^{ba}$ $(a,b=i,s,5)$, imposing the torsion-free condition

$$dV^a - \mathcal{B}^{ab}V_b = 0. \tag{2.8}$$

By comparison with the MCE's (2.6), one finds

[1] For details about the notations and conventions see the Appendix.

$$\mathcal{B}^{ij}=-\epsilon^{ij}\left[\omega^H+\left(c-\frac{a^2}{4c}\right)V^5\right],\quad \mathcal{B}^{5i}=\frac{a^2}{4c}\epsilon^{ij}V_j,$$

$$\mathcal{B}^{st}=-\epsilon^{st}\left[\omega^H-\left(c-\frac{b^2}{4c}\right)V^5\right],\quad \mathcal{B}^{5s}=-\frac{b^2}{4c}\epsilon^{st}V_t, \tag{2.9}$$

$$\mathcal{B}^{is}=0.$$

Consequently, the curvature two-form, defined as

$$R^{ab}=d\mathcal{B}^{ab}-\mathcal{B}^a{}_c\mathcal{B}^{cb}, \tag{2.10}$$

turns out to be

$$R_{ij}=\left(a^2-\frac{3}{16}\frac{a^4}{c^2}\right)V^iV^j+\frac{a^2b^2}{16c^2}\epsilon^{ij}\epsilon^{st}V_sV_t,$$

$$R^{st}=\left(b^2-\frac{3}{16}\frac{b^4}{c^2}\right)V^sV^t+\frac{a^2b^2}{16c^2}\epsilon^{st}\epsilon^{ij}V_iV_j,$$

$$R^{is}=\frac{a^2b^2}{16c^2}\epsilon^{ij}\epsilon^{st}V_jV_t, \tag{2.11}$$

$$R^{i5}=\frac{a^4}{16c^2}V^iV^5,$$

$$R^{s5}=\frac{a^4}{16c^2}V^sV^5.$$

The Ricci tensors are now easily computed. We find

$$R^i{}_k=\left(\frac{1}{2}a^2-\frac{a^4}{16c^2}\right)\delta^i_k,\quad R^s{}_t=\left(\frac{1}{2}b^2-\frac{b^4}{16c^2}\right)\delta^s_t,$$

$$R^5{}_5=\frac{a^4}{8c^2}. \tag{2.12}$$

In order to have an Einstein space with Ricci tensor

$$R^a{}_b=2e^2\delta^a_b, \tag{2.13}$$

we must have

$$a^2=b^2=6e^2,\quad \text{and}\quad c^2=\frac{9}{4}e^2. \tag{2.14}$$

An essential tool for the computation of the Laplace-Beltrami invariant operators on T^{11} is the covariant derivative $\mathcal{D}\equiv(\mathcal{D}_i,\mathcal{D}_s,\mathcal{D}_5)$. Starting from the definition

$$\mathcal{D}=d+\mathcal{B}^{ab}T_{ab}\equiv d+\mathcal{B}, \tag{2.15}$$

where T_{ab} are the $SO(5)$ generators written as matrices, $(T_{ab})^{cd}=-\delta^{cd}_{ab}$, setting $\mathcal{B}=\omega^H+M$, one can write

$$\mathcal{D}=\mathcal{D}^H+M, \tag{2.16}$$

where the H-covariant derivative is defined by

$$\mathcal{D}^H=d+\omega^H \tag{2.17}$$

and the matrix of one-forms M can be computed from Eq. (2.9)

$$M^{ij}=-\left(c-\frac{a^2}{4c}\right)V^5\epsilon^{ij},\quad M^{5i}=\frac{a^2}{4c}\epsilon^{ij}V_j,$$

$$M^{st}=\left(c-\frac{a^2}{4c}\right)V^5\epsilon^{st},\quad M^{5s}=-\frac{a^2}{4c}\epsilon^{st}V_t,$$

$$M^{is}=0. \tag{2.18}$$

The usefulness of the decomposition (2.16), (2.17), (2.18) lies in the fact that the action of $\mathcal{D}^H$ on the basic harmonic represented by the T^{11} coset representative L^{-1} can be computed algebraically. Indeed one has quite generally [30,31]

$$\mathcal{D}^H=-r(a)T_aV^a\equiv-a(T_iV^i+\hat{T}_sV^s)-cT_5V^5, \tag{2.19}$$

where $r(i)=r(s)=a$, $r(5)=c$ are the rescalings and T_a are the coset generators of T^{11}.

In summary, the covariant derivative on the basic harmonic L^{-1} can be written as follows:

$$\mathcal{D}L^{-1}=(-r(a)T_aV^a+M^{ab}T_{ab})L^{-1}, \tag{2.20}$$

or, in components, using Eq. (2.18),

$$\mathcal{D}_iL^{-1}=\left(-aT_i-\frac{a^2}{2c}\epsilon_i{}^jT_{5j}\right)L^{-1},$$

$$\mathcal{D}_sL^{-1}=\left(-aT_s+\frac{a^2}{2c}\epsilon_s{}^tT_{5t}\right)L^{-1}, \tag{2.21}$$

$$\mathcal{D}_5L^{-1}=\left(-cT_5-2\left(c-\frac{a^2}{4c}\right)(T_{12}-T_{34})\right)L^{-1}.$$

In a KK compactification, after the linearization of the equations of motion of the field fluctuations, one is left with a differential equation on the ten-dimensional fields $\phi^{[\Lambda]}_{[\lambda_1,\lambda_2]}(x,y)$

$$(\Box^{[\Lambda]}_x+\boxtimes^{[\lambda_1,\lambda_2]}_y)\phi^{[\Lambda]}_{[\lambda_1,\lambda_2]}(x,y)=0. \tag{2.22}$$

Here the field $\phi^{[\Lambda]}_{[\lambda_1,\lambda_2]}(x,y)$ transforms irreducibly in the representations $[\Lambda]\equiv[E_0,s_1,s_2]$ of $SU(2,2)\approx O(4,2)$ and $[\lambda_1,\lambda_2]$ of $SO(5)$ and it depends on the coordinates x of AdS_5 and y of T^{11}. $\Box_x$ is the kinetic operator for a field of quantum number $[\Lambda]$ in five dimensional AdS space and $\boxtimes_y$ is the kinetic operator for a field of spin $[\lambda_1,\lambda_2]$ in the internal space T^{11}. (In the following we omit the index $[\Lambda]$ on the fields.)

Expanding $\phi_{[\lambda_1,\lambda_2]}(x,y)$ in the harmonics of T^{11} transforming irreducibly under the isometry group of T^{11}, one is reduced to the problem of computing the action of $\boxtimes_y$ on the harmonics, whose eigenvalues define the AdS mass.

$\boxtimes_y$ is a Laplace-Beltrami operator on T^{11} and it is constructed, for every representation $[\lambda_1,\lambda_2]$, in terms of the covariant derivative on G/H. Since the covariant derivative acts algebraically on the basic vector or spinor harmonic L^{-1} (in terms of which any harmonic can be constructed), the problem of the mass spectrum computation is reduced, via Eqs. (2.20), (2.21) to a purely algebraic problem.

The explicit evaluation of the linearized equation (2.22) for the five-dimensional case has been given in [32] and we will adopt the same notations therein to denote the five-dimensional space-time fields appearing in the harmonic expansion. Note that Eq. (2.22) has been evaluated in [32] around the background solution presented in [8]:

$$\begin{aligned} &F_{abcde}=e\,\epsilon_{abcde}, \quad R^a{}_b=2e^2\delta^a_b, \\ &F_{mnpqr}=-e\,\epsilon_{mnpqr}, \quad R^m{}_n=-2e^2\delta^m_n, \\ &B=A_{MN}=0, \quad \psi_M=\chi=0, \end{aligned} \tag{2.23}$$

where the field F_{abcde} and F_{mnpqr} is the projection on T^{11} and AdS_5 of the ten-dimensional five-form F defined as $F=dA_4$, A_4 being the real self-dual four-form of type IIIB supergravity. The other fields of type IIB supergravity are the metric $G_{MN}(x,y)$ with internal and space-time components $g_{\alpha\beta}(y), g_{\mu\nu}(x)$ whose Ricci tensors in this background are given in Eq. (2.23) and the complex 0-form and 2-form B and A_{MN} [the fermionic fields ψ_M and λ are obviously zero in the background (2.23)].

The harmonics on the coset space T^{11} are labeled by two kinds of indices, the first labelling the particular representation of the isometry group $SU(2)\times SU(2)\times U_R(1)$ and the other referring to the representation of the subgroup $H\equiv U_H(1)$. The harmonic is thus denoted by $Y^{(j,l,r)}_{(q)}(y)$ where j, l are the spin quantum numbers of the two $SU(2)$ in a given representation, q is the $U_H(1)$ charge and r denotes the $U_R(1)$ quantum number associated to the generator T_5 orthogonal to T_H. We can identify r as the R-symmetry quantum number [13,28].

Now we observe that $U_H(1)$ is necessarily a subgroup of $SO(5)$, the tangent group of T^{11}. The embedding formula of $U_H(1)$ in a given representation of $SO(5)$ labeled by indices Λ, Σ is given by [30,31]

$$(T_H)^\Lambda{}_\Sigma=C^{ab}_H(T_{ab})^\Lambda{}_\Sigma, \tag{2.24}$$

where the structure constants C^{ab}_H are derived from the algebra (2.3) and T_{ab} are the $SO(5)$ generators.

In the vector representation of $SO(5)$ we find

$$(T_H)_{ab}=C_{Hab}=\begin{pmatrix} \epsilon_{ij} & & \\ & \epsilon_{st} & \\ & & 0 \end{pmatrix}, \tag{2.25}$$

while for the spinor representation we get

$$\begin{aligned}(T_H)&=C^{ab}_H(T_{ab})=-\frac{1}{4}C^{ab}_H(\gamma_{ab})=-\frac{1}{2}(\gamma_{12}+\gamma_{34}) \\ &=i\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \end{aligned} \tag{2.26}$$

where γ are the $SO(5)$ gamma matrices.

The above results imply that an $SO(5)$ field $\phi_{[\lambda_1,\lambda_2]}(x,y)$ can be split into the direct sum of $U_H(1)$ one-dimensional fragments labeled by the $U_H(1)$ charge q. From Eqs. (2.25) and (2.26) it follows that the five-dimensional and four-dimensional $SO(5)$ representations break under $U_H(1)$ as

$$\begin{aligned} &\mathbf{5}\to 1\oplus -1\oplus 1\oplus -1\oplus 0 \quad [\lambda_1,\lambda_2]=[1,0], \\ &\mathbf{4}\to 1\oplus -1\oplus 0\oplus 0 \quad [\lambda_1,\lambda_2]=[1/2,1/2]. \end{aligned} \tag{2.27}$$

From Eq. (2.27) we easily find the analogous breaking law for antisymmetric tensors ($[\lambda_1,\lambda_2]=[1,1]$), symmetric traceless tensors ($[\lambda_1,\lambda_2]=[2,0]$) and spin tensors ($[\lambda_1,\lambda_2]=[3/2,1/2]$) by taking suitable combinations:

$$\begin{aligned} &\mathbf{10}\to \pm1\oplus\pm1\oplus\pm2\oplus0\oplus0\oplus0\oplus0 \\ &\qquad [\lambda_1,\lambda_2]=[1,1], \\ &\mathbf{16}\to \pm2\oplus\pm2\oplus\pm1\oplus\pm1\oplus\pm1\oplus\pm1\oplus0\oplus0\oplus0\oplus0 \\ &\qquad [\lambda_1,\lambda_2]=\left[\frac{3}{2},\frac{1}{2}\right], \\ &\mathbf{14}\to \pm2\oplus\pm2\oplus\pm2\oplus\pm1\oplus\pm1\oplus0\oplus0\oplus0\oplus0 \\ &\qquad [\lambda_1,\lambda_2]=[2,0]. \end{aligned} \tag{2.28}$$

Actually it is often more convenient to write down the harmonic expansion in terms of the $SO(5)$ harmonics $Y^{(j,l)}_{[\lambda_1,\lambda_2]}$ whose fragments are the $Y^{(j,l,r)}_{(q)}$ introduced before.

The generic field $\phi_{[\lambda_1,\lambda_2]}(x,y)$ can be expanded in these harmonics as follows:

$$\phi_{ab\ldots}(x,y)=\sum_{(\nu)}\sum_{(m)}\phi_{(\nu)(m)}(x)Y^{(\nu)(m)}_{ab\ldots}(y), \tag{2.29}$$

where $a,b,\ldots$ are $SO(5)$ tensor (or spinor) indices of the representation $[\lambda_1,\lambda_2]$, (ν) is a shorthand notation for (j,l,r) and m labels the representation space of (j,l,r). In our case m coincides with the labelling of the $U_H(1)$ fragments. It is well known [30,31] that the irrepses of $SU(2)\times SU(2)$ appearing in the expansion (2.29) are only those which contain, when reduced with respect to $U_H(1)$, a charge q also appearing in the decomposition of $[\lambda_1,\lambda_2]$ under $U_H(1)$.

It is easy to see which are the constraints on j,l,r selecting the allowed representations (ν) appearing in Eq. (2.29). We write a generic representation of $SU(2)\times SU(2)$ in the Young tableaux formalism:

$$(j,l) \equiv \underbrace{\boxed{}\boxed{\;\dots\;}\boxed{}}_{2j} \otimes \underbrace{\boxed{}\boxed{\;\dots\;}\boxed{}}_{2l}. \tag{2.30}$$

A particular component of Eq. (2.30) can be written as

$$\underbrace{\boxed{1}\boxed{\dots}\boxed{1}}_{m_1}\underbrace{\boxed{2}\boxed{\dots}\boxed{2}}_{m_2} \otimes \underbrace{\boxed{1}\boxed{\dots}\boxed{1}}_{n_1}\underbrace{\boxed{2}\boxed{\dots}\boxed{2}}_{n_2} \tag{2.31}$$

and we have

$$\begin{aligned} &2j = m_1 + m_2, \quad 2l = n_1 + n_2, \\ &2j_3 = m_2 - m_1, \quad 2l_3 = n_2 - n_1. \end{aligned} \tag{2.32}$$

Furthermore, recalling the definitions (2.1), (2.2), we get

$$\begin{aligned} T_H Y^{(j,l,r)}_{(q)} &= iqY^{(j,l,r)}_{(q)} \equiv i(j_3 + l_3) Y^{(j,l,r)}_{(q)}, \\ T_5 Y^{(j,l,r)}_{(q)} &= irY^{(j,l,r)}_{(q)} \equiv i(j_3 - l_3) Y^{(j,l,r)}_{(q)}. \end{aligned} \tag{2.33}$$

Hence

$$\begin{aligned} 2j_3 &= q + r \equiv m_2 - m_1, \\ 2l_3 &= q - r \equiv n_2 - n_1. \end{aligned} \tag{2.34}$$

Now we observe that as long as $m_2 - m_1$ and $n_2 - n_1$ are even or odd, the same is true for $m_1 + m_2$ and $n_1 + n_2$. Therefore the parity of $2j$ and $2l$ is the same as that of $2j_3$ and $2l_3$ and since $2j_3 + 2l_3 = 2q$ can be even or odd, the same is true for $2j + 2l$. It follows that j and l must either be both integers or both half-integers. This means that the q value of any $U_H(1)$ fragment of the $SO(5)$ fields is always contained in any $SO(5)$ harmonic in the irrep (j,l) provided that j and l are both integers or half-integers. Since $q+r$ and $q-r$ are related to the third component of the ''angular momentum'' of the two $SU(2)$ factors, one also has the conditions $|q+r| \leqslant 2j$ and $|q-r| \leqslant 2l$. The two above conditions select the harmonics appearing in the expansion.

In order to be specific it is now convenient to list all the five-dimensional space-time fields appearing in the harmonic expansion together with the corresponding ten-dimensional fields, with AdS_5 indices and/or internal indices, following the notations of [32]. We group them according to the appropriate $SO(5)$ bosonic (Y) or fermionic (Ξ) harmonic.

Note that the ten-dimensional fields $h^{\mu}_{\mu}(x,y)$, $A_{\mu\nu\rho\sigma}(x,y)$, $A_{\mu\nu\rho a}(x,y)$ are not part of the above list since, as shown in [32], they appear algebraically in the linearized equations of motion and thus can be eliminated in terms of the other propagating fields.

To obtain the mass spectrum of the above fields we must apply the Laplace-Beltrami operator to the harmonic expansion. We list such operators for the $SO(5)$ harmonics[2] $Y^{(j,l)}_{[\lambda_1,\lambda_2]}$:

$$\boxtimes_y Y_{[0,0]} \equiv \Box Y, \tag{2.35a}$$

$$\boxtimes_y Y_{[1,0]} \equiv 2\mathcal{D}^a \mathcal{D}_{[a} Y_{b]}, \tag{2.35b}$$

$$\boxtimes_y Y_{[1,1]} \equiv \bigstar\, dY_{ab} V^a V^b, \tag{2.35c}$$

$$\boxtimes_y Y_{[2,0]} \equiv 3\mathcal{D}^c \mathcal{D}_{(c} Y_{ab)}, \tag{2.35d}$$

$$\boxtimes_y Y_{[1/2,1/2]} \equiv \not{\mathcal{D}}\Xi, \tag{2.35e}$$

$$\boxtimes_y Y_{[3/2,1/2]} \equiv \gamma^{abc} \mathcal{D}_b \Xi_c. \tag{2.35f}$$

The explicit computation of the mass matrices derived from the above Laplace-Beltrami differential operators will not be worked out here and we refer the interested reader to [27]. We can give however as an example the computation involving scalar harmonics $Y^{(j,l)}_{[0,0]} = Y^{j,l,r}_{q=0}$ which is straightforward. In this case the five-dimensional invariant operator is simply the covariant Laplacian:

$$\Box = \mathcal{D}^a \mathcal{D}_a \equiv \mathcal{D}^i \mathcal{D}_i + \mathcal{D}^s \mathcal{D}_s + \mathcal{D}^5 \mathcal{D}_5. \tag{2.36}$$

From Eq. (2.25) and the fact that $T_{ab} L^{-1} \equiv T_{ab} Y^{j,l,r}_{q=0} \equiv 0$, we obtain the following result:

$$\Box Y^{j,l,r}_{q=0} = (-a^2 (T_i T_i + T_s T_s) - c^2 T_5 T_5) Y^{j,l,r}_{q=0}. \tag{2.37}$$

Let us now evaluate Eq. (2.37). We set

$$\begin{aligned} &T_i = -\frac{i}{2}\sigma_i, \quad T_s = -\frac{i}{2}\hat{\sigma}_s, \\ &T_5 = T_3 - \hat{T}_3 = \frac{i}{2}(\hat{\sigma}_3 - \sigma_3), \end{aligned} \tag{2.38}$$

where σ and $\hat{\sigma}$ are ordinary Pauli matrices. Using the relations

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[2]Notice that the operator on the two-form $Y = Y_{ab} V^a V^b$ is of the first order, like the fermionic ones. Indeed it is the square root of the usual second order operator $\mathcal{D}^a \mathcal{D}_{[a} Y_{b]}$:

$$\mathcal{D}^a \mathcal{D}_{[a} Y_{bc]} V^b V^c = \frac{1}{3} * d * d(Y_{ab} V^a V^b),$$

where

$$* dY = \frac{1}{2} \epsilon_{ab}{}^{cde} \mathcal{D}_c Y_{de} V^a V^b.$$

Hence $\frac{1}{2}\epsilon_{ab}{}^{cde} \mathcal{D}_c Y_{de} = \pm i\sqrt{3}\sqrt{\mathcal{D}^c \mathcal{D}_{[c} Y_{ab]}}$.

TABLE I. Fields appearing in the harmonic expansion.

10D	$h_{\mu\nu}$	h_a^a	A_{abcd}	B	$A_{\mu\nu}$	
5D	$H_{\mu\nu}$	π	b	B	$a_{\mu\nu}$	Y
10D	$h_{a\mu}$	$A_{\mu abc}$	$A_{\mu a}$			
5D	B_μ	ϕ_μ	a_μ			Y_a
10D	$A_{\mu\nu ab}$	A_{ab}				
5D	$b^{\pm}_{\mu\nu}$	a				$Y_{[ab]}$
10D	h_{ab}					
5D	ϕ					$Y_{(ab)}$
10D	λ	$\psi_{(a)}$	ψ_μ			
5D	λ	$\psi^{(L)}$	ψ_μ			Ξ
10D	ψ_a					
5D	$\psi^{(T)}$					Ξ_a

TABLE II. Graviton multiplet. $E_0=1+\sqrt{H_0+4}$.

		(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
◇	★	(1,1)	E_0+1	r	$H_{\mu\nu}$	H_0
◇	★	(1,1/2)	$E_0+1/2$	$r-1$	ψ_μ^L	$-2+\sqrt{H_0+4}$
◇	★	(1/2,1)	$E_0+1/2$	$r+1$	ψ_μ^R	$-2+\sqrt{H_0+4}$
	★	(1/2,1)	$E_0+3/2$	$r-1$	ψ_μ^R	$-2-\sqrt{H_0+4}$
		(1,1/2)	$E_0+3/2$	$r+1$	ψ_μ^L	$-2-\sqrt{H_0+4}$
◇	★	(1/2,1/2)	E_0	r	ϕ_μ	$H_0+4-2\sqrt{H_0+4}$
		(1/2,1/2)	E_0+1	$r+2$	a_μ	H_0+3
	★	(1/2,1/2)	E_0+1	$r-2$	a_μ	H_0+3
		(1/2,1/2)	E_0+2	r	B_μ	$H_0+4+2\sqrt{H_0+4}$
		(1,0)	E_0+1	r	$b^+_{\mu\nu}$	$\sqrt{H_0+4}$
	★	(0,1)	E_0+1	r	$b^-_{\mu\nu}$	$-\sqrt{H_0+4}$
		(1/2,0)	$E_0+1/2$	$r+1$	λ_L	$1/2-\sqrt{H_0+4}$
	★	(0,1/2)	$E_0+1/2$	$r-1$	λ_R	$1/2-\sqrt{H_0+4}$
		(1/2,0)	$E_0+3/2$	$r-1$	λ_L	$1/2+\sqrt{H_0+4}$
		(0,1/2)	$E_0+3/2$	$r+1$	λ_R	$1/2+\sqrt{H_0+4}$
		(0,0)	E_0+1	r	B	H_0

TABLE III. Gravitino multiplet I. $E_0=\sqrt{H_0^-+4}-1/2$.

		(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
	★	(1,1/2)	E_0+1	r	ψ_μ^L	$-3+\sqrt{H_0^-+4}$
	★	(1/2,1/2)	$E_0+1/2$	$r+1$	ϕ_μ	$H_0^-+7-4\sqrt{H_0^-+4}$
	★	(1/2,1/2)	$E_0+3/2$	$r-1$	a_μ	$H_0^-+4-2\sqrt{H_0^-+4}$
●	★	(1,0)	$E_0+1/2$	$r-1$	$a_{\mu\nu}$	$2-\sqrt{H_0^-+4}$
		(1,0)	$E_0+3/2$	$r+1$	$b^+_{\mu\nu}$	$1-\sqrt{H_0^-+4}$
●	★	(1/2,0)	E_0	r	$\psi_L^{(T)}$	$-5/2+\sqrt{H_0^-+4}$
●	★	(1/2,0)	E_0+1	$r-2$	$\psi_L^{(T)}$	$-3/2+\sqrt{H_0^-+4}$
	★	(0,1/2)	E_0+1	r	λ_R	$3/2-\sqrt{H_0^-+4}$
		(1/2,0)	E_0+1	$r+2$	$\psi_L^{(T)}$	$-3/2+\sqrt{H_0^-+4}$
		(1/2,0)	E_0+2	r	$\psi_L^{(T)}$	$-1/2+\sqrt{H_0^-+4}$
●	★	(0,0)	$E_0+1/2$	$r-1$	a	$H_0^-+4-4\sqrt{H_0^-+4}$
		(0,0)	$E_0+3/2$	$r+1$	a	$H_0^-+1-2\sqrt{H_0^-+4}$

TABLE IV. Gravitino multiplet II. $E_0=5/2+\sqrt{H_0^++4}$.

(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
(1,1/2)	E_0+1	r	ψ_μ^L	$-3-\sqrt{H_0^++4}$
(1/2,1/2)	$E_0+1/2$	$r+1$	a_μ	$H_0^++4+2\sqrt{H_0^++4}$
(1/2,1/2)	$E_0+3/2$	$r-1$	B_μ	$H_0^++7+4\sqrt{H_0^++4}$
(1,0)	$E_0+1/2$	$r-1$	$b^+_{\mu\nu}$	$1+\sqrt{H_0^++4}$
(1,0)	$E_0+3/2$	$r+1$	$a_{\mu\nu}$	$2+\sqrt{H_0^++4}$
(1/2,0)	E_0	r	$\psi_L^{(T)}$	$-1/2-\sqrt{H_0^++4}$
(1/2,0)	E_0+1	$r-2$	$\psi_L^{(T)}$	$-3/2-\sqrt{H_0^++4}$
(0,1/2)	E_0+1	r	λ_R	$3/2+\sqrt{H_0^++4}$
(1/2,0)	E_0+1	$r+2$	$\psi_L^{(T)}$	$-3/2-\sqrt{H_0^++4}$
(1/2,0)	E_0+2	r	$\psi_L^{(T)}$	$-5/2-\sqrt{H_0^++4}$
(0,0)	$E_0+1/2$	$r-1$	a	$H_0^++1+2\sqrt{H_0^++4}$
(0,0)	$E_0+3/2$	$r+1$	a	$H_0^++4+4\sqrt{H_0^++4}$

TABLE V. Gravitino multiplet III. $E_0=-1/2+\sqrt{H_0^++4}$.

	(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
★	(1/2,1)	E_0+1	r	ψ_μ^R	$-3+\sqrt{H_0^++4}$
★	(1/2,1/2)	$E_0+1/2$	$r-1$	ϕ_μ	$H_0^++7-4\sqrt{H_0^++4}$
	(1/2,1/2)	$E_0+3/2$	$r+1$	a_μ	$H_0^++4-2\sqrt{H_0^++4}$
★	(0,1)	$E_0+1/2$	$r+1$	$a_{\mu\nu}$	$2-\sqrt{H_0^++4}$
★	(0,1)	$E_0+3/2$	$r-1$	$b^-_{\mu\nu}$	$1-\sqrt{H_0^++4}$
★	(0,1/2)	E_0	r	$\psi_R^{(T)}$	$-5/2+\sqrt{H_0^++4}$
★	(0,1/2)	E_0+1	$r+2$	$\psi_R^{(T)}$	$-3/2+\sqrt{H_0^++4}$
	(1/2,0)	E_0+1	r	λ_L	$3/2-\sqrt{H_0^++4}$
	(0,1/2)	E_0+1	$r-2$	$\psi_R^{(T)}$	$-3/2+\sqrt{H_0^++4}$
	(0,1/2)	E_0+2	r	$\psi_R^{(T)}$	$-1/2+\sqrt{H_0^++4}$
	(0,0)	$E_0+1/2$	$r+1$	a	$H_0^++4-4\sqrt{H_0^++4}$
	(0,0)	$E_0+3/2$	$r-1$	a	$H_0^++1-2\sqrt{H_0^++4}$

TABLE VI. Gravitino multiplet IV. $E_0=5/2+\sqrt{H_0^-+4}$.

	(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
★	(1/2,1)	E_0+1	r	ψ_μ^R	$-3-\sqrt{H_0^-+4}$
★	(1/2,1/2)	$E_0+1/2$	$r-1$	a_μ	$H_0^-+4+2\sqrt{H_0^-+4}$
	(1/2,1/2)	$E_0+3/2$	$r+1$	B_μ	$H_0^-+7+4\sqrt{H_0^-+4}$
★	(0,1)	$E_0+1/2$	$r+1$	$b^-_{\mu\nu}$	$1+\sqrt{H_0^-+4}$
★	(0,1)	$E_0+3/2$	$r-1$	$a_{\mu\nu}$	$2+\sqrt{H_0^-+4}$
★	(0,1/2)	E_0	r	$\psi_R^{(T)}$	$-1/2-\sqrt{H_0^-+4}$
★	(0,1/2)	E_0+1	$r+2$	$\psi_R^{(T)}$	$-3/2-\sqrt{H_0^-+4}$
	(1/2,0)	E_0+1	r	λ_L	$3/2+\sqrt{H_0^-+4}$
	(0,1/2)	E_0+1	$r-2$	$\psi_R^{(T)}$	$-3/2-\sqrt{H_0^-+4}$
	(0,1/2)	E_0+2	r	$\psi_R^{(T)}$	$-5/2-\sqrt{H_0^-+4}$
	(0,0)	$E_0+1/2$	$r+1$	a	$H_0^-+1+2\sqrt{H_0^-+4}$
	(0,0)	$E_0+3/2$	$r-1$	a	$H_0^-+4+4\sqrt{H_0^-+4}$

$$\sigma_1 \boxed{1} = \boxed{2} \qquad \sigma_2 \boxed{1} = -i \boxed{2} \qquad \sigma_3 \boxed{1} = \boxed{1} \tag{2.39}$$

$$\sigma_1 \boxed{2} = \boxed{1} \qquad \sigma_2 \boxed{2} = i \boxed{1} \qquad \sigma_3 \boxed{2} = -\boxed{2} \tag{2.40}$$

(the same is true for $\hat{\sigma}$) and observing that on a Young tableaux the σ's act like a derivative (Leibnitz rule), we find on the first tableaux of Eq. (2.31)

$$\begin{aligned}(\sigma_1\sigma_1 + \sigma_2\sigma_2)\begin{array}{|c|c|c|}\hline \; & \;\dots\; & \; \\ \hline\end{array} &= (2m_1(m_2+1) + 2m_2(m_1+1))\begin{array}{|c|c|c|}\hline \; & \;\dots\; & \; \\ \hline\end{array} = \\ &= 4(j(j+1) - (j_3)^2)\begin{array}{|c|c|c|}\hline \; & \;\dots\; & \; \\ \hline\end{array}.\end{aligned} \tag{2.41}$$

An analogous result holds when acting with $\hat{\sigma}_1\hat{\sigma}_1 + \hat{\sigma}_2\hat{\sigma}_2$ on the second tableaux of Eq. (2.31), with $j \leftrightarrow l$.

Furthermore, the eigenvalue of $(\hat{\sigma}_3 - \sigma_3)^2$ on Eq. (2.31) is

$$(m_2 - m_1 + n_2 - n_1)^2 = 4(j_3 + l_3)^2. \tag{2.42}$$

For a scalar, $q=0$ and so, from Eq. (2.34), we have

$$j_3 = -l_3 = r/2. \tag{2.43}$$

Therefore, we find

$$\Box Y_{(0)}^{(j,l,r)} = \left[a^2 j(j+1) + b^2 l(l+1) + (4c^2 - a^2 - b^2)\frac{r^2}{4} \right] Y_{(0)}^{(j,l,r)}. \tag{2.44}$$

Substituting the values of a, b, and c given in Eq. (2.14), we obtain

$$\Box Y_{(0)}^{(j,l,r)} = H_0(j,l,r) Y_{(0)}^{(j,l,r)}, \tag{2.45}$$

where

$$H_0(j,l,r) \equiv 6\left(j(j+1) + l(l+1) - \frac{r^2}{8} \right) \tag{2.46}$$

is the eigenvalue of the Laplacian. The same result was first given in [13] using differential methods.

When the harmonic is not scalar, $q \neq 0$, the computation of the Laplace-Beltrami operators is more involved since the covariant derivative (2.21) is valued in the $SO(5)$ Lie algebra in the given representation $[\lambda_1, \lambda_2]$.

B. Spectrum and multiplet structure

We begin by the spectrum deriving from the *scalar harmonic* that appears in the expansion of the ten-dimensional fields $h_{\mu\nu}(x,y)$, $B(x,y)$, $h_a^a(x,y)$, $A_{abcd}(x,y)$ and $A_{\mu\nu}$. The masses of the corresponding five-dimensional fields (see Table I) are thus given in terms of the scalar harmonic eigenvalue $H_0(j,l,r)$ given in Eq. (2.46). They are

$$m^2(H_{\mu\nu}) = H_0, \tag{2.47}$$

$$m^2(B) = H_0, \tag{2.48}$$

$$m^2(\pi, b) = H_0 + 16 \pm 8\sqrt{H_0+4}, \tag{2.49}$$

$$m^2(a_{\mu\nu}) = 8 + H_0 \pm 4\sqrt{H_0+4}. \tag{2.50}$$

TABLE VII. Vector multiplet I. $E_0 = \sqrt{H_0+4} - 2$.

			(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
◇		★	(1/2,1/2)	E_0+1	r	ϕ_μ	$H_0+12-6\sqrt{H_0+4}$
◇	●	★	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(L)}$	$7/2-\sqrt{H_0+4}$
◇		★	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(L)}$	$7/2-\sqrt{H_0+4}$
		★	(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(L)}$	$5/2-\sqrt{H_0+4}$
			(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(L)}$	$5/2-\sqrt{H_0+4}$
◇	●	★	(0,0)	E_0	r	b	$H_0+16-8\sqrt{H_0+4}$
	●	★	(0,0)	E_0+1	$r-2$	ϕ	$H_0+9-6\sqrt{H_0+4}$
			(0,0)	E_0+1	$r+2$	ϕ	$H_0+9-6\sqrt{H_0+4}$
			(0,0)	E_0+2	r	ϕ	$H_0+4-4\sqrt{H_0+4}$

TABLE VIII. Vector multiplet II. $E_0 = \sqrt{H_0+4} + 4$.

(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
(1/2,1/2)	E_0+1	r	B_μ	$H_0+12+6\sqrt{H_0+4}$
(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(L)}$	$5/2+\sqrt{H_0+4}$
(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(L)}$	$5/2+\sqrt{H_0+4}$
(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(L)}$	$7/2+\sqrt{H_0+4}$
(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(L)}$	$7/2+\sqrt{H_0+4}$
(0,0)	E_0	r	ϕ	$H_0+4+4\sqrt{H_0+4}$
(0,0)	E_0+1	$r-2$	ϕ	$H_0+9+6\sqrt{H_0+4}$
(0,0)	E_0+1	$r+2$	ϕ	$H_0+9+6\sqrt{H_0+4}$
(0,0)	E_0+2	r	π	$H_0+16+8\sqrt{H_0+4}$

TABLE IX. Vector Multiplet III. $E_0=\sqrt{H_0^{++}+4}+1$.

	(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
	(1/2,1/2)	E_0+1	r	a_μ	$H_0^{++}+3$
	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(T)}$	$+1/2-\sqrt{H_0^{++}+4}$
	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(T)}$	$-1/2+\sqrt{H_0^{++}+4}$
	(0,1/2)	$E_0 3/2$	$r-1$	$\psi_R^{(T)}$	$1/2+\sqrt{H_0^{++}+4}$
●	(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(T)}$	$-1/2-\sqrt{H_0^{++}+4}$
	(0,0)	E_0	r	a	$H_0^{++}+1-2\sqrt{H_0^{++}+4}$
	(0,0)	E_0+1	$r-2$	ϕ	H_0^{++}
●	(0,0)	E_0+1	$r+2$	ϕ	H_0^{++}
●	(0,0)	E_0+2	r	a	$H_0^{++}+1+2\sqrt{H_0^{++}+4}$

Note that while the Laplacian acts diagonally on the AdS_5 fields $H_{\mu\nu}(x)$ and $B(x)$, the eigenvalues for $\pi(x)$ and $b(x)$, which appear entangled in the linearized equations of motion [32,33], have been obtained after diagonalization of a two by two matrix. With an abuse of notation, in Tables II–X we will call π, b the linear combinations given by the plus or minus signs in Eq. (2.49).

For the *vector harmonic* we have found four eigenvalues

$$\lambda_{[1,0]}=\{3+H_0(j,l,r\pm 2),H_0+4\pm 2\sqrt{H_0+4}\}$$

and the mass spectrum of the sixteen vectors is thus

$$m^2(a_\mu)=\begin{cases}3+H_0(j,l,r\pm 2),\\ H_0+4\pm 2\sqrt{H_0+4},\end{cases} \tag{2.51}$$

$$m^2(B_\mu,\varphi_\mu)=\begin{cases}H_0(j,l,r\pm 2)+7\pm 4\sqrt{H_0+4},\\ H_0+12\pm 6\sqrt{H_0+4},\\ H_0+4\pm 2\sqrt{H_0+4}.\end{cases} \tag{2.52}$$

In fact, as the Laplace-Beltrami operator acts diagonally on the complex vector field $a_\mu(x)$ we get for it eight mass values. Furthermore, the vectors $B_\mu(x)$, $\varphi_\mu(x)$ get mixed in the linearized equations of motion, and upon diagonalization we find two extra masses for each eigenvalue. Here also we use the same names for the linear combinations with plus or minus sign respectively in the mass formulas (2.52).

TABLE X. Vector Multiplet IV. $E_0=\sqrt{H_0^{--}+4}+1$.

		(s_1,s_2)	$E_0^{(s)}$	R symm.	Field	Mass
	★	(1/2,1/2)	E_0+1	r	a_μ	$H_0^{--}+3$
●	★	(1/2,0)	$E_0+1/2$	$r-1$	$\psi_L^{(T)}$	$-1/2+\sqrt{H_0^{--}+4}$
	★	(0,1/2)	$E_0+1/2$	$r+1$	$\psi_R^{(T)}$	$+1/2-\sqrt{H_0^{--}+4}$
	★	(0,1/2)	$E_0+3/2$	$r-1$	$\psi_R^{(T)}$	$-1/2-\sqrt{H_0^{--}+4}$
		(1/2,0)	$E_0+3/2$	$r+1$	$\psi_L^{(T)}$	$1/2+\sqrt{H_0^{--}+4}$
●	★	(0,0)	E_0	r	a	$H_0^{--}+1-2\sqrt{H_0^{--}+4}$
●	★	(0,0)	E_0+1	$r-2$	B	H_0^{--}
		(0,0)	E_0+1	$r+2$	ϕ	H_0^{--}
		(0,0)	E_0+2	r	a	$H_0^{--}+1+2\sqrt{H_0^{--}+4}$

For the *antisymmetric tensor harmonics* we get six eigenvalues from the Laplace Beltrami operator $*d$

$$\lambda_{[1,1]}=\{i(1\pm\sqrt{H_0(j,l,r\pm 2)+4}),\pm i\sqrt{H_0+4}\}$$

and the masses

$$m^2(b_{\mu\nu})=\begin{cases}H_0+4,\\ H_0+4,\\ 5+H_0(j,l,r\pm 2)\pm 2\sqrt{H_0(j,l,r\pm 2)+4},\end{cases} \tag{2.53}$$

$$m^2(a)=\begin{cases}H_0+4\pm 4\sqrt{H_0+4},\\ H_0(j,l,r\pm 2)+1\pm 2\sqrt{H_0(j,l,r\pm 2)+4},\end{cases} \tag{2.54}$$

The *spinor harmonics* eigenvalues of $\mathcal{D}$ are synthetically

$$\lambda_{[1/2,1/2]}=\left\{\pm\frac{1}{2}\pm\sqrt{H_0(r\pm 1)+4}\right\}.$$

The masses for the spinors and gravitinos are given in terms of $\mathcal{D}$ by a numerical shift

$$\text{gravitino:}\quad m(\psi_\mu)=\mathcal{D}-\frac{5}{2},$$

$$\text{dilatino:}\quad m(\lambda)=\mathcal{D}+1, \tag{2.55}$$

$$\text{longitudinal spinors:}\quad m(\psi^{(L)})=\mathcal{D}+3.$$

We have not yet calculated either the eigenvalues of $\mathcal{D}$ corresponding to the vector-spinor harmonic Ξ_a which produce AdS_5 spinors $\psi^{(T)}$, or the eigenvalues of the symmetric traceless harmonic $Y^{(v)}_{(ab)}$. However, we know *a priori* how many states we obtain in these two cases, and by a counting argument we can circumvent the problem of the explicit computation of the eigenvalues of their mass matrices. For the vector spinors we have in principle a matrix of rank 20, that becomes 16×16 due to the irreducibility condition, and further gets to 12×12, once the transversality condition $\mathcal{D}^a\Xi_a=0$ is imposed. In this way we are left with 12 nontrivial (nonlongitudinal) eigenvalues and thus we expect 12 $\psi^{(T)}$ spinors. In an analogous way, the traceless symmetric tensor $Y^{(\nu)}_{(ab)}$ gives a 14×14 mass matrix out of which five eigenvalues are longitudinal leaving 9 nontrivial eigenvalues.

If we match the bosonic and fermionic degrees of freedom including the 12+12 (right) left-handed spinors $\psi^{(T)}$ and the 9 real fields ϕ of the traceless symmetric tensor we find 128 bosonic degrees of freedom and 128 fermionic ones. Therefore, once we have correctly and unambiguously assigned all the fields except the $\psi^{(T)}$ and ϕ to supermultiplets of $SU(2,2|1)$, the remaining degrees of freedom of $\psi^{(T)}$ and ϕ are uniquely assigned to the supermultiplets for their completion.

In Tables II–X we have arranged our results in $SU(2,2|1)$ supermultiplets by an exhaustion principle, starting from the highest spin of the supermultiplet. Each state of such multiplets is labeled by the $SU(2,2)$ quantum numbers (E_0,s_1,s_2)

other than the internal symmetry attributes (j,l,r). As explained in Sec. III, E_0, the AdS energy, is identified with the conformal dimension Δ. Taking into account the E_0 value of each state and its R symmetry, we are able to fit unambiguously every mass at the proper place. For this purpose it is essential to use the relations between the conformal weights Δ and the masses given by

$$\begin{aligned}
&\text{spin } 2: \quad \Delta=2+\sqrt{4+m_{(2)}^2},\\
&\text{spin } 3/2: \quad \Delta=2+|m_{(3/2)}+3/2|,\\
&\text{spin } 1: \quad \Delta=2+\sqrt{1+m_{(1)}^2},\\
&\text{two-form:} \quad \Delta=2+|m_{(2f)}|,\\
&\text{spin } 1/2: \quad \Delta_{\pm}=2\pm|m_{(1/2)}|,\\
&\text{spin } 0: \quad \Delta_{\pm}=2\pm\sqrt{4+m_{(0)}^2}
\end{aligned} \tag{2.56}$$

(where Δ is equal to the E_0 value of the state). The sign ambiguity in the spin (0, 1/2) dimensions is present because the unitarity bound $E_0 \geqslant 1+s$ allows the possibility $E_0<2$ for such states. The spin 0 case and its implications were analyzed in [33] and noticed also in [22]. There is no such ambiguity in all the other cases.

In the theory at hand, the chiral primary $\mathrm{Tr}(AB)$ has the scalars with $E_0=3/2$, $E_0+1=5/2$ coming from the $\Delta_{\pm}$ dimensions of the same $k=1$ mass value. The fermionic partner is massless so there are no fermions with $E_0<2$.

We have found nine families of supermultiplets: one graviton multiplet, four gravitino multiplets, and four vector multiplets which are reported in Tables II–X.

These are organized as follows.

In the first column we give the (s_1,s_2) spin quantum numbers of the state.

In the second column we give the E_0 value of the state, where, according to the standard nomenclature, the value of E_0 is referred to as the E_0 of the multiplet and belongs to a vector field, a spin 1/2 field or to a scalar field for the graviton, gravitino, and vector multiplets, respectively. The other states have an E_0 value shifted in a range of ± 2 (in 1/2 steps) with respect to the E_0 of the multiplet.

In the third column we write the R symmetry of the state where the value r is assigned to the highest spin state ($r=r^{h.s.}$), the other states having R symmetry shifted in a range of ± 2 (in integer steps).

In the fourth column we give the right association of that particular $SU(2,2|1)$ state to the field obtained from the KK spectrum, according to the notations explained above.

In the fifth column we give the mass of the state[3] in terms of the ubiquitous expression H_0, where H_0 is evaluated at a value r corresponding to that R symmetry of the multiplet defined as the R symmetry of the highest spin $r=r^{h.s.}$. We note that in all the formulas giving the mass spectrum (2.47)–(2.55), the R symmetry r refers to the particular state we are considering. There, H_0 appears to have dependence on the r of the state which is different for different states. However, when arranging the states in supermultiplets of $SU(2,2|1)$, it is convenient to express the r of the state in terms of the R symmetry of the supermultiplet $r=r^{h.s.}$, defined as the R symmetry of the highest spin. In this case, all the masses can be expressed in terms of an H_0 which has the same dependence on $r=r^{h.s.}$ for all the members of the multiplet. For the graviton multiplet and the first two families of vector multiplets all the masses are written in terms of $H_0 \equiv H_0(j,l,r)$; and for the last two families of vectors all the masses are given in terms of $H_0^{\pm}\equiv H_0(j,l,r\pm1)$ and for the last two families of vectors all the masses are given in terms of $H_0^{\pm\pm}\equiv H_0(j,l,r\pm2)$. Indeed, if we compute the conformal weight Δ of the state from the mass values, it turns out to be expressed in terms of $H_0, H_0^{\pm}, H_0^{\pm\pm}$ which are the same for every state of the multiplet, as it must be. Of course, the value of Δ in terms of $H_0, H_0^{\pm}, H_0^{\pm\pm}$ can be computed from Eq. (2.56) and we have given for each multiplet the conformal weight of the lowest state labeled by E_0 in terms of H_0.

The multiplets of Tables II–X are long multiplets of $SU(2,2|1)$ when the $SU(2)\times SU(2)$ quantum numbers j, l and the R-symmetry values are generic. However, it is well known from group theory [5,22] that shortening of the multiplets can occur in correspondence with particular values of the $SU(2,2|1)$ quantum numbers giving rise to chiral (●), semilong (⋆) or massless (◇) multiplets. The above symbols have been used in the columns at the left of the tables to denote the surviving states in the shortened multiplets. In particular, the absence of these symbols in Table IV means that no shortening of any kind can occur for the gravitino multiplet II. Notice that shortenings are indicated only for positive values of the (shifted) R symmetry r, namely when r satisfies the following inequalities (see Sec. IV):

$$\begin{aligned}
r&\geqslant 0 \quad \text{Tables II, VII, VIII},\\
r+1&\geqslant 0 \quad \text{Tables IV, V},\\
r-1&\geqslant 0 \quad \text{Tables III, VI},\\
r+2&\geqslant 0 \quad \text{Table IX},\\
r-2&\geqslant 0 \quad \text{Table X}.
\end{aligned} \tag{2.57}$$

In fact, these shortened multiplets are the most interesting in light of the correspondence with the CFT at the boundary. We give the discussion of the shortenings in Sec. IV, after a preliminary introduction to the representation of superconformal superfields in CFT and the discussion of the conformal operators of protected scaling dimensions.

III. CFT AND $SU(2,2|1)$ REPRESENTATIONS

A. $SU(2,2|1)$ conformal superfields

The AdS-CFT correspondence [1,2,3] gives a relation between the particle states in AdS_5, classified in this case by

[3]According to Eq. (2.56) we give here the mass for the fermion and two-form fields, while for all the other bosons we give the mass squared.

the $SU(2,2|1)$ superalgebra and the realization of the very same representations [2,3,12] in terms of conformal fields on the boundary $\widetilde{M}_4 = \partial \mathrm{AdS}_5$.

In this way, the highest weight representations of $SU(2,2|1)$ correspond to *primary* superconformal fields on the boundary and a generic state on the bulk, labeled by four quantum numbers [5,34,35] $\mathcal{D}(E_0, s_1, s_2|r)$ related to $U(1) \times SU(2) \times SU(2) \times U_R(1) \subset SU(2,2) \times U_R(1)$, is mapped to a primary conformal field $\mathcal{O}^{\Delta,r}_{(s_1,s_2)}(x)$ with scaling dimension $\Delta = E_0$, Lorentz quantum numbers (s_1, s_2) and R symmetry r. E_0 is the AdS energy level and its relation to the AdS mass depends on the spin of the state. We recall here the relevant cases [3,5,16]

$$
\begin{aligned}
\left(\frac{1}{2},\frac{1}{2}\right) \quad & m^2 = (E_0 - 1)(E_0 - 3), \\
(0,0) \quad & m^2 = E_0(E_0 - 4), \\
(1,0),(0,1) \quad & m^2 = (E_0 - 2)^2, \\
(1,1) \quad & m^2 = E_0(E_0 - 4), \\
\left.\begin{matrix}\left(\frac{1}{2},0\right),\left(0,\frac{1}{2}\right) \\ \left(\frac{1}{2},1\right),\left(1,\frac{1}{2}\right)\end{matrix}\right\} \quad & m = |E_0 - 2|.
\end{aligned}
\tag{3.1}
$$

It is crucial in our discussion to classify states corresponding to short multiplets because in this case the conformal dimension Δ is *protected* and it allows a stringent test between the supergravity theory and the conformal field theory realization. Here, protected means that Δ is related to the R charge which is quantized in terms of the isometry generator of $U_R(1)$ and therefore it is exact to all orders in the $N^{-1}, (g_s N)^{-1}$ expansion. However, we note that unlike the $\mathcal{N}=4$ theory [24,36], operators with protected dimensions have conformal dimension different from their free-field value.

$\mathcal{N}=1$ superfields with protected and unprotected dimensions have been discussed by many authors [3,5,22,37]. We would like to remind here just their field theory realization, which will become especially important in comparing conformal operators with the particular model described by the type IIB theory compactified on $\mathrm{AdS}_5 \times \mathrm{T}^{11}$.

A generic conformal primary superfield is classified by an $SL(2,\mathbb{C})$ representation (s_1, s_2), a dimension E_0 and an R-symmetry charge r. These are the quantum numbers of the $\vartheta = 0$ component of the superfield. All descendants are given by the ϑ expansion which also dictates their spin, R symmetry r and scaling dimension Δ, since ϑ_α has $(s_1, s_2) = (1/2,0)$, $\Delta = -1/2$, $r = 1$ (so $\overline{\vartheta}_{\dot\alpha}$ has $(s_1, s_2) = (0,1/2)$, $\Delta = -1/2$, $r = -1$). For a generic primary conformal field the dimension is not protected since it can take any value $\Delta \geqslant 2 + s_1 + s_2$ $(s_1 s_2 \neq 0)$ or $\Delta \geqslant 1 + s$ $(s_1 s_2 = 0)$ due to unitarity bounds of the irrepses of $SU(2,2)$ [44]. $SU(2,2|1)$ requires the additional unitarity bounds

$$
2 + 2s_1 - E_0 \leqslant \frac{3}{2} r \leqslant E_0 - 2 - 2s_2, \tag{3.2}
$$

$E_0 \geqslant 1 + s$ $(E_0 = \frac{3}{2}|r|)$, $E_0 = s_1 = s_2 = r = 0$ (identity representation), which restrict the allowed values of the R-symmetry charge [22,34,35].

Operators with protected dimensions fall in four categories (as discussed in [5,22,37]).

(1) *Chiral superfields*: S. They satisfy the condition

$$
\overline{D}_{\dot\alpha} S_{(\alpha_1 \ldots \alpha_{2s_1})}(x, \vartheta, \overline{\vartheta}) = 0. \tag{3.3}
$$

For them $s_2 = 0$ ($s_1 = 0$ if antichiral) and $r = \frac{2}{3}\Delta$ ($r = -\frac{2}{3}\Delta$ if antichiral). These superfields contain the (massless on the boundary) free *singleton* representations for $\Delta = 1 + s$. These multiplets have $4(2s+1)$ degrees of freedom.

(2) *Semichiral superfields*: $U_{\alpha_1 \ldots \alpha_{2s_1}, \dot\alpha_1 \ldots \dot\alpha_{2s_2}}$. They satisfy the condition

$$
\overline{D}_{(\dot\alpha} U_{\dot\alpha_1 \ldots \dot\alpha_{2s_2})\alpha_1 \ldots \alpha_{2s_1}}(x, \vartheta, \overline{\vartheta}) = 0, \tag{3.4}
$$

and for them $r = \frac{2}{3}(\Delta + 2s_2)$. If $s_2 = 0$ the above superfield becomes chiral. For example $s_2 = 1/2$ would correspond to semichiral superfield whose lowest component is a right-handed spin 1/2 and its highest spin is a vector field with $r = \frac{2}{3}\Delta - \frac{1}{3}$.

(3) *Conserved superfields*: $J_{(s_1,s_2)}$. They satisfy

$$
D^{\alpha_1} J_{\alpha_1 \ldots \alpha_{2s_1}, \dot\alpha_1 \ldots \dot\alpha_{2s_2}}(x, \vartheta, \overline{\vartheta}) = 0 \tag{3.5}
$$

and

$$
\overline{D}^{\dot\alpha_1} J_{\alpha_1 \ldots \alpha_{2s_1}, \dot\alpha_1 \ldots \dot\alpha_{2s_2}}(x, \vartheta, \overline{\vartheta}) = 0 \tag{3.6}
$$

(or $\overline{D}^2 J_{\alpha_1 \ldots \alpha_{2s_1}} = 0$ if $s_2 = 0$) and for them $r = \frac{2}{3}(s_1 - s_2)$, $\Delta = 2 + s_1 + s_2$.

(4) *Semiconserved superfields*: $L_{(s_1,s_2)}$. They satisfy

$$
\overline{D}^{\dot\alpha_1} L_{\alpha_1 \ldots \alpha_{2s_1}, \dot\alpha_1 \ldots \dot\alpha_{2s_2}}(x, \vartheta, \overline{\vartheta}) = 0 \tag{3.7}
$$

or

$$
\overline{D}^2 L_{\alpha_1 \ldots \alpha_{2s_1}}(x, \vartheta, \overline{\vartheta}) = 0 \quad \text{for } s_2 = 0. \tag{3.8}
$$

Their R symmetry is $r = \frac{2}{3}(\Delta - 2 - 2s_2)$. A semiconserved superfield becomes conserved if it is left and right semiconserved in which case $\Delta = 2 + s_1 + s_2$ and $r = \frac{2}{3}(s_1 - s_2)$.

Operators of type (1), (2) and (4) have protected (but anomalous) dimensions in a non-trivial conformal field theory. They are short or semishort because some of the fields in the ϑ expansion are missing. In the language of [22] the (1) and (2) superfields correspond to the shortening conditions $n_2^+ = 0$ $(n_1^+ = 0)$, (3) correspond to $n_1^- = n_2^- = 0$ and (4) to $n_2^- = 0$ $(n_1^- = 0)$.

In the AdS-CFT correspondence all these superfields correspond to KK states with multiplet shortening and typically they occur when there is a lowering in the rank of the mass matrix and rational values of E_0 are obtained. Conserved current multiplets correspond to massless fields in AdS_5. They can only occur for fields whose mass is protected by a symmetry (such as gauge fields) and there is only a finite number of them corresponding to the gauge fields of the $SU(2,2|1)\times SU(2)\times SU(2)$ algebra and possibly Betti multiplets [38,39]. While the massless vectors of the isometry group correspond to the $U_R(1)$ and flavor symmetry of the boundary gauge theory, the Betti multiplet, as recently shown by Klebanov and Witten [33], corresponds to the $U_b(1)$ baryonic current multiplet of the boundary CFT. There are also two complex moduli related to B and A_{ab} wrapped on a 2-cycle of T^{11} [9], giving two hypermultiplets with $E_0=3$ and $r=2$. Massive KK states with arbitrary irrational value of E_0 correspond to generic conformal field operators with anomalous dimension.

It is easy to relate operators of different type by superfield multiplication. By multiplying a chiral $(s_1,0)$ by an antichiral $(0,s_2)$ primary one gets a generic superfield with (s_1,s_2), $\Delta=\Delta^c+\Delta^a$ and $r=2/3(\Delta^c-\Delta^a)$. By multiplying a *conserved current* superfield $J_{\alpha_1\ldots\alpha_{2s_1},\dot\alpha_1\ldots\dot\alpha_{2s_2}}$ by a chiral scalar superfield one gets a semiconserved superfield with $\Delta=\Delta^c+2+s_1+s_2$ $[r=2/3(\Delta-2-2s_2)]$.

In a KK theory only particular values of (s_1,s_2) can occur, because the theory in higher dimensions has only spin 2, spin 3/2 fields and lower. This implies that for bosons only (0,0), (1,0), (0,1), (1/2,1/2), (1,1) representations and for fermions only (1/2,0), (0,1/2), (1,1/2), (1/2,1) representations can occur. This drastically limits the spin of conformal superfields. Indeed, for chiral ones $s=0,1/2$, while for nonchiral $s_1,s_2\leq 1/2$.

B. CFT analysis of $AdS_5\times T^{11}$ compactification

In the conformal field theory [9] which, at least for large N and g_sN, is *dual* to type IIB supergravity on $AdS_5\times T^{11}$ the basic superfields are the gauge fields[4] W_α of $SU(N)\times SU(N)$ and two doublets of chiral superfields *A, B* which are in the $(N,\bar N)$ and $(\bar N,N)$ of $SU(N)\times SU(N)$ and in the (1/2,0) $r=1$,(0,1/2) $r=1$ of the *global* symmetry group $SU(2)\times SU(2)\times U_R(1)$. At the conformal point these superfields have anomalous dimension $\Delta=3/4$ and R-symmetry $r=1/2$. The chiral W_α superfield has $\Delta=3/2$, $r=1$.

The superpotential [9] $W=\lambda\epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_iB_kA_jB_l)$ has $\Delta=3$, $r=2$ and plays an important role in the discussion, since it determines to some extent both the chiral spectrum as well as the marginal deformations of the SCFT. It is related to some of the $\Delta=3$ flavor singlet chiral operators which are discussed later.

Let us specify the superspace gauge transformations of the above superfields. Following [40], we introduce Lie algebra valued chiral parameters Λ_1,Λ_2 of the two factors of $\mathcal{G}=SU(N)\times SU(N)$. Then, under $\mathcal{G}$ gauge transformations

$$
\begin{aligned}
e^{V_1}&\to e^{i\Lambda_1}e^{V_1}e^{-i\bar\Lambda_1},\\
e^{V_2}&\to e^{i\Lambda_2}e^{V_2}e^{-i\bar\Lambda_2},\\
A&\to e^{i\Lambda_1}Ae^{-i\Lambda_2},\\
B&\to e^{i\Lambda_2}Be^{-i\Lambda_1}
\end{aligned}
\tag{3.9}
$$

and we define

$$
\begin{aligned}
W_{1\alpha}&=\overline{DD}(e^{V_1}D_\alpha e^{-V_1}),\\
W_{2\alpha}&=\overline{DD}(e^{V_2}D_\alpha e^{-V_2}),
\end{aligned}
\tag{3.10}
$$

where V_1 and V_2 are superfields Lie algebra valued in the two $\mathcal{G}$ factors and $V=V_1+V_2$. Gauge covariant combinations are therefore

$$W_\alpha(AB)^k=W^1_\alpha(AB)^k, \tag{3.11}$$

$$W_\alpha(BA)^k=W^2_\alpha(BA)^k, \tag{3.12}$$

$$Ae^V\bar Ae^{-V}=Ae^{V_2}\bar Ae^{-V_1}, \tag{3.13}$$

$$Be^V\bar Be^{-V}=Be^{V_1}Be^{-V_2}. \tag{3.14}$$

Formulas (3.11) and (3.13) transform as

$$X\to e^{i\Lambda_1}Xe^{-i\Lambda_1} \tag{3.15}$$

while Eqs. (3.12) and (3.14) transform as

$$Y\to e^{i\Lambda_2}Ye^{-i\Lambda_2}. \tag{3.16}$$

We can multiply Eqs. (3.13) and (3.14) as

$$Ae^{V_2}\overline{AB}e^{-V_2}B, \tag{3.17}$$

which transforms as X or

$$Be^{V_1}\overline{BA}e^{-V_1}A, \tag{3.18}$$

which transforms as Y and thus build gauge covariant combinations as $W^1_\alpha X$ or $W^2_\alpha Y$.

If a symmetry $A\leftrightarrow B$ is required, then symmetrization exchanging Eq. (3.11) with Eq. (3.12), Eq. (3.13) with Eq. (3.14) or Eq. (3.17) with Eq. (3.18) will occur.

We will now consider sets of towers of superfields, labeled by an integer number *k* which correspond to *chiral* and *(semi)conserved* gauge invariant superfields and having therefore protected dimensions. As we will see in the next section, these conformal operators are precisely those corresponding to AdS-KK states undergoing multiplet shortening.

Let us first consider chiral superfields. There are three infinite sequences of them, corresponding to *hypermultiplets* and *tensor multiplets* in the AdS bulk.

[4]Below we use standard superfield notations [40].

They are given as[5]

$$S^k = \mathrm{Tr}(AB)^k, \quad \Delta^k = \frac{3}{2}k, \quad r = k, \quad k>0, \tag{3.19}$$

$$T^k = \mathrm{Tr}(W_\alpha(AB)^k), \quad \Delta^k = \frac{3}{2}(k+1), \quad r = k+1, \quad k>0, \tag{3.20}$$

$$\Phi^k = \mathrm{Tr}(W^\alpha W_\alpha(AB)^k), \quad \Delta^k = 3 + \frac{3}{2}k, \quad r = k+2. \tag{3.21}$$

The series (3.19) was anticipated by Klebanov, Witten [9] and shown to occur in the KK modes of the supergravity theory by Gubser [13], who also discussed descendants of the series (3.21).

The series (3.20), (3.21) has been constructed by the knowledge of the full mass spectrum and the shortening conditions.[6]

It is useful to note that in the Eq. (3.20) and Eq. (3.21) towers, we find operators of the type

$$B^k_{\alpha\beta} = \mathrm{Tr}(F_{\alpha\beta}(AB)^k), \quad \Delta^k = 2 + \frac{3}{2}k \quad (k>0), \tag{3.22}$$

$$\phi^k = \mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta}(AB)^k), \quad \Delta^k = 4 + \frac{3}{2}k, \tag{3.23}$$

as descendants. $F_{\alpha\beta}, F_{\dot\alpha\dot\beta}$ refer in the spinor notation to the dual and anti-self-dual parts of the field strength $F_{\mu\nu}$.

Even more interesting is the appearance of (semi)conserved superfields corresponding in the language of [22] to semilong multiplets in AdS_5. These superfields explain the appearance of KK towers with (spin 1) vector fields and (spin 2) tensor fields with protected dimensions.

In superfield language such fields are given by superfields containing terms of the form

$$J^k_{\alpha\dot\alpha} = \mathrm{Tr}(J_{\alpha\dot\alpha}(AB)^k), \begin{cases} j = l = \dfrac{k}{2}, \quad r = k, \\[2ex] \Delta = 3 + \dfrac{3}{2}k, \end{cases} \tag{3.24}$$

$$J^k = \mathrm{Tr}(J(AB)^k), \begin{cases} j = l+1, l = \dfrac{k}{2}, \quad r = k, \\[2ex] \Delta = 2 + \dfrac{3}{2}k, \end{cases} \tag{3.25}$$

$$I^k = \mathrm{Tr}(JW^2(AB)^k) \begin{cases} j = l+1, l = \dfrac{k}{2}, \quad r = k+2, \\[2ex] \Delta = 5 + \dfrac{3}{2}k, \end{cases} \tag{3.26}$$

where

$$J_{\alpha\dot\alpha} = W_\alpha e^V \bar W_{\dot\alpha} e^{-V} \quad (\Delta = 3), \tag{3.27}$$

$$J = A(e^V \bar A)e^{-V} \quad (\Delta = 2), \tag{3.28}$$

and satisfying

$$\bar D^{\dot\alpha} J^k_{\alpha\dot\alpha} = 0, \quad \bar D \bar D J^k = 0, \quad \bar D \bar D I^k = 0. \tag{3.29}$$

Analogous structures appear with B replacing A in Eq. (3.28) and $j \leftrightarrow l$ in Eqs. (3.25) and (3.26). Note that the *non-gauge-invariant* operators in Eqs. (3.24)–(3.26) behave as if they would have conformal dimension 3 and 2 respectively when the gauge singlet is formed. This is because the shortening condition implies that operators starting with structures as in Eqs. (3.24), (3.25), and (3.26) have dimension given by $3 + 3/2k$, $2 + 3/2k$ and $5 + 3/2k$ respectively.

The highest spin states contained in Eqs. (3.24), (3.25), and (3.26) are *descendants* with spin 2 and $\Delta = 4 + 3/2k$, spin 1 with $\Delta = 3 + 3/2k$, and spin 1 with $\Delta = 6 + 3/2k$. These are massive recursions of the graviton, massless gauge boson, and massive vector fields respectively. The AdS masses of the above states are given by

$$\text{spin 2:} \quad M^k = \sqrt{\frac{3}{2}k\left(\frac{3}{2}k + 4\right)}, \tag{3.30}$$

$$\text{spin 1:} \quad M^k = \sqrt{\frac{3}{2}k\left(\frac{3}{2}k + 2\right)}, \tag{3.31}$$

$$\text{spin 1:} \quad M^k = \sqrt{\left(\frac{3}{2}k + 5\right)\left(\frac{3}{2}k + 3\right)}. \tag{3.32}$$

The first two masses vanish for the $k = 0$ level corresponding to the *conserved* currents $\mathrm{Tr}\, J_{\alpha\dot\alpha}, \mathrm{Tr}\, J$ of the superconformal

---

[5] Here and in what follows we always mean symmetrized trace and symmetrized $SU(2) \times SU(2)$ indices.

[6] Chiral operators of the type $\mathrm{Tr}(W_{\alpha_1} \dots W_{\alpha_p})$ cannot appear in the KK spectrum for $p>2$ since such operators have $\Delta = 3/2p$, $r = p$, $j = l = 0$ and therefore are incompatible with the spectrum of the $U_R(1)$ charge on T^{11} (see next section). For $p = 2$ the chiral operators $\mathrm{Tr}(W_{\alpha_1} W_{\alpha_2}(AB)^k)$ are allowed but they contain two irreducible parts: one symmetric [(1, 0) spin one] and the other antisymmetric [(0, 0) spin zero]. However, following an observation of Aharony (as quoted in [41]) only the scalar term is a chiral primary operator. This is due to the superspace identity

$$\bar D \bar D [e^V D_\alpha (e^{-V} W_\beta e^V) e^{-V}] = [W_\alpha, W_\beta],$$

where the symmetry of the left hand side derives from the following superspace Bianchi identity $e^V D^\alpha (e^{-V} W_\alpha e^V) e^{-V} = \bar D_{\dot\alpha}(e^V \bar W^{\dot\alpha} e^{-V})$. Therefore, the other term is not chiral primary since

$$\mathrm{Tr}(W_{(\alpha} W_{\beta)}(AB)^k) = \bar D \bar D \mathrm{Tr}(e^V D_\alpha (e^{-V} W_\beta e^V) e^{-V} (AB)^k).$$

field theory with flavor group $G=SU(2)\times SU(2)$, while the third mass does not vanish at $k=0$.

For the spin 3/2 massive tower we do not expect to get vanishing gravitino mass when $k=0$, since the massless gravitino is already contained in the graviton tower. In spite of this, there are semiconserved superfields corresponding to shortened massive gravitino towers.

These are

$$L_{\dot\alpha}^{1k}=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}(AB)^k)\begin{cases} j=l, & r=k-1,\\ \Delta=\dfrac{3}{2}+\dfrac{3}{2}k & (k>0),\end{cases} \tag{3.33}$$

$$L_{\dot\alpha}^{2k}=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}W^2(AB)^k)\begin{cases} j=l, & r=k+1,\\ \Delta=\dfrac{9}{2}+\dfrac{3}{2}k, & \end{cases} \tag{3.34}$$

$$L_{\alpha}^{3k}=\mathrm{Tr}(W_\alpha(Ae^V\overline{A}e^{-V})\times(AB)^k)\begin{cases} j=l+1, & r=k+1,\\ \Delta=\dfrac{7}{2}+\dfrac{3}{2}k, & \end{cases} \tag{3.35}$$

which satisfy $D^{\dot\alpha}L_{\dot\alpha}=0$ and $D^2L_\alpha=0$, respectively.

We note in particular that the tower analogous to Eq. (3.33), in type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$, is [3,16,17,26]

$$L_{\dot\alpha}^{1k}=\mathrm{Tr}(e^VW_{\dot\alpha}e^{-V}\phi_{(i_1\ldots\phi_{ik})}) \tag{3.36}$$

in the k-fold symmetric of $SU(3)$. For $k>1$ these superfields are semiconserved but for $k=1$, unlike in our case, they become conserved, corresponding to the fact that on S^5 an additional $SU(3)$ triplet of massless gravitinos is required by $\mathcal{N}=4$ supersymmetry.

In this case the exact operator $L_{\dot\alpha}^{11}$ is

$$L_{\dot\alpha}^{11}=\mathrm{Tr}[(e^V\overline{W}_{\dot\alpha}e^{-V}\phi_a)+\overline{D}_{\dot\alpha}(e^V\overline{\phi}^be^{-V})(e^V\overline{\phi}^ce^{-V})\epsilon_{abc}] \tag{3.37}$$

which satisfies

$$\overline{D}^{\dot\alpha}L_{\dot\alpha}^{11}=D^2L_{\dot\alpha}^{11}=0 \tag{3.38}$$

as a consequence of the equations of motion for W_α,ϕ_a and the identity

$$D^2[e^{-V}\overline{D}_{\dot\alpha}(e^V\overline{\phi}^ae^{-V})e^V]=[\overline{\phi}^a,\overline{W}_{\dot\alpha}]. \tag{3.39}$$

The above superfields (3.33)–(3.35) are the lowest nonchiral operators of more general towers with irrational scaling dimensions described by

$$O_{\dot\alpha}^{1nk}=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}(Ae^V\overline{A}e^{-V})^n(AB)^k), \tag{3.40}$$

$$O_{\dot\alpha}^{2nk}=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}(Ae^V\overline{A}e^{-V})^nW^2(AB)^k), \tag{3.41}$$

$$O_{\alpha}^{3nk}=\mathrm{Tr}(W_\alpha(Ae^V\overline{A}e^{-V})^n(AB)^k), \tag{3.42}$$

with G representation

$$O_{\dot\alpha}^{1nk}:\ \left(\frac{k}{2}+n,\frac{k}{2}\right),\ \ r=k-1, \tag{3.43}$$

$$O_{\dot\alpha}^{2nk}:\ \left(\frac{k}{2}+n,\frac{k}{2}\right),\ \ r=k+1, \tag{3.44}$$

$$O_{\alpha}^{3nk}:\ \left(\frac{k}{2}+n,\frac{k}{2}\right),\ \ r=k+1. \tag{3.45}$$

The multiplets in Eqs. (3.19)–(3.21), (3.24)–(3.26), and (3.33)–(3.35) are shortened multiplets with protected dimensions because of supersymmetry through nonrenormalization theorems. However we will see that a peculiar phenomenon of $\mathcal{N}=1$ which can be learned from the AdS-CFT correspondence is that there exist also infinite towers of long multiplets with rational dimensions, at least for N and g_sN large, which in principle are not expected to have protected dimensions.

A typical tower which is not expected to have protected dimension is the massive tower

$$Q^k=\mathrm{Tr}(W^2e^V\overline{W}^2e^{-V}(AB)^k), \tag{3.46}$$

which contains the descendant $\mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta}\overline{F}_{\dot\alpha\dot\beta}\overline{F}^{\dot\alpha\dot\beta}(AB)^k)$. Supergravity predicts for it $\Delta=8+3/2k$.

We just note that the analogous operator in type IIB on $\mathrm{AdS}_5\times\mathrm{S}^5$ was a descendant of a chiral primary (showing up at first at $p=4$ level [14,16,17,26]) and therefore having protected dimensions because of $\mathcal{N}=4$ supersymmetry [24,36,43].

The identification of such long multiplets with superconformal operators will be given in the next section. Operators whose R symmetry is not related to the top components of one of the two $SU(2)$ factors (see Sec. IV) are for instance towers of the form

$$\mathrm{Tr}[(Ae^V\overline{A}e^{-V})^{n_1}(e^V\overline{B}e^{-V}B)^{n_2}(AB)^k], \tag{3.47}$$

which have $j=k/2+n_l$, $l=k/2+n_2$ and $r=k$. These operators have all irrational dimensions unless n_1,n_2 are consecutive terms in a particular sequence described in [13].

It is worthwhile to point out that in this gauge theory we have no realization of the semichiral superfields described before and indeed we do not find on the supergravity side any shortened multiplet satisfying the $r=2/3(E_0+2s_2)$ condition $(s_2\neq 0)$. The reason is that such superfields correspond to nonunitary modules.

IV. AdS-CFT CORRESPONDENCE

In Secs. II and III we have described the KK spectrum with its multiplet structure and the CFT operators with protected dimensions. We would like now to present the multiplet shortening conditions and analyze the correspondence of these states with the boundary field theory operators shown in the last section. This is an important nontrivial check for the AdS-CFT correspondence. On the other hand, supergravity seems to suggest additional dynamical inputs to the extent that, in the large N, g_sN limit, it predicts that certain towers of long multiplets have rational dimensions, suggesting the presence of some hidden symmetry. This latter may perhaps be explained in the context of Born-Infeld theory which relates D-brane dynamics to AdS supergravity in the large N limit.

From the point of view of the $SU(2,2|1)$ multiplet structure, the shortening conditions correspond to saturation of some of the inequalities describing the unitarity bounds [22]. These become relations between E_0 and the other $SU(2,2|1)$ quantum numbers.

In the KK context, we do not know *a priori* the multiplet structure of the KK states and the shortening conditions merely derive from the disappearance of some harmonics in the field expansion. This reduces the rank of the mass matrices and thus some of the states drop from the multiplet. The relevant fact is that these shortening conditions must be in one to one correspondence with those deriving from the $SU(2,2|1)$ group theoretical analysis.

As discussed in the previous section, the shortening conditions can be read as the following relations on the $SU(2,2|1)$ quantum numbers already given in Sec. III A

$$\text{(anti) chiral}\quad E_0=+\frac{3}{2}r\ \left(-\frac{3}{2}r\right), \tag{4.1}$$

$$\text{conserved}\quad E_0=2+s_1+s_2,\quad (s_1-s_2)=\frac{3}{2}r, \tag{4.2}$$

$$\text{semiconserved}\quad E_0=\frac{3}{2}r+2s_2+2\quad (\text{or}\ \ s_2\to s_1, r\to -r). \tag{4.3}$$

This means that the corresponding conformal dimension must have a rational value. As it can easily be seen from the mass spectrum presented in Sec. II, this implies that only for specific G quantum numbers we can retrieve such short multiplets. Actually, a rational scaling dimension can be found only if $H_0(j,l,r)+4$ is a perfect square of a rational number. Two possible sets of values for which such a condition is satisfied are

$$j=l=\left|\frac{r}{2}\right|=\frac{k}{2}, \tag{4.4}$$

$$j=l-1=\left|\frac{r}{2}\right|=\frac{k}{2}\quad\text{or}\quad l=j-1=\left|\frac{r}{2}\right|=\frac{k}{2}. \tag{4.5}$$

We will also examine briefly the case

$$j=l=\frac{r-2}{2},\quad r\geqslant 2, \tag{4.6}$$

which for most multiplets leads to a violation of inequality (3.2), but in one case gives a consistent shortening of the vector multiplet III. We will show that these three cases are the relevant ones. Indeed, in the first case $H_0(j,l,r)=\frac{9}{4}r^2+6|r|$ and thus $H_0(j,l,r)+4=(3|r/2|+2)^2$, in the second $H_0(j,l,r)=\frac{9}{4}r^2+12|r|+12$ and thus $H_0(j,l,r)+4=(3|r/2|+4)^2$, while in the third case we have $H_0(j,l,r)=\frac{9}{4}r^2-6r$ and thus $H_0(j,l,r)+4=(3(r/2)-2)^2$.

Of course there are other possible solutions, but we will see that only those presented above correspond to multiplet shortening.

Looking at Tables II–X we see that for the graviton and type I and II vector multiplets (VM) E_0 is given in terms of $H_0(j,l,r)$ while for gravitino multiplet of type I, IV and II, III E_0 is given in terms of $H^{\mp}\equiv H_0(j,l,r\mp 1)$ respectively. Analogously, for the type III and IV VM, E_0 is given in terms of $H_0^{\pm\pm}\equiv H_0(j,l,r\pm 2)$ respectively. As a consequence the conditions for rational values of E_0 (protected dimensions) are different for different multiplets.

Let us examine the conditions (4.4), (4.5), and (4.6) separately.

Condition (4.4) for the various multiplets reads

$$\text{Gravition and type I and II VM}\quad j=l=\left|\frac{r}{2}\right|\equiv\frac{k}{2}, \tag{4.7}$$

$$\text{type I gravitino}\quad j=l=\left|\frac{r-1}{2}\right|\equiv\frac{k}{2}, \tag{4.8}$$

$$\text{type II gravitino}\quad j=l=\left|\frac{r+1}{2}\right|\equiv\frac{k}{2}, \tag{4.9}$$

$$\text{type III gravitino}\quad j=l=\left|\frac{r+1}{2}\right|\equiv\frac{k}{2}, \tag{4.10}$$

$$\text{type IV gravitino}\quad j=l=\left|\frac{r-1}{2}\right|\equiv\frac{k}{2}, \tag{4.11}$$

$$\text{type III VM}\quad j=l=\left|\frac{r+2}{2}\right|\equiv\frac{k}{2}, \tag{4.12}$$

$$\text{type IV VM}\quad j=l=\left|\frac{r-2}{2}\right|\equiv\frac{k}{2}. \tag{4.13}$$

Here $k\in\mathbb{Z}_+$ identifies the $SU(2)\times SU(2)$ representations of the multiplet; it is obvious that all the multiplets obeying condition (4.4) are in the irrep $(k/2,k/2)$.

Substituting in the E_0 value of the multiplet given in Tables II–X H_0+4, $H_0^{\pm}+4$, and $H_0^{\pm\pm}+4$ with $(3/2k+2)^2$ we find the following values of E_0 for the various multiplets:

$$\text{Gravition multiplet}\quad E_0=\frac{3}{2}k+3\equiv\pm\frac{3}{2}r+3, \tag{4.14}$$

$$\text{type I LH gravitino}\quad E_0=\frac{3}{2}k+\frac{3}{2}\equiv\begin{cases}\frac{3}{2}r,\\ -\frac{3}{2}r+3,\end{cases}\tag{4.15}$$

$$\text{type II DH gravitino}\quad E_0=\frac{3}{2}k+\frac{9}{2}\equiv\begin{cases}\frac{3}{2}r+6,\\ -\frac{3}{2}r+3,\end{cases}\tag{4.16}$$

$$\text{type III RH gravitino}\quad E_0=\frac{3}{2}k+\frac{3}{2}\equiv\begin{cases}\frac{3}{2}r+3,\\ -\frac{3}{2}r,\end{cases}\tag{4.17}$$

$$\text{type IV RH gravitino}\quad E_0=\frac{3}{2}k+\frac{9}{2}\equiv\begin{cases}\frac{3}{2}r+3,\\ -\frac{3}{2}r+6,\end{cases}\tag{4.18}$$

$$\text{type I VM}\quad E_0=\frac{3}{2}k\equiv\pm\frac{3}{2}r,\tag{4.19}$$

$$\text{type II VM}\quad E_0=\frac{3}{2}k+6\equiv\pm\frac{3}{2}r+6,\tag{4.20}$$

$$\text{type III VM}\quad E_0=\frac{3}{2}k+3\equiv\begin{cases}\frac{3}{2}r+6,\\ -\frac{3}{2}r,\end{cases}\tag{4.21}$$

$$\text{type IV VM}\quad E_0=\frac{3}{2}k+3\equiv\begin{cases}\frac{3}{2}r,\\ -\frac{3}{2}r+6,\end{cases}\tag{4.22}$$

where the upper and lower choices on the right hand side refer to positive or negative arguments of the absolute values in Eqs. (4.7)–(4.13).

Using Eqs. (4.1)–(4.3) we see that under condition (4.4) we obtain the following: a chiral tensor multiplet from type I LH gravitino (4.15) (or an antichiral one from type III RH gravitino); one hypermultiplet (for both signs of r) from type I VM (4.19), and another hypermultiplet from type IV VM (4.22) (or from type III VM if $r<-2$); a semilong graviton multiplet from Eq. (4.14) (for both signs of r), two semilong gravitino from type III and IV (or from type I if $r<1$ and type II if $r<-1$ respectively), and IV RH gravitino multiplets from the two equations (4.17) and (4.18); for $k=0$ (G-singlet), we also obtain from Eq. (4.14) a short massless graviton multiplet with $E_0=3$, $r=0$. In this case only four states survive: the massless graviton, two massless gravitini (with $r=\pm1$ depending on the chirality), and one massless vector. This latter, being an $SU(2)\times SU(2)\times U_R(1)$ singlet, must be identified with the R-symmetry Killing vector.

Note that Eqs. (4.16), (4.20), and (4.21) do not correspond to any shortening condition, yet we have a rational value of E_0 belonging to a long multiplet.

It is now easy to find the correspondence between the supermultiplets obeying condition (4.7)–(4.13) and the primary conformal superfields on the CFT side discussed in the previous section. Given the values of E_0 and k (or r) we have immediately that the two hypermultiplets from Eqs. (4.19) and (4.22) are in correspondence with the chiral superfields S^k and Φ^k (3.19) and (3.21); the tensor multiplet from Eq. (4.15) corresponds to the chiral superfield T^k of Eq. (3.20); the semilong graviton multiplet from Eq. (4.14), associated with the semiconserved superfield $J^k_{\alpha\dot\alpha}$ of Eq. (3.24) [in particular the massless graviton multiplet ($k=0$ in Eq. (4.14)) corresponds to the conserved superfield $J^0_{\alpha\dot\alpha}$]; finally, the two semilong graviton multiplets from Eqs. (4.17) and (4.18) can be put in correspondence with the semiconserved superfields $L^{1,k}_{\dot\alpha}$ and $L^{2k}_{\dot\alpha}$ of Eqs. (3.33) and (3.34).

We note that the type I vector series in Table VII for $j=l=r=0$, see Eq. (4.19), degenerates into the identity representation, since $E_0=0$. However, as follows from the same table, another unitary representation, a massless vector multiplet, appears in the spectrum. Indeed, for $j=l=r=0$, the multiplet bosonic mass squared eigenvalues are $m^2_{(1)}=0$, $m^2_{(0)}=0$, $m^2_{(0)}=-3$, $m^2_{(0)}=-4$. The eigenvalue $m^2_{(0)}=0$ gives two possible values for E_0: $E_0=0$ and $E_0=4$. If we choose the $E_0=0$ branch, the other modes (scalars with $E_0=1,2$ and vector with $E_0=1$) are gauge modes and decouple from the physical Hilbert space, thus the multiplet is a gauge module [44]. If we choose the $E_0=4$ branch, we get a unitary representation with a scalar with $E_0=2$ and a vector with $E_0=3$ as physical states, while the other modes (scalars with $E_0=3,4$) decouple from the physical Hilbert space. This massless vector multiplet is the so called *Betti multiplet* of KK supergravity, related to the fact that a $(p+1)$-form (in this case $p=3$) couples to a p-brane wrapped on a nontrivial p-cycle which in this case is related to $b_3=1$, the third Betti number of T^{11} [33,45]. The general occurrence of such Betti multiplets in the KK context was widely discussed in [38]. In the case of $AdS_4\times M^{111}$, such a multiplet is related to $b_2=1$ [39,46], corresponding to the M theory three-form with one component on AdS_4 and two components on M^{111} and it was found in the KK context in [20]. Incidentally, in the language of [47], the Betti massless vector $[D(3,1/2,1/2)]$

isa zero center module[7] of the conformal group $SU(2,2)$, since all the Casimir vanish $C_I = C_{II} = C_{III} = 0$ as is the case for the identity $D(0,0,0)$, the gauge module $D(1,1/2,1/2)$, the massless scalars $D(4,0,0)$ appearing in the hypermultiplet S^k for $k=0$ (3.19) and the spin one singleton $D(2,1,0) + D(2,0,1)$ representations [44,47,49]. The geometrical origin of this gauge field coupled to a wrapped $D3$ brane on T^{11} has recently been discussed in [33] together with its interpretation as baryon current in the AdS-CFT correspondence.

The boundary superfield corresponding to the Betti multiplet is

$$\mathcal{U} = \mathrm{Tr}\, A e^{V} \overline{A} e^{-V} - \mathrm{Tr}\, B e^{V} \overline{B} e^{-V} \quad (D^2 \mathcal{U} = \overline{D}^2 \mathcal{U} = 0). \tag{4.23}$$

Its $\vartheta = 0$ component is a scalar $\mathcal{U}|_{\vartheta=0} = A\overline{A} - B\overline{B}$ with $E_0 = 2(m^2_{(0)} = -4)$ and the baryon current is the $\theta\sigma_\mu\theta$ component with $\Delta = E_0 + 1 = 3(m^2_{(1)} = 0)$ [33]. Note that all KK states are neutral under the $U_B(1)$, and thus it lies outside the T^{11} isometry.

Beside shortened multiplets, there are CFT superconformal operators with rational dimensions that are associated with the long multiplets of Eqs. (4.16), (4.20), and (4.21). Indeed we may construct the following superfields[8] all in the $(k/2,k/2)$ of G:

$$P^k_\alpha = \mathrm{Tr}(W_\alpha e^{V} \overline{W}^2 e^{-V} (AB)^k) \quad \Delta = \frac{3}{2}k + \frac{9}{2},$$

$$r = k-1,\ k>0, \tag{4.24}$$

$$Q^k = \mathrm{Tr}(W^2 e^{V} \overline{W}^2 e^{-V} (AB)^k) \quad \Delta = \frac{3}{2}k + 6, \quad r = k, \tag{4.25}$$

$$R^k = \mathrm{Tr}(e^{V} \overline{W}^2 e^{-V} (AB)^k) \quad \Delta = \frac{3}{2}k + 3,$$

$$r = k-2,\ k>0. \tag{4.26}$$

Let us now discuss the shortening conditions when the G-quantum numbers satisfy condition (4.5).

In this case Eqs. (4.7)–(4.13) are replaced by the analogous equations

$$\text{Graviton and type I and II VM} \quad l = j-1 = \left|\frac{r}{2}\right| \equiv \frac{k}{2}, \tag{4.27}$$

$$\text{type I gravitino} \quad l = j-1 = \left|\frac{r-1}{2}\right| \equiv \frac{k}{2}, \tag{4.28}$$

$$\text{type II gravitino} \quad l = j-1 = \left|\frac{r+1}{2}\right| \equiv \frac{k}{2}, \tag{4.29}$$

$$\text{type III gravitino} \quad l = j-1 = \left|\frac{r+1}{2}\right| \equiv \frac{k}{2}, \tag{4.30}$$

$$\text{type IV gravitino} \quad l = j-1 = \left|\frac{r-1}{2}\right| \equiv \frac{k}{2}, \tag{4.31}$$

$$\text{type III VM} \quad l = j-1 = \left|\frac{r+2}{2}\right| \equiv \frac{k}{2}, \tag{4.32}$$

$$\text{type IV VM} \quad l = j-1 = \left|\frac{r-2}{2}\right| \equiv \frac{k}{2} \tag{4.33}$$

(or $j \leftrightarrow l$) where all the states have the representation $(k/2+1,k/2)$ if $j = l+1$ or in the $(k/2,k/2+1)$ if $l = j+1$.

Proceeding as before we now substitute H_0+4, $H_0^{\pm}+4$, $H_0^{\pm\pm}+4$ with $(3/2k+4)^2$ in the E_0 value of the various multiplets given in Tables II–X and we obtain for each multiplet the following rational values of E_0:

$$\text{Graviton multiplet} \quad E_0 = \frac{3}{2}k + 5 \equiv \frac{3}{2}r + 5, \tag{4.34}$$

$$\text{type I LH gravitino} \quad E_0 = \frac{3}{2}k + \frac{7}{2} \equiv \frac{3}{2}r + 2, \tag{4.35}$$

$$\text{type II LH gravitino} \quad E_0 = \frac{3}{2}k + \frac{13}{2} \equiv \frac{3}{2}r + 8, \tag{4.36}$$

$$\text{type III RH gravitino} \quad E_0 = \frac{3}{2}k + \frac{7}{2} \equiv \frac{3}{2}r + 5, \tag{4.37}$$

$$\text{type IV RH gravitino} \quad E_0 = \frac{3}{2}k + \frac{13}{2} \equiv \frac{3}{2}r + 5, \tag{4.38}$$

$$\text{type I VM} \quad E_0 = \frac{3}{2}k + 2 \equiv \frac{3}{2}r + 2, \tag{4.39}$$

$$\text{type II VM} \quad E_0 = \frac{3}{2}k + 8 \equiv \frac{3}{2}r + 8, \tag{4.40}$$

[7]A zero center module also appears in the graviton multiplet of the $OSp(6|4)$ superalgebra [47]. In fact this multiplet contains an $O(6)$ singlet massless vector other than the $O(6)$ gauge fields. This agrees with the geometrical interpretation of $\mathcal{N}=6$ supergravity as the low-energy limit of type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, the latter being obtained by Hopf reducing M-theory on $\mathrm{AdS}_4 \times \mathrm{S}^7$ [48].

[8]The Q^k massive tower was also considered in [13].

$$\text{type III VM} \quad E_0=\frac{3}{2}k+5\equiv\frac{3}{2}r+8, \tag{4.41}$$

$$\text{type IV VM} \quad E_0=\frac{3}{2}k+5\equiv\frac{3}{2}r+2, \tag{4.42}$$

where we have limited ourselves to the positive branch of the expressions in the absolute values appearing in Eqs. (4.28)–(4.33).

By Eq. (4.1) we see that there are no chiral supermultiplets when condition (4.5) holds. However we have that Eqs. (4.35), (4.39), and (4.42) give the condition (4.3) for semilong multiplets, all the other values of E_0 corresponding to long multiplets with rational dimensions.

Thus we have one semilong type I L.H. gravitino corresponding to the semiconserved superfield (3.35); one semilong type I VM corresponding to the semiconserved superfield J^k of Eq. (3.25) which, in the particular case $k=0$, becomes a conserved superfield J corresponding to the massless type I VM with $E_0=2$, $r=0$ [these correspond to the $SU(2)\times SU(2)$ Killing vectors]; one semilong type IV VM corresponding to the semiconserved superfield I^k of Eq. (3.26).

Furthermore we have long multiplets from Eqs. (4.34), (4.36), (4.37), (4.38), (4.40), (4.41) corresponding respectively to the following superconformal fields with rational dimensions:

$$C^k=\mathrm{Tr}(Ae^V\overline{A}e^{-V}J_{\alpha\dot\alpha}(AB)^k), \quad E_0=\frac{3}{2}k+5, \quad r=k, \tag{4.43}$$

$$D^k=\mathrm{Tr}(W_\alpha e^V\overline{W}^2e^{-V}Ae^V\overline{A}e^{-V}(AB)^k),$$
$$E_0=\frac{3}{2}k+\frac{13}{2}, \quad r=k-1, \tag{4.44}$$

$$E^k=\mathrm{Tr}(W^2e^V\overline{W}^2e^{-V}Ae^V\overline{A}e^{-V}(AB)^k),$$
$$E_0=\frac{3}{2}k+8, \quad r=k, \tag{4.45}$$

$$F^k=\mathrm{Tr}(e^V\overline{W}^2e^{-V}Ae^V\overline{A}e^{-V}(AB)^k),$$
$$E_0=\frac{3}{2}k+5, \quad r=k-2, \tag{4.46}$$

$$G^k=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}Ae^V\overline{A}e^{-V}(AB)^k),$$
$$E_0=\frac{3}{2}k+\frac{7}{2}, \quad r=k-1, \tag{4.47}$$

$$H^k=\mathrm{Tr}(e^V\overline{W}_{\dot\alpha}e^{-V}W^2Ae^V\overline{A}e^{-V}(AB)^k),$$
$$E_0=\frac{3}{2}k+\frac{13}{2}, \quad r=k+1. \tag{4.48}$$

It must be noted that G^k coincides with $O_{\dot\alpha}^{1nk}$ for $n=1$ and H^k coincides with O_α^{2nk} for $n=1$. Moreover, D^k coincides with the operator $\overline{O}_\alpha^{2nk}$ for $n=1$ and $k=0$.

Inspection of the above list shows that these families are the lowest dimensional operators of a given structure, with building blocks given by W_α, A, $\overline{A}$, B and $\overline{B}$.

It should also be stressed that, although these operators have given quantum numbers of $SU(2)\times SU(2)$, and of $SU(2,2|1)E_0,s_1,s_2,r$, we have not discussed the most general form of these operators due to further mixing in terms of the constituent singleton fields W_α, A, B. For instance, we have not written terms involving $D_\alpha A$ or $D_\alpha B$, which certainly occur in the completion of some of the above operators [for example the ones including $J^k_{\alpha\dot\alpha}$ which contain both $W_\alpha W_{\dot\alpha}$ and $D_\alpha A\overline{D}_{\dot\alpha}\overline{A}$ (or $A\leftrightarrow B$)].

Finally, we analyze the Eq. (4.6) condition. In this case the only multiplet which does not violate the Eq. (3.2) inequality is the type III vector multiplet, for which we get $E_0=\frac{3}{2}r+2$. This apparently could be interpreted as shortening to a semilong vector multiplet. However, the states of such multiplet do not appear in the KK expansion, while the states which are complementary to them form a chiral hypermultiplet which is allowed by the KK analysis.[9] Its lowest state is the ϕ field with $E_0^{(s)}=E_0+1=\frac{3}{2}r^{(s)}$, which is indeed the group theoretical condition for the shortening to a chiral multiplet of the type given in Eq. (3.19). The $k=0$ ($r^s=2$) chiral multiplet has as a last component a complex massless scalar related to the A_{ab} 2-form wrapped on the nontrivial 2-cycle of T^{11}, giving a second complex modulus other than the dilaton B for type IIB on $AdS_5\times T^{11}$. Note that there is another massless scalar in the series S^k (3.19) for $k=2$. This corresponds to the spin $j=l=1$ in the harmonic expansion in the internal metric h_{ab}.

We would also like to remark that there are many more operators in the gauge theory which do not correspond to any supergravity KK mode, even though these multiplets may have spin less than two. A typical example is the Konishi (massive vector) superfield [50]

$$K=\mathrm{Tr}(Ae^V\overline{A}e^{-V})+\mathrm{Tr}(Be^V\overline{B}e^{-V}), \tag{4.49}$$

with $r=0$ and in the G singlet $j=l=0$.

[9] Physically, the exclusion of the semilong multiplet can also be seen by the fact that it would contain an additional massless vector for $j=l=r=0$ which does not correspond to any symmetry besides the isometry and baryon symmetry.

This superfield has anomalous dimension [42]. However, inspection of the supergravity spectrum shows that the multiplets with $j=l=r=0$ must have rational dimension and indeed they were identified with $Q^{k=0}$ in Eq. (3.46) with $E_0=6$ and the Betti multiplet $\mathcal{U}$ in Eq. (4.23) with $E_0=2$.

This state of affairs is resolved by the fact that K is expected to have a divergent dimension Δ in the large N, g_sN limit, as presumably happens in the $\mathcal{N}=4$ theory so that it should correspond to a string state.

The Konishi multiplet [50] is a long multiplet whose $\bar{D}^2$ is a chiral superfield which is a linear combination of the superpotential $\mathcal{W}=\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}(A_iB_kA_jB_l)$ and $\mathrm{Tr}(W^\alpha W_\alpha)$. This implies that neither $\mathcal{W}$ nor $\mathrm{Tr}(W^\alpha W_\alpha)$ are chiral primaries but rather a combination orthogonal to $\overline{DD}K$. It is the latter superfield which appears in the supergravity spectrum and coincides with the chiral dilation multiplet Φ^k with $k=0$. This is an example of operator mixing alluded to before.

Finally we observe that the knowledge of the flavor and R-symmetry anomalies in the gauge theory allows one to completely fix the low energy effective action of type IIB supergravity on $\mathrm{AdS}_5\times \mathrm{T}^{11}$ at least in the sector of the massless vector multiplets [5]. In fact this relies on the computation of the bulk Chern-Simons term of the several gauge factors involved [51]

$$d_{\Lambda\Sigma\Delta}\int F^\Lambda\wedge F^\Sigma\wedge A^\Delta, \tag{4.50}$$

where $\lambda=1,\ldots,8$ with $U_R(1)$, $U_b(1)$ and $SU_A(2)\times SU_B(2)$ gauge factors.

Because of the AdS-CFT correspondence, the gauge variation of such Chern-Simons terms must precisely match, at least in leading order in N, the current anomalies of the boundary gauge theory [3,5,52,13,53]. Moreover, the mixed gravitational gauge Chern-Simons terms

$$c_\Lambda\int A^\Lambda\wedge \mathrm{Tr}\,R\wedge R \tag{4.51}$$

[where Λ here runs only over the $U(1)$ factors of the bulk gauge fields] should be nonleading since they are related to string corrections in the AdS-CFT correspondence [53]. Because of the particular matter content of the model [9], all coefficients are in principle proportional to N^2 and thus leading in the AdS-CFT duality.

So it is crucial that $c_\Lambda=0$, i.e., that $U_R(1),U_b(1)$ are traceless [13]. The only non-vanishing $d_{\Lambda\Sigma\Delta}$ coefficients are

$$d_{rAA}=d_{rBB}\,,\quad d_{bAA}=-d_{bBB}\,,\quad d_{rrr}\,,\quad d_{rbb} \tag{4.52}$$

and thus they determine (up to two derivatives) the low energy effective action.

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APPENDIX: NOTATIONS AND CONVENTIONS

Consider $\mathrm{AdS}_5\times T^{11}$. We call M, N the curved ten-dimensional indices, $\mu,\nu/m,n$ the curved or flat AdS_5 ones and $\alpha\beta/a,b$ the curved or flat T^{11} ones. In the four dimensional CFT $\alpha,\ldots$ and $\dot{\alpha},\ldots$ are spinorial indices.

Our ten-dimensional metric is the mostly minus $\eta=\{+-\cdots-\}$, so that the internal space has a negative definite metric. For ease of construction, we have also used a negative metric to raise and lower the $SU(2)\times SU(2)$ Lie-algebra indices.

Furthermore, for the $SU(2)$ algebras we have defined $\epsilon^{123}=\epsilon^{12}=1$.

The $SO(5)$ gamma matrices are

$$\gamma_1=\begin{pmatrix} & & & 1\\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},\quad \gamma_2=\begin{pmatrix} & & & -i\\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}, \tag{A1}$$

$$\gamma_3=\begin{pmatrix} & & 1 & \\ & & & -1\\ -1 & & & \\ & 1 & & \end{pmatrix},\quad \gamma_4=\begin{pmatrix} & & i & \\ & & & i\\ i & & & \\ & i & & \end{pmatrix}, \tag{A2}$$

$$\gamma_5=\begin{pmatrix} i & & & \\ & i & & \\ & & -i & \\ & & & -1\end{pmatrix}. \tag{A3}$$

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," hep-th/9905111.

[5] S. Ferrara and A. Zaffaroni, Phys. Lett. B **431**, 49 (1998).

[6] S. Kachru and E. Silverstein, Phys. Rev. Lett. **80**, 4855 (1998); A. Lawrence, N. Nekrasov, and C. Vafa, Nucl. Phys. **B533**, 199 (1998); M. Bershadsky, Z. Kakushadze, and C. Vafa, *ibid.* **B523**, 59 (1998); Z. Kakushadze, *ibid.* **B529**, 157 (1998); Y. Oz and J. Terning, *ibid.* **B532**, 163 (1998).

[7] A. Kehagias, Phys. Lett. B **435**, 337 (1998).

[8] L. J. Romans, Phys. Lett. **153B**, 392 (1985).
[9] I. R. Klebanov and E. Witten, Nucl. Phys. **B536**, 199 (1998).
[10] D. R. Morrison and M. R. Plesser, Adv. Theor. Math. Phys. **3**, 1 (1999).
[11] A. M. Uranga, J. High Energy Phys. **01**, 022 (1999); K. Dasgupta and S. Mukhi, Nucl. Phys. **B551**, 204 (1999); S. Gubser, N. Nekrasov, and S. Shatashvili, J. High Energy Phys. **05**, 003 (1999); R. de Mello Koch, K. Oh, and R. Tatar, Nucl. Phys. **B555**, 457 (1999); R. von Unge, J. High Energy Phys. **02**, 023 (1999); B. R. Greene, Nucl. Phys. **B525**, 284 (1998); G. Dall'Agata, Phys. Lett. B **460**, 79 (1999).
[12] M. Flato and C. Fronsdal, Lett. Math. Phys. **2**, 421 (1978); Phys. Lett. **97B**, 236 (1980); J. Math. Phys. **22**, 1100 (1981); Phys. Lett. B **172**, 412 (1986).
[13] S. S. Gubser, Phys. Rev. D **59**, 025006 (1999).
[14] M. Gunaydin and N. Marcus, Class. Quantum Grav. **2**, L11 (1985).
[15] M. Gunaydin, D. Minic, and M. Zagermann, Nucl. Phys. **B544**, 737 (1999); **B534**, 96 (1998).
[16] S. Ferrara, C. Fronsdal, and A. Zaffaroni, Nucl. Phys. **B532**, 153 (1998).
[17] L. Andrianopoli and S. Ferrara, Phys. Lett. B **430**, 248 (1998); Lett. Math. Phys. **46**, 265 (1998).
[18] D. Z. Freedman and H. Nicolai, Nucl. Phys. **B237**, 342 (1984).
[19] A. Ceresole, P. Fre, and H. Nicolai, Class. Quantum Grav. **2**, 133 (1985).
[20] D. Fabbri, P. Fre, L. Gualtieri, and P. Termonia, ''M-theory on $AdS_4\times M^{111}$: The Complete $Osp(2|4)\times SU(3)\times SU(2)$ spectrum from harmonic analysis,'' hep-th/9903036; Class. Quantum Grav. **17**, 55 (2000).
[21] S. Ferrara and A. Zaffaroni, talk given at Strings 98, Santa Barbara, California, 1998, hep-th/9807090.
[22] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner, ''Renormalization group flows from holography supersymmetry and a c theorem,'' hep-th/9904017.
[23] A. A. Tseytlin, Nucl. Phys. **B501**, 41 (1997).
[24] S. S. Gubser, A. Hashimoto, I. R. Klebanov, and M. Krasnitz, Nucl. Phys. **B526**, 393 (1998).
[25] S. R. Das and S. P. Trivedi, Phys. Lett. B **445**, 142 (1998).
[26] S. Ferrara, M. A. Lledo, and A. Zaffaroni, Phys. Rev. D **58**, 105029 (1998).
[27] A. Ceresole, G. Dall'Agata, and R. D'Auria, J. High Energy Phys. **11**, 009 (1999).
[28] D. P. Jatkar and S. Randjbar-Daemi, Phys. Lett. B **460**, 281 (1999).
[29] L. Castellani, L. J. Romans, and N. P. Warner, Ann. Phys. (N.Y.) **157**, 394 (1984).
[30] L. Castellani, R. D'Auria, and P. Fre, *Supergravity and Superstrings: A Geometric Perspective* (World Scientific, Singapore, 1991), Vols. 1 and 2.
[31] A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982).
[32] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, Phys. Rev. D **32**, 389 (1985).
[33] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).
[34] M. Flato and C. Fronsdal, Lett. Math. Phys. **8**, 159 (1984).
[35] V. K. Dobrev and V. B. Petkova, Phys. Lett. **162B**, 127 (1985).
[36] M. P. Dine and N. Seiberg, Phys. Lett. B **409**, 239 (1997).
[37] H. Osborn, Ann. Phys. (N.Y.) **272**, 243 (1999).
[38] R. D'Auria and P. Fre, Ann. Phys. (N.Y.) **162**, 372 (1985).
[39] R. D'Auria and P. Fre, Class. Quantum Grav. **1**, 447 (1984).
[40] J. Bagger and J. Wess, *Supersymmetry and Supergravity*, Princeton Series in Physics (Princeton University Press, Princeton, NJ, 1992).
[41] C. Vafa, ''Puzzles at large N,'' hep-th/9804172.
[42] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, Phys. Lett. B **394**, 329 (1997).
[43] I. Klebanov, W. I. Taylor, and M. Van Raamsdonk, ''Absorption of dilaton partial waves by D3-branes,'' hep-th/9905174.
[44] B. Binegar, C. Fronsdal, and W. Heidenreich, J. Math. Phys. **24**, 2828 (1983).
[45] S. S. Gubser and I. R. Klebanov, Phys. Rev. D **58**, 125025 (1998); P. Candelas and X. C. de la Ossa, Nucl. Phys. **B342**, 246 (1990).
[46] E. Witten, in Quantum Field Theory and the Fundamental Problems of Physics, Proceedings of Shelter Island II, 1983, Princeton report PRINT-83-1056.
[47] M. Flato and C. Fronsdal, *Essays in Supersymmetry*, Math. Phys. Stud. Vol. 8 (Reidel, Dordecht, 1986).
[48] B. E. W. Nilsson and C. N. Pope, Class. Quantum Grav. **1**, 499 (1984).
[49] S. Ferrara and C. Fronsdal, Class. Quantum Grav. **15**, 2153 (1998); Phys. Lett. B **433**, 19 (1998); Lett. Math. Phys. **46**, 157 (1998).
[50] K. Konishi, Phys. Lett. **135B**, 439 (1984).
[51] M. Gunaydin, G. Sierra, and P. K. Townsend, Nucl. Phys. **B242**, 244 (1984); **B253**, 573 (1985).
[52] M. Henningson and K. Skenderis, J. High Energy Phys. **07**, 023 (1998).
[53] D. Anselmi and A. Kehagias, Phys. Lett. B **455**, 155 (1999).