Spinning test particles in general relativity: Nongeodesic motion in the Reissner-Nordström spacetime

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The dynamics of a charged spinning test particle in general relativity is studied in the context of gravitoelectromagnetism. Various families of test observers and supplementary conditions are examined. The spin-gravity-electromagnetism coupling is investigated for motion in the background of a Reissner-Nordström black hole both in the exact spacetime and in the weak-field approximation. Results are compared with those of the theory.

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I. INTRODUCTION

In this paper we consider a charged massive test particle in the Dixon-Souriau model [1–9], which is a first order cutoff in the multipole expansion of the equations of motion of a small extended body. The model includes spin-electromagnetism and spin-gravity coupling terms, but it requires the assumption of additional supplementary conditions to be completed. For uncharged particles it reduces to the well-known Papapetrou model.

Here we briefly review the model and the different possibilities for the supplementary conditions, with their corresponding meaning in terms of the center-of-mass world line definition. We then provide the splitting of the equations of motion with respect to a frame associated with a generic four-velocity field u, and then specialize the results for certain preferred families of observers.

As an example, we consider the Reissner-Nordström background and study the evolution of a particle with respect to the family of static observers. We thus incorporate into a single framework the exact generalization of earlier results belonging to the purely electromagnetic case [10–12] and to the purely gravitational case [11,13,14]. We also discuss the weak field slow motion approximation and somewhat complete and generalize the picture given in the literature.

II. THE PAPAPETROU EQUATIONS OF MOTION

In general relativity, an extended body is described by its associated energy-momentum tensor. A small body can be studied by a multipole expansion method enabling it to be equivalently described by a set of multipole (energy-momentum) moments defined along a central line [1–3,15]. An analogous scheme is obtained by considering a singular energy-momentum tensor distribution defined along a single curve [6–8]. The cutoff at successive multipole orders defines a hierarchy of elementary multipole particles (see, e.g., [5,15–17]). The first cutoff yields a point particle (or single pole) governed by the geodesic equation of motion.

The next cutoff leads to the dipole (''spinning'') particle which interests us here. The equations of motion for such a particle were first derived in the purely gravitational case by Papapetrou [15] as

$$\frac{D}{d\tau_U}p^{\alpha}=\frac{1}{2}R_{\rho\sigma\beta}{}^{\alpha}S^{\rho\sigma}U^{\beta},$$

$$\frac{D}{d\tau_U}S^{\alpha\beta}=p^{\alpha}U^{\beta}-p^{\beta}U^{\alpha}, \tag{1}$$

where $R_{\alpha\beta\rho\sigma}$ is the Riemann tensor, p^{α} is the (generalized) momentum vector, $S^{\alpha\beta}$ is a (antisymmetric) spin tensor, $U=DX/d\tau_U$ is the unit tangent vector ($U^{\alpha}U_{\alpha}=-1$) of the ''center line'' l_U used to make the multipole reduction, and where $X=X(\tau_U)$ is the center point whose world line is l_U. The fields S, U, and p are defined only along l_U. Units are chosen here so that the speed of light in empty space satisfies $c=1$.

It is well known that the number of independent equations in Eq. (1) is less than that of the unknown quantities; three additional scalar supplementary conditions (SC) are needed for the scheme to be completed. Once a suitable choice has been made, l_U, p, and S can in principle be determined by the complete set of equations.

The various supplementary conditions which are considered in the literature are all of the form $\hat{u}^{\alpha}S_{\alpha\beta}=0$ for some timelike unit vector $\hat{u}$ along the world line l_U. According to the special relativistic analogy ([18], p. 161), this is equivalent to defining the central line l_U as the world line of the centroid of the body with respect to an observer family with four-velocity $\hat{u}$.

The supplementary conditions discussed in the literature are the following: CP $\hat{u}=u$ (Corinaldesi-Papapetrou condition: see, e.g., [14,19]), where u is a (known) preferred family of observers usually suggested by the background; T $\hat{u}=p/||p||=\bar{u}$ (Tulkzyjew's condition: see, e.g., [1,5,20,21]); and P $\hat{u}=U$ (Pirani's condition: see, e.g., [11,22,23]).

Clearly the fields X, U, p, and S all depend on the choice of supplementary conditions [21] so a more precise notation would be $X_{(\mathrm{SC})}$, $U_{(\mathrm{SC})}$, $p_{(\mathrm{SC})}$, $S_{(\mathrm{SC})}$, where the index values $\mathrm{SC}=CP,T,P$ correspond to these choices. This cumbersome notation will be avoided when possible, but it is essential to clarify certain relationships between different choices.

The existence and uniqueness of the center line of the body under different supplementary conditions has been studied by many authors (see, e.g., [21,24–27]). Clearly, U and $\bar{u}$ are the two four-velocities most naturally associated with the particle. Consequently, often in the literature, the associated centers, the P center and the T center respectively, are both referred to as the center of mass of the particle [21,27]. The P center has also recently been renamed the ''center of trace'' [23].

The case in which both gravitational and electromagnetic fields are present was studied by Dixon and Souriau [5,9]. In the following we will work with the Dixon-Souriau equations of motion (see Sec. IV) which reduce to the classic Papapetrou ones in the purely gravitational case.

III. THE SPLITTING OF THE SPACETIME AND GRAVITOELECTROMAGNETISM

In general relativity a reference frame is defined by a congruence of timelike curves, the set of world lines of a family of observers. We denote by u the unit tangent vector $(u\cdot u=-1)$ of the world lines of a generic reference frame, namely the observer four-velocity field. The splitting of the spacetime along u and its orthogonal local rest space (LRS_u) gives the measurement relative to u of any tensor field defined on a domain of the spacetime; similarly one can obtain the formulation relative to u of any tensor equation. For notations and conventions we follow [28] with a slight change; the subscript notation for spatial projections and relative observer quantities is adopted.

The measurement of the spacetime metric gives rise to the spatial metric $P_{(u)\alpha\beta}=g_{\alpha\beta}+u_\alpha u_\beta$, i.e., the spatial projection operator with respect to u; the temporal projection operator along u is $-u_\alpha u_\beta$. Analogously $\eta_{(u)\alpha\beta\sigma}=u^\rho\eta_{\rho\alpha\beta\sigma}$ is the only spatial field resulting from the measurement of the unit (oriented) volume four-form η; it defines the spatial cross product $\times_u$ as well as the spatial duality operation on LRS_u.

From the measurement of a p-form S only two distinct fields result: the purely spatial or ''magnetic'' part $S^{(M)}_{(u)}$ (a p-form) and the ''electric'' part $S^{(E)}_{(u)}$ [a $(p-1)$-form], the spatial projection of a contraction with a single factor of u:

$$S=u^\flat\wedge S^{(E)}_{(u)}+S^{(M)}_{(u)}, \tag{2}$$

where the completely covariant (contravariant) form of a generic tensor X is denoted by $X^\flat$ $(X^\sharp)$.

For a generic two-form S the component notation for this decomposition is

$$S_{\alpha\beta}=2u_{[\alpha}S^{(E)}_{(u)\ \beta]}+S^{(M)}_{(u)\ \alpha\beta}=2u_{[\alpha}S^{(E)}_{(u)\ \beta]}+\eta_{(u)\alpha\beta}{}^{\sigma}S^{(M)}_{(u)\ \sigma}, \tag{3}$$

where $S^{(M)}_{(u)\ \sigma}$ denotes the spatial dual of $S^{(M)}_{(u)\ \alpha\beta}$.

According to the measurement process, the spatial and temporal projection of spacetime derivative operators gives rise to the corresponding spatial and temporal counterparts. As described in [28], the spatial covariant derivative is defined as $\nabla_{(u)\alpha}=P_{(u)}P_{(u)}{}^{\beta}{}_{\alpha}\nabla_\beta$ (the first projection operator acts on the tensorial indices of the field after the derivative is applied) and the spatial Lie derivative along a generic field X is $\mathcal{L}_{(u)_X}=P_{(u)}\mathcal{L}_X$. Similarly one can construct temporal derivatives; the Lie temporal derivative $\nabla_{(\mathrm{lie},u)}=P_{(u)}\mathcal{L}_u$ and the Fermi-Walker temporal derivative $\nabla_{(\mathrm{fw},u)}=P_{(u)}\nabla_u$. The covariant derivative of u has the following decomposition:

$$\nabla_\alpha u^\beta=\eta_{(u)\alpha}{}^{\beta}{}_{\mu}\omega_{(u)}{}^{\mu}+\theta_{(u)\alpha}{}^{\beta}-u_\alpha a_{(u)}{}^{\beta}, \tag{4}$$

where $a_{(u)}=\nabla_{(\mathrm{fw},u)}u\in\mathrm{LRS}_u$ is the acceleration vector, $\theta_{(u)}\in\mathrm{LRS}_u\otimes\mathrm{LRS}_u$ is the (symmetric) deformation two-tensor and $\omega_{(u)}\in\mathrm{LRS}_u$ is the vorticity vector of the observer congruence.

If X is a tensor field defined only along the line l_U (parametrized by the proper time τ_U and having unit tangent U), then the ''measurement'' of $DX/d\tau_U$, the intrinsic derivative along U, by a family of observers with four-velocity u is

$$\frac{D_{(\mathrm{fw},U,u)}}{d\tau_{(U,u)}}X=[\nabla_{(\mathrm{fw},u)}+\nabla_{(u)\nu_{(U,u)}}]X, \tag{5}$$

where $U=\gamma_{(U,u)}[u+\nu_{(U,u)}]$, $d\tau_{(U,u)}=\gamma_{(U,u)}d\tau_U$ is the differential of the standard relative time parametrization along the line U, and the field X on the right hand side is some smooth extension of X to a neighborhood of the line.

It is convenient to introduce the composition of projection maps from the local rest space of one observer onto that of another:[1]

$$P_{(u,U)}=P_{(U)}P_{(u)}:\mathrm{LRS}_u\rightarrow\mathrm{LRS}_U.$$

We also introduce the relative gravitational field $F^{(G)}_{(\mathrm{fw},U,u)}$ (see [28]),

[1]This subscript notation for the projection $P_{(u,U)}$ differs from the argument notation $P_{(U,u)}$ of [28] by interchanging u and U so that the second subscript of the pair of four-velocities always refers to the space in which the result lives.

$$
\begin{aligned}
F^{(G)}_{(\mathrm{fw},U,u)} &= -\gamma^{-1}_{(U,u)} P_{(u)} \frac{Du}{d\tau_U} \\
&= -P_{(u)} \frac{Du}{d\tau_{(U,u)}} \\
&= \gamma_{(U,u)} \bigg[g_{(u)} + ||\nu_{(U,u)}|| \\
&\quad \times \left(\frac{1}{2} \hat{\nu}_{(U,u)} \times_u H_{(u)} - \theta_{(u)} \llcorner \hat{\nu}_{(U,u)} \right) \bigg],
\end{aligned} \tag{6}
$$

where $g_{(u)} = -a_{(u)}$ represents the gravitoelectric field while $H_{(u)} = 2\omega_{(u)}$ is the gravitomagnetic field and $\llcorner$ denotes right-contraction. The last equation of Eq. (6) gives the gravitational force a Lorentz-like form allowing the introduction of terms such as gravitoelectromagnetic force and gravitoelectromagnetism.

The electric, magnetic, and ‘‘mixed parts’’ of the Riemann tensor $\mathcal{E}_{(u)\alpha\beta}$, $\mathcal{H}_{(u)\alpha\beta}$, $\mathcal{F}_{(u)\alpha\beta}$ are

$$
\begin{aligned}
\mathcal{E}_{(u)\beta\delta} &= R_{\alpha\beta\gamma\delta} u^\alpha u^\gamma, \\
\mathcal{H}_{(u)\beta\delta} &= -\frac{1}{2} \eta_{(u)}{}^{\gamma\mu}{}_\delta R_{\alpha\beta\gamma\mu} u^\alpha, \\
\mathcal{F}_{(u)\beta\delta} &= \frac{1}{4} \eta_{(u)}{}^{\alpha\mu}{}_\beta \eta_{(u)}{}^{\gamma\sigma}{}_\delta R_{\alpha\mu\gamma\sigma}.
\end{aligned} \tag{7}
$$

The Riemann tensor may then be represented as

$$
\begin{aligned}
R_{\alpha\beta\rho\sigma} &= -2\mathcal{H}_{(u)[\alpha|\mu|} u_{\beta]} \eta_{(u)}{}^\mu{}_{\rho\sigma} - 2\mathcal{H}_{(u)[\rho|\mu|} u_{\sigma]} \eta_{(u)}{}^\mu{}_{\alpha\beta} \\
&\quad + 2u_\alpha u_{[\rho} \mathcal{E}_{(u)|\beta|\sigma]} - 2u_\beta u_{[\rho} \mathcal{E}_{(u)|\alpha|\sigma]} \\
&\quad + \eta_{(u)}{}^\nu{}_{\alpha\beta} \eta_{(u)}{}^\xi{}_{\rho\sigma} \mathcal{F}_{(u)\nu\xi},
\end{aligned}
$$

while the expressions for the Ricci tensor, the Einstein tensor, and the scalar curvature tensor are

$$
\begin{aligned}
R_{\beta\sigma} &= 4u_{(\beta}\mathcal{H}_{(u)\sigma)} - \mathcal{E}_{(u)\beta\sigma} + u_\beta u_\sigma \mathcal{E}_{(u)} - \mathcal{F}(u)_{\beta\sigma} \\
&\quad + P(u)_{\beta\sigma} \mathcal{F}_{(u)}, \\
R &= 2[-\mathcal{E}_{(u)} + \mathcal{F}_{(u)}], \\
G_{\alpha\beta} &= 4u_{(\alpha}\mathcal{H}_{(u)\beta)} - \mathcal{E}_{(u)\alpha\beta} - P(u)_{\alpha\beta}\mathcal{E}_{(u)} - \mathcal{F}_{(u)\alpha\beta} \\
&\quad + u_\alpha u_\beta \mathcal{F}_{(u)},
\end{aligned} \tag{8}
$$

where

$$
\mathcal{E}_{(u)} = \mathcal{E}_{(u)}{}^\mu{}_\mu, \quad \mathcal{F}_{(u)} = \mathcal{F}_{(u)}{}^\mu{}_\mu, \quad \mathcal{H}_{(u)}{}^\alpha = \tfrac{1}{2} \eta_{(u)}{}^{\alpha\beta\sigma} \mathcal{H}_{(u)\beta\sigma}.
$$

IV. THE DIXON-SOURIAU EQUATIONS OF MOTION

The Dixon-Souriau equations of motion of a charged spinning test particle of charge e in a given gravitational and electromagnetic background [6–9] are given by

$$
\begin{aligned}
\frac{D}{d\tau_U} p^\alpha &= \frac{1}{2} R_{\rho\sigma\beta}{}^\alpha S^{\rho\sigma} U^\beta + eF^\alpha{}_\beta U^\beta - \frac{1}{2}\lambda S^{\mu\nu} \nabla^\alpha F_{\mu\nu}, \\
\frac{D}{d\tau_U} S^{\alpha\beta} &= p^\alpha U^\beta - p^\beta U^\alpha + \lambda [S^{\alpha\mu} F_\mu{}^\beta - S^{\beta\mu} F_\mu{}^\alpha],
\end{aligned} \tag{9}
$$

where $F^{\alpha\beta}$ is the electromagnetic field and λ is an electromagnetic coupling scalar.[2] The spatial dual of the spin-electromagnetism coupling term $S^{\alpha\mu}F_\mu{}^\beta - S^{\beta\mu}F_\mu{}^\alpha$ appearing in the second of equations (9) coincides with the coupling term found by Bargman, Michel, and Telegdi [10] (see Sec. V). The classic Papapetrou scheme is obtained from Eq. (9) by assuming $F=0$, i.e., by neglecting the electromagnetic field. An alternative scheme, considered by Khriplovich and Pomeranski in [11], is obtained by neglecting the spin-electromagnetism interaction terms $S^{\mu\nu}\nabla^\alpha F_{\mu\nu}$ and $S^{\mu[\alpha}F_\mu{}^{\beta]}$ (see Secs. VI D and VI E).

It is convenient to introduce the following notation for the spin-gravity and spin-electromagnetism coupling terms:

$$
\begin{aligned}
\mathcal{R}_{\alpha\beta} &= \frac{1}{2} R_{\alpha\beta\mu\nu} S^{\mu\nu}, \\
Q_\alpha &= \frac{1}{2} S^{\mu\nu} \nabla_\alpha F_{\mu\nu}, \\
N^{\alpha\beta} &= S^{\alpha\mu} F_\mu{}^\beta - S^{\beta\mu} F_\mu{}^\alpha.
\end{aligned} \tag{10}
$$

Assume that (a) $p = ||p||\bar{u} = M_0 \bar{u}$, defined along l_U, is timelike: $\bar{u}^\alpha \bar{u}_\alpha = -1$, and that (b) U and $\bar{u}$ may be extended in a regular way in a neighborhood of the line l_U. We then have two timelike congruences $\bar{u}$ and U at our disposal. Both $\bar{u}$ and U are associated with the particle in a natural way and so are both candidates for defining the proper rest frame of the particle, but there is no agreement in the literature about which of them should be used for this purpose. This also leads to the problem of the definition of the center-of-mass world line as either the P center or the T center, respectively. In the following (see Sec. VI C) we will assume that all the P, T, and CP-center lines exist for our small test body in order to compare the associated spin-gravity and spin-electromagnetism interaction terms in a given background.

Let us now consider a generic observer field u. The splitting of U and p along u and LRS_u is

$$
\begin{aligned}
U &= \gamma_{(U,u)}[u + \nu_{(U,u)}], \\
p &= ||p||\bar{u} = E_{(p,u)} u + p(u) = M_0 \gamma_{(\bar{u},u)}[u + \nu_{(\bar{u},u)}],
\end{aligned} \tag{11}
$$

where $E_{(p,u)} = \gamma_{(\bar{u},u)}||p|| = \gamma_{(\bar{u},u)} M_0$. Let us introduce the following notation for the electric and magnetic parts of the

[2]In the following (Sec. V) we will determine λ by comparison with the purely electromagnetic case in flat spacetime, i.e., the Bargman, Michel, and Telegdi model. A detailed discussion of the choice of this factor can be found in [6,9]. See also [29].

antisymmetric two-tensor fields S, F, N, and $\mathcal{R}$ which appear in the equations $L_{(u)}=S^{(E)}_{(u)}$, $S_{(u)}=S^{(M)}_{(u)}$, $E_{(u)}=F^{(E)}_{(u)}$, $B_{(u)}=F^{(M)}_{(u)}$, $N_{(u)}=N^{(E)}_{(u)}$, $M_{(u)}=N^{(M)}_{(u)}$, $K_{(u)}=\mathcal{R}^{(E)}_{(u)}$, and $T_{(u)}=\mathcal{R}^{(M)}_{(u)}$, and extend the notation to the one-form Q, $Q_{(u)}=Q^{(M)}_{(u)}, q_{(u)}=Q^{(E)}_{(u)}$: $Q=Q_{(u)}+q_{(u)}u$, $q_{(u)}=-u\cdot Q$. In particular, we have

$$
\begin{aligned}
K_{(u)}{}^{\alpha}&=-\mathcal{E}_{(u)}{}^{\alpha}{}_{\sigma}L_{(u)}{}^{\sigma}+\mathcal{H}_{(u)}{}^{\alpha}{}_{\sigma}S_{(u)}{}^{\sigma},\\
T_{(u)}{}^{\alpha}&=\mathcal{F}_{(u)\mu}{}^{\alpha}S_{(u)}{}^{\mu}-\mathcal{H}_{(u)\mu}{}^{\alpha}L_{(u)}{}^{\mu},\\
N_{(u)}{}^{\alpha}&=[E_{(u)}\times_u S_{(u)}+B_{(u)}\times_u L_{(u)}]^{\alpha},\\
M_{(u)}{}^{\alpha}&=[B_{(u)}\times_u S_{(u)}+L_{(u)}\times_u E_{(u)}]^{\alpha},\\
Q_{(u)}{}^{\alpha}&=S_{(u)}{}^{\sigma}\nabla_{(u)}{}^{\alpha}B_{(u)\,\sigma}-L_{(u)}{}^{\sigma}\nabla_{(u)}{}^{\alpha}E_{(u)\,\sigma}\\
&\quad+[N_{(u)}\times_u\omega_{(u)}]^{\alpha}+\theta_{(u)}{}^{\alpha}{}_{\mu}N_{(u)}{}^{\mu},\\
q_{(u)}&=L_{(u)}{}^{\sigma}\nabla_{(\mathrm{fw},u)}E_{(u)\sigma}-S_{(u)}{}^{\sigma}\nabla_{(\mathrm{fw},u)}B_{(u)\sigma}\\
&\quad-N_{(u)}\cdot a_{(u)}.
\end{aligned}
\tag{12}
$$

The background is assumed to be completely known; in particular the electromagnetic field satisfies the Maxwell equations. It is also useful to write the following decomposition for $K(U)\in \mathrm{LRS}_U$:

$$K_{(U)}=P_{(U,u)}K_{(U)}+u[\nu_{(U,u)}\cdot P_{(U,u)}K_{(U)}],$$

where $P_{(U,u)}K_{(U)}=\gamma_{(U,u)}[K_{(u)}+\nu_{(U,u)}\times_u T_{(u)}]$. With these definitions, the equations of motion (9) are equivalent to the following set:

$$
\begin{aligned}
&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[E_{(p,u)}\nu_{(\bar{u},u)}]\\
&\quad=E_{(p,u)}F^{(G)}_{(\mathrm{fw},U,u)}+P_{(U,u)}[-K_{(U)}+eE_{(U)}]-\lambda Q_{(u)},
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}E_{(p,u)}\\
&\quad=E_{(p,u)}\nu_{(\bar{u},u)}\cdot F^{(G)}_{(\mathrm{fw},U,u)}+\nu_{(U,u)}\cdot P_{(U,u)}[-K_{(U)}+eE_{(U)}]\\
&\qquad-\lambda q_{(u)},
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}L_{(u)}\\
&\quad=S_{(u)}\times_u F^{(G)}_{(\mathrm{fw},U,u)}+\gamma_{(U,u)}E_{(p,u)}[\nu_{(U,u)}-\nu_{(\bar{u},u)}]\\
&\qquad+\lambda N(u),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}S_{(u)}\\
&\quad=F^{(G)}_{(\mathrm{fw},U,u)}\times_u L_{(u)}+\gamma_{(U,u)}E_{(p,u)}\nu_{(\bar{u},u)}\times_u\nu_{(U,u)}+\lambda M_{(u)}.
\end{aligned}
\tag{16}
$$

The reference frame field u appearing in Eqs. (13)–(16) is a generic one. If one specializes it to U, the following simplifications occur:

$$
\begin{aligned}
&D_{(\mathrm{fw},U,U)}/d\tau_{(U,U)}=\nabla_{(\mathrm{fw},U)},\\
&\nu_{(U,U)}=0,\quad \gamma_{(U,U)}=1,\\
&F^{(G)}_{(\mathrm{fw},U,U)}=-a_{(U)},
\end{aligned}
$$

so that we have

$$
\begin{aligned}
\nabla_{(\mathrm{fw},U)}[E_{(p,U)}\nu_{(\bar{u},U)}]&=-E_{(p,U)}a_{(U)}-K_{(U)}+eE_{(U)}-\lambda Q_{(U)},\\
\nabla_{(\mathrm{fw},U)}E_{(p,U)}&=-E_{(p,U)}\nu_{(\bar{u},U)}\cdot a_{(U)}-\lambda q_{(U)},\\
\nabla_{(\mathrm{fw},U)}L_{(U)}&=-S_{(U)}\times_U a_{(U)}-E_{(p,U)}\nu_{(\bar{u},U)}\\
&\quad+\lambda N_{(U)},\\
\nabla_{(\mathrm{fw},U)}S_{(U)}&=-a_{(U)}\times_U L_{(U)}+\lambda M_{(U)}.
\end{aligned}
\tag{17}
$$

If we instead specialize the reference frame field to $\bar{u}$, one has $\nu_{(\bar{u},\bar{u})}=0$ so that

$$
\begin{aligned}
&M_0F^{(G)}_{(\mathrm{fw},U,\bar{u})}\\
&\quad=-P_{(U,\bar{u})}[-K_{(U)}+eE_{(U)}]+\lambda Q_{(\bar{u})},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\gamma_{(U,\bar{u})}D_{(\mathrm{fw},U,\bar{u})}/d\tau_{(U,\bar{u})}M_0\\
&\quad=\nu_{(U,\bar{u})}\cdot P_{(U,\bar{u})}[-K_{(U)}+eE_{(U)}]-\lambda q_{(\bar{u})},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\gamma_{(U,\bar{u})}D_{(\mathrm{fw},U,\bar{u})}/d\tau_{(U,\bar{u})}L_{(\bar{u})}\\
&\quad=S_{(\bar{u})}\times_{\bar{u}}F^{(G)}_{(\mathrm{fw},U,\bar{u})}+\gamma_{(U,\bar{u})}E_{(p,\bar{u})}\nu_{(U,\bar{u})}+\lambda N_{(\bar{u})},
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\gamma_{(U,\bar{u})}D_{(\mathrm{fw},U,\bar{u})}/d\tau_{(U,\bar{u})}S_{(\bar{u})}\\
&\quad=F^{(G)}_{(\mathrm{fw},U,\bar{u})}\times_{\bar{u}}L_{(\bar{u})}+\lambda M_{(\bar{u})}.
\end{aligned}
\tag{21}
$$

Taking into account Eq. (18), one can rewrite Eq. (19) as an energy theorem

$$
\begin{aligned}
\gamma_{(U,\bar{u})}D_{(\mathrm{fw},U,\bar{u})}/d\tau_{(U,\bar{u})}M_0&=-M_0\nu_{(U,\bar{u})}\cdot F^{(G)}_{(\mathrm{fw},U,\bar{u})}\\
&\quad-\lambda[Q_{(\bar{u})}\cdot\nu_{(U,\bar{u})}+q_{(\bar{u})}].
\end{aligned}
\tag{22}
$$

The motion of the body relative to u can in principle be described by either the spatial velocity $\nu_{(U,u)}$ or the generalized one $\nu_{(\bar{u},u)}$, the two descriptions being inequivalent. From ordinary relativistic kinematics we have the following ‘‘addition of velocity law’’

$$\frac{\gamma_{(U,\bar{u})}}{\gamma_{(U,u)}}\nu_{(U,\bar{u})}=P_{(u,\bar{u})}[\nu_{(U,u)}-\nu_{(\bar{u},u)}]\tag{23}$$

or equivalently

$$\frac{\nu_{(U,u)}-\nu_{(\bar{u},u)}}{1-\nu_{(U,u)}\cdot\nu_{(\bar{u},u)}}=\gamma_{(\bar{u},u)}P^{-1}_{(u,\bar{u})}\nu_{(U,\bar{u})}.\tag{24}$$

Analogously, a generalized acceleration of the particle relative to u can also be described by the Fermi-Walker spatial derivative of either $\nu_{(U,u)}$ or $\nu_{(\bar{u},u)}$ and, again, the two are inequivalent. The acceleration corresponding to $\nu_{(\bar{u},u)}$ is given by Eq. (13), which includes terms corresponding to the gravitational field, the electromagnetic field, and a sum of spin-gravity-electromagnetism coupling terms. It is convenient to introduce the following notation for the (relative) acceleration terms, due entirely to the spin of the particle:

$$\begin{aligned}\gamma_{(U,u)}A_{(\bar{u},u)} &= -P_{(U,u)}K_{(U)} - \lambda Q_{(u)}\\ &= \gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[E_{(p,u)}\nu_{(\bar{u},u)}]\\ &\quad - E_{(p,u)}F^{(G)}_{(\mathrm{fw},U,u)} - eP_{(U,u)}E_{(U)}. \qquad (25)\end{aligned}$$

We also define an analogous term $A_{(U,u)}$, the weak field approximation of which will be useful in Sec. VI, as

$$\begin{aligned}\gamma_{(U,u)}A_{(U,u)} &= \gamma(U,u)D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[E_{(p,u)}\nu_{(U,u)}]\\ &\quad - E_{(p,u)}F^{(G)}_{(\mathrm{fw},U,u)} - eP_{(U,u)}E_{(U)}. \qquad (26)\end{aligned}$$

From Eq. (15), we have the law which allows us to shift from $\nu_{(\bar{u},u)}$ to $\nu_{(U,u)}$. Thus, by differentiation we have

$$\begin{aligned}A_{(U,u)} &= A_{(\bar{u},u)} - D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[\gamma^{-1}_{(U,u)}S_{(u)}\times_u F^{(G)}_{(\mathrm{fw},U,u)}]\\ &\quad - \lambda D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[\gamma^{-1}_{(U,u)}N_{(u)}]\\ &\quad + D^2_{(\mathrm{fw},U,u)}/d\tau^2_{(U,u)}L_{(u)}. \qquad (27)\end{aligned}$$

Clearly the relative acceleration (26) leads to different spin-gravity and spin-electromagnetism coupling terms than the acceleration (25).

The choice of the supplementary condition is fundamental for the following reasons: (1) it defines the world line l_U, support for all the fields we are dealing with; and (2) the equations are not formally invariant for different choices. In fact, from Eq. (11) in the P, T, and CP cases we have, respectively,

$$(P):\quad L_{(u)} = S_{(u)}\times_u \nu_{(U,u)}, \qquad (28)$$

$$(T):\quad L_{(u)} = S_{(u)}\times_u \nu_{(\bar{u},u)}, \qquad (29)$$

$$(CP):\quad L_{(u)} = 0. \qquad (30)$$

When substituted into Eq. (13) or Eq. (27) these three conditions lead to a total of six different expressions for the sum of the spin-gravity and the spin-electromagnetism couplings. We will illustrate the situation in the special case of the Reissner-Nordström background in Sec. VI.

V. MOTION IN FLAT SPACETIME

Let us now consider the special case of the motion of an electron in flat spacetime where u is a generic observer. The evolution equations reduce to

$$\begin{aligned}&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}[E_{(p,u)}\nu_{(\bar{u},u)}]\\ &\quad = E_{(p,u)}F^{(G)}_{(\mathrm{fw},U,u)} + eP_{(U,u)}E_{(U)} - \lambda Q_{(u)}, \qquad (31)\end{aligned}$$

$$\begin{aligned}&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}E_{(p,u)}\\ &\quad = E_{(p,u)}\nu_{(\bar{u},u)}\cdot F^{(G)}_{(\mathrm{fw},U,u)} + e\nu_{(U,u)}\cdot P_{(U,u)}E_{(U)} - \lambda q_{(u)},\end{aligned}$$

$$\begin{aligned}&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}L_{(u)}\\ &\quad = S_{(u)}\times_u F^{(G)}_{(\mathrm{fw},U,u)} + \gamma_{(U,u)}E_{(p,u)}[\nu_{(U,u)} - \nu_{(\bar{u},u)}]\\ &\qquad + \lambda N_{(u)},\end{aligned}$$

$$\begin{aligned}&\gamma_{(U,u)}D_{(\mathrm{fw},U,u)}/d\tau_{(U,u)}S_{(u)}\\ &\quad = F^{(G)}_{(\mathrm{fw},U,u)}\times_u L_{(u)} + \gamma_{(U,u)}E_{(p,u)}\nu_{(\bar{u},u)}\\ &\qquad \times_u \nu_{(U,u)} + \lambda M_{(u)},\end{aligned}$$

and in the special case $u=U$ we have

$$\begin{aligned}&\nabla_{(\mathrm{fw},U)}[E_{(p,U)}\nu_{(\bar{u},U)}]\\ &\quad = -E_{(p,U)}a_{(U)} + eE_{(U)} - \lambda Q_{(U)}, \qquad (32)\end{aligned}$$

$$\begin{aligned}&\nabla_{(\mathrm{fw},U)}E_{(p,U)}\\ &\quad = -E_{(p,U)}\nu_{(\bar{u},U)}\cdot a_{(U)} - \lambda q_{(U)}, \qquad (33)\end{aligned}$$

$$\begin{aligned}&\nabla_{(\mathrm{fw},U)}L_{(U)}\\ &\quad = -S_{(U)}\times_U a_{(U)} - E_{(p,U)}\nu_{(\bar{u},U)} + \lambda N_{(U)}, \qquad (34)\end{aligned}$$

$$\begin{aligned}&\nabla_{(\mathrm{fw},U)}S_{(U)}\\ &\quad = -a_{(U)}\times_U L_{(U)} + \lambda M_{(U)}. \qquad (35)\end{aligned}$$

The evolution of the spin is expressed by Eq. (35). Its right hand side includes the classic precession term $B_{(U)}\times_U S_{(U)}$ (which comes out from the splitting of the Bargman-Michel-Telegdi coupling term $N^{\alpha\beta}$), as well as other general relativistic corrections. With the additional conditions:

$$p = ||p||U = M_0 U, \quad (\bar{u} = U, \quad E_{(p,U)} = M_0),$$

$$Q = 0, \quad L_{(U)} = 0, \qquad (36)$$

the Bargman-Michel-Telegdi model holds exactly [10]:

$$M_0 a_{(U)} = eE_{(U)},$$

$$\nabla_{(\mathrm{fw},U)}M_0 = 0,$$

$$\nabla_{(\mathrm{fw},U)}S_{(U)} = \lambda M_{(U)} = \lambda B_{(U)}\times_U S_{(U)}. \qquad (37)$$

From these relations we have $\lambda = -e/M_0$. For the sake of simplicity, we assume this value for λ in what follows.

VI. MOTION IN THE REISSNER-NORDSTRÖM BACKGROUND

Now consider the motion of a spinning particle satisfying Eqs. (9) on the background of a static black hole of mass M and charge Q, described by the Reissner-Nordström line element (see, e.g., [18] p. 877) in Boyer-Lindquist coordinates as

$$ds^2=-\Delta r^{-2}dt^2+r^2\sin^2\theta d\varphi^2+r^2\Delta^{-1}dr^2+r^2d\theta^2, \tag{38}$$

where $\Delta=r^2-2Mr+Q^2$. The associated electromagnetic field is

$$A=-\frac{Q}{r}dt, \quad F=\frac{Q}{r^2}dr\wedge dt. \tag{39}$$

Next we choose the observer family to be the static observers with four-velocity, resulting in $m^\alpha=r\Delta^{-1/2}\delta^\alpha{}_t$. Then $F^{(G)}_{(\mathrm{fw},U,m)}=-\gamma_{(U,m)}a_{(m)}$, where

$$a_{(m)}=\frac{Mr-Q^2}{r^2\sqrt{\Delta}}n, \quad n^\sharp=\sqrt{\Delta}r^{-1}\partial_r \tag{40}$$

and n is the radial unit vector. Then we find

$$\mathcal{E}_{(m)\alpha}{}^\beta=\frac{Mr-Q^2}{r^4}P_{(m)}{}^\beta_\alpha-\frac{3Mr-4Q^2}{r^4}n_\alpha n^\beta,$$

$$\mathcal{F}_{(m)\,\alpha}{}^\beta=-\mathcal{E}_{(m)\,\alpha}{}^\beta+\frac{2Q^2}{r^4}n_\alpha n^\beta, \quad \mathcal{H}_{(m)\,\alpha}{}^\beta=0,$$

$$E_{(m)}{}^\alpha=\frac{Q}{r^2}n^\alpha, \quad B_{(m)}{}^\alpha=0, \tag{41}$$

and therefore

$$K_{(m)}=-\frac{Mr-Q^2}{r^4}L_{(m)}+\frac{3Mr-4Q^2}{r^4}[L_{(m)}\cdot n]n,$$

$$T_{(m)}=-\frac{Mr-Q^2}{r^4}S_{(m)}+\frac{3Mr-2Q^2}{r^4}[S_{(m)}\cdot n]n,$$

$$N_{(m)}=\frac{Q}{r^2}n\times_m S_{(m)}, \quad M_{(m)}=-\frac{Q}{r^2}n\times_m L_{(m)},$$

$$Q_{(m)}=-\frac{Q\sqrt{\Delta}}{r^4}L_{(m)}+3\frac{Q\sqrt{\Delta}}{r^4}[L_{(m)}\cdot n]n, \quad q_{(m)}=0. \tag{42}$$

A. Acceleration vectors

In the present context the acceleration formula (26),

$$A_{(u,m)}=D_{(\mathrm{fw},U,m)}/d\tau_{(U,m)}[E_{(p,m)}\nu_{(u,m)}]+E_{(p,m)}a_{(m)}-eE_{(m)}, \tag{43}$$

for $u=\bar{u}$ becomes explicitly

$$A_{(\bar{u},m)}=\frac{Mr-Q^2}{r^4}\nu_{(U,m)}\times_m S_{(m)}-\left[\frac{3Mr-4Q^2}{r^4}+3\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right]\times[L_{(m)}\cdot n]n+\left[\frac{Mr-Q^2}{r^4}+\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right]L_{(m)}-\frac{3Mr-2Q^2}{r^4}[S_{(m)}\cdot n]\nu_{(U,m)}\times_m n, \tag{44}$$

while the corresponding formula (27) for $u=U$ becomes

$$\begin{aligned}A_{(U,m)}=&A_{(\bar{u},m)}+\frac{\lambda Q}{\gamma^2r^2}[D_{(\mathrm{fw},U,m)}/d\tau_{(U,m)}\gamma]n\times_m S_{(m)}\\&+\frac{r^4}{\Delta}\left[\frac{Mr-Q^2}{r^4}+\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right]^2[L_{(m)}\times_m n]\times_m n\\&+D^2_{(\mathrm{fw},U,m)}/d\tau^2_{(U,m)}L_{(m)}+\left[\frac{Mr-Q^2}{r^4}+\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right]\\&\times\left\{\frac{r^2}{\sqrt{\Delta}}E_{(p,m)}[\nu_{(\bar{u},m)}\times_m\nu_{(U,m)}]\times_m n+S_{(m)}\times_m\nu_{(U,m)}\right\}\\&-\left[\frac{3Mr-4Q^2}{r^4}+\frac{(Mr-Q^2)^2}{r^4\Delta}+3\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right]\\&\times[\nu_{(U,m)}\cdot n]S_{(m)}\times_m n,\end{aligned} \tag{45}$$

where the abbreviated notation $\gamma=\gamma_{(U,m)}$ has been used. The following relation is useful in obtaining Eq. (45):

$$D_{(\mathrm{fw},U,m)}n/d\tau_{(U,m)}=\frac{\sqrt{\Delta}}{r^2}\{\nu_{(U,m)}-[\nu_{(U,m)}\cdot n]n\}. \tag{46}$$

Next we must deal with the three supplementary conditions (28), (29), and (30). Together with the two choices for four-velocity, we thus have a total of six possibilities. The situation will be clearer when the usual approximations are made.

B. Weak field approximation

Let us introduce the following approximation assumptions: A1. Slow motion: squares of $\nu_{(U,m)}$, $\nu_{(\bar{u},m)}$, $L_{(m)}$, $S_{(m)}$ are negligible and $E_{(p,m)}$ is nearly constant in time. A2. Large distance: factors of r^{-4} are negligible.

Under these assumptions we have

$$\left(\frac{Mr-Q^2}{r^4}+\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right)\cong\tfrac{1}{3}\left(\frac{3Mr-4Q^2}{r^4}\right)+\left(3\frac{\lambda Q}{\gamma r^2}\frac{\sqrt{\Delta}}{r^2}\right)\cong\frac{M+\lambda Q}{r^3}. \tag{47}$$

Using the abbreviations $v \cong \nu_{(U,m)}$, $\bar{v} \cong \nu_{(\bar{u},m)}$, $M_0 \cong E_{(p,m)}$, $S \cong S_{(m)}$, $L \cong L_{(m)}$, $()^{\cdot} \cong D_{(\mathrm{fw},U,m)}/d\tau_{(U,m)}$, we have:

$$K_{(m)} \cong \frac{3M}{r^3}[-\tfrac{1}{3}L + (L\cdot n)n],$$

$$T_{(m)} \cong \frac{3M}{r^3}[-\tfrac{1}{3}S + (S\cdot n)n],$$

$$Q_{(m)} \cong \frac{3Q}{r^3}[-\tfrac{1}{3}L + (L\cdot n)n], \tag{48}$$

and the two possible acceleration laws are then

$$\dot{\bar{v}} \cong -\frac{M+\lambda Q}{r^2}n - \frac{M^2-Q^2}{r^3}n + \frac{1}{M_0}a_{(\bar{u},m)},$$

$$\dot{v} \cong -\frac{M+\lambda Q}{r^2}n - \frac{M^2-Q^2}{r^3}n + \frac{1}{M_0}a_{(U,m)}, \tag{49}$$

where we have introduced the approximate spin-gravity-electromagnetism coupling terms

$$a_{(\bar{u},m)} = \frac{3M}{r^3}\left[-\tfrac{2}{3}v\times S + [S\cdot(n\times v)]n + (v\cdot n)n \times S + \frac{1}{3}L - (L\cdot n)n\right] + \frac{3\lambda Q}{r^3}[\tfrac{1}{3}L - (L\cdot n)n],$$

$$a_{(U,m)} = \frac{3M}{r^3}[-v\times S + [S\cdot(n\times v)]n + 2(v\cdot n)n \times S + \tfrac{1}{3}L - (L\cdot n)n] + \frac{3\lambda Q}{r^3}[-\tfrac{1}{3}v\times S + (v\cdot n)n \times S + \tfrac{1}{3}L - (L\cdot n)n] + \left(\frac{M+\lambda Q}{r^2} - \frac{Q^2}{r^3}\right) \times M_0(\bar{v}\times v)\times n + \ddot{L}, \tag{50}$$

such that $a_{(\bar{u},m)} \cong A_{(\bar{u},m)}$, $a_{(U,m)} \cong A_{(U,m)}$ and where

$$\bar{v} \cong v + \frac{1}{M_0}\left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)n\times S - \frac{1}{M_0}\dot{L}. \tag{51}$$

As in Sec. V, we have set $\lambda = -e/M_0$.

The evolution laws for the mass and the spin are then

$$\dot{M}_0 \cong -M_0\left(\frac{M}{r^2} + \frac{M^2-Q^2}{r^3}\right)(\bar{v}\cdot n) + \frac{eQ}{r^2}(v\cdot n) - \frac{3M}{r3}[-\tfrac{1}{3}(v\cdot L) + (L\cdot n)(v\cdot n)],$$

$$\dot{S} \cong \left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)L\times n + M_0\bar{v}\times v. \tag{52}$$

The approximate formulas

$$(n/r^2)^{\cdot} \cong (-3/r^3)[-v/3 + (v\cdot n)n], \quad (n/r^3)^{\cdot} \cong 0, \tag{53}$$

and the tensor algebra relation [14]

$$(S\cdot n)n\times v = (v\cdot n)n\times S + [S\cdot(n\times v)]n - v\times S \tag{54}$$

are useful in these calculations.

C. Supplementary conditions: a comparative analysis

The discussion so far holds for a generic center-of-mass world line. We now specialize it to a particular world line, corresponding to one of the three choices of supplementary conditions. To distinguish the spin-gravity-electromagnetism coupling terms which occur for different supplementary conditions (corresponding to the different expressions for L as a function of S and v), the explicit SC subscript notation for $a_{(\bar{u},m)(\mathrm{SC})}$ and $a_{(U,m)(\mathrm{SC})}$ is now necessary.

The case CP. Here $L\equiv 0$ so the picture is simplified as

$$\bar{v} \cong v + \frac{M+\lambda Q}{M_0 r^2}n\times S + \frac{M^2-Q^2}{M_0 r^3}n\times S \tag{55}$$

and

$$a_{(\bar{u},m)(CP)} = 3\frac{M}{r^3}\{-\tfrac{2}{3}v\times S + [S\cdot(n\times v)]n + (v\cdot n)n\times S\},$$

$$a_{(U,m)(CP)} = \frac{3M}{r^3}\{-v\times S + [S\cdot(n\times v)]n + 2(v\cdot n)n\times S\} + \frac{3\lambda Q}{r^3}[-\tfrac{1}{3}v\times S + (v\cdot n)n\times S]. \tag{56}$$

The mass and spin evolve according to

$$\dot{M}_0 \cong -M_0\left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)(v\cdot n),$$

$$\dot{S} \cong M_0\bar{v}\times v \cong \left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)(n\times S)\times v. \tag{57}$$

The case T. In this case $L = S\times\bar{v}$ and, consequently,

$$\bar{v} \cong v + \frac{2\lambda Q}{M_0 r^2} n \times S \tag{58}$$

and

$$a_{(\bar{u},m)(T)} = \frac{3M}{r^3}\{-v\times S + 2[S\cdot(n\times v)]n + (v\cdot n)n\times S\} + \frac{3\lambda Q}{r^3}\{-\tfrac{1}{3}v\times S + [S\cdot(n\times v)]n\},$$

$$a_{(U,m)(T)} = \frac{3M}{r^3}\{-v\times S + 2[S\cdot(n\times v)]n + (v\cdot n)n\times S\} + \frac{3\lambda Q}{r^3}\{-v\times S + [S\cdot(n\times v)]n + 2(v\cdot n)n\times S\}. \tag{59}$$

The mass and spin evolve according to

$$\dot{M}_0 \cong -M_0\left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)(v\cdot n),$$

$$\dot{S} \cong \left(\frac{M+\lambda Q}{r^2} + \frac{M^2-Q^2}{r^3}\right)(S\times v)\times n + M_0 \bar{v}\times v. \tag{60}$$

The case P. Here $L = S\times v$ and so

$$\bar{v} \cong v + \frac{M+\lambda Q}{M_0 r^2} n\times S + \frac{M^2-Q^2}{M_0 r^3} n\times S - \frac{1}{M_0}\dot{L} \tag{61}$$

and

$$a_{(\bar{u},m)(P)} = \frac{3M}{r^3}\{-v\times S + 2[S\cdot(n\times v)]n + (v\cdot n)n\times S\} + \frac{3\lambda Q}{r^3}\{-\tfrac{1}{3}v\times S + [S\cdot(n\times v)]n\}. \tag{62}$$

Thus both $a_{(\bar{u},m)(P)}$ and $a_{(\bar{u},m)(T)}$ have the same formal dependence on the fields S, n, r, v, but the latter are defined on different world lines for the two cases. An expression for the quantity $a_{(U,m)(P)}$ cannot be given without first solving the differential equations, which in turn is related to the fact that under the supplementary condition P, the full system of equations is of second order, as was pointed out in [20,30].

However, if by analogy with the (CP) and (T) cases, we ''*a priori*'' assume that A3. $a_{(U,m)(P)}$ and S are of the same order, then we find an expression for $a_{(U,m)(P)}$ which is formally identical to that of $a_{(U,m)(T)}$:

$$a_{(U,m)(P)} = 3\frac{M}{r^3}\{-v\times S + 2[S\cdot(n\times v)]n + (v\cdot n)n\times S\} + \frac{3\lambda Q}{r^3}\{-v\times S + [S\cdot(n\times v)]n + 2(v\cdot n)n\times S\}. \tag{63}$$

The same thing happens for the mass and spin evolution equations, which are formally identical to Eq. (60).

The formal identity of the equations for the P and T cases would suggest that

$$X_{(T)} \cong X_{(P)}. \tag{64}$$

However, to make this relation consistent with the model we have to introduce the following additional assumption:

A4. The variables M_0 and S can each be transported from any SC center to another in such a way that the difference between their values on one world line and the values transported to it from another are always negligible in the sense of A1 and A2.

We make the assumption A4 from now on so that we can consistently drop the SC subscript notation for these variables. A3 and A4 were also implicitly assumed in [14], where the same relation (64) was obtained in the special case of the Schwarzschild background. According to Eq. (64) and A4, the P and T models can be considered completely equivalent and therefore the six acceleration laws of our general picture reduce to four.

We are also allowed to compare the spin-gravity-electromagnetism coupling terms defined at $X_{(T)}$ (or $X_{(P)}$) with those defined at $X_{(CP)}$, provided we also introduce a law for the shift of these centers (we will do this for the Schwarzschild case in Sec. VI E).

Two of the acceleration vectors of the present description, namely $a_{(U,m)(CP)}$ and $a_{(\bar{u},m)(P)}$, were previously calculated for the purely gravitational case (Papapetrou's equations) by Barker and O'Connell in [14] where they were denoted by $a_{S(CP)}$ and $a_{S(P)}$, respectively. In the purely gravitational case, it is interesting to note that the same expression for $a_{(\bar{u},m)(T)}$ (or $a_{(\bar{u},m)(P)}$) is the result of a completely different approach, that of the post-Minkowskian calculation of the spin-orbit interaction of the two-body problem, as carried out by Damour in [31]; it suffices to consider the limiting case when one of the bodies is very heavy.

D. The further approximation of Khriplovich and Pomeransky

Recently Khriplovich and Pomeransky [11] considered a simplified (covariant) model, which is obtained from Eqs. (9) by neglecting the spin-electromagnetism interaction terms $S^{\mu\nu}\nabla^{\alpha}F_{\mu\nu}$ and $S^{\mu[\alpha}F_{\mu}{}^{\beta]}$. In the Reissner-Nordström background and under the same assumptions A1 and A2, this model leads to a different set of accelerations, which are

$$a_{(\bar{u},m)}{}^{(KP)}_{(CP)}=3\frac{M}{r^3}\{-\tfrac{2}{3}v\times S+(v\cdot n)n\times S+[S\cdot(n\times v)]n\},$$

$$a_{(U,m)}{}^{(KP)}_{(CP)}=3\frac{M}{r^3}\{-v\times S+2(v\cdot n)n\times S+[S\cdot(n\times v)]n\},$$

$$a_{(\bar{u},m)}{}^{(KP)}_{(T)}=3\frac{M}{r^3}\{-v\times S+(v\cdot n)n\times S+2[S\cdot(n\times v)]n\},$$

$$a_{(U,m)}{}^{(KP)}_{(T)}=3\frac{M}{r^3}\{-v\times S+(v\cdot n)n\times S+2[S\cdot(n\times v)]n\}$$

$$+\frac{3\lambda Q}{r^3}[\tfrac{1}{3}v\times S-(v\cdot n)n\times S], \qquad (65)$$

where the superscript KP is used to avoid further confusion with the corresponding terms of the Dixon-Souriau model. Here the P case also reduces to the T case provided we make an assumption analogous to A3. In the original paper [11] only the P condition is implemented, and the corresponding acceleration [Eq. (65)] is discussed only in the limiting cases of vanishing M or Q. Of course in this latter case both the Dixon-Souriau and the Khriplovich-Pomeransky models reduce to the Papapetrou one. Later Khriplovich and Pomeransky introduced a noncovariant model for the spin-gravity-electromagnetism interaction [12] which they consider more appropriate.

E. Nonrelativistic center of mass

The classical interaction of spin with the gravitational field is usually obtained from the spin-orbit interaction potential

$$V_{(G)}=\frac{3}{2}\frac{M}{r^2}S\cdot(n\times v)$$

(see, e.g., [13,14,11]); the corresponding acceleration is

$$a_{(G)}=\frac{3M}{r^3}\{-v\times S+\tfrac{3}{2}(n\cdot v)n\times S+\tfrac{3}{2}[S\cdot(n\times v)]n\}. \qquad (66)$$

Similarly, for the interaction of spin with the electromagnetic field, using the Thomas interaction potential

$$V_{(E)}=-\frac{1}{2}\frac{\lambda Q}{r^2}S\cdot(n\times v),$$

we have the following expression for the acceleration [11]:

$$a_{(E)}=\frac{3\lambda Q}{r^3}\{\tfrac{1}{3}v\times S-\tfrac{1}{2}(n\cdot v)n\times S-\tfrac{1}{2}[S\cdot(n\times v)]n\}. \qquad (67)$$

We will compare the classic results (66) and (67) with those coming from relativistic models.

In the Schwarzschild limit $(Q\to 0)$ it is possible to define an auxiliary center point $X_{\rm NR}$ which behaves like a nonrelativistic center of mass; that is, its evolution involves a spin-gravity coupling term formally identical to $a_{(G)}$ [14]. To this end we introduce

$$X_{\rm NR}=X_{(CP)}-\frac{1}{2M_0}v_{(CP)}\times S \qquad (68)$$

and the associated evolution equation [analogous to Eqs. (49)]

$$\ddot{X}_{\rm NR}=\dot{v}_{\rm NR}\cong-\frac{M+\lambda Q}{r_{\rm NR}^2}n_{\rm NR}-\frac{M^2-Q^2}{r_{\rm NR}^3}n_{\rm NR}+\frac{1}{M_0}a_{\rm NR}, \qquad (69)$$

where $r_{\rm NR}=|X_{\rm NR}|$. Equation (69) is in fact a definition for $a_{\rm NR}$. It leads to $a_{\rm NR}\cong a_{(G)}$, provided one replaces r and v in the expression (66) with $r_{\rm NR}$ and $v_{\rm NR}$, respectively.

Moreover, in the Schwarzschild case, the following ‘‘shift of the center of mass law’’ was found consistent with the Papapetrou equations by Barker and O'Connell in [14][3]

$$X_{(CP)}\cong X_{(T)}+\frac{1}{M_0}v_{(CP)}\times S. \qquad (70)$$

This relation allows $X_{\rm NR}$ to be interpreted as the midpoint between $X_{(CP)}$ and $X_{(T)}$: $X_{\rm NR}\cong(1/2)[X_{(CP)}+X_{(T)}]$. Therefore Eq. (68) can be written equivalently as

$$X_{\rm NR}=X_{(T)}+\frac{1}{2M_0}v_{(CP)}\times S. \qquad (71)$$

Unfortunately, in the Dixon-Souriau model, relations (68), (70), and (71) cannot be generalized to the Reissner-Nordström case.

With the Khriplovich-Pomeransky simplified model, provided one defines $X_{\rm NR}$ by Eq. (71) [and not by Eq. (69)], relation (49) still holds in the Reissner-Nordström case. As an aside we remark that Khriplovich and Pomeransky hypothesize that the true origin of Eq. (71) is a Foldy-Wouthuysen transformation, which is necessary to obtain the classical limit of a spin 1/2 Dirac field [11]. However, Eq. (70) again fails to be true except in the Schwarzschild case. Therefore this midpoint interpretation in both the Dixon-Souriau model and the Khriplovich-Pomeransky model is only valid in the Schwarzschild case.

Finally in the purely gravitational case it is worth noting that under assumptions A1 and A2 only, $a_{(\bar{u},m)(CP)}$ is formally identical (but for a numerical factor) to $a_{(G)}$, i.e., we have:

$$a_{(\bar{u},m)(CP)}=\tfrac{2}{3}a_{(G)}, \qquad (72)$$

[3]The original law by Barker and O'Connell had $X_{(P)}$ in place of $X_{(T)}$ (see [14]), which is equivalent if the identification $X_{(P)}\cong X_{(T)}$ is assumed. However, the formulation in terms of $X_{(T)}$ is preferable for two reasons: (a) it is the general relativistic analogue of Moller's exact special relativistic formula [32]; (b) some calculations at $X_{(P)}$ need the further hypothesis A3, while those at $X_{(T)}$ do not.

provided one replaces r and v in the expression (66) with $r_{(CP)}$ and $v_{(CP)}$, respectively. This relation does not seem to have been considered previously in the literature. It can analogously be extended to the Reissner-Nordström case as follows:

$$a_{(\bar{u},m)(CP)}=\frac{2M}{3M-\lambda Q}[a_{(G)}+a_{(E)}]. \tag{73}$$

We thus note that the result of the nonrelativistic theory is also recovered at $X_{(CP)}$ modulo a factor, without the necessity of introducing the auxiliary point $X_{\rm NR}$; it suffices to follow the evolution of the momentum vector $\bar{u}$ instead of that of U. A problem with this alternative approach is that in the limit $M\to 0$ one obtains $a_{(\bar{u},m)(CP)}=0$ instead of $a_{(E)}$ and that the singular case $3M=\lambda Q$ is excluded.

VII. CONCLUDING REMARKS

With the present study the motion of a charged spinning test particle has been given a general 1+3 formulation in the framework of gravitoelectromagnetism, a language for discussing the spacetime splitting itself, for any choice of supplementary conditions. The ''center line'' of the body with the associated timelike unit tangent vector as well as the direction of the generalized momentum vector of the body are quite natural to be considered as the four-velocity of observers at rest with the body itself. Both of them are used in the definition of supplementary conditions to be added to the equations of motion in order to determine a complete model. Of course any observer can be used either to describe the motion of the body or to induce some supplementary conditions; in the particular case of motion in the Schwarzschild background, the pioneering work by Papapetrou and Corinaldesi was done by using the family of static (Killing) observers. In this paper the 1+3 version of the equations of motion for a charged spinning test particle has been given for a generic observer, generic observer-induced supplementary conditions, and for various models for the gravity-electromagnetism coupling studied in the literature.

As an application, the motion of a charged spinning test particle in the Reissner-Nordström spacetime has been studied. The general framework is helpful in understanding what results can be extended from the well-known case of the Schwarzschild background. For example, one is interested in how the classical expression for the spin-orbit interaction acceleration is modified. In the case of the Schwarzschild spacetime, as first shown by Barker and O'Connell, one can introduce the midpoint between the T and the CP centers and interpret it as a nonrelativistic center of mass. In the case of Reissner-Nordström it is shown here that only in the special case of the Khriplovich and Pomeransky model for spinning test particles is it still possible to introduce a point playing the role of a center of mass but where the midpoint interpretation is lost. For generic spacetimes and more general models it is not even possible to introduce such a point.

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