Pure-radiation gravitational fields with a simple twist and a Killing vector

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(Received 19 October 1999; published 23 February 2000)

Pure-radiation solutions are found exploiting the analogy with the Euler-Darboux equation for aligned colliding plane waves and the Euler-Tricomi eqution in hydrodynamics of two-dimensional flow. They do not depend on one of the spacelike coordinates and comprise the Hauser solution as a special subcase.

PACS number(s): 04.20.Jb

I. INTRODUCTION

There exist many papers dealing with algebraically special, expanding and twisting pure-radiation solutions of the Einstein equations. An extensive bibliography up to 1980 exists in Ref. [1]. Further results on pure-radiation fields can be found in Refs. [2–7]. The standard form of the metric is [1]

$$ds^2=\frac{2d\zeta d\bar{\zeta}}{\rho\bar{\rho}P^2}-2\Omega[dr+Wd\zeta+\bar{W}d\bar{\zeta}+H\Omega],$$

$$\Omega=du+Ld\zeta+\bar{L}d\bar{\zeta}. \tag{1}$$

Here r is the coordinate along the null congruence of geodesics, u is the retarded time, while $\zeta,\bar{\zeta}$ span a two-dimensional surface. The metric components are determined by the r-independent real functions P,m,M and the complex function L:

$$2i\Sigma=P^2(\bar{\partial}L-\partial\bar{L}), \tag{2}$$

$$\rho=-\frac{1}{r+i\Sigma}, \tag{3}$$

$$W=\rho^{-1}L_u+i\partial\Sigma, \tag{4}$$

$$H=-r(\ln P)_u-(mr+M\Sigma)\rho\bar{\rho}+\frac{K}{2}, \tag{5}$$

$$K=2P^2Re[\partial(\bar{\partial}\ln P-\bar{L}_u)], \tag{6}$$

where $\partial=\partial_\zeta-L\partial_u$ and Σ is the twist. The functions mentioned above satisfy the system of equations

$$(\partial-3L_u)(m+iM)=0, \tag{7}$$

$$P^{-3}M=\mathrm{Im}\,\partial\partial\bar{\partial}\bar{\partial}V, \tag{8}$$

$$n^2=-2P^3[P^{-3}(m+iM)]_u+2P^3(\partial\partial\bar{\partial}\bar{\partial}V)_u-2P^2(\partial\partial V)_u(\bar{\partial}\bar{\partial}V)_u, \tag{9}$$

where $V_u=P$, n is the energy density of pure radiation and the Newton constant is set to 1. Equations (7)–(9) are in fact Eqs. (26.32) and (26.33) from Ref. [1].

It has been noticed in different contexts that the condition $M=0$ simplifies the equations [4–6,8,9]. In a previous paper [10] we explored this condition with the help of Stephani's method [2] for some algebraically special, axisymmetric, expanding and twisting gravitational fields. They depend on u and $\sigma=\zeta\bar{\zeta}$ intrinsically; i.e., the u dependence cannot be taken away by applying the gauge transformations (25.27) from Ref. [1]. In the present paper we find solutions for fields which depend on u and the real part of ζ, $x=(1/\sqrt{2})(\zeta+\bar{\zeta})$ and have the simplest possible twist. Many of the well known algebraically special exact solutions possess this kind of symmetry [1,3,4]. Among them are the only known seed vacuum Robinson-Trautman solution of type III, the static C metric and the only known expanding vacuum solution of type N with twist, namely the Hauser solution, given respectively by Eqs. (24.15), (24.23) and (25.71) from Ref. [1].

In Sec. II Eqs. (7)–(9) are reformulated in terms of an invariant potential which leads to the $L_u=0$ gauge. In Sec. III solutions with separated variables u and x are found. In Sec. IV the main Eq. (8) for simplest twist is shown to be equivalent to a case of the Euler-Darboux equation, somewhat different from the equation derived in Ref. [10]. On its turn it is a complex version of the central equation in the theory of aligned colliding plane waves (CPW). Three new solutions are found. In Sec. V a homogenous hypergeometric solution is derived, exploiting the similarity between Eq. (8) and the Euler-Tricomi equation. It contains as a special case the Hauser solution of type N. Section VI contains some conclusions.

II. FIELD EQUATIONS IN THE $L_u=0$ GAUGE

Following [2,10] we introduce the invariant complex potential ϕ which solves Eq. (7):

$$m+iM=\phi_u^3, \tag{10}$$

$$L=\frac{\phi_x}{\sqrt{2}\phi_u}. \tag{11}$$

When $M=0$ we can apply the transformation

$$u'=f(u,x), \tag{12}$$

$$(m+iM)'=f_u^{-3}(m+iM) \tag{13}$$

to make m a positive or a negative constant m_0, so that

$$\phi = m_0^{1/3}[u + iq(x)], \tag{14}$$

$$L = \frac{i}{\sqrt{2}} q_x \tag{15}$$

with real q. Obviously $L_u = 0$. This gauge differs from the usual gauge $P_u = 0$ but is very suitable when the Newman-Unti-Tamburino (NUT) parameter M vanishes. Equations (8) and (9) simplify

$$\partial\partial\bar{\partial}\bar{\partial}V = \bar{\partial}\bar{\partial}\partial\partial V, \tag{16}$$

$$n^2 = 6m_0 P^{-1} P_u + 2P^3 \partial\partial\bar{\partial}\bar{\partial}P - 2P^2 \partial\partial P \bar{\partial}\bar{\partial}P \tag{17}$$

with $\partial = (1/\sqrt{2})(\partial_x - iq_x\partial_u)$. The second equation is in fact an inequality. When $P_u \neq 0$, n^2 can be made positive by the choice of m_0 at least for some region of spacetime [1,4,8]. The expressions for the metric components simplify too, e.g., the gauge invariants Σ and K become

$$\Sigma = \frac{1}{2} q_{xx} P^2, \tag{18}$$

$$K = P^2(\bar{\partial}\partial + \partial\bar{\partial}) \ln P. \tag{19}$$

When $m + iM = 0$ (Petrov types III and N) Eq. (7) is an identity but still a potential ϕ may be introduced with the property $\partial\phi = 0$ and the subclass of solutions satisfying Eq. (14) (with $m_0 = 1$) can be studied. One should put $m_0 = 0$ in all other equations.

The main equation (16), which is of fourth order with respect to V, becomes in both cases a linear second order equation for P. Equation (18) shows that q must be at least quadratic in x for a non-trivial twist. Let us choose the simplest possibility, $q = x^2/2$, $L = (i/\sqrt{2})x$. Then Eqs. (16) and (17) read

$$x^2 P_{uu} + P_{xx} = 0, \tag{20}$$

$$2n^2 = 12m_0 P^{-1} P_u - 3P^3 P_{uu} - 4x^4 P^2 P_{uu}^2 - P^2(P_u + 2xP_{ux})^2. \tag{21}$$

The last two terms in Eq. (21) are definitely negative, so the first term must be necessarily positive for a type II solution and the second term must be positive for a type III solution.

III. SOLUTION WITH SEPARATED VARIABLES

Suppose that $P = F(x)G(u)$. Equation (20) splits into two parts,

$$G_{uu} = cG, \tag{22}$$

$$F_{xx} + cx^2 F = 0, \tag{23}$$

where c is an arbitrary constant. There are three types of solutions, depending on the sign of c. If $c = 0$ we have

$$P = ux, \tag{24}$$

$$n^2 = \frac{6m_0}{u} - \frac{9}{2} u^2 x^4. \tag{25}$$

Type III solutions have negative energy while type II solutions have negative energy density for $x \to \infty$. If $c > 0$, P contains Bessel functions, e.g.,

$$P = \sqrt{x} J_{1/4}\left(\frac{\sqrt{c}}{2} x^2\right) e^{-\sqrt{c}u}, \tag{26}$$

$$4n^2 = -24\sqrt{c} m_0 - 2c(3 + 4cx^4) F^4 e^{-4\sqrt{c}u} - 2cP^2(F + 2xF_x)^2 e^{-2\sqrt{c}u}. \tag{27}$$

The first term can be made positive when $m_0 < 0$, but the second has a negative pole when $x \to \infty$ because $(xF)^4 \sim x^2$. There are no solutions with positive n^2 everywhere. The third case $c < 0$ gives a generic solution with modified Bessel functions like

$$P = \sqrt{x} K_{1/4}\left(\frac{\sqrt{-c}}{2} x^2\right) \sin\sqrt{-c}u, \tag{28}$$

$$4n^2 = 24\sqrt{-c} m_0 \cot(\sqrt{-c}u) - 2c(3 + 4cx^4) F^4 \sin^4\sqrt{-c}u + 2cP^2(F + 2xF_x)^2 \cos^2\sqrt{-c}u. \tag{29}$$

The first term changes sign and has poles in u, thus type II solutions are unphysical. Type III solutions have negative n^2 when, e.g., $u = \pi/\sqrt{-c}$, although the second term doesn't have poles in u.

Equations (24) and (26) have been obtained by another method in a more general form in Ref. [4] but the energy density has not been discussed in detail.

If we separate the variables like $P = F(x) + G(u)$ then

$$G = -\frac{cu^2}{2} + c_1 u + c_2, \tag{30}$$

$$F = \frac{cx^4}{12} + c_3 + c_4, \tag{31}$$

where c_i are constants. Numerous negative terms arise and the region of positivity of n^2 is rather complicated due to the many arbitrary constants. A more detailed discussion in a general setting of this type of separation of variables may be found in Ref. [11].

IV. REDUCTION TO THE EULER-DARBOUX EQUATION

Equation (20) with x^2 replaced by x^l where l is an integer has been studied in the past [12,13]. Green functions for different boundary problems have been found. The basis of these results is the change of variables which transforms Eq.

(20) into the Euler-Darboux equation

$$4P_{\varphi\omega}+\frac{1}{\varphi+\omega}(P_{\varphi}+P_{\omega})=0, \tag{32}$$

$$\varphi=u+\frac{i}{2}x^2, \tag{33}$$

where $\omega=-\bar{\varphi}$. This equation is similar to Eq. (24) from Ref. [10] and the main equation for aligned colliding plane waves [14]—one needs only to replace the multiplier 4 by 2. This, however, is not a trivial change and can't be achieved by scaling the variables or P. Nevertheless, one may use the numerous techniques developed in the search for CPW solutions and applied in Ref. [10] for axisymmetric fields. We shall find analogues of these solutions. As pointed out in Ref. [10] the reality of P must be ensured because the variables φ,ω are complex.

The simplest solution

$$P=i^{-1/2}(\varphi+\omega)^{1/2}=x \tag{34}$$

is a time-independent vacuum solution of Kerr-Schild type.

A solution with separated variables $P=F(\varphi)G(\omega)$ exists. Replacement in Eq. (32) yields

$$P=A[(\sigma+\varphi)(\sigma+\bar{\varphi})]^{-1/4}, \tag{35}$$

where A and σ are constants. A is ignorable and σ can be hidden in u to obtain a real solution:

$$P=B^{-1/4}, \tag{36}$$

$$B\equiv u^2+\frac{1}{4}x^4. \tag{37}$$

This solution is analogous to the solution given by Eqs. (27) and (28) from Ref. [10] and has a number of nice features. The energy density is

$$n^2=-\frac{3m_0}{B^{3/2}}u+\frac{3}{16B^3}(x^4-6u^2)$$
$$-\frac{1}{128B^5}[4x^4(6u^2-x^4)^2+u^2(9x^4-4u^2)^2]. \tag{38}$$

In the following we suppose that the retarded time satisfies the condition $u>u_0>0$ for some constant u_0. Equation (38) is regular in x unlike many other solutions, plagued by singular pipes for $x=0$ or $x=\pm\infty$ [1]. When $m_0<0$ the first term dominates over the others if $|m_0|$ is big enough and consequently n^2 is positive. Unfortunately, type III solutions are not with positive n^2 for any x because the second term changes sign. The gauge invariants (18) and (19) are regular in x and vanish when $u\to\infty$:

$$\Sigma=\frac{1}{2}P^2, \tag{39}$$

$$K=-\frac{x^2}{4B^{3/2}}. \tag{40}$$

The same is true for the Weyl scalars [1,15] with leading terms given by

$$\Psi_2=m_0\rho^3, \tag{41}$$

$$\Psi_3=-\rho^2P^3\partial I+O(\rho^3), \tag{42}$$

$$\Psi_4=\rho P^2 I_u+O(\rho^2), \tag{43}$$

$$I=P^{-1}\overline{\partial\partial}P. \tag{44}$$

An exception is Ψ_2 which approaches $-m_0/r^3$ as $u\to\infty$.

Let us present next an analogue of the $\cosh^{-1}$ solution found in Ref. [10]. We substitute the ansatz $P=P(a)$,

$$a=\frac{i(\varphi+\bar{\varphi})}{\varphi-\bar{\varphi}}=\frac{2u}{x^2}, \tag{45}$$

into Eq. (32). The result is an elliptic integral of the first kind $F(\psi,\kappa)$:

$$P=\sqrt{2}F\left(\psi,\frac{1}{\sqrt{2}}\right), \tag{46}$$

$$\psi=\arccos(1+a^2)^{-1/4}, \tag{47}$$

$$P_a=(1+a^2)^{-3/4}. \tag{48}$$

The function P is bounded: $0\leqslant P\leqslant\sqrt{2}F(\pi/2,1/\sqrt{2})$. The solution possesses regular characteristics like the previous one:

$$n^2=\frac{12m_0|x|}{P(4B)^{3/4}}+\frac{18u|x|P^3}{(4B)^{7/4}}-\frac{9x^2P^2}{4B^{3/2}}, \tag{49}$$

$$\Sigma=\frac{1}{2}P^2, \tag{50}$$

$$K=-\frac{2}{B^{1/2}}. \tag{51}$$

All terms in Eq. (49) are regular. The first term is positive for $m_0>0$ and type II solutions with positive n^2 exist for big enough m_0. The second term is positive too but does not always dominate over the third one. The Weyl scalars are also regular and Ψ_3,Ψ_4 vanish when $u\to\infty$.

Let us transform now the Euler-Darboux equation (32) into its canonical form. Introducing the new variables $\tau=x^2$, $\lambda=2u$ we obtain

$$P_{\tau\tau}+\frac{1}{2\tau}P_{\tau}+P_{\lambda\lambda}=0. \tag{52}$$

This is an analogue of Eq. (25) from Ref. [10] and its solution is given by the Bessel functions from Sec. III. We can go further, utilizing the coordinates for the first Yurtsever solution [10,14,16]:

$$\tau = \nu \sin \eta, \tag{53}$$

$$\lambda = \nu \cos \eta, \tag{54}$$

$$\nu^2 = 4B, \tag{55}$$

$$\cos \eta = \left(1 + \frac{x^4}{4u^2}\right)^{-1/2}. \tag{56}$$

Then Eq. (52) becomes

$$P_{\nu\nu} + \frac{1}{\nu^2} P_{\eta\eta} + \frac{1}{2}\left(\frac{3}{\nu} P_\nu + \frac{1}{\nu^2} \cot \eta \, P_\eta\right) = 0. \tag{57}$$

It has a separable solution of the kind $P = \nu^l Y(\eta)$. Y, instead of being a Legendre function of the first or second kind, satisfies the equation

$$(1 - w^2) Y_{ww} - \frac{3}{2} w Y_w + l\left(l + \frac{1}{2}\right) Y = 0, \tag{58}$$

where $w = \cos \eta$. Its solution is a hypergeometric function and

$$P = (4B)^{l/2} F\left(\varepsilon, \sigma, -\frac{3}{4}, X\right), \tag{59}$$

where $\varepsilon + \sigma = -\frac{5}{2}$, $\varepsilon\sigma = l(l + \frac{1}{2})$ and

$$X = \frac{1}{2}(1 + uB^{-1/2}). \tag{60}$$

P is real because $0 \leqslant X \leqslant 1$. The hypergeometric function is reducible to a Legendre function:

$$F\left(\varepsilon, \sigma, -\frac{3}{4}, X\right) = \Gamma\left(-\frac{3}{4}\right)\left(\frac{x^4}{4B}\right)^{7/8} P^{7/4}_{-(1+\varepsilon)/2}(-uB^{-1/2}). \tag{61}$$

If $l = -\frac{1}{2}$, either ε or σ vanishes and formula (59) degenerates to the rational function given by Eq. (36).

V. HYDRODYNAMICAL ANALOGY

Equation (20) has certain similarities with the Euler-Tricomi equation

$$x\Phi_{uu} - \Phi_{xx} = 0. \tag{62}$$

It appears in hydrodynamics in the study of a two-dimensional flow of compressible fluid with velocity near the velocity of sound [17]. It is a limiting case of the more complex Chapligin equation which has integrals among the hypergeometric functions [17,18]. The Euler-Tricomi equation is invariant under the transformations $u^2 \to cu^2$, $x^3 \to cx^3$ which leads to a homogeneous hypergeometric solution. It is discussed at length in Ref. [17]. In our case Eq. (20) is invariant under $u \to c^2 u$, $x \to cx$ and we can try a homogenous solution

$$P = u^k F(z), \tag{63}$$

$$z = -\frac{x^4}{4u^2}, \tag{64}$$

where k is the degree of homogeneity. Then Eq. (20) becomes

$$z(z-1)F_{zz} + \left[\left(\frac{3}{2} - k\right) z - \frac{3}{4}\right] F_z + \frac{k(k-1)}{4} F = 0. \tag{65}$$

Once again this is a hypergeometric equation and one of its fundamental solutions leads to

$$P = u^k F\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{3}{4}, z\right). \tag{66}$$

In fact, the hypergeometric function in Eq. (66) degenerates to a Legendre function for any k

$$P = 2^{-1/4} \Gamma\left(\frac{3}{4}\right) u^k (-z)^{1/8} (1-z)^{k/2 - 1/8} P^{1/4}_{-k-3/4}[(1-z)^{-1/2}]. \tag{67}$$

It becomes a rational function in some cases. Thus if $k = \frac{1}{4}$

$$P = \left(\frac{u}{2}\right)^{1/4} [1 + (1-z)^{1/2}]^{1/4}, \tag{68}$$

and if $k = -\frac{3}{4}$

$$P = \left(\frac{2}{u^3}\right)^{1/4} (1-z)^{-1/2} [1 + (1-z)^{1/2}]^{1/4}. \tag{69}$$

Exploiting only Eq. (64) one finds the following identities:

$$kP = uP_u + \frac{1}{2} x P_x, \tag{70}$$

$$P_{uu} = P_1 B_0^{-1}, \tag{71}$$

$$P_u + 2xP_{xu} = (4k-3)u^{-1}\left(kP - \frac{1}{2} x P_x\right) - 4uP_1 B_0^{-1}, \tag{72}$$

$$P_1 \equiv 4k(k-1)P - (4k-3)xP_x, \tag{73}$$

where $B_0 = 4B$. They hold for any solution of Eq. (65). By plugging Eqs. (70)–(73) into Eq. (21) one can study the properties of the energy density. It simplifies drastically when $k = \frac{3}{4}$:

$$P=u^{3/4}F\left(-\frac{3}{8},\frac{1}{8},\frac{3}{4},z\right), \tag{74}$$

$$\frac{1}{6}n^2=m_0P^{-1}P_u=\frac{m_0}{4u}\left[3-\frac{zF(\frac{5}{8},\frac{9}{8},\frac{7}{4},z)}{2F(-\frac{3}{8},\frac{1}{8},\frac{3}{4},z)}\right]. \tag{75}$$

When $0\leqslant|z|\leqslant1$ it can be checked with MAPLEV that the right-hand side of Eq. (75) is positive for $m_0>0$. In the region $1\leqslant|z|\leqslant\infty$ an analytic continuation of the hypergeometric functions should be performed. Thus

$$F\left(-\frac{3}{8},\frac{1}{8},\frac{3}{4},z\right)=A_1(-z)^{3/8}F\left(-\frac{3}{8},-\frac{1}{8},\frac{1}{2},\frac{1}{z}\right)+A_2(-z)^{-1/8}F\left(\frac{1}{8},\frac{3}{8},\frac{3}{2},\frac{1}{z}\right), \tag{76}$$

$$A_1=\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{8})\Gamma(\frac{9}{8})},$$

$$A_2=\frac{\Gamma(\frac{3}{4})\Gamma(-\frac{1}{2})}{\Gamma(-\frac{3}{8})\Gamma(\frac{5}{8})},$$

and similarly for the other hypergeometric function in Eq. (75). A check with MAPLEV confirms again the positivity of n^2. The energy density is regular in u and z. When $m_0=0$ a vacuum solution is obtained. It has $\Psi_2=0$ and then the higher terms in Eq. (42) vanish [1]. From Eq. (44) it follows that

$$I=\frac{3}{4(x^2-2iu)} \tag{77}$$

and $\partial I=0$, $I_u\neq0$. Therefore $\Psi_3=0$, while $\Psi_4\neq0$ and the solution is of type N. In fact, this is the Hauser solution [1,19] in the $L_u=0$ gauge.

Let us show this in detail. We have used the transformation (13) to bring ϕ to the simple form (14) [with $m_0=1$ since the field is of type N] and then have dropped the primes. Restoring them, Eqs. (12)–(14) show that

$$f=u'=Cx^2u, \tag{78}$$

where C is an arbitrary constant which we fix to $C=1/2$. Under transformation (12) P and L change as

$$P'=f_u^{-1}P, \tag{79}$$

$$L'=f_uL-\frac{1}{\sqrt{2}}f_x, \tag{80}$$

[see Eq. (25.27) from Ref. [1]]. P' is given by Eq. (74) while $L'=ix/\sqrt{2}$. Then the original P and L are given by

$$P=x^{7/2}F(u), \tag{81}$$

$$L=\frac{\sqrt{2}}{x}(u+i), \tag{82}$$

$$F(u)=\frac{1}{2}A_1F\left(-\frac{3}{8},-\frac{1}{8},\frac{1}{2},-u^2\right)+\frac{1}{4}A_2uF\left(\frac{1}{8},\frac{3}{8},\frac{3}{2},-u^2\right). \tag{83}$$

We have used formula (76) to derive the expression for $F(u)$. Equations (81)–(83) give exactly the Hauser solution [19] as written out in Refs. [1] and [20]. Equation (83) is a specific linear combination of the even and odd solutions given in Ref. [20] which fixes the one-parameter freedom of the Hauser solution. This ends the proof of our assertion.

VI. CONCLUSION

We have shown that when the NUT parameter M vanishes and the gauge $L_u=0$ is used, the main equation (20) for expanding pure radiation fields with a simple twist and a special symmetry becomes a tractable second order linear equation for P. It is reducible to a case of the Euler-Darboux equation, somewhat different from the central equation in the theory of aligned colliding plane waves. We have found regular solutions, separating the variables in different coordinate systems. In some cases the regions of positivity of the energy density were investigated. Another analogy with the Euler-Tricomi equation, appearing in the hydrodynamics of two-dimensional fluid flow, has been exploited to find homogenous solutions. Interestingly enough, the Hauser vacuum solution of type N is an exceptional member of the family of type II solutions with degree of homogeneity $\frac{3}{4}$.

ACKNOWLEDGMENTS

This work was supported by the Bulgarian National Fund for Scientific Research under Contract No. F-632.

[1] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
[2] H. Stephani, Gen. Relativ. Gravit. **15**, 173 (1983).
[3] J. Lewandowski, P. Nurowski, and J. Tafel, Class. Quantum Grav. **8**, 493 (1991).
[4] J. Tafel, P. Nurowski, and J. Lewandowski, Class. Quantum Grav. **8**, L83 (1991).
[5] A. Grundland and J. Tafel, Class. Quantum Grav. **10**, 2337 (1993).
[6] D. Kramer and U. Hähner, Class. Quantum Grav. **12**, 2287 (1995).
[7] U. von der Gönna and D. Kramer, Class. Quantum Grav. **15**, 2017 (1998).
[8] H. Stephani, J. Phys. A **12**, 1045 (1979).
[9] E. Herlt and H. Stephani, Class. Quantum Grav. **1**, 95 (1984).

[10] B. V. Ivanov, Phys. Rev. D **60**, 104005 (1999).
[11] B. V. Ivanov, J. Math. Phys. **40**, 5849 (1999).
[12] E. Holmgren, Ark. Mat., Astron. Fys. B **19**, 1 (1926).
[13] S. Gellerstedt, Ark. Mat., Astron. Fys. A **25**, 1 (1936).
[14] J. B. Griffiths, *Colliding Plane Waves in General Relativity* (Clarendon, Oxford, 1991).
[15] D. W. Trim and J. Wainwright, J. Math. Phys. **15**, 535 (1974).
[16] U. Yurtsever, Phys. Rev. D **37**, 2790 (1988).
[17] L. D. Landau and E. M. Lifshitz, *Hydrodynamics* (Nauka, Moscow, 1988).
[18] R. Mises, *Mathematical Theory of Compressible Fluid Flow* (Academic, New York, 1958).
[19] I. Hauser, Phys. Rev. Lett. **33**, 1098 (1974).
[20] D. E. Novoseller, Phys. Rev. D **11**, 2330 (1975).