Some cubic couplings in type IIB supergravity on $AdS_5 \times S^5$ and three-point functions in four-dimensional super Yang-Mills theory at large N

G. Arutyunov*
Sektion Physik, Munich University, Theresienstrasse 37, D-80333 Munich, Germany

S. Frolov†
Department of Physics and Astronomy, University of Alabama, Box 870324, Tuscaloosa, Alabama 35487-0324
(Received 27 September 1999; published 23 February 2000)

All cubic couplings in type IIB supergravity on $AdS_5 \times S^5$ that involve two scalar fields s^I that are mixtures of the five form field strength on S^5 and the trace of the graviton on S^5 are derived by using the covariant equations of motion and the quadratic action for type IIB supergravity on $AdS_5 \times S^5$. All corresponding three-point functions in SYM_4 are calculated in the supergravity approximation. It is pointed out that the scalars s^I correspond not to the chiral primary operators in the $\mathcal{N}=4$ SYM but rather to a proper extension of the operators.

PACS number(s): 04.50.+h, 11.25.Hf

I. INTRODUCTION

According to the AdS-conformal field theory (CFT) correspondence [1–3], the generating functional of Green functions in $D=4$, $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM_4) at large N and at strong 't Hooft coupling λ coincides with the on-shell value of the type IIB supergravity action on $AdS_5 \times S^5$. For this reason, to calculate an n-point Green function one has to know the supergravity action up to the nth order. In particular, the normalization constants of two- and three-point Green functions [4–25] are determined by the quadratic and cubic actions for physical fields of supergravity.

The particle spectrum of type IIB supergravity on $AdS_5 \times S^5$ [26,27] contains scalar fields s^I that are mixtures of the five form field strength on S^5 and the trace of the graviton on S^5. The transformation properties of the scalars with respect to the superconformal group of SYM_4 allow one to conclude that they correspond to chiral primary operators (CPOs) of SYM_4. In [12] the quadratic and cubic actions for the scalars s^I have been found and used to calculate all three-point functions of normalized CPOs. These three-point functions appeared to coincide with the three-point functions of CPOs computed in free field theory for generic values of conformal dimensions of CPOs. However, there is an apparent contradiction. As was noted in [28] (see also [25]) a three-point function of CPOs calculated in the AdS-CFT framework vanishes, if the sum of conformal dimensions of any of the two operators equals the conformal dimension of the third operator, because of the vanishing of the cubic couplings of the corresponding scalar fields. Thus we are forced to conclude that the scalars s^I used in [12] cannot correspond to CPOs. Another way to come to the conclusion is that the scalars from [12] do not coincide with the original scalars that are mixtures of the five-form and the graviton but depend nonlinearly on the original scalars and their derivatives. Thus the scalars used in [12] do not transform with respect to the superconformal group in a proper way and cannot correspond to CPOs.

In this paper we show that a scalar s^I used in [12] corresponds to an operator which is the sum of a CPO and non-chiral composite operators. The non-chiral operators are normal-ordered products of CPOs and their descendants, i.e., so-called double- and multi-trace operators.

The knowledge of correlation functions of the chiral primary operators allows one to compute correlation functions of all their descendants, in particular, the correlation functions of the stress energy tensor and R-symmetry currents. To compute four-point functions[1] of the chiral operators one has to know the s^I-dependent quartic terms and all cubic terms that involve two scalar fields s^I. In the present paper, as the first step in this direction, we determine all such cubic terms. It is sufficient to consider only the sector of type IIB supergravity that depends on the graviton and the four-form potential. There are four different types of vertices describing interaction of two scalars s^I with symmetric tensor fields of the second rank coming from the AdS_5 components of the graviton, with vector fields, with scalar fields coming from the S^5 components of the graviton, and with scalar fields t^I that are mixtures of the trace of the graviton on the sphere and the five form field strength on the sphere.

To this end we apply an approach similar to the one used in [12]. Namely, we use the quadratic action for type IIB supergravity on $AdS_5 \times S^5$ recently obtained in [40] and the covariant equations of motion of [41–43]. Just as it was in the case of cubic couplings of three scalars s^I [12], to get rid

*On leave of absence from Steklov Mathematical Institute, Gubkin str. 8, GSP-1, 117966, Moscow, Russia. Email address: arut@theorie.physik.uni-muenchen.de

†On leave of absence from Steklov Mathematical Institute, Gubkin str. 8, GSP-1, 117966, Moscow, Russia. Email address: frolov@bama.ua.edu

[1]Some results on four-point functions have been obtained in [28–39].

of higher-derivative terms we will have to redefine the original gravity fields. Thus the fields entering the final action correspond not to descendants of CPOs but to extended operators involving products of CPOs and their descendants. However, we expect that for generic values of conformal dimensions of these operators, their three-point functions coincide with the three-point functions of the corresponding descendants of CPOs. Let us note in passing that the only way to find an action depending on the fields that correspond directly to CPOs and their descendants seems to be to derive the action starting from the covariant action of [44,45]. In this way one probably should obtain a nonvanishing cubic couplings of scalars s^I corresponding to CPOs whose conformal dimensions satisfy the relation $\Delta_1+\Delta_2=\Delta_3$. These cubic terms seem to be of the form suggested in [28]. Unfortunately, the lack of covariance of the gauge-fixed action of [44,45] makes the analysis extremely complicated.

The paper is organized as follows. In Sec. II we suggest the operators that correspond to the scalars s^I from [12]. In Sec. III we recall equations of motion for the graviton and the four-form potential, and the quadratic actions for the fields under consideration, and introduce notations. In Sec. IV we obtain cubic couplings of two scalars s^I with a scalar t^I, and with scalars ϕ^I coming from the graviton on the sphere, and calculate their three-point functions by using results obtained in [7]. In Sec. V cubic couplings of two scalars s^I with symmetric second rank tensor fields are derived and the corresponding three-point functions are found. In Sec. VI we obtain cubic vertices of two scalars s^I and a vector field, and calculate their three-point functions. Note that three-point functions of two scalars with a massive vector field, or a massive symmetric second rank tensor, were not considered in the literature before. In the Conclusion we discuss the results obtained, and open problems. In the Appendix we recall the definitions of scalar, vector and tensor spherical harmonics.

II. EXTENDED CHIRAL PRIMARY OPERATORS

In this section we recall the definition of chiral primary operators and introduce a notion of extended chiral primary operators.

According to [12], CPOs have the form

$$O^I(\vec{x})=\frac{(2\pi)^k}{\sqrt{k\lambda^k}}C^I_{i_1\cdots i_k}\,\mathrm{tr}\,(:\phi^{i_1}(\vec{x})\cdots\phi^{i_k}(\vec{x}):), \qquad (1)$$

where $C^I_{i_1\cdots i_k}$ are totally symmetric traceless rank k orthonormal tensors of $SO(6)$: $\langle C^I C^J\rangle = C^I_{i_1\cdots i_k}C^J_{i_1\cdots i_k}=\delta^{IJ}$, ϕ^i are scalars of SYM_4, and $:A_1\cdots A_n:$ means the normal-ordered product of the operators A_i.

The two- and three-point functions of CPOs computed in free theory are [12]

$$\langle O^I(\vec{x})O^J(\vec{y})\rangle=\frac{\delta^{IJ}}{|\vec{x}-\vec{y}|^{2k}}, \qquad (2)$$

$$\langle O^{I_1}(\vec{x})O^{I_2}(\vec{y})O^{I_3}(\vec{z})\rangle=\frac{1}{N}\frac{\sqrt{k_1k_2k_3}\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{|\vec{x}-\vec{y}|^{2\alpha_3}|\vec{y}-\vec{z}|^{2\alpha_1}|\vec{z}-\vec{x}|^{2\alpha_2}}, \qquad (3)$$

where $\alpha_i=\frac{1}{2}(k_j+k_l-k_i)$, $j\neq l\neq i$, and $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ is the unique $SO(6)$ invariant obtained by contracting α_1 indices between C^{I_2} and C^{I_3}, α_2 indices between C^{I_3} and C^{I_1}, and α_3 indices between C^{I_2} and C^{I_1}. According to the AdS-CFT conjecture, there should exist fields of type IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ that correspond to CPOs. The transformation properties of CPOs and supergravity fields with respect to the superconformal group of SYM_4 show that these fields seem to be scalar fields s^I, that are mixtures of the five form field strength on S^5 and the trace of the graviton on S^5.[2] To calculate the three-point functions of CPOs in the framework of the AdS-CFT correspondence the quadratic and cubic actions for the scalars s^I were found in [12]. Then, it was shown that for generic values of conformal dimensions of CPOs the normalized three-point functions computed using the actions precisely coincide with the free field theory result (3). On the other hand, as was pointed out in [28] the cubic couplings of scalars s^I satisfying one of the three relations:

$$k_1+k_2=k_3,\quad k_2+k_3=k_1,\quad k_3+k_1=k_2, \qquad (4)$$

vanish, and, therefore, the three-point functions of the operators corresponding to scalars s^I vanish too. Thus, scalars s^I used in [12] do not correspond to CPOs. We can explain this by noting that the scalars s^I from [12] differ from the original scalars that are mixtures of the graviton and the five-form on S^5. The original scalars s^I satisfy equations which depend on higher-derivative terms. To remove the derivative terms the following field redefinition was made in [12]

$$s^{I_1}=s'^{I_1}+\sum_{I_2,I_3}(J_{I_1I_2I_3}s'^{I_2}s'^{I_3}+L_{I_1I_2I_3}\nabla^a s'^{I_2}\nabla_a s'^{I_3}). \qquad (5)$$

Namely for the scalars s'^I the cubic couplings mentioned above vanish. Because of the redefinition (5) new scalars s'^I do not transform with respect to the superconformal group in a proper way, and, therefore, cannot correspond to CPOs.

From the computational point of view these cubic couplings have to vanish because if, say, $k_1+k_2=k_3$ then the three-point function (3) is nonsingular at $x=y$, but gravity calculations with a nonvanishing on-shell bulk cubic coupling always lead to a function singular at $x=y$, $x=z$ and $y=z$. By the same reason we expect that n-point functions of operators corresponding to scalars s'^I [with an additional field redefinition which is required to remove higher-

[2] Strictly speaking this correspondence between CPOs and scalars s^I may be valid only at linear order in supergravity fields. The reason is that the local supersymmetry transformations of supergravity fields are nonlinear, and, one should expect that the induced superconformal transformations are nonlinear too. Thus the original gravity fields seem to depend nonlinearly on fields with the linear transformation law.

derivative terms from the $(n-1)$th order equations of motion for s^I] would vanish if, say, $k_n = k_1 + \cdots + k_{n-1}$. Study of the general scalar exchange performed in [28] seems to confirm the conclusion.

Thus, scalars s^I (here and in what follows we omit the primes on redefined fields) correspond to properly extended CPOs which have vanishing three-point functions if Eq. (4) is fulfilled. Indeed one can easily find such an extension of CPOs. Namely, we define the extended CPOs that correspond to scalars s^I as

$$\tilde{O}^{I_1}(\vec{x}) = O^{I_1}(\vec{x}) - \frac{1}{2N} \sum_{I_2+I_3=I_1} C^{I_1 I_2 I_3} : O^{I_2}(\vec{x}) O^{I_3}(\vec{x}) :, \tag{6}$$

where $C^{I_1I_2I_3} = \sqrt{k_1k_2k_3}\langle C^{I_1}C^{I_2}C^{I_3}\rangle$. It is not difficult to verify that in the large N limit these operators have the normalized two-point functions (2), the three-point functions (3) if Eq. (4) is not satisfied, and vanishing three-point functions if Eq. (4) takes place. Indeed, for generic values of conformal dimensions the second term on the right-hand side (RHS) of Eq. (6) gives a contribution of order $1/N^2$ to a three-point function. Only if one of the relations Eq. (4) is fulfilled, e.g. $k_1 = k_2 + k_3$ the correlator

$$\frac{1}{N} C^{I_1I_2I_3} \langle O^{I_1}(\vec{x}) : O^{I_2}(\vec{y}) O^{I_3}(\vec{y}) : \rangle$$

does not vanish, and gives exactly the same function as in Eq. (3).

However, these operators will require a further modification to be consistent with all n-point functions computed in the framework of the AdS-CFT correspondence. In general, an extended CPO is the sum of a CPO and non-chiral composite operators which are normal-ordered products of CPOs and their descendants. Nevertheless, we expect that in the large N limit an n-point function of extended CPOs coincides with n-point functions of CPOs for generic values of conformal dimensions of the operators. As we will discuss in the following sections a similar modification is required for operators corresponding to other supergravity fields.

III. EQUATIONS OF MOTION AND QUADRATIC ACTIONS

To obtain cubic couplings of two scalars s^I with other type IIB supergravity fields it is sufficient to consider only the graviton and the four-form potential. To this end we apply the method of [12], and use the covariant equations of motion [41–43] and the quadratic action for type IIB supergravity on $AdS_5 \times S^5$ [40]. The equations of motion of the 4-form potential and the graviton are

$$F_{M_1 \ldots M_5} = \frac{1}{5!} \varepsilon_{M_1 \ldots M_{10}} F^{M_6 \ldots M_{10}}, \tag{7}$$

$$R_{MN} = \frac{1}{3!} F_{MM_1 \ldots M_4} F_N^{M_1 \ldots M_4}. \tag{8}$$

Here $M,N,\ldots,=0,1,\ldots 9$ and we use the following notations:

$$F_{M_1 \ldots M_5} = 5\partial_{[M_1} A_{M_2 \ldots M_5]} = \partial_{M_1} A_{M_2 \ldots M_5} + 4 \text{ terms},$$

i.e., all antisymmetrizations are with ''weight''1. The dual forms are defined as

$$\varepsilon_{01\ldots 9} = \sqrt{-G}; \quad e^{01\ldots 9} = -\frac{1}{\sqrt{-G}}$$

$$\varepsilon^{M_1 \ldots M_{10}} = G^{M_1N_1} \cdots G^{M_{10}N_{10}} \varepsilon_{N_1 \ldots N_{10}}$$

$$(F^*)_{M_1 \ldots M_k} = \frac{1}{k!} \varepsilon_{M_1 \ldots M_{10}} F^{M_{k+1} \ldots M_{10}}$$

$$= \frac{1}{k!} \varepsilon^{N_1 \ldots N_{10}} G_{M_1N_1} \cdots G_{M_kN_k} F_{N_{k+1} \ldots N_{10}}.$$

In the units in which the radius of S^5 is set to be unity, the $AdS_5 \times S^5$ background solution looks as

$$ds^2 = \frac{1}{x_0^2}(dx_0^2 + \eta_{ij} dx^i dx^j) + d\Omega_5^2 = g_{MN} dx^M dx^N$$

$$R_{abcd} = -g_{ac}g_{bd} + g_{ad}g_{bc}; \quad R_{ab} = -4g_{ab}$$

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}; \quad R_{\alpha\beta} = 4g_{\alpha\beta}$$

$$\bar{F}_{abcde} = \varepsilon_{abcde}; \quad \bar{F}_{\alpha\beta\gamma\delta\varepsilon} = \varepsilon_{\alpha\beta\gamma\delta\varepsilon}, \tag{9}$$

where $a,b,c,\ldots$ and $\alpha,\beta,\gamma,\ldots$ are the AdS and the sphere indices respectively and η_{ij} is the 4-dimensional Minkowski metric. We represent the gravitational field and the 4-form potential as

$$G_{MN} = g_{MN} + h_{MN}; \quad A_{MNPQ} = \bar{A}_{MNPQ} + a_{MNPQ};$$

$$F = \bar{F} + f.$$

Then the self-duality equation (7) decomposed up to the second order looks as

$$f-f^{*}+T^{(1)}+T(h,f^{*})+T(h)=0. \tag{10}$$

Here we introduced the following notations:

$$T^{(1)}_{M_1\ldots M_5}=\frac{1}{2}h\bar{F}_{M_1\ldots M_5}-5h^{K}_{[M_1}\bar{F}_{M_2\ldots M_5]K},$$

$$h=h^{K}_{K}$$

$$T_{M_1\ldots M_5}(h,f^{*})=\frac{1}{2}hf^{*}_{M_1\ldots M_5}-5h^{K}_{[M_1}f^{*}_{M_2\ldots M_5]K}$$

$$T_{M_1\ldots M_5}(h)=\frac{5}{2}hh^{K}_{[M_1}\bar{F}_{M_2\ldots M_5]K}-\left(\frac{1}{8}h^2+\frac{1}{4}h^{ML}h_{ML}\right)\bar{F}_{M_1\ldots M_5}-10h^{K_1}_{[M_1}h^{K_2}_{M_2}\bar{F}_{M_3M_4M_5]K_1K_2}. \tag{11}$$

Decomposing the Einstein equation (8) up to the second order, we get

$$R^{(1)}_{MN}+R^{(2)}_{MN}=-\frac{4}{3!}h^{KL}\bar{F}_{MKM_1M_2M_3}\bar{F}^{M_1M_2M_3}_{NL}+\frac{1}{3!}(f_{MM_1\ldots M_4}\bar{F}^{M_1\ldots M_4}_{N}+\bar{F}_{MM_1\ldots M_4}f^{M_1\ldots M_4}_{N})+\frac{4}{3!}h^{KL}h^{S}_{L}\bar{F}_{MKM_1M_2M_3}\bar{F}^{M_1M_2M_3}_{NS}+\frac{2\times 3}{3!}h^{K_1S_1}h^{K_2S_2}\bar{F}_{MK_1K_2M_1M_2}\bar{F}^{M_1M_2}_{NS_1S_2}-\frac{4}{3!}h^{KS}\times(f_{MKM_1M_2M_3}\bar{F}^{M_1M_2M_3}_{NS}+f_{NKM_1M_2M_3}\bar{F}^{M_1M_2M_3}_{MS})+\frac{1}{3!}f_{MM_1\ldots M_4}f^{M_1\ldots M_4}_{N}. \tag{12}$$

Here

$$R^{(1)}_{MN}=\nabla_K h^{K}_{MN}-\frac{1}{2}\nabla_M\nabla_N h^{L}_{L}$$

$$R^{(2)}_{MN}=-\nabla_K(h^{K}_{L}h^{L}_{MN})+\frac{1}{2}\nabla_N(h_{KL}\nabla_M h^{KL})+\frac{1}{2}h^{K}_{MN}\nabla_K h^{L}_{L}-h^{L}_{MK}h^{K}_{NL} \tag{13}$$

and we introduce a notation

$$h^{K}_{MN}=\frac{1}{2}(\nabla_M h^{K}_{N}+\nabla_N h^{K}_{M}-\nabla^{K}h_{MN}). \tag{14}$$

In Eqs. (10)–(14) and in what follows indices are raised and lowered by means of the background metric, and the covariant derivatives are with respect to the background metric, too.

The gauge symmetry of the equations of motion allows one to impose the de Donder gauge:

$$\nabla^{\alpha}h_{a\alpha}=\nabla^{\alpha}h_{(\alpha\beta)}=\nabla^{\alpha}a_{M_1M_2M_3\alpha}=0;\qquad h_{(\alpha\beta)}\equiv h_{\alpha\beta}-\frac{1}{5}g_{\alpha\beta}h^{\gamma}_{\gamma}. \tag{15}$$

This gauge choice does not remove all the gauge symmetry of the theory, for a detailed discussion of the residual symmetry see [26]. As was shown in [26], the gauge condition (15) implies that the components of the 4-form potential of the form $a_{\alpha\beta\gamma\delta}$ and $a_{a\alpha\beta\gamma}$ can be represented as follows:

$$a_{\alpha\beta\gamma\delta}=\varepsilon_{\alpha\beta\gamma\delta\varepsilon}\nabla^{\varepsilon}b;\quad a_{a\alpha\beta\gamma}=\varepsilon_{\alpha\beta\gamma\delta\varepsilon}\nabla^{\delta}\phi^{\varepsilon}_{a}. \tag{16}$$

It is also convenient to introduce the dual 1- and 2-forms for a_{abcd} and $a_{abc\alpha}$:

$$a_{abcd}=-\varepsilon_{abcde}Q^{e};\quad a_{abc\alpha}=-\varepsilon_{abcde}\phi^{de}_{\alpha}. \tag{17}$$

Then the solution of the first-order self-duality equation can be written as

$$Q^{a}=\nabla^{a}b,\quad \phi^{ab}_{\alpha}=\nabla^{[a}\phi^{b]}_{\alpha}. \tag{18}$$

The quadratic action for physical fields of type IIB supergravity was found in [40]. To write down the action we need to expand fields in spherical harmonics, and make some fields redefinition. We begin with the scalar fields b and $\pi\equiv h^{\alpha}_{\alpha}$. Expanding them into a set of scalar spherical harmonics[3]

$$\pi(x,y)=\sum \pi^{I_1}(x)Y^{I_1}(y);$$

$$b(x,y)=\sum b^{I_1}(x)Y^{I_1}(y);$$

[3] Here and in what follows we suppose that the spherical harmonics of all types are orthonormal.

$$\nabla_\beta^2 Y^k = -k(k+4)Y^k,$$

and making the fields redefinition [12][4]

$$\pi_k = 10ks_k + 10(k+4)t_k; \quad b_k = -s_k + t_k \tag{19}$$

we write the quadratic actions for the scalars s^I and t^I in the form

$$S(s) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum \frac{32k(k-1)(k+2)}{k+1} \times\left(-\frac{1}{2}\nabla_a s_k \nabla^a s_k - \frac{1}{2}k(k-4)s_k^2\right), \tag{20}$$

$$S(t) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum \frac{32(k+2)(k+4)(k+5)}{k+3} \times\left(-\frac{1}{2}\nabla_a t_k \nabla^a t_k - \frac{1}{2}(k+4)(k+8)t_k^2\right). \tag{21}$$

Now we expand the graviton on AdS_5 in scalar spherical harmonics

$$h_{ab}(x,y) = \sum h_{ab}^{I_1}(x)Y^{I_1}(y)$$

and make the following shift of the gravitational fields:

$$h_{ab}^k = \phi_{(ab)}^k + \nabla_{(a}\nabla_{b)}\zeta_k + \frac{1}{5}g_{ab}\left(\phi_{ck}^c - \frac{3}{5}\pi_k\right), \tag{22}$$

where

$$\zeta_k = \frac{4}{k+1}s_k + \frac{4}{k+3}t_k. \tag{23}$$

Then the zero mode $\phi_{ab}^0 \equiv \phi_{ab}$ describes a graviton on AdS_5 with the standard action

$$S(\phi_{ab}) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\left(-\frac{1}{4}\nabla_c\phi_{ab}\nabla^c\phi^{ab} + \frac{1}{2}\nabla_a\phi^{ab}\nabla^c\phi_{cb} - \frac{1}{2}\nabla_a\phi_c^c\nabla_b\phi^{ba} + \frac{1}{4}\nabla_c\phi_a^a\nabla^c\phi_b^b + \frac{1}{2}\phi_{ab}\phi^{ab} + \frac{1}{2}(\phi_a^a)^2\right) \tag{24}$$

and the action for the traceless symmetric tensor fields ϕ_{ab}^k has the form

$$S(\phi_{(ab)}^k) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum\left(-\frac{1}{4}\nabla_c\phi_{(ab)}^k\nabla^c\phi_k^{(ab)} + \frac{1}{2}\nabla_a\phi_k^{(ab)}\nabla^c\phi_{(cb)}^k - \frac{1}{4}(k^2+4k-2)\phi_{(ab)}^k\phi_k^{(ab)}\right). \tag{25}$$

As was shown in [26] the fields ϕ_{ck}^c are nondynamical and vanish on shell at the linearized level.

Expanding vector fields $h_{a\alpha}$ and $\phi_{a\alpha}$ into a set of vector spherical harmonics

$$h_{a\alpha}(x,y) = \sum hh_a^{I_5}(x)Y_\alpha^{I_5}(y);$$

$$\phi_{a\alpha}(x,y) = \sum \phi_a^{I_5}(x)Y_\alpha^{I_5}(y);$$

$$(\nabla_\beta^2 - 4)Y_\alpha^k = -(k+1)(k+3)Y_\alpha^k,$$

and making the change of variables [26]

$$A_a^k = h_a^k - 4(k+3)\phi_a^k; \quad C_a^k = h_a^k + 4(k+1)\phi_a^k \tag{26}$$

we present the actions for the vector fields in the form

$$S(A) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum\frac{k+1}{2(k+2)}\left(-\frac{1}{4}(F_{ab}(A^k))^2 - \frac{1}{2}(k^2-1)(A_a^k)^2\right) \tag{27}$$

$$S(C) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum\frac{k+3}{2(k+2)}\left(-\frac{1}{4}(F_{ab}(C^k))^2 - \frac{1}{2}(k+3)(k+5)(C_a^k)^2\right), \tag{28}$$

where $F_{ab}(A) = \partial_a A_b - \partial_b A_a$. Finally, expanding the graviton on the sphere in tensor harmonics

$$h_{(\alpha\beta)}(x,y) = \sum \phi^{I_{14}}(x)Y_{(\alpha\beta)}^{I_{14}}(y);$$

$$(\nabla_\gamma^2 - 10)Y_{(\alpha\beta)}^k = -(k^2+4k+8)Y_{(\alpha\beta)}^k,$$

we write the action for the scalars ϕ_k in the form

$$S(\phi) = \frac{4N^2}{(2\pi)^5}\int d^5x\sqrt{-g_a}\sum\left(-\frac{1}{4}\nabla_a\phi_k\nabla^a\phi_k - \frac{1}{4}k(k+4)\phi_k^2\right). \tag{29}$$

[4]We often denote π^{I_1} as π_k and a similar notation for other fields.

IV. CUBIC COUPLINGS OF SCALARS

The aim of this section is to find the cubic couplings of the scalar fields t_k and ϕ_k with a pair of scalars s_k. This can be achieved by finding the quadratic contribution of the scalars s_k to the equations of motion for t_k and ϕ_k respectively with a subsequent reconstruction of the corresponding Lagrangian vertex.

A. Cubic couplings of t_k

Since t_k appear as the mixture of fields π_k and b_k we begin by considering the equations of motion for these fields. Restricting in Eq. (13) indices M and N to the sphere and taking into account the gauge conditions (15), (16) we find that Einstein equation (13) results in

$$\frac{1}{10}g_{\alpha\beta}((\nabla_M\nabla^M-32)\pi+80\nabla_\gamma\nabla^\gamma b)+\frac{1}{2}\nabla_\alpha\nabla_\beta\phi_a^a=+\frac{1}{10}g_{\alpha\beta}\nabla_a(h^{ab}\nabla_b\pi)+\frac{3}{100}\nabla_\alpha\pi\nabla_\beta\pi+\frac{3}{50}\pi\nabla_\alpha\nabla_\beta\pi$$
$$+\frac{1}{4}\nabla_\alpha h_{ab}\nabla_\beta h^{ab}+\frac{1}{2}h_{ab}\nabla_\alpha\nabla_\beta h^{ab}+8\nabla_\alpha\nabla^a b\nabla_\beta\nabla_a b-4g_{\alpha\beta}\left(\nabla_\gamma\nabla^a b\nabla^\gamma\nabla_a b+\nabla_\gamma^2 b\nabla_\delta^2 b+\frac{2}{5}\pi^2-\frac{8}{5}\pi\nabla_\gamma^2 b\right.$$
$$\left.-\frac{1}{200}\nabla_\gamma(\pi\nabla^\gamma\pi)\right), \tag{30}$$

where $\phi_a^a=h_a^a+\frac{3}{5}\pi$ in accordance with Eq. (22). Note that we have omitted all the linear terms that are projected out under the projection onto the spherical harmonics $\nabla_{(\alpha}\nabla_{\beta)}Y^I$ or Y^I and accounted only for the quadratic terms that contain after the field redefinition (19) and (22) two scalars s_k. In particular the scalars s_k appear after redefinition (22) for the gravitational field h_{ab}.

Equation (30) implies then the following two equations:

$$\nabla_{(\alpha}\nabla_{\beta)}\phi_a^a=\frac{3}{50}\nabla_{(\alpha}\pi\nabla_{\beta)}\pi+\frac{3}{25}\pi\nabla_{(\alpha}\nabla_{\beta)}\pi+\frac{1}{2}\nabla_{(\alpha}h_{ab}\nabla_{\beta)}h^{ab}+h_{ab}\nabla_{(\alpha}\nabla_{\beta)}h^{ab}+16\nabla_{(\alpha}\nabla^a b\nabla_{\beta)}\nabla_a b \tag{31}$$

and

$$(\nabla_M\nabla^M-32)\pi+80\nabla_\gamma\nabla^\gamma b+\nabla_\alpha\nabla^\alpha\phi_a^a=\nabla_a(h^{ab}\nabla_b\pi)+\frac{13}{50}\nabla_\alpha\pi\nabla^\alpha\pi+\frac{8}{25}\pi\nabla_\alpha\nabla^\alpha\pi+\frac{1}{2}\nabla_\alpha h_{ab}\nabla^\alpha h^{ab}$$
$$+h_{ab}\nabla_\alpha\nabla^\alpha h^{ab}-24\nabla_\alpha\nabla^a b\nabla^\alpha\nabla_a b-40\nabla_\gamma^2 b\nabla_\delta^2 b-16\pi^2+64\pi\nabla_\gamma^2 b \tag{32}$$

that are obtained by decoupling from Eq. (30) the trace part. Projecting Eq. (31) onto $\nabla_\alpha\nabla_\beta Y^I$ one can solve it for ϕ_a^a and substituting the result in Eq. (32) obtain the close equation for π and b.

According to [26] the second equation involving the fields π and b is found by considering the component of the self-duality equation (10) involving one sphere and four AdS indices, and the component with five AdS indices. In our case these components read as

$$\nabla_\alpha(a_{a_1\ldots a_4}+\varepsilon_{a_1\ldots a_5}\nabla^{a_5}b)=\varepsilon_{a_1\ldots a_4 a}\left(\frac{3}{5}\pi\nabla^a\nabla_\alpha b+h^{ab}\nabla_b\nabla_\alpha b\right) \tag{33}$$

and

$$5\nabla_{[a_1}a_{a_2\ldots a_5]}=\varepsilon_{a_1\ldots a_5}\left(\nabla_\gamma^2 b+\frac{1}{2}\phi_a^a-\frac{4}{5}\pi-\frac{4}{5}\pi\nabla_\gamma^2 b-\frac{1}{4}h_{ab}h^{ab}+\frac{37}{100}\pi^2\right). \tag{34}$$

Projecting Eq. (33) onto $\nabla_\alpha Y^I$ one finds $a_{a_1\ldots a_5}$. Substituting then $a_{a_1\ldots a_5}$ as well as previously found ϕ_a^a into Eq. (34) one obtains the equation for π and b.

The required equation for t_k is then obtained by substituting the redefinition (19) in Eqs. (32)–(34) and by eliminating all the terms linear in s_k. Skipping all the computational details we write down the equation for t^I that is found to be of the form

$$(\nabla_a\nabla^a-(k_3+4)(k_3+8))t^{I_3}=D_{123}s^{I_1}s^{I_2}+E_{123}\nabla^a s^{I_1}\nabla_a s^{I_2}+F_{123}\nabla_{(a}\nabla_{b)}s^{I_1}\nabla^{(a}\nabla^{b)}s^{I_2}.$$

To remove the derivative terms we perform the appropriate redefinition of t^I similar to Eq. (5):

$$t^{I_3}=t'^{I_3}+\sum_{I_1,I_2}(J_{I_1I_2I_3}s'^{I_1}s'^{I_2}+L_{I_1I_2I_3}\nabla^a s'^{I_1}\nabla_a s'^{I_2}).$$

Introducing the notation $a_{123}=\int Y^{I_1}Y^{I_2}Y^{I_3}$ we quote the final answer

$$(\nabla_a\nabla^a-(k_3+4)(k_3+8))t^{I_3}=-t_{I_1I_2I_3}s^{I_2}s^{I_3},$$

$$t_{I_1I_2I_3}=a_{123}\frac{4(\Sigma+4)(\alpha_1+2)(\alpha_2+2)\alpha_3(\alpha_3-1)(\alpha_3-2)(\alpha_3-3)(\alpha_3-4)}{(k_1+1)(k_2+1)(k_3+2)(k_3+4)(k_3+5)},$$

where $\alpha_3=\frac{1}{2}(k_1+k_2-k_3)$, $\Sigma=k_1+k_2+k_3$.

Taking into account the normalization of the quadratic action for t_k fields (21) we obtain the corresponding vertex

$$S_{tss}=\frac{4N^2}{(2\pi)^5}T_{I_1I_2I_3}\int\sqrt{-g_a}s^{I_1}s^{I_2}t^{I_3}$$

with

$$T_{I_1I_2I_3}=a_{123}\frac{2^7(\Sigma+4)(\alpha_1+2)(\alpha_2+2)\alpha_3(\alpha_3-1)(\alpha_3-2)(\alpha_3-3)(\alpha_3-4)}{(k_1+1)(k_2+1)(k_3+3)}. \tag{35}$$

B. Cubic couplings of ϕ_k

To find equations of motion for the fields ϕ_k coming from the graviton on the sphere we again consider Eq. (13) for the indices $M=\alpha$, $N=\beta$:

$$(\nabla_M\nabla^M-2)h_{(\alpha\beta)}=\frac{3}{50}\nabla_{(\alpha}\pi\nabla_{\beta)}\pi+\frac{3}{25}\pi\nabla_{(\alpha}\nabla_{\beta)}\pi+\frac{1}{2}\nabla_{(\alpha}h_{ab}\nabla_{\beta)}h^{ab}+h_{ab}\nabla_{(\alpha}\nabla_{\beta)}h^{ab}+16\nabla_{(\alpha}\nabla^a b\nabla_{\beta)}\nabla_a b,$$

where this time all the linear terms that are projected out under the projection on $Y_{(\alpha\beta)}$ were omitted.

Introducing the notation $p_{123}=\int\nabla^\alpha Y^{I_1}\nabla^\beta Y^{I_2}Y^{I_3}_{(\alpha\beta)}$ and projecting both sides of the last equation on $Y_{(\alpha\beta)}$ we get an equation for ϕ:

$$(\nabla_a\nabla^a-k_3(k_3+4))\phi^{I_3}=p_{123}\left(-\frac{3}{50}\pi^{I_1}\pi^{I_2}-\frac{1}{2}h^{I_1}_{ab}h^{ab}_{I_2}+16\nabla^a b^{I_1}\nabla_a b^{I_2}\right).$$

Finally leaving on the RHS only the contribution of the scalars s_k we obtain

$$(\nabla_a\nabla^a-k_3(k_3+4))\phi^{I_3}=-\frac{p_{123}}{5(k_1+1)(k_2+1)}\times(48k_1k_2(k_1+1)(k_2+1)s^{I_1}s^{I_2}-80(k_1+1)(k_2+1)\nabla_a s^{I_1}\nabla^a s^{I_2}+40\nabla_{(a}\nabla_{b)}s^{I_1}\nabla^{(a}\nabla^{b)}s^{I_2}).$$

Performing again a shift of ϕ^I to get rid of the derivative terms one arrives at

$$(\nabla_a\nabla^a-k_3(k_3+4))\phi^{I_3}=-\frac{8p_{123}\Sigma(\Sigma+2)}{(k_1+1)(k_2+1)}(\alpha_3-1)(\alpha_3-2)s^{I_1}s^{I_2}.$$

Taking into account the normalization of the quadratic action for ϕ_k we can read off the corresponding vertex $S_{ss\phi}$:

$$S_{ss\phi}=\frac{4N^2}{(2\pi)^5}\Phi_{I_1I_2I_3}\int\sqrt{-g_a}s^{I_1}s^{I_2}\phi^{I_3}, \tag{36}$$

where

$$\Phi_{I_1I_2I_3}=\frac{4p_{123}\Sigma(\Sigma+2)}{(k_1+1)(k_2+1)}(\alpha_3-1)(\alpha_3-2).$$

C. Three-point functions

Recall that two- and three-point correlation functions of operators $\mathcal{O}_\Delta$ in a boundary conformal field theory corresponding to scalar fields on AdS are given by [7]

$$\langle \mathcal{O}_\Delta(\vec{x})\mathcal{O}_\Delta(\vec{y})\rangle = \frac{2}{\pi^2}\frac{\theta(\Delta-1)(\Delta-2)^2}{|\vec{x}-\vec{y}|^{2\Delta}}, \tag{37}$$

$$\langle \mathcal{O}_{\Delta_1}(\vec{x})\mathcal{O}_{\Delta_2}(\vec{y})\mathcal{O}_{\Delta_3}(\vec{z})\rangle = \frac{\lambda_{123}}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{x}-\vec{z}|^{\Delta_1+\Delta_3-\Delta_2}|\vec{y}-\vec{z}|^{\Delta_3+\Delta_2-\Delta_1}}, \tag{38}$$

where λ_{123} is given by

$$\lambda_{123} = -\varphi_{123}\frac{\Gamma\left[\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3-4)\right]\Gamma[\bar{\Delta}_1]\Gamma[\bar{\Delta}_2]\Gamma[\bar{\Delta}_3]}{2\pi^4\Gamma(\Delta_1-2)\Gamma(\Delta_2-2)\Gamma(\Delta_3-2)}$$

and $\bar{\Delta}_1=\frac{1}{2}(\Delta_2+\Delta_3-\Delta_1)$. Here φ_{123} stands for the coupling of scalar fields (that is a doubled interaction vertex for the fields we consider) and θ denotes the normalization constant of their quadratic action. Taking into account that a scalar t^{I_3} (ϕ^{I_3}) corresponds to a YM operator $\mathcal{O}_{\Delta_3}$ with the conformal weight $\Delta_3=k_3+8$ ($\Delta_3=k_3+4$), we, therefore find correlation functions of two extended CPOs with this operator. The constant λ_{123} reads for both cases as follows:

$$\lambda_{123}(t) = -\frac{4N^2}{(2\pi)^5}\frac{2^8}{\pi^4}\frac{\Gamma\left(\frac{1}{2}\Sigma+3\right)\Gamma(\alpha_1+4)\Gamma(\alpha_2+4)\Gamma(\alpha_3+1)(\alpha_1+2)(\alpha_2+2)}{(k_1+1)(k_2+1)(k_3+3)\Gamma(k_1-2)\Gamma(k_2-2)\Gamma(k_3+6)}a_{123}$$

and

$$\lambda_{123}(\phi) = -\frac{4N^2}{(2\pi)^5}\frac{2^4}{\pi^4}\frac{\Gamma\left(\frac{1}{2}\Sigma+2\right)\Gamma(\alpha_1+2)\Gamma(\alpha_2+2)\Gamma(\alpha_3)}{(k_1+1)(k_2+1)\Gamma(k_1-2)\Gamma(k_2-2)\Gamma(k_3+2)}p_{123}.$$

Taking into account the normalization of the two-point functions one can introduce the normalized extended CPO [12]:

$$O_\Delta = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{8(k-1)(k-2)}\left(\frac{k+1}{k(k+2)}\right)^{1/2}\mathcal{O}_\Delta \tag{39}$$

as well as the normalized gauge theory operator corresponding to scalar t_k:

$$O_\Delta = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{8(k+6)}\left(\frac{k+3}{(k+2)(k+4)(k+5)(k+7)}\right)^{1/2}\mathcal{O}_\Delta, \quad \Delta=k+8$$

and to scalar ϕ_k:

$$O_\Delta = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{(k+3)^{1/2}(k+2)}\mathcal{O}_\Delta, \quad \Delta=k+4.$$

With these formulas at hand we can finally write down the normalized constants:

$$\lambda_{123}^{norm}(t) = -\frac{(2\pi)^{5/2}}{N}\frac{1}{(2\pi)^{5/2}}\left(\frac{k_1k_2(k_3+1)(k_3+7)}{(k_3+3)(k_3+4)(k_3+5)}\right)^{1/2}\times\frac{\Gamma(\alpha_1+4)(\alpha_1+2)}{\alpha_1!}\frac{\Gamma(\alpha_2+4)(\alpha_2+2)}{\alpha_2!}\frac{k_3!}{\Gamma(k_3+8)}\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$$

and

$$\lambda_{123}^{norm}(\phi) = -\frac{(2\pi)^{5/2}}{N}\frac{(\alpha_1+1)(\alpha_2+1)}{4(2\pi)^{5/2}}\left(\frac{k_1k_2}{(k_3+1)(k_3+2)(k_3+3)}\right)^{1/2}P_{123}.$$

Here we used explicit expressions for a_{123} and p_{123} from the Appendix.

V. CUBIC COUPLINGS OF SECOND RANK TENSORS WITH S^I

A. Cubic couplings

Clearly the coupling of the symmetric second rank tensor $\phi^k_{(ab)}$ with a pair of scalars s_k can be found by studing the corrected equation of motion for $\phi^k_{(ab)}$. The most simple way consists however in finding the equations of motion for the field s_k corrected by the quadratic terms each containing one field $\phi^k_{(ab)}$ and s_k. This is explained by noting that the field $\phi^k_{(ab)}$ is transverse on-shell and therefore the interaction term, being in the latter case a Lorentz scalar does not contain derivatives acting on $\phi^k_{(ab)}$. As a consequence the additional shift needed to get rid of derivative terms is not required.

Since the field s_k appear as the mixture (19) of π and b, the equation for s_k again follows from the system (31)–(34). Clearly this time Eqs. (31) and (32) read as

$$\nabla_{(\alpha}\nabla_{\beta)}\phi^a_a=\nabla_{(\alpha}\phi_{(ab)}\nabla_{\beta)}\nabla^a\nabla^b\zeta+\phi_{(ab)}\nabla_{(\alpha}\nabla_{\beta)}\nabla^a\nabla^b\zeta+\nabla_a\nabla_b\zeta\nabla_{(\alpha}\nabla_{\beta)}\phi^{(ab)} \tag{40}$$

and

$$(\nabla_M\nabla^M-32)\pi+80\nabla_\gamma\nabla^\gamma b+\nabla_\alpha\nabla^\alpha\phi^a_a=\nabla_a(\phi^{(ab)}\nabla_b\pi)+\nabla_{(\alpha}\phi_{(ab)}\nabla_{\beta)}\nabla^a\nabla^b\zeta+\phi_{(ab)}\nabla_{(\alpha}\nabla_{\beta)}\nabla^a\nabla^b\zeta+\nabla_a\nabla_b\zeta\nabla_{(\alpha}\nabla_{\beta)}\phi^{(ab)} \tag{41}$$

where we have used representation (22) for the graviton field h_{ab} and left only the terms contributing to the vertex under consideration. By this reason the coefficients ζ_k in $\zeta=\int\zeta^I Y^I$ are reduced now to $\zeta_k=[4/(k+1)]s_k$ in comparison with Eq. (23).

Again projecting Eq. (40) onto $\nabla_\alpha\nabla_\beta Y^I$ one solves for ϕ^a_a and after substitution of the solution into Eq. (41) one obtains a closed form equation for π and b.

The second equation for π and b follows from Eqs. (33) and (34) that now acquire the form

$$\nabla_\alpha(a_{a_1\dots a_4}+\varepsilon_{a_1\dots a_5}\nabla^{a_5}b)=\varepsilon_{a_1\dots a_4a}(\phi^{(ab)}\nabla_b\nabla_\alpha b) \tag{42}$$

and

$$5\nabla_{[a_1}a_{a_2\dots a_5]}=\varepsilon_{a_1\dots a_5}\left(\nabla^2_\gamma b+\frac{1}{2}\phi^a_a-\frac{4}{5}\pi-\frac{1}{2}\phi_{(ab)}\nabla^a\nabla^b\zeta\right). \tag{43}$$

Omitting the straightforward but lengthy algebraic manipulations we write down the final answer for the Lagrangian vertex describing the interaction of the symmetric second rank tensor $\phi_{(ab)}$ with scalars s^I:

$$S_{ssg}=\frac{4N^2}{(2\pi)^5}G_{I_1I_2I_3}\int\sqrt{-g_a}\nabla^a s^{I_1}\nabla^b s^{I_2}\phi^{I_3}_{(ab)},$$

where $G_{I_1I_2I_3}$ is found to be

$$G_{I_1I_2I_3}=\frac{4(\Sigma+2)(\Sigma+4)\alpha_3(\alpha_3-1)}{(k_1+1)(k_2+1)}a_{123}.$$

B. Three-point functions

Denote by $\mathcal{T}^I_{ij}$ the operator in SYM of the conformal weight $\Delta_G=k+4$ that corresponds to the AdS field $\phi_{(ab)}$. To compute the three-point correlation function of this operator with extended CPOs in the boundary conformal field theory one needs the bulk-to-boundary propagator for the field $\phi^I_{(ab)}$. In principle this can be extracted from the momentum space results of [24]. In the case of three-point correlators it is however more convenient to deal directly with the x-space propagator.

Recall that the linearized equations of motion for $\phi^I_{(ab)}$ read as

$$\nabla_c\nabla^c\phi^I_{(ab)}+(2-k^2-4k)\phi^I_{(ab)}=0,\quad \nabla^b\phi^I_{(ab)}=0. \tag{44}$$

Now one can easily check that the following function:

$$G_{abij}(\omega_0,\vec{x})=\frac{\Delta_G+1}{\Delta_G-1}\omega_0^2\mathcal{K}_{\Delta_G}(\omega,\vec{x})J_{ak}(\omega-\vec{x})J_{bl}(\omega-\vec{x})\mathcal{E}_{ij,kl} \tag{45}$$

is the bulk-to-boundary Green function for Eq. (44). Here $\mathcal{E}_{ij,kl}$ denotes the traceless symmetric projector:

$$\mathcal{E}_{ij,kl}=\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj})-\frac{1}{4}\delta_{ij}\delta_{kl},$$

$\mathcal{K}_\Delta(\omega,\vec{x})$ is a bulk-to-boundary propagator for a scalar field corresponding to an operator of conformal dimension Δ:

$$\mathcal{K}_\Delta(\omega,\vec{x})=c_\Delta\frac{\omega_0^\Delta}{(\omega_0^2+(\vec{\omega}-\vec{x})^2)^\Delta},\quad c_\Delta=\frac{\Gamma(\Delta)}{\pi^2\Gamma(\Delta-2)}, \tag{46}$$

and $J_{ab}(x)=\delta_{ab}-2(x_ax_b/x^2)$.

Note that function (45) satisfies the transversality condition $\nabla^a G_{abij}=0$. The normalization constant $(\Delta_G+1)/(\Delta_G-1)$ in Eq. (45) is fixed by requiring the corresponding solution of Eq. (44) to reproduce correctly the boundary data in the limit $\omega_0\to 0$. In the case of vanishing AdS mass Eq. (45) turns into the graviton bulk-to-boundary propagator [8].

Having discussed the propagator for $\phi_{(ab)}$ we come back to the three-point correlator that now reads as

$$\langle \mathcal{O}^{I_1}(\vec{x})\mathcal{O}^{I_2}(\vec{y})\mathcal{T}^{I_3}_{ij}(\vec{z})\rangle = -\frac{8N^2}{(2\pi)^5}G_{I_1I_2I_3}\int\frac{d^5\omega}{\omega_0^5}\,\omega_0^4\nabla_a\nabla_b\mathcal{K}_{\Delta_1}(\omega,\vec{x})\mathcal{K}_{\Delta_2}(\omega,\vec{y})G^{I_3}_{abij}(\omega,\vec{z}). \tag{47}$$

By the conformal symmetry this correlator is defined up to the normalization constant β_{123}:

$$\langle \mathcal{O}^{I_1}(\vec{x})\mathcal{O}^{I_2}(\vec{y})\mathcal{T}^{I_3}_{ij}(\vec{z})\rangle = \frac{\beta_{123}}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_G}|\vec{x}-\vec{z}|^{\Delta_1+\Delta_G-\Delta_2}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_G-\Delta_1}}\left(\frac{Z_iZ_j}{Z^2}-\frac{1}{d}\delta_{ij}\right),$$

where

$$Z_i = \frac{(\vec{x}-\vec{z})_i}{(\vec{x}-\vec{z})^2}-\frac{(\vec{y}-\vec{z})_i}{(\vec{y}-\vec{z})^2}. \tag{48}$$

This constant is then found by explicit evaluation of integral (47):

$$\beta_{123} = -\frac{4N^2}{(2\pi)^5}4\pi^2 c_{\Delta_1}c_{\Delta_2}c_{\Delta_G}G_{I_1I_2I_3}\frac{\Delta_G+1}{\Delta_G-1}\frac{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2+\Delta_G-2)\right)}{\Gamma(\Delta_G+2)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_G-\Delta_2+2)\right)\Gamma\left(\frac{1}{2}(\Delta_2+\Delta_G-\Delta_1+2)\right)\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2-\Delta_G+2)\right)}{\Gamma(\Delta_1)\Gamma(\Delta_2)}. \tag{49}$$

Substituting here the normalization constants and $G_{I_1I_2I_3}$ we finally find

$$\beta_{123} = -\frac{4N^2}{(2\pi)^5}\frac{64}{\pi^4}\left(\frac{k_3+2}{(k_1+1)(k_2+1)}\right)\frac{\Gamma\left(\frac{1}{2}\Sigma+3\right)\Gamma(\alpha_1+3)\Gamma(\alpha_2+3)\Gamma(\alpha_3+1)}{\Gamma(k_1-2)\Gamma(k_2-2)\Gamma(k_3+5)}a_{123}.$$

The two-point correlation function of the YM operator $\mathcal{T}_{ij}$ corresponding to the symmetric second rank tensor field $\phi_{(ab)}$ was computed in [24]

$$\langle \mathcal{T}^I_{ij}(\vec{x})\mathcal{T}^J_{kl}(\vec{y})\rangle = \frac{4N^2}{(2\pi)^5}\frac{1}{\pi^2}(\Delta_G-2)^2(\Delta_G+1)\frac{\delta^{IJ}}{|\vec{x}-\vec{y}|^{2\Delta_G}}\mathcal{E}_{iji'j'}J_{i'k}(\vec{x}-\vec{y})J_{j'l}(\vec{x}-\vec{y}).$$

Therefore, introducing the normalized operator

$$T^I_{ij} = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{(\Delta_G-2)(\Delta_G+1)^{1/2}}\mathcal{T}^I_{ij}$$

one obtains the correlation function of two normalized CPO's and T^I_{ij} with the constant β^{norm}_{123}:

$$\beta^{norm}_{123} = -\frac{(2\pi)^{5/2}}{N}\frac{1}{2^{3/2}\pi^{5/2}}(k_1k_2(k_3+1)(k_3+2)(k_3+5))^{1/2}\times\frac{(\alpha_1+1)(\alpha_1+2)(\alpha_2+1)(\alpha_2+2)}{(k_3+1)(k_3+2)(k_3+3)(k_3+4)(k_3+5)}\langle\mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle,$$

where the explicit expression for a_{123} was used. Note that the variable α_3 completely dissappeared from the final answer.

VI. CUBIC COUPLINGS OF TWO SCALARS s^I WITH VECTOR FIELDS

A. Cubic couplings

To obtain cubic couplings of two scalars s^I with vectors fields we need equations of motion for the vector fields up to the second order. The equations of motion for the vector fields ϕ^α_a can be derived from the following components of the self-duality equation:

$$f_{\alpha abcd}-f^*_{\alpha abcd}+T^{(1)}_{\alpha abcd}+T(h,f^*)_{\alpha abcd}+T(h)_{\alpha abcd}=0, \tag{50}$$

$$f_{\alpha\beta abc}-f^*_{\alpha\beta abc}+T^{(1)}_{\alpha\beta abc}+T(h,f^*)_{\alpha\beta abc}+T(h)_{\alpha\beta abc}=0. \tag{51}$$

From the definition of f we have

$$f_{\alpha\beta abc}=2\nabla_{[\alpha}a_{\beta]abc},$$
$$f^{*}_{\alpha\beta abc}=\varepsilon_{abcde}(\nabla^d\nabla_\alpha\phi^e_\beta-\nabla^d\nabla_\beta\phi^e_\alpha).$$

Here we omitted all terms dependent on the components of the 4-form potential of the form $a_{ab\alpha\beta}$ which are not relevant for the cubic couplings under consideration. From the definition of the tensors T (11) we can easily see that

$$T^{(1)}_{\alpha\beta abc}=T(h,f^*)_{\alpha\beta abc}=T(h)_{\alpha\beta abc}=0,$$

if we keep only terms which may give a contribution to the cubic couplings. Thus Eq. (51) does not get relevant quadratic corrections, and, therefore,

$$a_{\alpha abc}=\varepsilon_{abcde}\nabla^d\phi^e_\alpha. \tag{52}$$

Taking into account Eq. (52) and formulas (11) for the tensors T, one can rewrite Eq. (50) in the form

$$(\nabla^2_b+\nabla^2_\beta-4)\phi^a_\alpha-\nabla_b\nabla^a\phi^b_\alpha-h^a_\alpha+\frac{1}{2}h^b_b\nabla^a\nabla_\alpha b$$
$$-\frac{3}{10}\pi\nabla^a\nabla_\alpha b-h^{ab}\nabla_b\nabla_\alpha b=0. \tag{53}$$

Here we have omitted all terms that are projected out under the projection onto Y_α. Expanding all the fields in spherical harmonics and using Eqs. (19)–(23), we obtain equations of motion for the vector fields ϕ^I_a

$$\nabla^2_b\phi^3_a-\nabla^b\nabla_a\phi^3_b-(k_3+1)(k_3+3)\phi^3_a-h^3_a$$
$$=-t_{123}\left(\frac{4k_2(k_2+2)}{k_2+1}s_2\nabla_a s_1+\frac{4}{k_2+1}\nabla_a\nabla_b s_2\nabla^b s_1\right), \tag{54}$$

where $t_{123}\equiv t_{I_1I_2I_3}=\int\nabla^\alpha Y^{I_1}Y^{I_2}Y^{I_3}_\alpha$, ϕ_3 means ϕ^{I_3} and so on, and summation over 1 and 2 is assumed.

Now we proceed with the equations of motion for h^α_a. These equations can be derived from the a,α components of Eq. (12). Omitting all intermediate calculations, we present the equations in the form

$$\nabla^2_b h^3_a-\nabla^b\nabla_a h^3_b-((k_3+1)(k_3+3)+8)h^3_a-16(k_3+1)(k_3+3)\phi^3_a$$
$$=2t_{123}f(k_1,k_2)s_1\nabla_a s_2-16t_{123}\frac{k_2-5}{k_1+1}\nabla_a\nabla_b s_1\nabla^b s_2+\frac{8t_{123}}{(k_1+1)(k_2+1)}\nabla_a\nabla_b\nabla_c s_2\nabla^b\nabla^c s_1, \tag{55}$$

where

$$f(k_1,k_2)=\frac{4k_1(16+4k_1-2k_1^2+10k_2+4k_1k_2-2k_1^2k_2-2k_2^2-k_1k_2^2)}{(k_1+1)(k_2+1)}.$$

The equations of motion for vector fields A and C are linear combinations of the two above and can be written in the form

$$\nabla^2_b V^3_a-\nabla^b\nabla_a V^3_b-m^2_3V^3_a=\nabla_a V^3+D_{123}s_1\nabla_a s_2+E_{123}\nabla^b s_1\nabla_a\nabla_b s_2+F_{123}\nabla^b\nabla^c s_1\nabla_a\nabla_b\nabla_c s_2, \tag{56}$$

where V may be either A or C, and the constants D, E, F are antisymmetric with respect to the permutation of the indices 1 and 2. We can remove the higher-derivative terms from the equation by means of the following field redefinition:

$$V^3_a\to V^3_a-\frac{1}{m^2_3}\nabla_a\tilde{V}^3+J_{123}s_1\nabla_a s_2+L_{123}\nabla^b s_1\nabla_a\nabla_b s_2, \tag{57}$$

where

$$2L_{123}=F_{123}$$

$$2J_{123}+L_{123}(m_1^2+m_2^2-m_3^2-12)=E_{123}$$

$$\tilde{V}^3=V^3-(J_{123}-2L_{123})m_1^2s_1s_2-L_{123}m_1^2\nabla_b s_1\nabla^b s_2.$$

Then Eq. (56) acquires the form

$$\nabla^2_b V^{I_3}_a-\nabla^b\nabla_a V^{I_3}_b-m^2_3V^{I_3}_a+\sum_{I_1,I_2}v_{I_1I_2I_3}s^{I_1}\nabla_a s^{I_2}=0, \tag{58}$$

where

$$v_{I_1I_2I_3}=-D_{I_1I_2I_3}+J_{I_1I_2I_3}(m_1^2+m_2^2-m_3^2)$$
$$-2L_{I_1I_2I_3}(m_1^2+m_2^2). \tag{59}$$

A straightforward calculation of the constants v gives

$$v_{I_1I_2I_3}(A)=\frac{4(\alpha_3-1/2)(\Sigma-1)(\Sigma+1)(\Sigma+3)}{(k_1+1)(k_2+1)}t_{123} \tag{60}$$

$$v_{I_1I_2I_3}(C)=\frac{16(\alpha_3-1/2)(\alpha_3-3/2)(\alpha_3-5/2)(\Sigma+3)}{(k_1+1)(k_2+1)}t_{123}. \tag{61}$$

Taking into account the normalization of the quadratic actions (27) and (28), we get the corresponding cubic terms

$$S_{ssv}=\frac{4N^2}{(2\pi)^5}V_{I_1I_2I_3}\int\sqrt{-g_a}s^{I_1}\nabla^a s^{I_2}V_a^{I_3}, \tag{62}$$

where

$$V_{I_1I_2I_3}(A)=\frac{2(k_3+1)(\alpha_3-1/2)(\Sigma-1)(\Sigma+1)(\Sigma+3)}{(k_1+1)(k_2+1)(k_3+2)}t_{123} \tag{63}$$

$$V_{I_1I_2I_3}(C)=\frac{8(k_3+3)(\alpha_3-1/2)(\alpha_3-3/2)(\alpha_3-5/2)(\Sigma+3)}{(k_1+1)(k_2+1)(k_3+2)}t_{123}. \tag{64}$$

B. Three-point functions

Denote by $\mathcal{R}_i^{I_3}$ the operator in SYM that corresponds to $V_i^{I_3}$ on the gravity side. Then the three-point function of two scalars and a vector field is given by the integral

$$\langle\mathcal{O}^{I_1}(\vec{x})\mathcal{O}^{I_2}(\vec{y})\mathcal{R}_i^{I_3}(\vec{z})\rangle=\frac{8N^2}{(2\pi)^5}V_{I_1I_2I_3}\int\frac{d^5\omega}{\omega_0^5}\,\omega_0^2\mathcal{K}_{\Delta_1}(\omega,\vec{x})\partial_b\mathcal{K}_{\Delta_2}(\omega,\vec{y})G_{bi}^{I_3}(\omega,\vec{z}). \tag{65}$$

Here $\mathcal{K}_\Delta(\omega,\vec{x})$ with $\Delta=k$ is a bulk-to-boundary propagator (46) for s^I and $G_{ai}(\omega,\vec{x})$ is a bulk-to-boundary propagator for a massive vector field $V_a^{I_3}$ with a mass $m(V)$:

$$G_{ai}(\omega,\vec{x})=\frac{\Delta_v}{\Delta_v-1}\,\omega_0^{-1}\mathcal{K}_{\Delta_v}(\omega,\vec{x})J_{ai}(\omega-\vec{x}),$$

where $J_{ab}(x)=\delta_{ab}-2(x_ax_b/x^2)$.

In the last formula $\Delta_v=2+\sqrt{1+m^2(V)}$ and, thus $\Delta_v=k+2$ for the field A_a^I and $\Delta_v=k+6$ for C_a^I. Note that G_{ai} obeys the transversality condition $\nabla^aG_{ai}=0$.

The condition of the conformal covariance defines the correlator (65) uniquely up to the coefficient λ_{123}:

$$\langle\mathcal{O}^{I_1}(\vec{x})\mathcal{O}^{I_2}(\vec{y})\mathcal{R}_i^{I_3}(\vec{z})\rangle=\frac{\lambda_{123}}{|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_v}|\vec{x}-\vec{z}|^{\Delta_1+\Delta_v-\Delta_2}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_v-\Delta_1}}\left(\frac{|\vec{x}-\vec{z}||\vec{y}-\vec{z}|}{|\vec{x}-\vec{y}|}Z_i\right), \tag{66}$$

with

$$Z_i=\frac{(\vec{x}-\vec{z})_i}{(\vec{x}-\vec{z})^2}-\frac{(\vec{y}-\vec{z})_i}{(\vec{y}-\vec{z})^2}.$$

Applying the inversion method of [7] to integrate Eq. (65) one finds for λ_{123} the following answer:

$$\lambda_{123}=\frac{8N^2}{(2\pi)^5}\frac{1}{\pi^4}V_{I_1I_2I_3}\frac{(\Delta_v-2)\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_2+\Delta_v-3)\right)}{\Gamma(\Delta_v)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_v-\Delta_2+1)\right)\Gamma\left(\frac{1}{2}(\Delta_2+\Delta_v-\Delta_1+1)\right)\Gamma\left(\frac{1}{2}(\Delta_2+\Delta_2-\Delta_v+1)\right)}{\Gamma(\Delta_1-2)\Gamma(\Delta_2-2)}.$$

For the field A^I the last formula reads as

$$\lambda_{123}(A) = \frac{4N^2}{(2\pi)^5}\frac{2^5}{\pi^4}\frac{\Gamma\left(\frac{1}{2}\Sigma+\frac{5}{2}\right)}{(k_1+1)(k_2+1)(k_3+2)}\frac{\Gamma(\alpha_1+3/2)\Gamma(\alpha_2+3/2)\Gamma(\alpha_3+1/2)}{\Gamma(k_1-2)\Gamma(k_2-2)\Gamma(k_3)}t_{123}$$

while for C^I:

$$\lambda_{123}(C) = \frac{4N^2}{(2\pi)^5}\frac{2^5}{\pi^4}\frac{\Gamma\left(\frac{1}{2}\Sigma+\frac{5}{2}\right)(k_3+3)(k_3+4)}{(k_1+1)(k_2+1)(k_3+2)}\frac{\Gamma(\alpha_1+7/2)\Gamma(\alpha_2+7/2)\Gamma(\alpha_3+1/2)}{\Gamma(k_1-2)\Gamma(k_2-2)\Gamma(k_3+6)}t_{123}.$$

The two-point correlator corresponding to a massive vector field on the AdS space was found in [10]:

$$\langle \mathcal{R}_i^I(\vec{x}),\mathcal{R}_j^J(\vec{y})\rangle = \frac{2}{\pi^2}\theta\Delta_v(\Delta_v-1)^2\frac{\delta^{IJ}}{|\vec{x}-\vec{y}|^{2\Delta_v}}J_{ij}(\vec{x}-\vec{y}), \tag{67}$$

where the constant θ accounts our normalization of the quadratic action for the vector fields and is equal to $\theta = [4N^2/(2\pi)^5][(k+1)/2(k+2)]$ for the field A and to $\theta = [4N^2/(2\pi)^5][(k+3)/2(k+2)]$ for C, respectively. We introduce a normalized operator R_i^I with the two-point correlation function

$$\langle R_i^I(\vec{x}),R_j^J(\vec{y})\rangle = \frac{\delta^{IJ}}{|\vec{x}-\vec{y}|^{2\Delta_v}}J_{ij}(\vec{x}-\vec{y}).$$

Explicitly R_i^I is given by

$$R_i^I = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{(k+1)^{3/2}}\mathcal{R}_i^I$$

for the YM operator corresponding to A_a^I and

$$R_i^I = \frac{(2\pi)^{5/2}}{2N}\frac{\pi}{(k+5)}\left(\frac{k+2}{(k+3)(k+6)}\right)^{1/2}\mathcal{R}_i^I$$

for C_a^I. By using these formulas, the definition (39) of the normalized CPO, and the expression for t_{123} from the Appendix, one gets the correlation functions of normalized operators

$$\lambda_{123}^{norm}(A) = \frac{(2\pi)^{5/2}}{N}\frac{1}{4\pi^{5/2}}\left(\frac{k_1k_2}{k_3+2}\right)^{1/2} \times\frac{k_3(\alpha_1+1/2)(\alpha_2+1/2)}{(k_3+1)^2}T_{123},$$

$$\lambda_{123}^{norm}(C) = \frac{(2\pi)^{5/2}}{N}\frac{1}{4\pi^{5/2}}\left(\frac{k_1k_2(k_3+3)}{(k_3+1)(k_3+6)}\right)^{1/2}\frac{k_3+4}{k_3+5} \times\frac{k_3!\Gamma(\alpha_1+7/2)\Gamma(\alpha_2+7/2)}{\Gamma(k_3+6)(\alpha_1-1/2)!(\alpha_2-1/2)!}T_{123}.$$

VII. CONCLUSION

In this paper we obtained the cubic couplings in type IIB supergravity on $AdS_5\times S^5$ involving two scalar fields s^I and the corresponding three-point functions by using the covariant equations of motion and the quadratic action. Since all the fields we considered correspond to operators which are descendants of CPOs, one may, in principle, derive the same results directly from the superconformal invariance. This would require a detailed study of superconformal Ward identities in SYM_4 which, to our knowledge, has not been carried out yet.

In most cases to find a cubic coupling of two scalars s^I with a field F we used a corrected equation of motion for the field F. We saw that to get rid of higher-derivative terms we had to make a field redefinition of the form (5). By this reason, a field F corresponds not to a descendant of a CPO, but to a properly extended operator which includes products of CPOs.

In fact one can obtain the cubic couplings by using corrected equations of motion for scalars s^I as it was done in the case of the graviton couplings. We have done that to derive the cubic couplings of two scalars s^I with the scalars ϕ^I, and with the vector fields, and we have certainly obtained the same results (36) and (62). The fact that we derived the same vertices from different equations also confirms the correct normalization of the quadratic action for type IIB supergravity [40]. It is worth noting that contrary to the graviton case considered in Sec. V, in these cases to remove higher-derivative terms from the corrected equations of motion for s^I we had to make the following redefinitions of the scalars s^I:

$$s_1 \rightarrow s_1 + J_{123}s_2\phi_3 + L_{123}\nabla_a s_2\nabla^a\phi_3$$

and

$$s_1 \rightarrow s_1 + J_{123}\nabla_a s_2 V_3^a + L_{123}\nabla_a\nabla_b s_2\nabla^b V_3^a.$$

This implies that the extended CPOs corresponding to the scalars s^I that were discussed in Sec. II have to depend on products of CPOs and their descendants. Unfortunately, the knowledge of the three-point functions obtained in the paper does not allow one to fix the explicit form of the extended CPOs uniquely.

It is worth noting that the cubic couplings of three scalars s vanish when any of the $\alpha' s$ vanish [12]. The cubic couplings studied in this paper vanish if α_3 takes special values, and in most of the cases there are several such values of α_3. Since α_1 and α_2 have to be non-negative the cubic couplings have no zeros at α_1 and α_2. However, in all of the three-point functions considered zeros of the cubic couplings are cancelled by poles in the general expressions for the three-point functions, just as in the case of the three-point functions of extended CPOs. This gives us a reason to believe that for generic values of conformal dimensions the three-point functions obtained coincide with the three-point functions of CPOs and their descendants.

The next natural step is to find quartic couplings of scalars s^I, and to compute four-point functions of extended CPOs. We expect that the quartic couplings vanish if, say, $k_4 = k_1 + k_2 + k_3$, because in this case there is no exchange diagram, and all contributions to the four-point functions may be given only by the quartic couplings. However, the four-point functions in this case are nonsingular at $\vec{x}_1 = \vec{x}_2$, and it seems to be impossible to reproduce such a coordinate dependence via supergravity with a nonzero on-shell quartic coupling.

Finally, it would be interesting to find the supergravity fields that correspond to CPOs, and to compute their cubic couplings. A similar problem exists in the case of AdS compactifications of 11-dimensional supergravity, where analogous cubic couplings [46,47] also have zeros. In the 11-dimensional case the problem seems to be simpler because the covariant action is known.

ACKNOWLEDGMENTS

We would like to thank Professor S. Theisen and Professor J. Wess for kind hospitality at the University of Munich. We are grateful to Professor S. Theisen and to S. Kuzenko for valuable discussions and to F. Bastianelli and R. Zucchini for pointing out the factor 2 in the Green functions missed in the first version of the paper. The work of G.A. was supported by the Alexander von Humboldt Foundation and in part by the RFBI Grant N96-01-00608, and the work of S.F. was supported by the U.S. Department of Energy under Grant No. DE-FG02-96ER40967 and in part by the Alexander von Humboldt Foundation.

APPENDIX

We follow [12] describing spherical harmonics on S^5. The scalar spherical harmonics Y^I are defined by

$$Y^I = z(k)^{-1/2} C^I_{i_1 \ldots i_k} x^{i_1} \cdots x^{i_k} \tag{A1}$$

where $C^I_{i_1 \cdots i_k}$ are totally symmetric traceless rank k orthonormal tensors of $SO(6)$: $\langle C^I C^J \rangle = C^I_{i_1 \cdots i_k} C^J_{i_1 \cdots i_k} = \delta^{IJ}$, x^i are the Cartesian coordinates of the $\mathbf{R}^6$ in which S^5 is embedded, and

$$z(k) = \frac{\pi^3}{2^{k-1}(k+1)(k+2)}.$$

The scalar spherical harmonics are orthonormal and satisfy the relation

$$\int Y^{I_1} Y^{I_2} Y^{I_3} = a_{123},$$

$$a_{123} = (z(k_1) z(k_2) z(k_3))^{-1/2} \frac{\pi^3}{\left(\frac{1}{2}\Sigma + 2\right)! \, 2^{(1/2)(\Sigma - 2)}} \times \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!} \langle C^{I_1} C^{I_2} C^{I_3} \rangle, \tag{A2}$$

where $\alpha_i = \frac{1}{2}(k_j + k_l - k_i)$, $j \neq l \neq i$, and $\langle C^{I_1} C^{I_2} C^{I_3} \rangle$ is the unique $SO(6)$ invariant obtained by contracting α_1 indices between C^{I_2} and C^{I_3}, α_2 indices between C^{I_3} and C^{I_1}, and α_3 indices between C^{I_2} and C^{I_1}.

A vector spherical harmonic is defined as a tangent component of the following vector:

$$Y^I_m = z(k)^{-1/2} C^I_{m; i_1 \ldots i_k} x^{i_1} \cdots x^{i_k} \tag{A3}$$

where the tensor $C^I_{m; i_1 \ldots i_k}$ is symmetric and traceless with respect to $i_1, \ldots, i_k$, and its symmetric part vanishes. The tensors are orthonormal

$$C^I_{m; i_1 \ldots i_k} C^J_{n; i_1 \ldots i_k} = \delta^{IJ} \delta_{mn}.$$

The vector spherical harmonics are orthonormal and satisfy the relation

$$\int \nabla^\alpha Y^{I_1} Y^{I_2} Y^{I_3}_\alpha = t_{123},$$

$$t_{123} = \frac{\pi^3}{k_3 + 1} \frac{(z(k_1) z(k_2) z(k_3))^{-1/2}}{\left(\frac{1}{2}(\Sigma + 3)\right)! \, 2^{(1/2)(\Sigma - 3)}} \times \frac{k_1! k_2! k_3!}{\left(\alpha_1 - \frac{1}{2}\right)! \left(\alpha_2 - \frac{1}{2}\right)! \left(\alpha_3 - \frac{1}{2}\right)!} T_{123} \tag{A4}$$

where

$$T_{123} = C^{I_1}_{m i_1 \ldots i_{p_2} j_1 \ldots j_{p_3}} C^{I_2}_{j_1 \ldots j_{p_3} l_1 \ldots l_{p_1}} C^{I_3}_{m; l_1 \ldots l_{p_1} i_1 \ldots i_{p_2}} - C^{I_1}_{i_1 \ldots i_{p_2+1} j_1 \ldots j_{p_3}} C^{I_2}_{j_1 \ldots j_{p_3} l_1 \ldots l_{p_1 - 1} m} \times C^{I_3}_{m; l_1 \ldots l_{p_1 - 1} i_1 \ldots i_{p_2 + 1}} \tag{A5}$$

and $p_1 = \alpha_1 + \frac{1}{2}$, $p_2 = \alpha_2 - \frac{1}{2}$, $p_3 = \alpha_3 - \frac{1}{2}$.

A tensor spherical harmonic is defined as a projection of the following six-dimensional tensor onto the sphere:

$$Y_{mn} = z(k)^{-1/2} C^I_{mn; i_1 \ldots i_k} x^{i_1} \cdots x^{i_k}, \tag{A6}$$

where the tensor $C^I_{mn;i_1\ldots i_k}$ is symmetric and traceless with respect to $i_1,\ldots,i_k$, and m,n, and its symmetric part vanishes, i.e.

$$C^I_{mn;i_1\ldots i_k}+C^I_{mi_1;n\ldots i_k}+\cdots+C^I_{mi_k;i_1\ldots i_{k-1}n}=0.$$

The tensors are orthonormal

$$C^I_{m_1n_1;i_1\ldots i_k}C^J_{m_2n_2;i_1\ldots i_k}=\delta^{IJ}\delta_{m_1n_1;m_2n_2}.$$

Then, we get that the tensor spherical harmonics are orthonormal and satisfy the relation

$$\int \nabla^\alpha Y^{I_1}\nabla^\beta Y^{I_2}Y^{I_3}_{(\alpha\beta)}=p_{123,}$$

$$p_{123}=(z(k_1)z(k_2)z(k_3))^{-1/2}\frac{\pi^3}{\left(\frac{1}{2}\Sigma+1\right)!2^{(1/2)\Sigma}}\times\frac{k_1!k_2!k_3!}{\alpha_1!\alpha_2!(\alpha_3-1)!}P_{123}, \tag{A7}$$

where

$$P_{123}=C^{I_1}_{mi_1\ldots i_{p_2}j_1\ldots j_{p_3}}C^{I_2}_{nj_1\ldots j_{p_3}l_1\ldots l_{p_1}}C^{I_3}_{mn;l_1\ldots l_{p_1}i_1\ldots i_{p_2}} \tag{A8}$$

and $p_1=\alpha_1,\ p_2=\alpha_2,\ p_3=\alpha_3-1$.

In deriving the equations of motions for scalar fields t_k and for tensor $\phi^k_{(ab)}$ one comes across a number of integrals of scalar spherical harmonics, all of them can be reduced to a_{123}. Introducing the concise notation $f(k)=k(k+4)$ we present below the corresponding formulas:

$$\int \nabla^\alpha Y^{I_1}Y^{I_2}\nabla_\alpha Y^{I_3}=\frac{1}{2}(f(k_1)+f(k_3)-f(k_2))a_{123},$$

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}\nabla_\alpha Y^{I_2}\nabla_\beta Y^{I_3} =\left(\frac{1}{10}f(k_1)f(k_2)+\frac{1}{10}f(k_1)f(k_3)+\frac{1}{2}f(k_2)f(k_3) -\frac{1}{4}f(k_2)^2-\frac{1}{4}f(k_3)^2+\frac{3}{20}f(k_1)^2\right)a_{123},$$

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}Y^{I_2}\nabla_\alpha\nabla_\beta Y^{I_3} =\frac{1}{2}\left(-f(k_1)f(k_2)-f(k_2)f(k_3)+\frac{3}{5}f(k_1)f(k_3) +\frac{1}{2}f(k_1)^2+\frac{1}{2}f(k_2)^2+\frac{1}{2}f(k_3)^2 -4(f(k_1)+f(k_3)-f(k_2))\right)a_{123}.$$

Analogously, when computing the interaction vertex S_{ssv} from equations of motion for scalars s_k one finds two integrals involving the vector harmonics Y^I_α. Both of them are expressed via t_{123}:

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}Y^{I_2}\nabla_\alpha Y^{I_3}_\beta=\frac{1}{2}((k_3+1)(k_3+3)-8 +f(k_1)-f(k_2))t_{123}$$

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}\nabla_\alpha Y^{I_2}Y^{I_3}_\beta=\frac{1}{2}\left(f(k_2)+\frac{3}{5}f(k_1) -(k_3+1)(k_3+3)\right)t_{123}.$$

Finally the derivation of the $S_{ss\phi}$-vertex from the equations of motion for scalars s_k requires the knowledge of the following integrals:

$$\int \nabla^{(\alpha}\nabla^{\gamma)}Y^{I_1}\nabla^\beta\nabla_\gamma Y^{I_2}Y^{I_3}_{(\alpha\beta)}=\frac{1}{10}(3f(k_1)+5f(k_2)-5k_3^2 -20k_3-30)p_{123}$$

$$\int \nabla^{(\alpha}\nabla^{\beta)}Y^{I_1}\nabla_\gamma Y^{I_2}\nabla^\gamma Y^{I_3}_{(\alpha\beta)}=\frac{1}{2}(f(k_1)-f(k_2)-k_3^2 -4k_3-8)p_{123}.$$

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] G. G. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] I. Ya. Aref'eva and I. V. Volovich, ''On large N conformal field theories, field theories in anti-de Sitter space and singletons,'' hep-th/9803028.
[5] M. Henningson and K. Sfetsos, Phys. Lett. B **431**, 63 (1998).
[6] W. Mück and K. S. Viswanathan, Phys. Rev. D **58**, 041901 (1998).
[7] D. Freedman, S. Mathur, A. Matusis, and L. Rastelli, Nucl. Phys. **B546**, 96 (1999).
[8] H. Liu and A. A. Tseytlin, Nucl. Phys. **B553**, 88 (1998).
[9] G. Chalmers, H. Nastase, K. Schalm, and R. Siebelink, Nucl. Phys. **B540**, 247 (1999).
[10] W. Mück and K. S. Viswanathan, Phys. Rev. D **58**, 106006 (1998).
[11] A. Ghezelbash, K. Kaviani, S. Parvizi, and A. Fatollahi, Phys.

Lett. B **435**, 291 (1998).
[12] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, Adv. Theor. Math. Phys. **2**, 697 (1998).
[13] G. Arutyunov and S. Frolov, Nucl. Phys. **B544**, 576 (1999).
[14] G. Arutyunov and S. Frolov, Phys. Lett. B **441**, 173 (1998).
[15] E. D'Hoker, D. Freedman, and W. Skiba, Phys. Rev. D **59**, 045008 (1999).
[16] P. S. Howe, E. Sokatchev, and P. C. West, Phys. Lett. B **444**, 341 (1998).
[17] S. Corley, Phys. Rev. D **59**, 086003 (1999).
[18] A. Volovich, J. High Energy Phys. **09**, 022 (1998).
[19] W. S. l'Yi, ''Generating functionals of correlation functions of *p*-form currents in the AdS/CFT correspondence,'' hep-th/9809132.
[20] W. Mück and K. S. Viswanathan, ''The graviton in the AdS/CFT correspondence: Solution via the Dirichlet boundary value problem,'' hep-th/9810151.
[21] L. Chekhov, ''AdS/CFT correspondence on torus,'' hep-th/9811146.
[22] A. Koshelev and O. Rytchkov, Phys. Lett. B **450**, 368 (1999).
[23] G. Arutyunov and S. Frolov, Phys. Rev. D **60**, 026004 (1999).
[24] A. Polishchuk, J. High Energy Phys. **07**, 007 (1999).
[25] H. Liu and A. A. Tseytlin, J. High Energy Phys. **10**, 003 (1999).
[26] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, Phys. Rev. D **32**, 389 (1985).
[27] M. Gunaydin and N. Marcus, Class. Quantum Grav. **2**, L11 (1985).
[28] E. D'Hoker and D. Freedman, Nucl. Phys. **B550**, 612 (1999).
[29] H. Liu and A. A. Tseytlin, Phys. Rev. D **59**, 086002 (1999).
[30] D. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, Phys. Lett. B **452**, 61 (1999).
[31] G. Chalmers and K. Schalm, Nucl. Phys. **B554**, 215 (1999).
[32] E. D'Hoker and D. Freedman, Nucl. Phys. **B544**, 612 (1999).
[33] J. H. Brodie and M. Gutperle, Phys. Lett. B **445**, 296 (1999).
[34] H. Liu, Phys. Rev. D **60**, 106005 (1999).
[35] E. D'Hoker, D. Freedman, S. Mathur, A. Matusis, and L. Rastelli, ''Graviton exchange and complete 4-point functions in the AdS/CFT correspondence,'' hep-th/9903196.
[36] E. D'Hoker, D. Freedman, and L. Rastelli, ''AdS/CFT 4-point functions: How to succeed at *z*-integrals without really trying,'' hep-th/9905049.
[37] B. Eden, P. S. Howe, C. Schubert, E. Sokatchev, and P. C. West, Phys. Lett. B **466**, 20 (1999).
[38] S. Sanjay, Mod. Phys. Lett. A **14**, 1413 (1999).
[39] M. Bianchi, S. Kovacs, G. Rossi, and Y. S. Stanev, J. High Energy Phys. **08**, 020 (1999).
[40] G. Arutyunov and S. Frolov, J. High Energy Phys. **08**, 024 (1999).
[41] J. H. Schwarz, Nucl. Phys. **B226**, 269 (1983).
[42] J. H. Schwarz and P. C. West, Phys. Lett. **126B**, 301 (1983).
[43] P. S. Howe and P. C. West, Nucl. Phys. **B238**, 181 (1984).
[44] G. Dall'Agata, K. Lechner, and D. Sorokin, Class. Quantum Grav. **14**, L195 (1997).
[45] G. Dall'Agata, K. Lechner, and M. Tonin, J. High Energy Phys. **07**, 017 (1998).
[46] R. Corrado, B. Florea, and R. McNees, Phys. Rev. D **60**, 085001 (1999).
[47] F. Bastianelli and R. Zucchini, Phys. Lett. B **467**, 61 (1999).