Inflation and gauge hierarchy in Randall-Sundrum compactification

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We obtain the general inflationary solutions for the slab of five-dimensional AdS spacetime where the fifth dimension is an orbifold S^1/Z_2 and two 3-branes reside at its boundaries, of which the Randall-Sundrum model corresponds to the static limit. The investigation of the general solutions and their static limit reveals that the RS model recasts both the cosmological constant problem and the gauge hierarchy problem into the balancing problem of the bulk and the brane cosmological constants.

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The huge gap between the electroweak scale M_W and the Planck scale M_P, $M_P/M_W \sim 10^{16}$, has been a long standing puzzle in unifying the standard model and gravity. Recently the extra dimensional models addressing it have drawn much attention [1,2]. Introducing the extra dimensions has a long history from Kaluza-Klein theory to string theory. What is new is that ordinary matter is confined to our four-dimensional brane while gravity propagates in the whole spacetime. One motivation for such models is the heterotic M theory, whose field theory limit is the 11-dimensional supergravity compactified on S_1/Z_2 with supersymmetric Yang-Mills fields residing on two boundaries [3].

The large extra dimension model [1] brings the fundamental gravitational scale around the weak scale to solve the gauge hierarchy problem and reduce the strength of the four-dimensional gravity by having large extra dimensions, while avoiding conflict with experiments by confining the standard model fields to a 3-brane in the extra dimensions. This model translates the gauge hierarchy to the hierarchy between the fundamental scale and the size of the extra dimensions.

More recently, Randall and Sundrum (RS) proposed a five-dimensional model with nonfactorizable geometry supported by negative bulk cosmological constant and oppositely signed boundary 3-brane cosmological constants. In this model, the gauge hierarchy problem can be explained by the exponential warp factor even for the small extra dimension. The model is quite interesting and has drawn much attention because it might be realizable in supergravity and superstring compactifications [3–6]. However, its geometry is based on the very specific relation between the bulk and the brane cosmological constants. Then the question arises as to how precisely this relation should hold to preserve the necessary geometry. In this paper, we try to answer this question by finding cosmological inflationary solutions with general sets of the bulk and the brane cosmological constants and comparing them with the static solution given by RS. The cosmological aspect of the extra dimensional models has been discussed by many authors [7–9]. Especially the inflationary solutions were obtained for flat bulk geometry [10], and for AdS bulk geometry [11,12], where the condition is imposed among parameters such that the extra dimension does not inflate. Here we obtain the general inflationary solutions for the AdS bulk geometry and focus on the connection to the gauge hierarchy problem.

We consider the five-dimensional spacetime with coordinates (τ, x^i, y) where τ and x^i, $i=1,2,3$, denote the usual four-dimensional spacetime and y is coordinate of the fifth dimension, which is an orbifold S^1/Z_2 where the Z_2 action identifies y and $-y$. We choose the range of y to be from $-1/2$ to $1/2$. We consider two 3-branes extending in the usual four-dimensional spacetime residing at two orbifold fixed points $y=0$ and $y=1/2$, so that they form the boundaries of five-dimensional spacetime. This five-dimensional model is described by the action

$$S=\int_M d^5x\sqrt{-g}\left[\frac{M^3}{2}R-\Lambda_b\right] +\sum_{i=1,2}\int_{\partial M^{(i)}} d^4x\sqrt{-g^{(i)}}[\mathcal{L}_i-\Lambda_i], \tag{1}$$

where M is the fundamental gravitational scale of the model, Λ_b and Λ_i are the bulk and the brane cosmological constants, and $\mathcal{L}_i$ are the Lagrangians for the fields confined in the branes. Since we are interested in the cosmological solution, we assume that the three-dimensional spatial section is homogeneous and isotropic. Further we consider it to be flat for simplicity. The most general metric satisfying this can be written as

$$ds^2=-n^2(\tau,y)d\tau^2+a^2(\tau,y)\delta_{ij}dx^i dx^j+b^2(\tau,y)dy^2. \tag{2}$$

For the above action and metric, we obtain the following Einstein equations corresponding to (00), (ii), (55), (05) components, respectively:

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$$\frac{3}{n^2}\frac{\dot a}{a}\left(\frac{\dot a}{a}+\frac{\dot b}{b}\right)-\frac{3}{b^2}\left[\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right]$$
$$=M^{-3}\left[\Lambda_b+\frac{\delta(y)}{b}(\Lambda_1+\rho_1)+\frac{\delta\left(y-\frac{1}{2}\right)}{b}(\Lambda_2+\rho_2)\right], \tag{3}$$

$$\frac{1}{n^2}\left[2\frac{\ddot a}{a}+\frac{\ddot b}{b}-\frac{\dot a}{a}\left(2\frac{\dot n}{n}-\frac{\dot a}{a}\right)-\frac{\dot b}{b}\left(\frac{\dot n}{n}-2\frac{\dot a}{a}\right)\right]-\frac{1}{b^2}\left[\frac{n''}{n}+2\frac{a''}{a}\right.$$
$$\left.+\frac{a'}{a}\left(2\frac{n'}{n}+\frac{a'}{a}\right)-\frac{b'}{b}\left(\frac{n'}{n}+2\frac{a'}{a}\right)\right]$$
$$=M^{-3}\left[\Lambda_b+\frac{\delta(y)}{b}(\Lambda_1-p_1)+\frac{\delta\left(y-\frac{1}{2}\right)}{b}(\Lambda_2-p_2)\right], \tag{4}$$

$$\frac{3}{n^2}\left[\frac{\ddot a}{a}-\frac{\dot a}{a}\left(\frac{\dot n}{n}-\frac{\dot a}{a}\right)\right]-\frac{3}{b^2}\frac{a'}{a}\left(\frac{n'}{n}+\frac{a'}{a}\right)=M^{-3}\Lambda_b, \tag{5}$$

$$3\left(\frac{\dot a}{a}\frac{n'}{n}+\frac{\dot b}{b}\frac{a'}{a}-\frac{\dot a'}{a}\right)=0, \tag{6}$$

where the overdot and the prime represent the derivatives with respect to τ and y, respectively. The equations with bulk and boundary sources are equivalent to the equations with bulk sources and proper boundary conditions. To give a nonsingular geometry, n, a, and b must be continuous along the extra dimension. But Eqs. (3) and (4) imply that n' and a' are discontinuous at $y=0,\pm\frac{1}{2}$ so that n'' and a'' have delta function singularities there. Applying $\int_{0^-}^{0^+}dy$ and $\int_{\frac{1}{2}^-}^{\frac{1}{2}^+=-\frac{1}{2}^+}dy$ to Eqs. (3) and (4), we obtain the boundary conditions

$$\left.\frac{n'}{n}\right|_{0^-}^{0^+}=-\frac{b(\tau,0)}{3M^3}(\Lambda_1-2\rho_1-3p_1),$$

$$\left.\frac{a'}{a}\right|_{0^-}^{0^+}=-\frac{b(\tau,0)}{3M^3}(\Lambda_1+\rho_1),$$

$$\left.\frac{n'}{n}\right|_{-1/2}^{+1/2}=+\frac{b\left(\tau,\frac{1}{2}\right)}{3M^3}(\Lambda_2-2\rho_2-3p_2),$$

$$\left.\frac{a'}{a}\right|_{-1/2}^{+1/2}=+\frac{b\left(\tau,\frac{1}{2}\right)}{3M^3}(\Lambda_2+\rho_2). \tag{7}$$

Since b'' does not appear in the equations, no boundary condition is imposed on b'.

Equations (3)–(6) and the boundary conditions (7) constitute the starting point of the cosmology of the five-dimensional model considered in this paper. It is difficult to solve the whole bulk equations with generic sources, but at the brane boundaries, the (55) and (05) equations give the Friedman-like equation

$$\left(\frac{\dot a}{a}\right)^2=\left(\frac{\Lambda_b}{6M^3}+\frac{\Lambda_i^2}{36M^6}\right)+\frac{\Lambda_i}{18M^6}\rho_i+\frac{1}{36M^6}\rho_i^2. \tag{8}$$

The implications of this equation are quite interesting [7–9], but we will not pursue them here.

In this paper, we consider the cosmological constant dominated cases, and neglect the matter and radiation energy densities of the branes. Then the above equations allow a static solution, the so-called RS solution [2], when the bulk cosmological constant is negative ($\Lambda_b<0$) and related to the brane cosmological constants by

$$k=k_1=-k_2, \tag{9}$$

where $k=(-\Lambda_b/6M^3)^{1/2}$ and $k_i=\Lambda_i/6M^3$. We take the brane with positive cosmological constant to be at $y=0$. The metric of the static solution is given by

$$ds^2=e^{-2kb_0|y|}\eta_{\mu\nu}dx^\mu dx^\nu+b_0^2dy^2, \tag{10}$$

where b_0 is a constant which determines the length of the extra dimension. In this model, the four-dimensional Planck scale is given by

$$M_{\rm P}^2=\frac{M^3}{k}[1-e^{-kb_0}], \tag{11}$$

and for $\frac{1}{2}kb_0\gtrsim 1$ it is k rather than $\frac{1}{2}b_0$ that determines it. RS [2] argued that, due to the warp factor e^{-kb_0y} which has different values at the hidden brane ($y=0$) and at the visible brane ($y=\frac{1}{2}$), any mass parameter m_0 on the visible brane corresponds to a physical mass $m=m_0e^{(-1/2)kb_0}$ and the moderate value $\frac{1}{2}kb_0\sim 37$ can produce the huge ratio $M_P/M_W\sim 10^{16}$. Thus the gauge hierarchy problem is converted to the problem related to geometry, fixing the size of the extra dimension. Then it is an important question as to how precise the relation (9) must be in order for the RS solution to work since the exact relation is assumed *a priori* to get Eq. (10).

Now we try to answer this question by solving the Einstein equations with the bulk and the brane cosmological constants, but without the fine-tuned condition (9). Boundary condition (7) suggests $n=a$, and we first try a separable function $n=a=g(\tau)f(y)$. Then the (05) equation yields $b=b(y)$. After separate coordinate transformations of τ and y, we come to an ansatz

$$n=f(y),\quad a=g(\tau)f(y),\quad b=b_0, \tag{12}$$

where b_0 is a constant. Now subtracting the (ii) equation by the (00) equation, we obtain $(\dot{g}/\dot{g})=0$. So we define $(\dot{g}/g) \equiv H_0 = \text{const}$. Then the (55) equation gives

$$\left(\frac{f'}{b_0}\right)^2 = H_0^2 + k^2 f^2, \tag{13}$$

and the (00) and (ii) equations just give a redundant equation. For $\Lambda_b<0$, the solution to this equation consistent with the orbifold symmetry is

$$f=\frac{H_0}{k}\sinh(-kb_0|y|+c_0). \tag{14}$$

The boundary condition (7) imposes

$$k_1 = k\coth(c_0), \quad -k_2 = k\coth\left(-\frac{1}{2}kb_0+c_0\right). \tag{15}$$

Therefore, the solution is allowed when k_1, k_2, and k satisfy $k<k_1<-k_2$ and they are related to the length of the extra dimension $L_5=\frac{1}{2}b_0$ by

$$L_5=\frac{1}{2}b_0=\frac{1}{2k}\ln\left[\frac{-k_2-k}{k_1-k}\,\frac{k_1+k}{-k_2+k}\right]. \tag{16}$$

The metric of this solution is

$$ds^2=\left(\frac{H_0}{k}\right)^2\sinh^2(-kb_0|y|+c_0)[-d\tau^2+e^{2H_0\tau}\delta_{ij}dx^i dx^j] + b_0^2 dy^2. \tag{17}$$

We arrive at the static limit by taking $H_0\to 0$ and $c_0\to\infty$ while keeping the ratio $(H_0/2k)e^{c_0}\to 1$ fixed, and the metric (17) becomes (10). For $H_0\neq 0$, H_0 is not a physical quantity and can be set to k which corresponds to the shift of the initial value of τ. Then by the coordinate transformation $dy=d\tilde{y}/\sinh(kb_0|\tilde{y}|+\tilde{c}_0)$, the metric becomes [11]

$$ds^2=\frac{-d\tau^2+e^{2k\tau}\delta_{ij}dx^i dx^j+b_0^2 d\tilde{y}^2}{\sinh^2(kb_0|\tilde{y}|+\tilde{c}_0)}. \tag{18}$$

The metric (17) describes the inflation of the spatial dimensions with the length of extra dimension fixed. At a given y, we can perform a four-dimensional coordinate transformation to make the four-dimensional metric be in the form $ds_{(4)}^2=-dt^2+e^{2H(y)t}\delta_{ij}dx^i dx^j$. Then we get the Hubble parameter

$$H(y)=k\,\mathrm{csch}(-kb_0|y|+c_0). \tag{19}$$

Especially, at each boundary, we have

$$H(0)=\sqrt{k_1^2-k^2}, \quad H\left(\frac{1}{2}\right)=\sqrt{k_2^2-k^2}, \tag{20}$$

respectively. We see that inflation occurs when the bulk and the brane cosmological constants deviate from the relation (9), which can be easily seen from the boundary equation (8). The condition (16) means that to keep the length of the extra dimension fixed while the spatial dimensions inflate, we must fine-tune the bulk and the brane cosmological constants; in other words, we must put two branes a distance L_5 apart for given k_1, k_2, and k.

For the more general case that the distance between two branes is larger or smaller than L_5, the solution can be obtained with the nonseparable function

$$n(\tau,y)=a(\tau,y)=\frac{1}{\tau f(y)+g_0}, \quad b(\tau,y)=kb_0\tau a(\tau,y), \tag{21}$$

where b_0 and g_0 are constants. The metric for this general case can be found to be

$$ds^2=\frac{-d\tau^2+\delta_{ij}dx^i dx^j+(kb_0\tau)^2 dy^2}{[k\tau\sinh(kb_0|y|+c_0)+g_0]^2}, \tag{22}$$

where b_0 and c_0 are given by

$$c_0=\cosh^{-1}\left(\frac{k_1}{k}\right),$$

$$kb_0=2\left[\cosh^{-1}\left(\frac{-k_2}{k}\right)-\cosh^{-1}\left(\frac{k_1}{k}\right)\right]. \tag{23}$$

When $g_0=0$, this metric becomes Eq. (18) by the coordinate transformation $(1/\tau)\,d\tau=-k\,d\tilde{\tau}$. When $g_0>0$ $(g_0<0)$, the distance between two branes is smaller (larger) than L_5 and the above metric describes the situation that two branes come close and finally meet (move away from each other) while the spatial dimensions are inflating. The boundary condition is not affected by the presence of g_0, and $H(0)$ and $H(\frac{1}{2})$ are the same Eq. (20), as seen in Eq. (8).

For the special case $k_1=-k2>k$, L_5 becomes 0 and the above solutions cannot cover. Instead, we can find a solution with a different ansatz $n(\tau,y)=a(\tau,y)=b(\tau,y)$. The metric is given by

$$ds^2=\frac{-d\tau^2+\delta_{ij}dx^i dx^j+dy^2}{[-(k_1^2-k^2)^{1/2}\tau+ky+c_0]^2}, \tag{24}$$

where c_0 is a constant. This metric describes inflation in both the spatial dimensions and the extra dimension.

Let us consider the connection between the inflationary solutions and the RS static solution. The static limit that both k_1 and $-k_2$ approach k in the inflationary solutions corresponds to the RS solution. Suppose that the five-dimensional universe underwent inflation in the early epoch and finally settles down to the static RS model. Then the relation (9) does not hold exactly but approximately now. This situation is most likely described by the static limit of the inflationary solutions. Then, the current observations on the Hubble constant restrict the visible brane Hubble parameter $H(\frac{1}{2})$:

$$H\left(\frac{1}{2}\right)=\sqrt{k_2^2-k^2}\lesssim 10^{-60}M_{\mathrm{P}}, \tag{25}$$

where $k=\mathcal{O}(M)=\mathcal{O}(M_{\mathrm{P}})$ is assumed. Therefore, the bulk and the visible brane cosmological constants must cancel each other up to very high precision. This is a five-dimensional version of the well-known cosmological constant problem [12] and the RS condition (9) is nothing but the condition for the vanishing four-dimensional effective cosmological constant at both branes. We do not attempt to solve this notorious problem in this paper. Instead, we concentrate on the gauge hierarchy problem within this context.

The key point of the RS solution to the gauge hierarchy problem is the size of the extra dimension appearing in the exponential warp factor. If the RS condition (9) exactly holds, it is not determined in this framework and remains as a flat direction in the moduli space. For the bulk and the brane cosmological constants that do not satisfy the RS condition, there is a critical value L_5, for which the extra dimension size remains constant. Our general solution shows that this is an unstable stationary configuration. A slight deviation will make the extra dimension size shrink or grow. This is a generic consequence of gravity theory, and there is a way to overcome it by including extra dynamics beyond simple gravity for the modulus b. This is a five-dimensional version of the modulus stabilization problem. Toward the solution of this problem, an attempt using the bulk scalar field was made recently in [13] and an earlier attempt within the compactified heterotic M theory without the bulk cosmological constant was done in [14] using the membrane instanton effects and the racetrack mechanism. We will not discuss it because it is beyond the scope of this paper.

With this in mind, let us look at the RS solution for the gauge hierarchy problem. In the static limit $k_1, -k_2 \to k$ of the inflationary solutions with fixed extra dimension size, we can see from Eq. (16) that the extra dimension size does not have a unique value but varies depending on how k_1 and $-k_2$ approach k. This is just what is said by the modulus stabilization problem. Near the static limit, the length of the extra dimension is expressed in terms of the Hubble parameters of two branes by

$$kL_{\mathrm{RS}}=\frac{1}{2}kb_{\mathrm{RS}}\simeq \ln\frac{H(\frac{1}{2})}{H(0)}. \tag{26}$$

Since $H(\frac{1}{2})$ is very small, to keep $\frac{1}{2}kb_{\mathrm{RS}}$ of order 1, we also have to adjust the bulk and the hidden brane cosmological constants to the same accuracy as we did for the bulk and visible brane cosmological constants. The RS solution for the gauge hierarchy problem demands $\frac{1}{2}kb_{\mathrm{RS}}\sim 37$, and this number seems quite moderate at first sight. However, in fact, it requires further fine-tuning of the hidden brane Hubble parameter than the visible one:

$$H(0)=\sqrt{k_1^2-k^2}\simeq 10^{-16}H\left(\frac{1}{2}\right)\lesssim 10^{-76}M_{\mathrm{P}}. \tag{27}$$

Thus, to solve the gauge hierarchy problem in the context of the RS model, the balance between the bulk and the hidden brane cosmological constants should take place with 10^{16} times more accuracy than that between the bulk and the visible cosmological constants. At any rate, the RS model again converts the gauge hierarchy to the fine-tuning of the bulk and the brane cosmological constants. It will be interesting to look at the problem in terms of four-dimensional effective theory and see whether there is a nice stabilization mechanism by which the above argument can be avoided using additional interactions of the b modulus other than the gravity.

In summary, we found the general inflationary solutions for the slab of five-dimensional AdS spacetime with two boundary 3-branes and viewed the RS model as the static limit of those solutions. Then both the cosmological constant problem and the gauge hierarchy problem are recast into the fine-tuning problem of the bulk and the brane cosmological constants. The cosmological constant problem appears as the fine-tuning between the bulk and the visible brane cosmological constants, and the gauge hierarchy problem as the more severe fine-tuning between the bulk and the hidden brane cosmological constants. The inclusion of matter at the brane boundaries might alter the above conclusion, but we expect that it does by not much. The real solution of those problems surely requires ingredients addition to the five-dimensional model considered in this paper, but the implications of this simple model are so interesting that it deserves further study. The magic bullet for the cosmological constant problem and the gauge hierarchy problem may be found in the correlation mechanism of the bulk and brane cosmological constants, which reminds us of the recent development in string theory, holography. We also leave the modulus stabilization problem which is connected to the problem addressed in this paper for future work.

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